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(revised version)

Singularity Theorems with an emphasis on the interplay of low regularity and energy conditions

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Abstract

The presented thesis aims to give a biased overview of the singularity theorems in general relativity. It will emphasise in particular the interplay between low regularity spacetimes (that is for us mostly C^1) and the formulation of physically reasonable energy conditions. Nevertheless it tries to give a mostly self-contained proof for the classical singularity theorems by R.Penrose and S.W.Hawking. Those results will then lie the foundation on which we will explore some possible generalizations. While generalizations of the energy conditions (here we will mainly follow [1]) can be formulated quite directly, to handle low regularity we will need to introduce new tools. In particular we will give a brief overview of distributions on manifolds which allows us to construct the Ricci-tensor even in C^1-spacetimes. This distributional language will then provide a natural formulation of some important prior discussed energy conditions. Finally we aim to formulate a C^1-singularity theorem assuming only a weakened distributional strong energy condition.

Abstract

Die vorliegenden Arbeit verfolgt das Ziel, einen eher parteiischen Überblick der Singularitäten Theoreme in der Allgemeinen Relativitätstheorie zu geben. Sie wird ihr Interesse vor allem auf das Zusammenspiel von niedriger Regularität der Metrik (damit ist meistens C^1-Regularität gemeint) und der Formulierung von physikalisch sinnvollen Energie-Bedingungen richten. Nichtsdestotrotz wird versucht, eine in sich geschlossene Darstellung der klassichen Singularitäten Theoreme von R.Penrose und S.W.Hawking zu erhalten. Diese Resultate werden dann als Grundlage für die Behandlung von möglichen Erweiterungen der Theoreme dienen. Während sich Verallgemeinerungen der Energie-Bedingungen (hier werde wir vor allem [1] folgen) ohne viel zusätzlichen Aufwand formulieren lassen, benötigen wir für das Behandeln von niedriger Regularität neue Werkzeuge. Insbesondere werden wir eine kurze Einführung in die Theorie der Distributionen auf Mannigfaltigkeiten geben, die es uns dann erlauben wird die Ricci-Krümmung auch bei niedriger Regularität zu definieren (hier folgen wir im Besonderen [2]). Es wird sich herausstellen , dass die Sprache der Distributionen eine natürliche Formulierung einiger der Energie-Bedingungen erlaubt. Schlussendlich wird versucht ein C^1-Singularitäten Theorem unter Anahme einer abgeschwächten distributionellen Energie-Bedingung zu formulieren.

Contents

1 Why do we need Singularity Theorems?

It is a part of human nature to ask for the past and reasons of our present existence. It is thus no surprise that the past of everything, that is the history of our universe always stimulated philosophical and scienti c endeavours. Only in the last century the theory of General Relativity gave us a new language in which to formulate the question. It is by now of popular belief that our universe started in a Big Bang singularity. Indeed, one argument for the above, is that the assumption of spatial homogeneity and isotropy leads to FLRW-Models which predict a Singularity under appropriate energy conditions. However the validity of this arguments su ers form the fact, that our universe for sure does not obey these exact symmetry assumptions. Here the Singularity theorems, which are often referred to as the rst genuine post-Einsteinian result in general relativity [1], take their role. Through formulating general conditions which necessarily lead to singularities they break loose from symmetry conditions associated to the speci c problem. In fact, the theorems themselves are formulated within the general language of spacetime as a Lorentzian Manifold and therefore do not even depend explicitly on the Einstein equations (thus they also apply to modi ed theories of gravity as Brans Dicke theory). Singularities, as they are believed to occur not only at the Big Bang but also when describing the collapse of massive stars, are generally believed to point at a breakdown of the theory which predicted them. Indeed singularities may be (depending on the exact de nition of a singularity) accompanied by strong gravitational elds and small radii of curvature. Here quantum e ects should become important and therefore demonstrate the need for a consistent theory of quantum gravity. The very general framework, given by the Singularity theorems thereby helps to identify the character of spacetime which leads to singularities and has to be xed in a di erent more general theory.
In fact this framework may be made precise by formulating the so called Pattern singularity theorem introduced by J.M.M.Senovilla [3]. It aims to summarize the general structure underlying most of the modern singularity theorems:

Theorem. (Pattern singularity theorem, see ([3], Theorem 6.1))
If our spacetime satis es :

(1) a causality condition
(2) an energy condition
(3) a boundary or initial condition

then it contains at least an incomplete causal geodesics.

Inspired by this general structure the following thesis will rst introduce the theory of causality, which thus enables us to express and understand condition (1). As we will see this leads us quite naturally to the study of the maximality of geodesics, which then again forces us to de ne and discuss energy conditions in detail. Here we will introduce generalizations of the standard energy inequalities, which were proposed in [1] by C.Fewster and E.A.Kontou. The boundary and initial conditions will be brie y established which thus nally prepares us to formulate the classical singularity theorems by R.Penrose and S.W.Hawking. Though 'secretly' we are changing the original formulation of the theorems such that various prior results concerning the generalized energy conditions may be applied directly. We thus will simultaneously have demonstrated the singularity theorems with weakened energy conditions as presented in [1]. In the background we will always concern the question for an appropriate regularity of the metric. Already in Sec.3.3 we will see that most of the desired global features of causality remain true in C^1- regularity. In fact it is mainly the energy condition not the causality or boundary conditions which will cause us problems when trying to formulate a C^1- singularity theorem. In Sec.6 we will devote our full attention to this problem . After shortly introducing the theory of distributions on manifolds, we use this new tool to deriveC^1-versions of Hawking's singularity theorems as presented in [2] by M.Graf. Since the language of distributions provides us with a natural framework in which to formulate energy conditions in C^1-regularity, we will proceed by trying to motivate a distributional version of the generalized energy conditions. At this point we will have proven singularity theorems of either low regularity or weakened energy conditions. In conclusion we shall therefore try to formulate a C^1- singularity theorem which to a certain extent incorporates both generalizations.

[1] https://ui.adsabs.harvard.edu/abs/2015CQGra..32l4008S/abstract

2 The Arena

As described in Sec.1 general relativity gave us a new language to speak about space and time. Since all the results discussed in this thesis are written in this language, an outline of the general framework will follow.
The basic structure under consideration is a smooth n-dimensional connected manifold ($n \geq 2$), denoted by M.

Definition 2.1. (Smooth manifold) A hausdorff, second countable topological space M is a smooth manifold if there exists a maximal Atlas $\mathcal{A} = \{(V_\alpha, \varphi_\alpha) : \alpha \in A\}$ of charts, with $V_\alpha \subseteq M$ and homeomorphisms $\varphi_\alpha : V_\alpha \to \mathbb{R}^n$ such that :

> $\forall \alpha, \beta \in A$ with $V_\alpha \cap V_\beta \neq \emptyset$: $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(V_\alpha \cap V_\beta) \to \varphi_\beta(V_\alpha \cap V_\beta)$ is a C^∞-diffeomorphism of open subsets in $\mathbb{R}^n$

For two manifolds, M with maximal atlas $\mathcal{A}$ and N with maximal atlas $\mathcal{B}$, we say $F : M \to N$ is C^k $:\Leftrightarrow$ for all $p \in M$ and for all charts (V, φ) in $\mathcal{A}$ at p and (V', φ') in $\mathcal{B}$ at $F(p)$ there exists an open neighborhood U of p, $U \subseteq V$ such that : $F(U) \subseteq V'$ and $\varphi' \circ F \circ \varphi^{-1} : \varphi(U) \to \varphi'(V')$ is C^k.

Definition 2.2. (submanifold) Let M be a smooth manifold. A subset $S \subseteq M$ is a s-dimensional C^k-submanifold of M if:

> $\forall p \in S$ there exists a C^k-map with C^k inverse (that is a C^k-diffeomorphism between manifolds) $\varphi : U \to \mathbb{R}^n$ with U an open neighbourhood of p and : $\varphi(S \cap U) = \varphi(U) \cap (\mathbb{R}^s \times 0^{s-n})$.

We will in the following denote a C^∞-submanifold just as a submanifold and add the suffix C^k- only at lower regularity.

In the end our model should locally approximate special relativity which is formulated in the context of vector spaces. It is therefore necessary to have some kind of local vector-space structure on top of our manifold which may be viewed in the simplest case as a linear approximation of the manifold at each point. We then impose properties on each of these fibers which will always be motivated by the case of special relativity. Starting with the tangent bundle, we will then generalize this concept to its dual, tensor products of it and finally to arbitrary vector bundles. What follows can be viewed only as reminder and short summary of notations used throughout the thesis. For a more rigorous and general approach I refer to [4].

Definition 2.3. (tangent space)

> First we define $C_p^\infty M := \{f \mid f : U \to \mathbb{R}$ smooth $\mid p \in U \subseteq M$, U open$\}/\sim$ where $f_1 \sim f_2 :\Leftrightarrow \exists\, U_{12} \subseteq U_1 \cap U_2$ such that $f_{1|U_{12}} = f_{2|U_{12}}$.
> $C_p^\infty M$ is naturally an $\mathbb{R}$-Algebra.
>
> Now we define the tangent space $T_pM := \{D : C_p^\infty M \to \mathbb{R} \mid$ such that $D([f_1][f_2]) = D([f_{1|U_{12}} f_{2|U_{12}}]) = f_1(p)D([f_2]) + f_2(p)D([f_1])\}$. In fact T_pM is a n-dimensional real subspace of the dual $(C_p^\infty M)^*$.

For $(V;\varphi)$ a chart as in Def. 2.1 with $p \in V$ we have $\partial_i|_p \in T_pM$ defined by $\partial_i|_p(f) := \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}\Big|_{\varphi(p)}$ for all $0 \leq i \leq n-1$

For $\gamma : I \subseteq \mathbb{R} \to M$ a piecewise C^1 we have $\dot\gamma(t_0) \in T_{\gamma(t_0)}M$ defined by $\dot\gamma(t_0)(f) := \frac{d(f \circ \gamma)}{dt}\Big|_{t_0}$

Definition 2.4. (tangent bundle) We call $\pi : TM \to M$ such that $\pi(v_p) = p$ the tangent bundle of M where $TM = \bigsqcup T_pM$ is by itself a smooth manifold. For $(V;\varphi)$ a chart as in Def. 2.1 we have $\psi : \pi^{-1}(V) \to V \times \mathbb{R}^n$ given by $\psi(v_p = v^i \partial_i|_p) := (p; (v_p)^i)$ as trivializations.

In the above definition just like in the remaining of this thesis the Einstein sum-notation has been used. Based on the above we are able to define the dual-tangent-bundle which subsequently will lead us to tensor-bundles.

Definition 2.5. (dual-tangent-bundle) Analogously we call $\pi^* : T^*M \to M$ the dual bundle such that $\pi^*(\omega_p) = p$ and $T^*M = \bigsqcup T_p^*M$ being itself a smooth manifold. For $(V;\varphi)$ a chart we have $\psi^* : (\pi^*)^{-1}(V) \to V \times \mathbb{R}^n$ given by $\psi^*(\omega_p = w_i\, dx^i|_p) := (p; (w_p)_i)$ as trivializations.

Definition 2.6. (tensor-bundle) We call $\pi^{(r,s)} : T^{(r,s)}M \to M$ with $T^{(r,s)}M = \bigsqcup T_pM^{\otimes r} \otimes T_p^*M^{\otimes s}$, again a smooth manifold, the (r,s) tensor bundle. We have $\psi^{(r,s)} : (\pi^{(r,s)})^{-1}(V) \to V \times \mathbb{R}^{n(r+s)}$ given by $\psi^{(r,s)}(T_p = T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} \partial_{i_1} \otimes \dots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}|_p) := (p; (T_p)^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s})$ as trivializations.

To define a structure on each tangent space one often imposes the existence of a certain tensor which then can act on all tangent vectors in a sense as a measurement device. Anticipating that every tangent space, at every point of our manifold will need such a structure, which furthermore should vary in a regular way, leads directly to the concept of tensor fields.

Definition 2.7. (tensor fields) We call $T \in \Gamma^k(T^{(r,s)}M) := \{\tilde T : M \to T^{(r,s)}M \mid C^k \text{ and } \pi^{(r,s)} \circ \tilde T = id_M\}$ a section of the tensor bundle or a $(r;s)_k$-tensor field on M.

Remark 2.8. Importantly we have an Isomorphism of $C^\infty(M)$-modules :

$$\Gamma(T^{(1;s)}M) \cong \text{Mult}(\Gamma(TM); \dots; \Gamma(TM); \Gamma(TM)) \qquad (1)$$

where $\text{Mult}(\Gamma(TM); \dots; \Gamma(TM); \Gamma(TM))$ describes the $C^\infty(M)$-multilinear forms $\Gamma(TM)^s \to \Gamma(TM)$.

As we mentioned earlier, we aim in general to require only low regularities for our tensor fields. Though as we will see later this restriction destroys some essential techniques of causality. The method used to avoid these problems, will be to approximate low regularity tensor fields with C^∞ fields. Hence we also need a concept for convergence of tensor fields (discussed in more detail in Sec.6.1.1).

Definition 2.9. (metric on tensor fields restricted to a compact subset) For $T \in \Gamma^0(T^{(r,s)}M)$ and $A \subset M$ compact, we define :

$$\|T\|_{\infty,A} := \sup\{|T|_x(\omega_1,\dots,\omega_r,X_1,\dots,X_s)| : x \in A, \omega_i \in T^*_xM, \|\omega\|_h = 1, \tag{2}$$
$$X_j \in T_xM, \|X_j\|_h = 1\} \tag{3}$$

Here h denotes a complete Riemannian backround metric (see Sec.3.2). We are now able to define locally uniform convergence of tensor fields:

Definition 2.10. (locally uniform convergence of tensor fields) We define locally uniform convergence that is convergence C^0_{loc} by convergence with respect to $\|.\|_{\infty,A}$ for all compact $A \subset M$.

It is often convenient to express the convergence in coordinates. In fact locally convergence of a net $\{T_\alpha\}$ of (r,s) tensor fields to T in C^0_{loc} is equivalent to the convergence of $(T_\alpha)^{i_1,\dots,i_r}_{j_1,\dots,j_s}$ to $(T)^{i_1,\dots,i_r}_{j_1,\dots,j_s}$ uniformly on $V \cap A$ for every chart (V,φ) and compact set $A \subset M$. Thus C^0_{loc} convergence independent of our choice for h.
The riemannian backround metric h furthermore induces an unique connection which is symmetric and metric (see Sec.3.2); the Levi-Cevita connection. With it we can naturally define convergence in C^k_{loc}.

Definition 2.11. (C^k_{loc} convergence) Let $\{T_\alpha\}$ be a net of $(r,s)_k$ tensor fields. We say $T_\alpha \to T$ in C^k_{loc} if $({}^h\nabla^i)T_\alpha \to ({}^h\nabla^i)T$ in C^0_{loc} for $i = 0,\dots,k$.

Similarly to above we can characterize the C^k_{loc} convergence equivalently in coordinates by the local uniform convergence of $\partial_k(T_\alpha)^{i_1,\dots,i_r}_{j_1,\dots,j_s} \to \partial_k(T)^{i_1,\dots,i_r}_{j_1,\dots,j_s}$ for all charts (V,φ): Thus convergence in C^k_{loc} is independent of the connection forms and therefore independent of a choice for h.

All of the above structures can be viewed as special cases of vector bundles which can be defined as:

Definition 2.12. (vector bundle) A real vector bundle (E,M,π) of rank $k \in \mathbb{N}$ over a smooth manifold of dimension M is defined as :

1. A manifold E and a surjective smooth map $\pi : E \to M$ such that $d\pi_p : T_pE \to T_{\pi(p)}M$ defined as $d\pi_p(v)(f) := v(f \circ \pi)$ is surjective
2. every fiber $E_p := \pi^{-1}(p)$ is a k-dimensional real vector space
3. there exists an open cover $\{U_i\}_{i \in I}$ of M and a family of trivializations $\{\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k\}$ that is smooth dieomorphisms such that $\varphi_i(E_p) = \{p\} \times \mathbb{R}^k$ and $\varphi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a linear Isomorphism.

Remark 2.13. For two trivializations φ_1 and φ_2 of the same vector bundle (E,M,π) it follows from the definition above that there exists a smooth map called transition function $A_{12} : (U_1 \cap U_2) \to GL_k(\mathbb{R})$ such that:

$$\varphi_2 \circ \varphi_1^{-1}(p,v) = (p, A_{12}(p) \cdot v) \tag{4}$$

Similarly as we did for tensor-bundles we can define sections of any general vector bundle.

Definition 2.14. (sections of vector bundles) We call $\sigma \in \Gamma^k(M;E) := \{\sigma' : M \to E \mid C^k \text{ and } \pi \circ \sigma' = id_M\}$ a C^k-section of E. If $k = \infty$ we will just write $\Gamma(M;E)$.

Also the prior definition of local uniform convergence may be generalised to arbitrary vector bundles. This will be discussed explicitly in 6.1.1.

These rather numerous definitions motivate us now to start our voyage into causality by first studying the geometry of the vector spaces T_pM at each $p \in M$. In Sec.3.2 we will then discuss, in how far our hope of locally transferring properties onto the manifold will be fulfilled.

3 Causality

3.1 Lorentz vector spaces

This section follows in its essence Chapter 5 of [5], to which we refer the reader for a more detailed analysis.

Definition 3.1. (Lorentz vector space) We define a Lorentz vector space V as a vector space equipped with a non-degenerate $(0;2)$ tensor $g_V \in V^* \otimes V^*$ of index 1. Here index 1 describes the maximal dimension of a subspace where g_V is negative definite.

The choice of $index(g_V) = 1$, can be described equivalently with the requirement that for any orthonormal basis the number of negative signs in the representation matrix of g_V is one. In special relativity, where $V = \mathbb{R}^4 = T_p\mathbb{R}^4$ and $g_V = \eta = diag(-1;1;1;1)$, this is clearly fulfilled. The above definition therefore represents a natural extension to general vector spaces. A central idea of special relativity has been to fix the speed of light in vacuum to a universal finite value. In fact the universality of this constant can be used in combination with homogeneity of Minkowski space to derive special relativity from the ground[2]. In special relativity this can be expressed as $\eta(\dot{\gamma}, \dot{\gamma}) = 0$ for every world-line γ and 4-velocity $\dot{\gamma}(t) \in T_{\gamma(t)}M$ describing a massless particle. If an arbitrary world-line γ fulfills $\eta(\dot{\gamma}, \dot{\gamma}) \leq 0$ it is called causal, hinting at slower than light travel and restricting the speed of causality to a fixed constant. Keeping in mind that T_pM will be the vector space of later interest, we are motivated to make the following definition.

[2] Though the Lorentz transformations are also derivable from the absence of preferred reference systems, the group structure of them and using Maxwell's equations to fix the only remaining free parameter c to speed of light (cf. [6]). Either way the motion of particles with positive mass is thus restricted by this universal constant.

Definition 3.2. (Causal character of vectors) In a Lorentz vector space V we say $v \in V$ is :

causal if	$g_V(v,v) \leq 0$ and $v \neq 0$
timelike if	$g_V(v,v) < 0$
null if	$g_V(v,v) = 0$ and $v \neq 0$
spacelike if	$g_V(v,v) > 0$ or $v = 0$

This can be further generalized to subspaces of Lorentz vector spaces.

Definition 3.3. (Causal character of subspaces) We call a subspace $W \subset V$:
spacelike if : g_W is positive definite
timelike if : g_W is non degenerate of index 1
null or lightlike if : g_W is degenerate

For later convenience we will need two further lemmas given in [5] to characterize timelike and lightlike subspaces in a more practical useful way.

Lemma 3.4. For a subspace $W \subset V$ of a Lorentz vector space of dimension $m \geq 2$ the following three statements equivalently characterize its causal character:

(1) W is timelike.
(2) W contains two linearly independent null vectors.
(3) W contains a timelike vector.

Proof. (1) $\Rightarrow$ (2): For an arbitrary orthonormal basis $(e_0, e_1, \ldots, e_{m-1})$ of W with e_0 being timelike we get $e_0 \pm e_1$ as two linearly independent null vectors.
(2) $\Rightarrow$ (3): Let u, v be two independent null vectors. Furthermore fix $(e_0, e_1, \ldots, e_{n-1})$ an orthonormal basis of V such that e_0 is timelike for the remaining proof. If $g_V(u,v) = -u^0v^0 + u^iv_i = 0$ (Einstein summation, i=1,...n) it would follow that $u^0v^0 = u^iv_i$. But since u, v are both null vectors and thus $u^0v^0 = (u^iu_i)^{\frac{1}{2}}(v^iv_i)^{\frac{1}{2}} \geq u^iv_i$ it follows from the Cauchy Schwarz inequality and again using that both vectors are null, that u and v would have to be linear dependent. Therefore $g_V(u,v) \neq 0$ which implies that either $u + v$ or $u - v$ is timelike.
(3) $\Rightarrow$ (1): Let z be a timelike vector in W implying $z^0 > (z^iz_i)^{\frac{1}{2}}$. Let $u \in V$ be causal. Then $u^0 \geq (u^iu_i)^{\frac{1}{2}}$ and therefore again with the Cauchy Schwarz inequality $u^0z^0 > (u^iu_i)^{\frac{1}{2}}(z^iz_i)^{\frac{1}{2}} \geq u^iz_i$. This implies that $z^\perp = \{u \in V \mid g_V(u,z) = 0\}$ is spacelike and therefore also $W^\perp \subset z^\perp$. Analogously follows for a spacelike subspace that its orthogonal complement is timelike. A classical calculation for general scalar products of arbitrary index shows that $W = (W^\perp)^\perp$ which in our case is therefore timelike. □

With the help of Lemma 3.4 one can readily proof (cf. [5] 5/Lemma28):

Lemma 3.5. For a subspace $W \subset V$ of a Lorentz vector space the following three statements equivalently characterize its causal character: (1) W is lightlike.
(2) W contains a null vector but not a timelike vector.

(3) $W \cap \mathcal{N} = L \setminus 0$, where L is a one dimensional subspace and $\mathcal{N}$ the subspace consisting of all nullvectors in V.

The concept of causality, philosophical and physical has been used hand in hand with notions of future and past. We will therefore proceed by defining timecones in Lorentz vector spaces. In special relativity they can be viewed as representing the causal future and past. In Sec.3.2 we will see in how far this concept can be preserved in our general setting of a Lorentzian manifold.

Definition 3.6. (timecone, see Fig.1) Let T be the set of all timelike vectors in a Lorentz vector space V. We define $C(u) := \{v \in T \mid g_V(u; v) < 0\}$, the timecone of V containing $u \in T$.

Figure 1: (Timecone in Minkowski space, Fig.17.11/ [7]): The timelike tangent vector corresponds to u and T is the interior of the upper cone.

There are essentially three properties which manifest the importance of timecones. First we notice that timecones are convex, since for $v; w \in C(u)$ and $a; b > 0$ $av + bw \in C(u)$. The second property helps us to better understand the condition for timelike vectors to be in the same cone. In fact the following lemma shows, that being in the same timecone actually defines a partition of T.

Lemma 3.7. The relation $u \sim v :\iff g_V(u; v) < 0$ is an equivalence relation on T. Therefore: $u \in C(v) \iff v \in C(u) \iff C(u) = C(v)$.

Proof. That $\sim$ is reflexive and symmetric follows directly from its definition. For transitivity let $u \sim v$ and $v \sim w$. We have to show $u \sim w$ that is $g_V(u; w) < 0$.

The definition of timecones does not depend on scaling with positive constants, which is why we can without loss of generality assume that $g_V(v,v) = -1$. Since $g_V(u,v) < 0$ that is $u \notin v^{\perp}$ we can write $u = av + x$ with $a > 0$ $x \in v^{\perp}$. Analogously we can write $w = bv + y$ with $b > 0$ and $y \in v^{\perp}$. Hence $g_V(u,w) = -ab + g_V(x,y)$. From $g_V|_{v^{\perp}}$ being positive definite it follows by using the Cauchy Schwarz inequality, that $|g_V(x,y)| \leq \|x\|\|y\| < ab$. Where the last inequality follows from u,w being timelike. Therefore $g_V(u,w) < 0$ which we aimed to prove. □

We therefore have constructed a partition of $\mathcal{T}$ into future and past pointing vectors. We have already mentioned, that for a timelike vector $v \in \mathcal{T}$: $v^{\perp}$ is spacelike. It thus follows that for an arbitrary causal vector u : $g_V(u,v) \neq 0$. The above proof thus demonstrates simultaneously that any timelike vector $v \in \mathcal{T}$ partitions the set of all causal vectors $\overline{\mathcal{T}}$ into $\overline{C(v)}$ and $\overline{C(-v)}$. This will be crucial for later defining causality on our Lorentzian manifold.
The last property, demsontrates the counterintuitive nature of Lorentzian geometry. Indeed it can be interpreted as the widely known 'Twin-Paradox' in special relativity.

Lemma 3.8. For $v,w \in \mathcal{T}$, both in the same timecone we have: $|v| + |w| \leq |v + w|$ where for $u \in \mathcal{T}$ we define $|u| := (-g_V(u,u))^{\frac{1}{2}}$.

In Sec.3.2 we will prove this in a more general setting.

3.2 Local Lorentzian Causality

We repeatedly emphasized the importance of general relativity as a new language of spacetime. Though not everyone would agree with this preferenced status of space and time. It is often stated, that causality may be even more fundamental than space and time. Indeed in [8] it is for example proven, that the class of curves on which massive particles can move, that is timelike curves, already determine the topology of spacetime. Furthermore spacetime[3] singularities, which is the topic of our interest in this thesis, often occur when the causal future of some region in spacetime is 'trapped'. It is thus no surprise that most parts of the singularity theorems actually are causality theorems. Having discussed briefly Lorentz vector spaces we are now motivated to analyse which properties of causality can be preserved when changing our setting to a manifold. Though not any manifold: as indicated earlier we want to impose a Lorentz vector space structure on each T_pM. Put in a more rigorous way:

Definition 3.9. (Lorentzian metric) We call a symmetric non-degenerate $(0,2)$ tensor field with index 1 : $g \in \Gamma^k(T^{(0,2)}M) = \{g : M \to T^{(0,2)}M \mid C^k$ and $\pi^{(r,s)} \circ g = \mathrm{id}_M\}$ a Lorentzian metric on M.

Remark 3.10. For simplicity we will write $g \in C^k$ for $g \in \Gamma^k(T^{(0,2)}M)$ as above. Also we will call a spacetime (M,g) a C^k-spacetime if $g \in C^k$.

Up until Sec.6 the metric is assumed to be C^1. For later purposes already touched upon in Sec.1 we will nevertheless aim to prove theorems, if not directly possible for the low-regularity case, by emphasizing those parts which need to be fixed using the methods we will introduce in Sec.6. In fact most of the global features of causality will be maintained via approximations by smooth metrics and therefore are direct consequences of the theorems proven in the C^1-case. In the following our spacetime model satisfying the above properties will be represented by the tuple $(M; g)$. For later purposes we also want to fix a complete smooth Riemmanian backround (index 0) metric denoted by h. Indeed [9] shows, there always exists a Riemannian metric h, such that $(M; h)$ is complete (geodesics as in Def. 3.11 are defined on $\mathbb{R}$). It is interesting to notice, that contrary to the above, the existence of a smooth Lorentzian metric by itself already restricts our possible choices for spacetimes through topological properties. Explicitly Prop.37 in Chapter 5 of [5] tells us:

Proposition. For a smooth manifod M the following are equivalent :

1. There exists a Lorentzian metric on M
2. There exists a time-orientable (cf.Def. 3.13) Lorentzian metric on M.
3. There is a nonvanishing vector field on M.
4. Either M is noncompact, or M is compact and has Euler number $\chi(M) = 0$

By the fundamental Theorem of Pseudo-Riemannian geometry there exists a unique affine connection [4](called Levi-cevita connection) $\nabla : \Gamma^1(TM) \times \Gamma^1(TM) \to \Gamma^0(TM)$ which is $C^1(M)$-linear in the first argument, obeys the Leibniz rule in the second, is symmetric ($\nabla_X Y - \nabla_Y X = [X; Y]$) and metric ($\nabla g = 0$ for the induced connection on the tensor-bundle). With the connection induced by our Lorentzian structure we can naturally extend the notion of derivatives in special relativity, that is in Minkowski space, onto manifolds. At this point our spacetime model does not yet predict anything concrete. That is we never mentioned how exactly things move in spacetime. Surprisingly enough, establishing this also introduces the key ingredient to answer our question of describing the manifold locally by a fiber in the tangent bundle. As in classical (Newtonian) Physics we need an Axiom analogous to Newton's first law of motion. In general relativity it is often stated as:

Axiom. (Geodesic principle) Any freeling falling test point particles moves along a causal geodesic

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[3]When defining spacetime singularities one has to be rather cautious. In Sec.5 we will discuss this further.

[4]For $F : N \to M$ a C^1-map of manifolds we will use the notation ${}^F\nabla$ for the unique Pull-back covariant derivative on the Pull-back bundle such that ${}^F\nabla_u(\xi \circ F) = \nabla_{d_qFu}\xi$, where $u \in T_qN$ and $\xi \in \Gamma^1(TM)$ (see [4] Satz 9.15).

Definition 3.11. (Geodesic). For (M, g) a spacetime with g being at least C^1 we call a C^2 cuve $\gamma : I \to M$:

- a geodesic if $\nabla_{\dot\gamma}\dot\gamma \equiv \nabla_{\partial_t}\dot\gamma = 0$. We can write this in coordinates as $\ddot\gamma^k = -\Gamma^k_{ij}\dot\gamma^i\dot\gamma^j$, with $\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ the Christoffel symbols.

- a pre-geodesic if $\nabla_{\dot\gamma}\dot\gamma = \lambda\dot\gamma$ for $\lambda : I \to \mathbb{R}$ a contionous function. Every pre-geodesic can be parameterized such that it is a geodesic[5]

For later purposes we also need the definition of an extendible geodesic:

- an extendible geodesic, if γ is a geodesic such that there exists $J \supsetneq I$ and a geodesic $\tilde\gamma : J \to M$ with $\tilde\gamma|_I = \gamma$.

- an inextendible geodesic, if γ is geodesic, that is not extendible.

Remark 3.12. For $g \in C^1$ standard results of the theory of ordinary differential equations (Picard-Lindelöf) prove the existence of a unique geodesic on an inextendible Interval, for a given initial value and velocity, that is tangent vector. For $g \in C^1$ the situation changes. In fact the existence is still given by use of the Peano-Theorem, though in general the uniqueness ceases to be true.

We stated the geodesic principle as an axiom. Interestingly S.Weinberg showed in [11] the equivalence principle which lies at the heart of general relativity is already enough to imply the geodesic principle for test particles. Furthermore in [12] R.Geroch and J.O.Weatherall showed that for real particles described by a tensor distribution supported on a timelike curve (which represents its energy momentum tensor) the geodesic principle follows from local energy conservation and an energy condition. Even for extended bodies they could construct conditions which imply the geodesic motion of sufficiently small free bodies on a background spacetime (M, g). We will therefore quite confidently take the preceding axiom as given and interpret the results of this thesis always by reference to it. In Def. 3.6 we defined timecones. By use of Lemma 3.7 we are thus able to define a causal future and past for each T_pM by arbitrarily choosing one of the equivalence classes to be the future. Could this choice be made in a smooth way?

Definition 3.13. (time-orientable) We call a Lorentzian manifold time-orientable if there exists a timelike vector field $u \in \Gamma(TM)$: That is at every point we $p \in M$ we can smoothly choose the causal future to be $C^+(u(p))$.

The above definition allows us to define what it means for an observer to be travelling into the future, a crucial property for defining global causality. It is therefore natural to impose our Lorentzian manifold to be time-orientable. In

[5] cf. [10]Def.6.2/p.158

fact we can assume this to be true without loss of generality (cf. [5],7/Lemma 17) by instead studying $(\tilde{M}, \tilde{g})$ the time-orientable double cover with the pulled back metric.

Definition 3.14. (spacetime) We call a time-orientable Lorentzian manifold a spacetime. In the subsequent discussions let $X \in \Gamma(TM)$ always denote a time-orienting timelike vector field on a given spacetime (M, g).

Finally we are prepared to venture out trying to demonstrate our now often promised connection between properties of Lorentz vector spaces and the local structure of the Lorentzian manifold.

Definition 3.15. (exponential map) For a Lorentzian manifold M let D be the set of vectors $v \in T_pM, p \in M$ such that the inextendible geodesic γ_v ($\dot{\gamma}_v(0) = v$) is defined on $[0, 1]$. We define :
$\mathrm{Exp} : D \to M \times M$ as $\mathrm{Exp}(v_p) := (p, \gamma_v(1))$.

Remark 3.16. (1) D is open.
(2) It is often convenient to define for every $p \in M$: $\exp_p := \mathrm{Exp}|_{D \cap T_pM}$: $D \cap T_pM \to M$. In the literature $\exp_p$ is generally called the exponential map.

The mapping of a subspace of T_pM onto the manifold is done in such a way that straight lines running through the origin (which are geodesics in minkowski space (T_pM, g_p)) are mapped to geodesics in (M, g). A classical result of riemmanian geometry, which can be generalized directly to the Lorentzian case, is the Gauß Lemma. It heavily depends on the above property and proves that the exponential map ($\exp_p$) can be described as a partial (radial) isometry from the Lorentz vectorspace (T_pM, g_p), viewed as a manifold, onto (M, g).

Lemma 3.17. (Gauß Lemma) Let $p \in M$ and $0 \neq x \in D \cap T_pM$. For arbitrary $w_x \in T_x(T_pM)$: $g_{\exp_p(x)}(d\exp_p(x), d\exp_p(w_x)) = g_p(x, w_x)$.

Remark 3.18. In the formulation of Lemma 3.17 we secretly used that $T_x(T_pM) = T_pM$ which implies that x and w_x can be viewed as an element of either of them.

Proof. Since the Lemma is a standard result in differential geometry we refer to [13] (Hilfssatz 4.39) for the Riemmanian and [5] (Chapter 5/Lemma 1) for the analogous Lorentzian case. □

Until now, $\exp_p$ only gave a map from the tangent space into the manifold. The claim is that its actually a C^∞-diffeomorphism in a sufficiently small neighbourhood of $(0_p) \in D$. Indeed since $(d\exp_p)_{0_p} v = \frac{d(\exp_p(vt))}{dt}\big|_{t=0} = \frac{d(\gamma_v(t))}{dt}\big|_{t=0} = v$ for an arbitrary $v \in D \cap T_pM$ we have $(d\exp_p)_{0_p} = 1_{T_pM}$. Therefore the claim follows directly from the inverse function theorem. By further restricting the neighbourhood $\tilde{V}$ of 0_p, we can assume that $\tilde{V}$ is starshaped about 0_p. Furthermore we can choose an isomorphism $x : \tilde{V} \to U \subset \mathbb{R}^n$ such that the coordinate system of the chart $x \circ \exp_p^{-1} : V \to U$ is orthonormal at p. We call V a normal neighbourhood of $p \in M$. We can generalize the result readily to Exp.

Lemma 3.19. For every $x \in D \cap T_pM$ we have the following implication: $(d\exp_p)_x$ invertible $\Rightarrow$ $(d\mathrm{Exp})_x$ is invertible.

Proof. Let us assume, there exists a $v \in T_x(TM)$ such that $(d\mathrm{Exp})_x v = 0$. From the definition of Exp (3.15) we know that $\pi_1 \circ \mathrm{Exp} = \pi_{TM}$ (π_1 the projection onto the first factor). Hence $(d\pi_1)(d\mathrm{Exp})_x v = (d\pi_{TM})_x v = 0$. This implies that v is tangent to T_pM. Since $\mathrm{Exp}_{|T_pM} = \exp_p$ it would follow $0 = (d\mathrm{Exp})_x v = (d\mathrm{Exp}_{|T_pM})_x v = (d\exp_p)_x v$. □

Using the above lemma, applying again the inversion theorem and reducing the neighbourhood if needed (to maintain convexity) one can prove the following:

Lemma 3.20. Every point $p \in M$ has a convex neighbourhood. That is a neigbourhood Z such that for every $q \in Z$, Z is a normal neighbourhood of q.

Proof. We sketched the proof above. More details can be found in [5] chapter 5/Proposition 7. □

The same reason, because of which normal neighbourhoods are of such technical importance also manifests their interpretation. The following lemma demonstrates, that normal neighbourhoods can be interpreted as freely falling reference frames. In fact it shows that in such coordinate systems first order effects due to a varying metric tensor g vanish along freely falling observers.

Lemma 3.21. (further standard properties of normal neighbourhoods) Let V be a normal neighbourhood of $p \in M$. It follows in normal coordinates $x \equiv \exp_p^{-1} : V \to U$:

(1) The geodesic with initial tangent vector $v_p \in \tilde{V}$ has the coordinate expression $\gamma : x(\gamma_{v_p}) = t x(v_p)$:

(2) For every $v_p \in \tilde{V}$ we have: $\Gamma^k_{ij}(\exp_p(tv_p)) v_p^i v_p^j = 0$

(3) $\nabla_{\partial_i}\partial_j(p) = \Gamma^k_{ij}\,\partial_k(p) = 0_p$

Proof. (Sketch)

(1): The uniqueness and existence property of geodesics implies that :

$$\exp_p(vt) = \gamma_{vt}(1) = \gamma_v(t) \tag{5}$$

for all $t \in [0,1]$ and $v \in \tilde{V}$. This implies (1).

(2): Writing the geodesic equation with (1) in coordinates implies (2)

(3): Evaluating (2) for $t = 0$, using that $\tilde{V}$ is star shaped and $\Gamma^k_{ij} = \Gamma^k_{ji}$ since our connection is symmetric, (3) follows. □

The structure, underlying the connection between the Lorentz vector space T_pM and a neighbourhood of $p \in M$ is constructed. Now it is time, to analyze how this structure carries causality from T_pM into M. Causality in special relativity (minkowski space) is determined by the spacetime interval, that is the constant

Lorentz metric η. We are therefore inspired to study an analogous construct in normal neighbourhoods. That is we will examine how far, causality determined by a constant metric tensor can be preserved in normal neighbourhoods with a general varying metric. Later we will refine the concept of a spacetime interval, until finally arriving at a global 'time-seperation' function, which will be a crucial ingredient for proving the singularity theorems.

Definition 3.22. Let V be a normal neighbourhood of $p \in M$, such that $\exp_p : \tilde{V} \to V$ is a C^1-diffeomorphism. We define :
(1) $\tilde{q} : \tilde{V} \to \mathbb{R}$ by $\tilde{q}(v) := g_p(v; v)$:
This leads to a locally defined function on M :
(2) $q : V \to \mathbb{R}$, $q := \tilde{q} \circ \exp_p^{-1}$.

Definition 3.23. We also define the radial postition vector field P on a normal neighbourhood by :
(1) $\tilde{P}(v) := v \in T_v(T_pM)$ for all $v \in \tilde{V}$
(2) $P(\exp_p(v)) := (d\exp_p)_v(\tilde{P}(v)) \in T_{\exp_p(v)}M$.

Definition 3.23 is useful since using the gauß lemma one can show that from $\mathrm{grad}(\tilde{q}) = 2\tilde{P}$ it follows that $\mathrm{grad}(q) = 2P$. This will be used in the following fundamental lemma. But first we need to define the objects we want to study:

Definition 3.24. Let $\gamma : I \to M$ be a piecewise C^2 curve (: $C^{2;pc}$).
We call γ causal and future directed (fd) : $\Longleftrightarrow$
(1) For all $t \in I$: $\dot{\gamma}(t)$ is causal in $T_{\gamma(t)}M$ and for a prior chosen future timelike vector field u : $g_{\gamma(t)}(\dot{\gamma}(t); u(\gamma(t))) < 0$ where γ is smooth.
(2) At each break $t_i \in I$ we have $g_{\gamma(t)}(\dot{\gamma}(t_i^-); \dot{\gamma}(t_i^+)) < 0$.

We call it timelike-fd if it is causal-fd and $\dot{\gamma}(t)$ is timelike in $T_{\gamma(t)}M$.

Lemma 3.25. Let V as before denote a normal neighbpurhood of $p \in M$. Furthermore let $\tilde{\gamma} : [0; b] \to \tilde{V}$ be a piecewise C^2 curve such that $\gamma := \exp_p \circ \tilde{\gamma}$ is causal ,future- directed and starts at p. We now have the following characterisation, by studying the timecones of T_pM : The curve $\tilde{\gamma}$ will stay inside
(1) $\overline{C(u_p)}$ for all $t \in I$.
(2) $C(u_p)$ from the point on γ is not a null geodesic.
(3) $\partial C(u_p) \Longleftrightarrow \gamma$ is a null (pre-)geodesic

Proof. (cf. [3] Prop. 2.1.)

The strategy is the following :

(I) For γ initially timelike : $\tilde{\gamma} \subseteq C(u_p)$.
(II) If $\tilde{\gamma}$ enters $C(u_p)$ at one point it cannot leave it anymore.
(III) For γ causal : $\tilde{\gamma} \subseteq \overline{C(u_p)}$
(IV) γ null (pre-)geodesic $\Longleftrightarrow \tilde{\gamma} \subseteq \partial C(u_p)$.

This would prove (1) and (3) directly. Also (2) follows, since from the point on is not a null (pre-)geodesic anymore, (3) implies it can not stay on $@C(u_p)$ after that point. Thus because of (1) it has to enter and because of (II) never leaves $C(u_p)$.

(I) :
It has to be proven that for all t 2 $(0; b]$ the following inequality holds:

$$(g_p)_{ij} \quad {}^i(t) \quad {}^j(t) < 0: \qquad (6)$$

To prove that indeed is contained in $C(u_p)$ and not in the opposite timecone $C(\ \ u)$ one then only has to show that it is initially in $C(u_p)$ (that is for some arbitrary $t > 0$).
Based on our assumption that is initially timelike we have:

$$g_p(\_(0);\ \_(0)) = (\ g_p)_{ij} \ \_^i(0)\ \_^j(0) < 0 \qquad (7)$$

Now since $(0) = 0_p$ we can write $\_(0) = \lim_{t!\ 0} \frac{(t)}{t}$ and therefore are allowed to assume that (6) holds on $(0;\ ]$. The same argument shows that $\_$ being in $C(u_p)$ implies that is initially in $C(u_p)$ too.
Our rst goal is to prove, that (6) actually holds until t_1, the rst break of .
Indeed using the gauÿ lemma 3.17 we have:

$$\frac{d(q\ \ \ )}{dt}(t) = g_p(\text{grad}(q);\ \_)(t) = 2\, g_p(P(\ \ );\ \_) = 2\, g_{\ (t)}(P;\ \_) \qquad (8)$$

which we can equivalently write as :

$$\frac{d(q\ \ \ )}{dt}(t) = 2(\ g_p)_{ij} \quad {}^i(t)\ \_^j(t) = 2(\ g_{\ (t)})_{ij} \quad {}^i(t)\ \_^j(t) \qquad (9)$$

Again using the gauÿ lemma it follows that for for all t with $q < 0$ and being future pointing at (t) :

$$0 > (g_p)_{ij} \quad {}^i(t) \quad {}^j(t) = (\ g_{\ (t)})_{ij} \quad {}^i(t) \quad {}^j(t) \qquad (10)$$

Therefore $d\exp_p\ (t)$ is timelike. If we remember now that $\_(t)$ ($\_^j(t)$ in normal coordinates) is causal and future pointing we conclude with (9) :

$$\frac{d(q\ \ \ )}{dt}(t) < 0 \qquad (11)$$

Thus $(q\ \ \ )$ is strictly decreasing when t 2 $(0;\ ]$ which implies again by (11) that (6) is true for $= t_1$.
Now we use point (2) in Def. 3.24 : Since

$$g_{\ (t)}(\_(t\ \ );\ \_(t^+)) < 0 \qquad (12)$$

we can conclude that stays in the same timecone at its breaks which therefore in turn implies (11) also for t . Thus by the same reasoning as before stays in $C(u_p)$ even after the break. Repeating this argument for all the breaks of we conclude that (6) is true for all t 2 $(0; b]$.

(II) :

This follows directly from the proof of (I). Indeed if enters $C(u_p)$ it has to stay there since (11) tells us that q becomes strictly monotonically decreasing from there on.

(III) :

For a causal curve we cannot use the argument above, since (7) does not hold. Since we also only want to show $\overline{C(u_p)}$ it is a quite natural approach to work with close approximations to . In fact our strategy now is to construct a net ( ) of timelike curves arbitrary close to . The simplest attempt to construct such curves is the following :

$$(t) := (t) + (t) \qquad (13)$$

$$(t) := \exp_p((t)) \qquad (14)$$

, being a piecewise C^2 curve in ∇.

We furthermore define: $\tilde{}(t) := d\exp_p \dot{}(t)$. Since finally our goal for is to be timelike we just calculate:

$$g\,(\dot{}\,;\dot{}\,) = g\,(d\exp_p(\dot{}(t) + \dot{}(t); d\exp_p(\dot{}(t) + \dot{}(t)) \qquad (15)$$

$$= g\,(\dot{}(t) + \tilde{}(t); \dot{}(t) + \tilde{}(t)) \qquad (16)$$

$$= g\,(\dot{}(t); \dot{}(t)) + 2\, g\,((\dot{}(t); \tilde{}(t)) + {}^2 g\,(\tilde{}(t); \tilde{}(t)) \qquad (17)$$

Since for $= 0$:

$$g\,(\dot{}\,;\dot{}\,)|_{=0} = g\,(\dot{}\,;\dot{}\,) \quad 0 \qquad (18)$$

, we only need to find a (t) such that :

$$\left.\frac{dg\,(\dot{}\,;\dot{}\,)}{d}\right|_{=0} < 0 \qquad (19)$$

Again a direct calculation shows in normal coordiantes:

$$\left.\frac{dg\,(\dot{}\,;\dot{}\,)}{d}\right|_{=0} = {}^k\partial_k((g\,)_{ij}\,)\,\dot{}^i\,\dot{}^j + 2(g\,)_{ij}\,\dot{}^i\,\dot{}^j \qquad (20)$$

The question arises now, what choice of could be sufficient to fulfill (19). For ! a (piecewise)C^2-future directed-timelike vector field on we would have : $2(g\,)_{ij}\,\dot{}^i\,!^j < 0$, since is future pointing causal. We therefore only need to fix the first term in (20). We can eliminate this term by demanding:

$$\dot{}^j = !^j \quad \frac{1}{2}(g\,)^{jl}\ {}^k\partial_k((g\,)_{lm}\,)\,\dot{}^m \quad j \quad (0) = 0 \qquad (21)$$

By standard results of ordinary di erential equations we can choose as the unique (piecewise) C^2 solution of (21). Thus (19) holds for our choice of approximations. This proves that $\overline{C(u_p)}$ since every neighbourhood of an arbitrary point on contains some part of (for small enough) which is contained in $C(u_p)$ due to (II).

(IV) :

(=)): The forward direction is proven rather directly. Every null geodesic starting at p, if a nely paramterized, can be written in normal coordinates as : $^i(t) = tv^i$, with v being a nullvector (= $_v$). Consequently, using that $@C(u_p) = f\, v\ 2\ V j\ g_p(v; v) = 0$ and $g_p(u; v) < 0 g$, the claim follows.

((=) : The condition $@C(u_p)$ can be expressed in normal coordinates as:

$$0 = (g_p)_{ij} \ ^i(t) \ ^j(t) = (g_{\ (t)})_{ij} \ ^i(t) \ ^j(t) \qquad (22)$$

As numerous times before in the second step, the gauÿ lemma came to help. Importantly (t) 6= 0 for t > 0 since, as we will prove in Theorem 3.35(III) the h-arclength of all causal curves contained in V is bounded. Thus we cannot have a closed causal curve inside V. It follows, that $dexp_p(\ )$ is null along . If we di erentiate the rst term of (22) we get :

$$(g_p)_{ij} \ ^i(t) \ \dot{}^j(t) = (g_{\ (t)})_{ij} \ ^i(t) \ \dot{}^j(t) = 0 \qquad (23)$$

If _ were timelike (23) could not hold. Therefore _ is null. Two null vectors which are orthogonal are necessary proportional to each other. Thus, there exists a $C^{2;pc}$ function (t) : [0; b] ! R ((0) = 0 ; (t) > 0 8t 2 (0; b]) such that:

$$^i(t) = (\ t) \ \dot{}^i(t) \qquad (24)$$

Di erentiating implies :

$$\dot{}^i(t) = \dot{}(t)\ \dot{}^i(t) + (\ t)\ \ddot{}^i(t) \ =) \ (\ t)\ \ddot{}^i(t) = \dot{}^i(t)(1 \ \dot{}(t)) \qquad (25)$$

Lemma 3.21 implies for t 2 (0; b] :

$$^k_{ij}(\ (t)) \ ^i(t) \ ^j(t) = 0 \qquad (26)$$

by using $v_p = (t)=t$ and then multiplying (2) of 3.21 by t^2.

Using (25) and (26) we therefore arrive at :

$$(r_{\_}\ \_)^k(t) = \ddot{}^k + \ ^k_{ij}(\ (t))\ \dot{}^i(t)\ \dot{}^j(t) \qquad (27)$$

$$= \ddot{}^k + \ ^2(t)\ ^k_{ij}(\ (t)) \ ^i(t) \ ^j(t) \qquad (28)$$

$$= \frac{1 \ \dot{}(t)}{(\ t)} \ \dot{}^k \qquad (29)$$

Thus $r_{\_}\ \_$ is proportional to _ (for t = 0 this follows by continuity) and therefore a (null) pre-geodesic.

□

After this rather tedious prove, we are now in a position to reap the fruits. We start by proving the promised generalization of Lemma 3.8. In the context of normal neighbourhoods (V of $p \in M$). Lemma 3.8 states that the radial timelike geodesic always has a longer length (measured in the Lorentz vector space) than piecewise geodesics connecting the same points p and $q \in C(u_p) \cap V$. We now want to genralize this to arbitrary piecewise smooth timelike curves and a length definition which takes the varying metric into account.

Definition 3.26. (length functional)
For an arbitrary causal piecewise C^2 curve $\gamma : I \to M$ we define the length of γ by :

$$L_g(\gamma) := \int_I |\dot\gamma(t)|_g dt = \int_I \sqrt{-g(\dot\gamma(t), \dot\gamma(t))}\, dt \qquad L_g(\gamma) \in \mathbb{R} \cup \infty \tag{30}$$

Lemma 3.27. (twin paradox)
Let V be a normal neighbourhood of $p \in M$. For $q \in \overline{C(u_p)}$ the radial geodesic curve is the unique (up to reparametrization) longest C^2-piecewise causal curve from p to q.

Proof. (cf. [5] 5/Prop.34 for the timelike case)

(I) γ timelike , $q \in C(u_p)$
Let γ be an arbitrary timelike curve from p to q. By Lemma 3.25 we know, that $\gamma \subset C(u_p)$, for $\gamma(t) = \exp_p(\sigma(t))$. We now use the position vector field P (Def. 3.23) on $C(u_p)$. By the gauß lemma (3.17) P is timelike and future pointing on $C(u_p)$. Also

$$g_x(P(x), P(x)) = g_p(x, x) \tag{31}$$

for all $x \in C(u_p)$. Thus we can construct a normalized future directed timelike vector field on $C(u_p)$ by :

$$U(x) := \frac{P(x)}{r(x)} \quad ; r(x) := |P(x)|_g \tag{32}$$

It follows that:

$$N := \dot\gamma + g(U, \dot\gamma)U \tag{33}$$

is orthogonal to U and therefore spacelike.
The length of γ is therefore given by :

$$L_g(\gamma) := \int_I |\dot\gamma(t)| dt = \int_I |N - g(U, \dot\gamma)U| dt = \int_I (g(U, \dot\gamma)^2 - g(N, N))^{\frac{1}{2}} dt \tag{34}$$

$$\geq \int_I (-g(U, \dot\gamma))\, dt = \int_I \frac{d(r \circ \gamma)}{dt'}(t)\, dt = r(q) \tag{35}$$

For the radial timelike geodesic : $L_g(\ \exp\ ^{1}(q)) = r(q)$. Hence $L_g(\ )$ $L_g(\ \exp\ ^{1}(q))$ for all timelike curves . Also if $L_g(\ ) = L_g(\ \exp\ ^{1}(q)) = r(q)$ it follows that $N = 0$. That is :

$$\dot{\ }^{i} = (\ t)\ ^{i} \quad jt > 0 \tag{36}$$

Analogous as in the proof of 3.25 it follows that is pre- geodesic and therefore a reparametrization of the radial geodesic. In fact one can show, that $(t) = (\frac{r(\ (t))}{r(q)})$ (cf. [5] 5/Lemma 14).

(II) causal $q\ 2\ C(u_p)$

If instead is a causal curve but not a null (pre-) geodesic, then there is some time t_1 at which enters $C(u_p)$ and never leaves it (see 3.25 (II)). Hence we can use the argument from above for $t > t_1$ to obtain $L_g(\ )\ \ r(q)$.

(III) causal $q\ 2\ @C(u_p)$

Now let $q\ 2\ @C(u_p)$ and be a causal curve from p to q contained in the normal neighbourhood V. If would enter $C(u_p)$ at some point, it would have to stay inside $C(u_p)$ that is it could not reach $q\ 2\ @C(u_p)$. Hence has to stay inside $@C(u_p)$ which therefore implies again due to 3.25 that is a null-pregeodesic, that is a reparametrization of the radial null geodesic.

□

The length of a causal curve (Def. 3.26) is often interpreted as the proper time of an observer, that is the time measured by an observer traveling along this curve. In light of this interpretation the previously proven twin paradox tells us, that locally an observer travelling along a geodesic, that is a freely falling observer, always measures more time to have passed than an accelerating one.

3.3 Global Causality

We already mentioned, that de ning singularities properly is a complicated task. One problem lies therein that singularities are easily constructed by a local modi cation of spacetime, which then can be equally easily removed. Those are not the singularities of our interest. We aim to prove singularities occuring due to global properties of spacetime which as we hope should be more stable under local modi cations. To de ne appropriate global properties, we rst need a general framework of global causality in which to formulate them. In the preceding discussion we have characterized the local existence of causal and timelike curves. We now aim to make global de nitions describing the existence of such curves, which will thereafter serve as the basic relations of causality.

3.3.1 Causality Relations

We begin by de ning the two fundamental relations of causality, characterising if two points of spacetime can causally in uence another .

Definition 3.28. (Causality Relations) Let $p, q \in M$, we write :
(1) $p << q :\Longleftrightarrow$ there is a future directed timelike curve from p to q.
(2) $p < q :\Longleftrightarrow$ there is a causal curve from p to q.
(3) $p \leq q :\Longleftrightarrow$ if either $p < q$ or $p = q$

Remark 3.29. (i) The relations above are all transitive. Though in most spacetimes the above do not define partial orders on M. In fact assuming non reflexivity for $<<$ is an assumption which arguably should be fulfilled by any reasonable spacetime.
(ii) The dual concept of past relations are defined analogously

With the above relations we are able to define the causal influence of sets in spacetime.

Definition 3.30. Let $A \subset M$, we call :

(1) $I^+(A) = I^+(A; M) := \{q \in M \mid \exists p \in A \text{ with } p << q\} = \bigcup \{I^+(p) \mid p \in M\}$ the chronological future of A.

(2) $J^+(A) = J^+(A; M) := \{q \in M \mid \exists p \in A \text{ with } p \leq q\} = \bigcup \{J^+(p) \mid p \in M\}$ the causal future of A.

(3) $E^+(A) = E^+(A; M) := J^+(A) \setminus I^+(A)$ the future horismos of A.

It is now our goal to derive general properties of these sets. First we start by recalling Lemma 3.25 in this new language:

Lemma 3.31. For Z a convex open neighbourhood in M, the following characterization of the causality relations hold :

(1) For $p \neq q$ in Z: $q \in J^+(p; Z) \Longleftrightarrow \mathrm{Exp}^{-1}(p, q) = \exp_p^{-1}(q)$ is future pointing causal in T_pM (analogously for I^+)
(2) For $p \neq q$ in Z: $q \in E^+(p; Z) \Longleftrightarrow \exp_p^{-1}(q)$ is future pointing null.
(3) $I^+(p; Z)$ is open in Z.
(4) $J^+(p; Z) = \overline{I^+(p; Z)}$
(5) The relation $\leq_Z$ is closed on Z: $(p_n) \to p \in Z$, $(q_n) \to q \in Z$:
$q_n \in J^+(p_n; Z) \Longrightarrow q \in J^+(p; Z)$.

Proof. (1),(2),(3),(4) essentially are 3.25.
If $p = q$ then (5) is clear, else :

$$q_n \in J^+(p_n; Z) \overset{(1)}{\Longleftrightarrow} \mathrm{Exp}^{-1}(p_n, q_n) \in \overline{C(u_p)}$$
$$\overset{def}{\Longleftrightarrow} g_{p_n}(\mathrm{Exp}^{-1}(p_n, q_n), \mathrm{Exp}^{-1}(p_n, q_n)) \leq 0 \quad (37)$$

in combination with

$$g_{p_n}(\mathrm{Exp}^{-1}(p_n, q_n), u(p_n)) < 0 \quad (38)$$

we deduce :

$$g_p(\mathrm{Exp}^{-1}(p;q);\mathrm{Exp}^{-1}(p;q)) \le 0 \tag{39}$$
$$g_p(\mathrm{Exp}^{-1}(p;q);u(p)) \le 0 \tag{40}$$

The last step follows from Exp^{-1}, g, u being contionous. Also $u(p)$ is timelike which implies the strict inequality:
$g_p(\mathrm{Exp}^{-1}(p;q);u(p)) < 0$. Thus (5) is proven. □

The remaining of this section essentially consists of examining when the above properties (2),(3),(4),(5) hold in general spacetimes or subsets of it.

We proceed by collecting some fundamental properties and relations of the causal sets, which hold in arbitrary C^1 -spacetimes.

Lemma 3.32. a) For $p \in M$ we have the following properties of the causal sets:
(a.1) $I^+(p)$ is open.
(a.2) $E^+(p) = \{q \in M \setminus I^+(p) \mid$ there exists a null geodesic from p to $q\}$
(a.3) $I^+(J^+(p)) = I^+(p)$ (Push up Lemma)

b) For an arbitrary set $\Sigma \subset M$ we have the following properties:
(b.1) $I^+(\Sigma)$ is open.
(b.2) $I^+(\overline{\Sigma}) = I^+(\Sigma)$
(b.3) $\overline{I^+(\Sigma)} = \{x \in M \mid I^+(x) \subset I^+(\Sigma)\}$.
(b.4) $J^+(\Sigma) \subset \overline{I^+(\Sigma)}$
(b.5) $\overline{J^+(\Sigma)} = \overline{I^+(\Sigma)}$; $\partial J^+(\Sigma) = \partial I^+(\Sigma)$; $\mathrm{int}(J^+(\Sigma)) = I^+(\Sigma)$

Proof. (cf. [3] Prop.2.15)
a)

(a.1): Choose for $q \in I^+(p)$ a timelike curve γ connecting both points. Then choose a convex neighboubhood Z of $q \in M$. Thus, γ has to meet Z. Choose one such point $r \in \gamma \cap Z$, which is therefore in $I^+(p) \cap \exp_q(C^-(u_q) \cap \tilde{Z})$. Hence $p \in \exp_r(C^+(u_r) \cap \tilde{Z})$ which is an open neighbourhood of p. Also, it is contained in $I^+(p)$ since we can connect p to every point $x \in \exp_r(C^+(u_r) \cap \tilde{V})$ by first going to r and then in V to x.

(a.2) : The statement is proven if we show :
For $q \in J^+(p)$ such that there exists an causal curve from p to q not being a null geodesic, then $q \in I^+(p)$, that is there exists a timelike curve between those two points.
Let $\gamma : [0;b] \to M$ be such a curve. Since $\gamma([0;b])$ is compact we can cover it with finitely many convex neighbourhoods say Z_i for $i = 1;\dots;N$ each containing a point $x_i = \gamma(t_i)$ with $t_i \in [0;b]$. We can also assume that $\overline{Z_i}$ is compact in a larger convex neighbourhood Z_i^0. Since γ is not a null geodesic there exists an $r_0 = \gamma(t_{r_0})$ where γ is either not a null geodesic on an open neighbourhood of

t_{r_0} or has a break. In either case there exists a $j_1 \in \{1, \ldots, N\}$ such that $r_0 \in Z_{j_1}$. Now let r_1^- be the last point in ∂Z_{j_1} of before r_0 and similarly r_1^+ the first one after r_0 (If those do not exist, that is either p or q are contained in Z_{j_1} choose $r_1^- = p$ respective $r_1^+ = q$). Since $|_{[t_{r_1^-}, t_{r_1^+}]}$ is not a null geodesic and contained in $Z^0_{j_1}$ Lemma 3.25 implies that there exists a timelike curve (geodesic) connecting $(t_{r_1^-})$ and $(t_{r_1^+})$. We call this segment $_1$. Now either $r_1^+ = q$ or r_1^+ is contained in Z_{j_2} for an $j_2 \in \{1, \ldots, N\}$ different to j_1. In the second case we again consider r_2^-, r_2^+ as before by taking r_1^+ as our new x and the concatenation of $_1$ and (denoted as $_1$) instead of . Now similarly since $_1$ is not a null geodesic we can construct a timelike curve from r_2^- to r_2^+, denoted $_2$. Thus by concatenating $_1$ up to r_2^- with $_2$ we have constructed a timelike curve from r_1^- to r_2^+. Repeating this argument (without loss of generality finitely many times) until we arrive at q, we can construct a timelike curve from r_1^- to q. We similarly can reiterate the above construction into the past of r_0. Thus by concatenating those two timelike cuves at r_1^- we have a timelike curve connecting p to q and (a.2) is proven.

(a.3): Follows from the proof of (a.2). If there is a causal curve from p to q which is timelike at some points there is also a timelike curve.

<u>b)</u>

(b.1): Since union of open sets are open the claim follows from part a.1).

(b.2): We have $I^+( ) \subseteq I^+(\bar{\ })$. Also for $q \in I^+(\bar{\ })$ there exists $p \in \bar{\ }$ such that $q \in I^+(p)$. Thus $p \in I^-(q)$ an open neighbourhood of p which therefore meets . This in turn implies that $q \in I^+( )$.

(b.3): Let x be in $\overline{I^+( )}$ and $r \in I^+(x)$. Thus $x \in I^-(r)$ an open neighbourhood of x which therefore meets $I^+( )$. As in (2) this implies $r \in I^+( )$. To prove the other direction let x be in M such that $I^+(x) \subseteq I^+( )$. Let B be any open neighbourhood of x. Any timelike geodesic starting at x therefore has to meet B which implies that $I^+(x) \cap B \neq \emptyset$. Hence $I^+( ) \cap B \neq$ which implies that $x \in \overline{I^+( )}$.

(b.4): This is implied by (b.3) when considering the Push up Lemma (a.3).

(b.5): Follows as a direct consequence of (b.4).

□

For the further study of global causality, we first need to present a different viewpoint, on it. In fact the tools developed in the next subsection will be of crucial importance for the later development of causality theory.

3.3.2 Limit-curves

Up to this point we viewed causal sets as point-sets. Most of the times this seems to be a convenient approach since a canonical topology is given by the subspace-topology. On the other hand the preceding sections demonstrated that the 'atoms' of causality theory are causal curves not individual points. It is thus not of much surprise, that in some cases, instead directly de ning a topology on the space of causal curves leads to a much clearer view on causality. This is what we aim to do in the following section.
In the preceding sections, it always turned out helpful to assume piecewise C^2 regularity to de ne causality. Though we did not mention any physical reason for this restriction. In fact the length functional which is of crucial importance for causality and its physical interpretation could be easily generalized to merely locally Lipschitz continuous (: LLC) curves. Here Lipschitz continuity is de ned by using the induced distance function by the Riemannian backround metric h. Importantly (cf. [14] Prop.2.3.1) any function being LLC for one background metric h is also LLC for any other distance function induced by a Riemmanian metric. By Rademacher's theorem, every LLC curve is almost everywhere di erentiable (cf. [14] Theorem 3.2.3). Thus Def. 3.26 can be used similarly for LLC curves. Furthermore to instead base causality on LLC curves has some important advantages when studying limits of curves. This is of little surprise, since proving that a limit curve which exists also is of a high di erentiable order is a much harder task than only proving local lipschitz continuity. Indeed, most of the limit curve theorems are based on the Arzelà Ascoli theorem which preserves local lipschitz continuity under some mild restrictions. Though in general it does not tell us much about higher di erentiability of the limit curve.
To our bene t the main results of the previous sections, can be shown to remain true even when considering only LLC curves. A general discussion can be found in [15]. Here it is shown (cf. [15] Sec.4 for an overview) that for g 2 C^1 the following equality holds:

$$I_{C^2_{pc}}(p) = I_{LLC}(p) \tag{41}$$

Thus the fundamental causal relation << is independent of our choice between LLC and piecewise C^2 curves. We will later prove ([2] Prop.2.13) which shows that in globally hyperbolic (Def.3.45) spacetimes also the length functional is independent of our choice and:

$$J_{C^2_{pc}}(p) = J_{LLC}(p) \tag{42}$$

We are thus con dent in choosing LLC curves as the 'atoms' of causality $(I(p) := I_{LLC}(p)$, $J(p) := J_{LLC}(p)$) and take advantage of the limit curve theorems we are therefore able to prove subsequently.

To begin with we aim to prove a general limit curve theorem from [16] (Theorem 1.5) concerning general sequences of causal curves. This will help us to characterize the now often used term of global hyperbolicity. Thereafter a more specialised theorem which only considers causal geodesics will be proven. Both will be a crucial tool for proving the C^1 singularity theorems.

We start by collecting some facts and definitions about locally lipschitz continous curves. A detailed discussion can be found in [14].

Definition 3.33. Let $\gamma : [a;b) \to M$ be LLC curve with $b \in \mathbb{R} \cup \infty$ such that $\dot\gamma \neq 0$ almost everywhere.

1. We call γ causal, if $\dot\gamma(t)$ is causal in $T_{\gamma(t)}M$ almost everywhere. Similarly it is defined to be future directed.
2. We call γ future-extendible if it has a future endpoint that is $\lim_{t \to b} \gamma(t)$ exists.

Facts 3.34. Let $\gamma : [a;b) \to M$ be LLC curve with $b \in \mathbb{R} \cup \infty$ such that $\dot\gamma \neq 0$ almost everywhere.

1. Due to Radmacher's Theorem (cf. [14] Theorem 2.3.2) we are able to define :
$$s(t) := \int_a^t |\dot\gamma|_h \, dt' \tag{43}$$
which by our assumptions is a strictly increasing (and continous) function. Thus $s(t)$ is bijective and has a continuous inverse. Furthermore $s(t)$ is locally Lipschitz continuous with Lipschitz constants smaller or equal than those of γ. We can define:
$$\hat\gamma := \gamma \circ s^{-1} \tag{44}$$
which fulfills : $|\dot{\hat\gamma}|_h = 1$ almost everywhere. This can be used to show that $\hat\gamma$ is lipschitz continuous (:LC) with lipschitz constant 1.
2. (cf. [14] Lemma 2.5.2/2.5.3) Let $\hat\gamma : [c;d) \to M$ be constructed as above and $d < \infty$. If a sequence t_k with $t_k \to d$ exists such that $\lim_{k\to\infty} \hat\gamma(t_k) = p \in M$, we can extend $\hat\gamma$ to a LC curve :
$\hat\gamma : [c;d] \to M$. This follows from :
$$\mathrm{dist}_h(\hat\gamma(t_k); \hat\gamma(s)) \leq |t_k - s| \tag{45}$$
in the limes $k \to \infty$
$$\mathrm{dist}_h(p; \hat\gamma(s)) \leq |d - s| \tag{46}$$
Thus $\lim_{t\to d} \hat\gamma(t) = p$ and by defining $\hat\gamma(d) := p$ we get a lipschitz continuous curve.
3. (cf. [14] Theorem 2.5.5) The h-parameterized curve is extendedible if and only if $d < \infty$.
Indeed if $d < \infty$, since $\hat\gamma$ is LC with constant $L = 1$ it follows that :
$$\hat\gamma([c;d)) \subseteq \overline{B^h(\gamma(c); d - c)} \tag{47}$$

where $B^h(\ (c); d \ \ c)$ describes the open unit ball at (c) with radius $d \ \ c$. The closure of open balls is compact by the Hopf-Rinow theorem. Thus there exists a sequence such t_k such that $\lim_{k!1} \ \hat{}(t_k) = p \ 2 \ \overline{B^h(\ (c); d \ \ c)}$. Which implies that ^ can be extended by (2.) to d.
On the other hand if $d = 1$, assume that $\lim_{t!1} \ (t) = q \ 2 \ M$ exists. This would mean that we nd an a $t_0 \ 2 \ R$ such that for all $t \ \ t_0$: $(t) \ 2 \ \overline{B^h(q; \ )}$. Now choose > 0 small enough, such that $B^h(q; \ ) \ \ Z$ a precompact normal neighbourhood contained in a larger one $\tilde{Z}$ such that $grad(x_0)$ is past directed timelike on $\overline{Z}$. In the proof of Theorem 3.35 (IV), we will show that all h-arclengths of causal curves contained in Z are bounded. The fact that is h-parameterized, that is $j \_ j_h$ thus yields a contradiction since $L_h(\ _{jt \ \ t_0}) = 1$.

4. Since the reparametrization is strictly monotonously increasing and continuous, it thus follows for any causal curve that it is extendible, if and only if its h-parametrization ^ is only de ned up to a nite time.

The above facts, will be needed in the limit-curve theorems when we want to handle inextendible curves.

Theorem 3.35. (limit curve theorem I) (cf. [16] Theorem 1.5 p.5 and [14] Prop. 2.6.1/2.6.7 p.34)
Let $(\ _n)_n$ be a sequence of C^0 LC -causal curves, such that $_n(0)! \ \ p \ 2 \ M$. If furthermore one of the following is given:

1. all $_n$ are proportional to h-arclength parameterized, are de ned on the interval $[0; 1]$ and have bounded h-arclengths from both sides $C^0 > L_h(\ _n) > C > 0$.[6].
2. all $_n$ are inextendible

then there exists a curve starting at p such that there is a subsequence $(\ _{n_k})_k$ which converges to uniformly on compact sets. In the inextendible case we rst have to reparameterize all $_n$ to h-arc length. Furthermore can be (continuously) parameterized to be a fd-causal curve.
In the rst case this implies uniform convergence on $[0; 1]$. If the second condition is ful lled instead, it follows that is inextendible too.

Proof. (I) Construction of the subsequence and limit-curve:

We begin by constructing a subsequence which converges uniformly on an arbitrary compact subset K of the domain of . We can assume without loss of generality $K = [\ c; d]$. As already mentioned before, the key theorem needed is the Arzelà Ascoli theorem.

[6] We believe without the bound from below the limit curve may become constant instead of causal. Though this case is not discussed in [16] or [14]

Theorem. (Arzelà Ascoli) Let X be a compact Hausdorff space, $C(X)$ the space of continuous real valued functions equipped with the sup-norm. Then :
$F \subset C(X)$ is relatively compact $\Leftrightarrow$ F is equicontinuous and pointwise bounded.

If all curves in $(\gamma_n)_n$ have uniformly bounded lipschitz constants :

$$d_h(\gamma_n(t);\gamma_n(s)) \leq L|t-s| \tag{48}$$

for an $L \in \mathbb{R}_{>0}$. This implies equicontinuity. Since K is compact, we can cover it with a finite number of charts V_i with compact closure. Furthermore we can assume, that there exists an $m \in \mathbb{N}$ such that for all $t \in K$ there exists an i such that : $[t-\frac{1}{m}; t+\frac{1}{m}] \subset V_i$. Recursively we can make the following reasoning. First, convergence of $\gamma_n|_{[c;c+\frac{1}{m}]}$ in $\overline{V_1}$ is equivalent to the convergence of each restricted coordinate expression $\gamma_n^{j_1}$ in $\mathbb{R}$. The existence of a subsequence $\gamma_{n_{k_1}}^{j_1} \to \gamma_1^{j_1}$ (uniform) follows from $\gamma_n^{j_1}$ being equicontinuous and bounded since $\overline{V_1}$ is bounded. Similarly we can choose a subsequence $\gamma_{n_{k_2}}$ of $\gamma_{n_{k_1}}$ such that $\gamma_{n_{k_2}}^{j_2}$ also converges on $[c+\frac{1}{m}; c+\frac{2}{m}]$ to an $\gamma_2^{j_2}$ (in general in a different chart $(V_2;\varphi_2)$) which agrees with $\gamma_1^{j_1}$ at $c+\frac{1}{m}$ after a coordinate change. Recursively repeating the above argument we arrive at a subsequence γ_{n_k} which converges on $[c;d]$ uniform to an γ which is defined by:

$$\gamma(t) = \varphi_i^{-1}(\gamma^{j_i})(t)$$

(49)

if $t \in [c+\frac{i-1}{m}; c+\frac{i}{m}]$ and φ_i denoting the corresponding chart, choosen recursively in the construction. The above is a well defined map since the construction assures that :

$$\varphi_{i+1}^{-1}(\gamma^{j_{i+1}})(c+\frac{i}{m}) = \varphi_i^{-1}(\gamma^{j_i})(c+\frac{i}{m}) \tag{50}$$

In the first case we can choose $[c;d] = [a;b]$ since the minimal lipschitz constants are bounded from above by C^0. We thus have constructed a curve to which a subsequence of γ_n converges uniformly. In the second case, we can reparameterize the inextendible curves γ_n by h-distance as described in Facts 3.34 with lipschitz constant 1 and defined on $\mathbb{R}$. We can now construct a subsequence $\gamma_{n_{k_1}}|_{[-1;1]}$ as above of the sequence $\gamma_n|_{[-1;1]}$ converging uniformly to a curve $\gamma^{(1)}$ defined on $[-1;1]$. Now choose a subsequence $\gamma_{n_{k_2}}$ of $\gamma_{n_{k_1}}$ which converges uniformly on $[-2;2]$ to a curve $\gamma^{(2)}$ defined on $[-2;2]$. Furthermore our choice of $\gamma_{n_{k_2}}$ as a subsequence of $\gamma_{n_{k_1}}$ implies that $\gamma^{(2)}|_{[-1;1]} = \gamma^{(1)}$. Now repeat this argument for all $m \in \mathbb{N}$ and finally define a subsequence : $(\gamma_{n_k})_k = (\gamma_{nk_k})_k$. This converges uniformly on every compact subset to the curve :

$$\gamma(t) := \gamma^{(k)}(t) \tag{51}$$

for jtj k.
We still have to prove (in both cases (1) and (2)) that is LC and causal and in the case (2) for the limit curve to be inextendible.

(II) can be parameterized as fd causal:

We only have convergence of $_n$ and not of the derivatives. Thus we somehow need a connection between the causal character of and to use the fact that is the limit curve of causal curves. This is exactly what Lemma 3.25 does. The problem here, is that we only have proven it for the case of being piecewise C^2. In [14] Prop 2.4.5 it is shown that a direct generalization to lipschitz continuous of 3.25 is possible. In the following we sketch a version restricted to convex neighbourhoods, since this will be su cient to prove that is causal and reduces the length signi cantly. Thus let Z be a convex neighbourhood of $(0) = p \; 2 \; M$, for $: [0; b]! \; Z$ a LC curve. In 3.25 we started by proving for initially timelike curves starting at p, that $= \exp_p^{\;1}(\;)\quad C^+(u(p))$. Now assume for the LC curve , that it is di erentiable at $t = 0$ and timelike. Just as in 3.25 it follows that :

$$(g_p)_{ij} \quad {}^i(t) \; {}^j(t) < 0 \tag{52}$$

$$(g_p)_{ij} \quad {}^i(t) u_p^j(t) < 0 \tag{53}$$

for all $t \; 2 \; (0; \;]$ with u again being a global future timelike vector eld. Furthermore the Gauÿ lemma implies that when and thus is di erentiable and (52),(53) hold at t then:

$$\frac{d(\mathrm{e}\!\!\!\!-\quad)}{dt}(t) = 2(\,g_p)_{ij} \quad {}^i(t) \,\underline{\;}^j(t) = 2(\,g_{\;(t)})_{ij} \quad {}^i(t) \,\underline{\;}^j(t) < 0 \tag{54}$$

Thus the above follows almost everyhwere on $(0; \;]$, which implies :

$$\mathrm{e}\!\!\!\!-\quad(s) = \int_0^s \frac{d(\mathrm{e}\!\!\!\!-\quad)}{dt}(t^0)dt^0 \tag{55}$$

is monotonously decreasing and thus negative on $(0; \;]$, that is $_{j(0;\;]}$ is contained in $C^+(u(p))$. Now let

$$s_{max} := \max f\, t \; 2 \; [0; b] : \quad \text{is di erentiable, timelike and } \mathrm{e}\!\!\!\!-\quad(t) < 0 g \tag{56}$$

Since $\mathrm{e}\!\!\!\!-\quad(s_{max}) < 0$ we have from the above reasoning that $\frac{d(\mathrm{e}\!\!\!\!-\;)}{dt}(s_{max}) < 0$
From the de nition of a derivative we have:

$$\mathrm{e}\!\!\!\!-\quad(s_{max} + \quad) = \mathrm{e}\!\!\!\!-\quad(s_{max}) + \frac{d(\mathrm{e}\!\!\!\!-\quad)}{dt}(s_{max}) \quad + o(\;) \tag{57}$$

Hence there exists an open interval of s_{max} where $\mathrm{e}\!\!\!\!-$ remains negative if $s_{max} < b$. This would be a contradiction, thus $s_{max} = b$. Now let be an LC causal curve which starts at p such that $([0; \;]) \quad Z$. Furthermore let $q_r \; 2 \; I \quad (p; Z) \setminus \; B^h(p; r)$ where $B^h(p; r)$ is the r open ball with respect to the backround metric h. Thus for t 2 [0;) there exists an initially timelike LC curve from q_r to (t) (geodesic top concatenated with). The previous argument for initially timelike LC curves thus implies that (t) 2 $C^+(u(q_r))$. This can be expressed as:

$$(g_{q_r})_{i_r j_r} \;{}^{i_r}(t)\;{}^{j_r}(t) < 0 \tag{58}$$

$$(g_{q_r})_{i_r j_r} \;{}^{i_r}(t) u^{j_r}(q_r) < 0 \tag{59}$$

Here the subindex r is added to indicate that those coordinate expressions are given in the chart $\exp_{q_r}{}^{1}$. Thus :

$${}^{i_r}(t) = \mathrm{Exp}\;{}^{1}(q_r; \;(t)) \tag{60}$$

Now since g and Exp ^{1} are continuous we get in the limit r ! 0:

$$(g_p)_{ij} \;{}^{i}(t)\;{}^{j}(t) \quad 0 \tag{61}$$

$$(g_p)_{ij} \;{}^{i}(t) u^{j}(p) \quad 0 \tag{62}$$

Just as in 3.25 : (t) 6 0, which follows from all h-arclengths being bounded on Z (see part (III)). This can be used in the following to show that our constructed limit curve is causal:

Let Z be a convex neighbourhood of (t) such that $([t_1; t_2])$ Z where $t_1 < t < t_2$. Since $_n$! on $[t_1; t_2]$, we also have $_n([t_1; t_2])$ Z for large enough n 2 N. Now the previous which generalized Lemma 3.25 can be used just as in Lemma 3.31 to show that the relation $_Z$ is closed. Hence from $_n(t_1)$ $_Z$ $_n(t_2)$ follows that (t_1) $_Z$ (t_2). In case (1) we have assumed that $L_h(\;_n) > C$ and all $_n$ being parameterized proportional to h-arclength hence $j\_j_h > C$ almost everywhere. We therefore conclude:

$$\mathrm{dist}_h(\;_n(t_1); \;_n(t_2)) \quad C|t_2 \quad t_1| \tag{63}$$

$$\Longrightarrow \quad \mathrm{dist}_h(\;(t_1); \;(t_2)) = \lim_{n \to 1} \mathrm{dist}_h(\;_n(t_1); \;_n(t_2)) \quad C|t_2 \quad t_1| \tag{64}$$

Thus (t_1) 6 (t_2), that is $(t_1) <_Z (t_2)$ and therefore by using the previously generalized lemma we obtain that (t_1) can be connected to (t_2) by a fd-causal geodesic contained in Z. Since t and Z were arbitrary we have therefore proven that is a continuous causal curve[7]. By a result given in [10] (Theorem 2.12) we can therefore parameterize to be a causal LLC curve. If the curves $_n$ are inextendible instead (that is the case (2)), we have after reparameterizing to h-arclength, $j\_j_h = 1$ almost everywhere. Hence the same argument as above can be applied for C = 1.

It therefore only remains to prove that if all $_n$ are inextendible (case (2)) then is inextendible too.

[7] We call : I! M a continuous causal curve if for all convex sets Z intersecting and $t_1 < t_2$ such that $([t_1; t_2])$ Z , the points (t_1) and (t_2) can be connected by a fd-causal geodesic contained in Z.

(III) In case (2) is inextendible:

We partly sketch the proof, more detials are described in [14] Prop.2.6.4 which we will follow quite closely. First, it is important to notice that even if all $_n$ are h-parameterized in general this is not the case for the limit curve . Though we do know, that is de ned on R (since all $_n$ are due to their inextendible character (Facts 3.34)). Now reparameterize by h-arclength. We need to show that ^ is still de ned on R. Assume this is not the case. Without loss of generality we can assume that ^ is not inextendible to the future that is : $\hat{}_{j t \ 0}$: $[0; a)!$ M . For readability we will denote the above reparameterized curve restricted to positive times with ^{0}. The Facts 3.34 tell us, that ^{0} can be extended to a. The extended curve will still be denoted as ^{0}. Let V be a pre-compact normal neighbourhood contained in larger one V^0 such that grad(x_0) is past directed timelike on $\overline{V}$. We note that $\overline{V}$ is compact and for all causal unit h-length tangent vectors v_p 2 T_pM with p 2 $\overline{V}$ we have : $jg_p(grad(x_0); v_p)j > 0$. It thus follows that there exists a positive constant C such that:

$$jg_p(grad(x_0); v_p)j \quad Cjv_pj_h \tag{65}$$

for all p 2 $\overline{Z}$ and v_p 2 T_pM . From here we are able to deduce that for a causal curve contained in Z , h-length is bounded by a constant l. This will be generalized later to so called 'non-totally imprisoning' global spacetimes. In our case this follows from (65) since:

$$L_h(\ ) = \int_{s_1}^{s_2} j \_ j_h dt \quad \int_{s_1}^{s_2} \frac{jg_{\ (t)}(grad(x_0); \_)j}{C} dt = \frac{jx_0(\ (s_2)) \quad x_0(\ (s_1))j}{C} \tag{66}$$

We therefore arrive at :

$$L_h(\ ) \quad l := \frac{2}{C} \sup_Z (x_0) < 1 \tag{67}$$

For n large enough there always exists an s_n such that : $_n(s_n)$ 2 Z . The above result implies that $_n(s_n + \ )$ 2= Z if $> l$. This contradicts that $_n$ converge to which has an endpoint in Z (that is $\lim_{t!1}$ (t) = ^{0}(a)). Hence a = 1 and is inextendible.

□

The proof above heavily depends on g being at least C^2 since the existence of convex neighbourhoods played a crucial role in it. For C^1 Lorentzian metrics we do not have this tool anymore and therefore need a di erent approach than trying to directly generalize the theorem above. As we already mentioned before, it turns out, that most of the global causality results can be generalized to C^1 metrics by approximation methods. We now nally begin to introduce those concepts. For those approximating smooth metrics to be of any use for studying causality theory we need a relation on the space of Lorentzian metrics, which compares the (local) causal relations induced by them.

Definition 3.36. (lightcones comparison) Let g and g' be two lorentzian metrics. We write :

$$g' \prec g :\iff \left(g'(v,v) \leq 0, v \neq 0 \implies g(v,v) < 0. \right) \tag{68}$$

The interpretation becomes clear by remembering Lemma 3.7. Thus $g' \prec g$ tells us that the timecones induced on every tangent space by g' which locally describe the causal structure of the manifold are strictly 'smaller' than those induced by g. Thus for $(g_\epsilon) \to g$ in C^0_{loc} with $g_\epsilon \prec g$ we can think of local timecones approximating those of g from inside. This will be used extensively in the proofs which will follow. The next definitions are inspired by the fact, that for a causal curve with respect to g' and $g' \prec g$ we directly also have a timelike curve with respect to g. Since g' can be arbitrarily close to g (we will later construct such approximations) one could hope, that the causal future defined by timelike curves in g which are also timelike curves for any $g' \prec g$ approximate the standard causal future of g. In formulas:

Definition 3.37. .

1. Let γ be a LLC curve. We call γ <u>locally uniformly timelike</u> (lut.) if there exists a smooth metric g' which has smaller lightcones than g ($g' \prec g$) and $g'(\dot\gamma, \dot\gamma) < 0$ almost everywhere.
2. We define the local uniform chronological future of a set $A \subset M$:

$$\check{I}^+(A) := \{q \in M \mid \text{there exists a future directed lut.-curve} \tag{69}$$
$$\text{starting in } A \text{ which ends at } q\} \tag{70}$$

Remark 3.38. .

1. We have the equality : $\check{I}^+(A) = \bigcup_{g' \prec g} I^+_{g'}(A)$. Thus $\check{I}^+(A)$ is open.
2. In [17] Prop.1.21 it is proven that for lipschitz continuous metrics : $\check{I}^+(A) = I^+(A)$. Therefore if g is at least lipschitz continuous, $I^+(A)$ is open.

So far we only considered approximations from inside, though we could equally study approximations with metrics of larger timecones than g. This seems to be the more interesting case for the generalization of Theorem 3.35 since any sequence of causal curves $(\gamma_n)_n$ with respect to g is also a sequence of causal curves measured by g' if $g' \succ g$. Thus for a sequence of g-causal curves the classical Theorem 3.35 can be applied for every smooth $g' \succ g$. The problem one faces now, is that this strategy only implies a causal limit curve with respect to g' from which we can not deduce a causal character for g. Nevertheless since $g' \succ g$ can be arbitrary close to g (see 6.15 for an explicit construction) one could hope that that a causal character for all such $g' \succ g$ also implies g-causality.

Indeed [17] proves exactly this property in Proposition 1.5 (p.5). Though we will need a result of similar essence, the proof will not be shown here. Instead we will present a shortened proof already adjusted to our conditions. We are now prepared to formulate and prove the second limit-curve theorem which not only extends Theorem 3.35 but also describes a little more general setting as discussed above.

Theorem 3.39. (limit curve theorem II) (cf. [16] Theorem 1.5 p.5)
Let $(g_n)_n$ be a sequence of smooth metrics such that $g \prec g_{n+1} \prec g_n$ for all $n \in \mathbb{N}$ with $g \in C^0$. Also assume that $g_n \to g$ in C^0_{loc}.
Let $(\gamma_n)_n$ be a sequence of LLC curves, such that for each $n \in \mathbb{N}$, γ_n is causal with respect to g_n. Furthermore let $\gamma_n(0) \to p \in M$. If additionally one of the following is given:

1. all γ_n are proportional to h-arclength parameterized, are defined on the interval $[0;1]$ and have bounded h-arclengths from both sides: $C' > L_h(\gamma_n) > C > 0$.
2. all γ_n are inextendible and parameterized by h-arclength.

then there exists a curve γ starting at p such that there is a subsequence $(\gamma_{n_k})_k$ which converges to γ uniformly on compact sets. Furthermore γ can be (continuously) parameterized to be a fd-causal curve.
In the first case this implies uniform convergence on $[a;b]$. If the second condition is fulfilled instead, it follows that γ is inextendible too.

Proof. The main work has already been done by proving Theorem 3.35. In fact since $(\gamma_n)_n$ is a sequence of causal curves with respect to g_0, (3.35) implies the existence of a LLC limit-curve γ which is g_0-causal and the uniform limit of a subsequence $(\gamma_{n_k})_k$. Hence it only remains to show, that γ is causal when measured by g.

We start by noticing that for every $m \in \mathbb{N}$ one can analogously examine the sub-sequence $(\gamma_{n_k})_{k \geq k_m}$ with $(k \geq k_m \Rightarrow n_k \geq m)$. This is a sequence of g_m-causal curves, which therefore has a g_m-causal limit-curve by Theorem 3.35. This limit-curve must be equal to γ since we have chosen a sub-sequence of $(\gamma_{n_k})_k$ which converges to γ. Thus we have proven that γ is g_n-causal for every $n \in \mathbb{N}$. The question of causality of a curve is a local question. Therefore we can without loss of generality restrict our consideration to a compact Interval $[c;d]$ in the domain of γ. Also we can reparameterize γ to h-arclength such that $|\dot\gamma|_h = 1$ (γ is g_n-causal which implies being almost everywhere non 0). Since $\gamma([c;d])$ is compact and $g_n \to g$ in C^0_{loc} there exists an $n_0 \in \mathbb{N}$ for every $\epsilon > 0$ such that for $n \geq n_0$:

$$\|g - g_n\|_{1;\gamma([c;d])} < \epsilon \tag{71}$$

$$\Rightarrow \quad g(\dot\gamma,\dot\gamma) < g_n(\dot\gamma,\dot\gamma) + \epsilon|\dot\gamma|_h^2 = g_n(\dot\gamma,\dot\gamma) + \epsilon \tag{72}$$

Hence $g(\dot\gamma,\dot\gamma) \leq 0$ almost everywhere on $[c;d]$, that is γ is g-causal.

□

Finally we have proven the limit-curve theorem for C^1 metrics which will be a key tool when discussing further global causality properties. Nonetheless we should not forget our final destination : The Singularity theorems In Sec.3.2 we mentioned the crucial physical role geodesics play in general relativity. The mathematical importance of them, due to illuminating the local lorentzian geometry has been illustrated by extensive use of the exponential map. The following discussions will reveal that, their importance does not end locally. In fact since the claim of the singularity theorems is the existence of incomplete geodesics, they will be crucial when finally proving the theorems. It is thus naturally to ask, weather one could similarly to the preceding discussion, formulate limit-curve theorems which are restricted to geodesics. We already mentioned at the beginning of this section, that proving higher differentiability of the limit-curves poses a real problem. Thus we need to adjust our strategy. First, we will consider geodesics as paths in the manifold TM, that is : $\dot\gamma : I \to TM$. By demanding certain conditions for those velocity-curves (which are reasonable for geodesics at least) we are allowed to use Arzelà Ascoli and therefore obtain a limit-curve which certainly is C^1. To prove C^2 differentiability, and more importantly the geodesic character of the limit curve we need to make use of the geodesic equation which serves as a connection between the convergence of first and second derivatives. This already quite precisely outlined the strategy to prove the following geodesic limit-curve theorems.

Lemma 3.40. (geodesic-limit-curves, cf. [2] Lemma 2.7/p.9)
Let $B \subseteq TM$ be compact and $(g_n)_n$ a sequence of C^1 metrics converging locally uniform to g. For $(\dot\gamma_n : [0, a_n] \to B)_n$ a sequence of g_n-geodesics such that $\dot\gamma_n(a_n) \in \partial B$ and $\dot\gamma_n(0) \to v \in B$, there exists $0 < a \leq \infty$ and a g-geodesic $\dot\gamma_v : [0, a) \to B$ to which a subsequence of $(\dot\gamma_n)_n$ converges uniformly on compact subsets of $[0, a)$.

To prove this theorem, we first need to collect some facts about geodesics in C^1-spacetimes.

Facts 3.41. (cf. [2] Section 2/p.7) It is important notice, that even though the Peano-existence Theorem does not provide uniqueness of geodesics, it still preserve some important properties:

1. A geodesic is inextendible if and only if $\dot\gamma$ leaves every compact subset of TM.
2. If a causal geodesic $\gamma : [a, b) \to M$ is continuously extendible to b it is extendible as a geodesic.
3. A geodesic has a fixed causal character.

Proof. (Lemma 3.40)
In the following we denote the christoffel symbols for each g_n by Γ^n and for g by Γ. We start with some simplifications which arise, due to B being compact.

That is, by choosing sub-sequences of $(\dot\gamma_n)_n$ we can assume that $a_n \to a \in [0,\infty]$ and $\dot\gamma_n(a_n) \to q \in \partial B$. Importantly $a \neq 0$. This follows from $v \in B$ and $q \in \partial B$. Indeed let $B^h(v,\epsilon) \subset B$ and n_0 big enough such that: $\dot\gamma_{n_0}(0) \in B^h(v,\frac{\epsilon}{3})$ and $dist_h(q,\dot\gamma_n(a_{n_0})) < \frac{\epsilon}{3}$. This implies for $n \geq n_0$: $dist_h(\dot\gamma_n(0),\dot\gamma_n(a_n)) > \frac{\epsilon}{3}$. Thus $\ddot\gamma$ would have to be unbounded if $a = 0$. This contradicts our assumptions. In fact by using the geodesic equation: $\ddot\gamma^i_n = -(\Gamma_n)^i_{jk}\dot\gamma^j_n\dot\gamma^k_n$ and that $(\Gamma_n)^i_{jk} \to (\Gamma)^i_{jk}$ uniformly on $\pi(B)$, we conclude that $\ddot\gamma_n$ has to be bounded. Hence we have shown that $a > 0$. By proving that $\ddot\gamma_n$ is bounded we have demonstrated that $\dot\gamma_n$ is equicontinous. Therefore we can use Arzelà Ascoli and conclude for any compact interval of $[0,a-\frac{1}{m}]$ uniform convergence of a sub-sequence to a function $f_m : [0,a-\frac{1}{m}] \to B$. Thus similarly as in the proof of Theorem 3.35 we can construct a sub-sequence which converges uniformly on compact subsets to a limit-curve $f : [0,a) \to B$. It only remains to show that f is a geodesic. Indeed by the geodesic equation ($\ddot\gamma^i_n = -(\Gamma_n)^i_{jk}\dot\gamma^j_n\dot\gamma^k_n$) also the $\ddot\gamma^i_n$ converge uniformly on compact sets to a function g. Thus f is differentiable with $\dot f = g$. We have therefore constructed $\dot\gamma = f$ as described in Theorem 3.40. □

The above lemma will play a critical role later since it will help us to formulate a limit-curve theorem for inextendible geodesics. Here we can not make use of our previous Theorem 3.39. The problem which prevents us from doing so is the affine parametrization which as we saw in the previous proof is crucial for geodesic limit-curve theorems. In 3.39 we could without any loss h-parameterize all curves of interest. For geodesics this procedure is not so innocent anymore. We briefly sketch where problems arise: One could start by considering a sequence of inextendible geodesics as curves in TM, that is $\dot\gamma_n : [0,t_n) \to TM$. Since TM is a manifold we can analogously as we did for M, view it as a complete Riemmanian Manifold. Therefore we can h-parameterize the curves above to get a sequence: $\dot\gamma_n : [0,\infty) \to TM$. Thus the proof of 3.39 would tell us the existence of an inextendible limit-curve $\dot\gamma : [0,\infty) \to TM$. Though due to the reparametrization which depends on the second derivatives of the geodesic equation does not directly help us anymore in proving the convergence of $\ddot\gamma^i_n$. In ([14] Prop.2.6.8) this problem can be circumnavigated by use of $C^{1,1}$ regularity of the metric which implies uniqueness of geodesics and continuous dependence on initial values. Since we consider C^1-metrics we need the following theorem considering inextendible geodesics.

Theorem 3.42. (inextendible geodesic limit-curves, cf. [2] Lemma 2.8) *Let $(g_n)_n$ be a sequence of C^1 metrics which converge locally uniform to g. For any sequence of inextendible g_n-geodesics $(\dot\gamma_n : [0,t_n) \to TM)_n$ such that $t_n \leq N$ and $\dot\gamma_n(0) \to v \in TM$, there exists a subsequence which converges uniformly on compact subsets to an inextendible g-geodesic $\dot\gamma : [0,b) \to TM$ with $\dot\gamma(0) = v$ and $b \leq N$.*

Proof. The proof follows a similar strategy as the limit-curve theorems before. Indeed one repeatedly use Lemma 3.40 on ever growing compact sets K_m, which cover TM. Since $\dot\gamma_n(0) \to v$ we can assume that all $\dot\gamma_n(0) \in K_1$ and all $\dot\gamma_n$ start in K_1. Since all $\dot\gamma$ are inextendible they leave every compact set K_m (Facts

3.41). Thus one always finds $a_n^m \leq N$ as in Lemma 3.40 where the curves meet ∂K_m and $\mathrm{im}(\gamma_n|_{[0;a_n^m)}) \subseteq K_m$. We therefore find ourselves for every K_m in the situation of Lemma 3.40.
Recursively one now constructs for each $m \in \mathbb{N}$ a sub-sequence $(\gamma_{n_k^m})_k$ of $(\gamma_{n_k^{m-1}})_k$ which converges to a limit g-geodesic $\gamma^m : [0; a^m) \to TM$. The diagonal sequence $(\gamma_{n_k^k})_k$ converges therefore to a g-geodesic $\gamma : [0; b) \to M$ for $b = \lim_{m \to \infty} a^m \leq N$.
Furthermore γ is inextendible, because it leaves every compact set as $t \to b$. □

The singularity theorems fundamentally consist of proving the existence of incomplete geodesics, for the generalization to $g \in C^1$ one would therefore hope to have some connection between geodesic completeness in a spacetime $(M; g)$ and for smooth approximations $(M; g_n)$. This is exactly what the following Theorem states.

Theorem 3.43. (implications of geodesic completeness for approximations, cf. [2] Prop.2.9)
Let (g_ε) be a net of C^1 metrics such that $g_\varepsilon \to g$ in C^1_{loc}. Furthermore let $K \subseteq TM$ be compact such that every geodesic which starts in K is defined on $[0; T)$. Then for any $N < T$ there exists an $\varepsilon_0(N; K)$ such that if $\varepsilon \leq \varepsilon_0(N; K)$ all g_ε geodesics which start in K are defined on $[0; N]$.

Proof. The theorem is a direct consequence of Theorem 3.42. Suppose such an $\varepsilon_0(N; K)$ would not exist, that is there would exist a sequence of inextendible g_{ε_n}-geodesics $\gamma_n : [0; t_n) \to TM$ with $t_n \leq N$, starting in K and $g_{\varepsilon_n} \to g$. Since K is compact we can assume $\gamma_n(0) \to v \in K$. Thus we can apply (3.42) which implies the existence of an inextendible limit g-geodesic $\gamma : [0; b) \to TM$ with $b \leq N$. This contradicts our assumptions since $N < T$ and γ starts in K. □

Those are all the limit-curve theorems we will need to further study global causality and then prove the singularity theorems. Nevertheless since they play such a key role in causality theory in general it is important to find their most general versions and formulating them in a lucid way. In fact such a review is given in [18] by E.Minguzzi to which we refer for more details.
We will end this subsection by a corollary of Theorem 3.43. Later when proving the C^1-singularity theorems it will tell us that the subset of TM which will be of interest, is compact and therefore restricting to it helps us to control the error in our approximations.

Corollary 3.44. Let $\varepsilon \mapsto g_\varepsilon$ be a continuous map from $[0; 1]$ to the space of C^1-metrics with respect to C^1_{loc} convergence. Now assume as in Theorem 3.43 that $K \subseteq TM$ is compact and that all g-geodesics starting in K are defined on $[0; T)$. Then for any $N < T$ and $\varepsilon_0(K; N)$ as in 3.43, the set :

$$F_{\varepsilon_0;K;N} := \bigcup_{0 \leq \varepsilon \leq \varepsilon_0(K;N)} F_{\varepsilon;K;N} \tag{73}$$

is compact. Here $F_{;K;N}$ is de ned as :

$$F_{;K;N} := \bigcup_{f\ \_,\ g\text{-geodesic with } \_(0) 2 K\ g} \mathrm{im}(\_{}_{[0;N]}) \quad TM \tag{74}$$

Proof. (I) $F_{0;K;N}$ is bounded (cf. [2] Prop.2.11)

Assume (I) would be false. Hence there exists a sequence $(\_{}_n(t_n))_n$ $F_{0;K;N}$ such that for $p_n := \_{}_n(t_n)$ leaves every compact set K_m. Here we have again chosen a compact exhaustion $(K_m)_m$ of TM as in 3.42. Furthermore we can simplify our situation by choosing a sub-sequence instead, such that : ${}_n\ !\ \ 2\ [0;\ (K;N)]$, $t_n\ !\ t_0\ 2\ [0;N]$, $\_{}_n(0)\ !\ v\ 2\ K$. Additionally, since $7!\ g$ is continuous $g\ !\ g$ in C^1_{loc}. The proof now is very similar to the one for Theorem 3.42. In fact since p_n leaves every compact subset we can as in 3.42 always nd a^m_n N such that $\_{}_n(a^m_n)\ 2\ @K_m$. Exactly as in 3.42 we obtain an inextendible limit g-geodesic $\_ : [0;b)!\ TM$ with b N. This contradicts Theorem 3.43.

(II) $F_{0;K;N}$ is closed

Let $(p_n)_n$ $F_{0;K;N}$ with $p_n = \_{}_n(t_n)$ and $p_n\ !\ p\ 2\ (@F_{0;K;N}$ $K)$. As in (I) we may assume that ${}_n\ !\ \ 2\ [0;\ (K;N)]$, $t_n\ !\ t_0\ 2\ [0;N]$, $\_{}_n(0)\ !\ v\ 2\ K$. Once more we have $g\ !\ g$ in C^1_{loc}. Instead of using the strategy of Theorem 3.42 we now proceed similar as in Lemma 3.40. First we have $t\ 6=\ 0$. This follows from $p\ 2\ (@F_{0;K;N}$ $K)$ and K being compact, which implies that $\mathrm{dist}_h(K;p) > 0$. Thus if $t = 0$ just as in 3.40 it would follow that $\bullet_{n\,j[0;t_n]}$ is not bounded. Though we have shown in (I) that $F_{0;K;N}$ is bounded, thus $(\_{}_{n\,j[0;t_n]})_n$ is a bounded sequence and as in 3.40 we can conclude from the geodesic equation that also $\bullet^i_n$ is bounded. Hence $t\ 6=\ 0$. Furthermore $(\_{}_{n\,j[0;t_n]})_n$ is bounded and equicontinuos. Applying the Arzelà Ascoli Theorem we get a limit g-geodesic $\_ : [0;t_0)\ !\ TM$. Now since $\lim_{t!\ t_0} \_(t) = \lim_{n!1} \_{}_n(t_n) = p$ is in $\overline{F_{0;K;N}}$ which is compact $\_$ is extendible as an geodesic (Facts 3.41). Thus $p = \_(t_0)\ 2\ F_{0;K;N}$ which is therefore compact. □

3.3.3 Causal Ladder

In 3.3.1 we collected some general local causality properties in C^1 - spacetimes and showed that at least some of them can be maintained globally. Subsequently in 3.3.2 we focused on expanding our toolbox to describe causal features of spacetime. Therefore we have now come to a point at which we need to de ne those causal properties which we would like to have in our spacetime and thereafter can be characterized using the limit curves Theorems of 3.3.2.

One could go on and motivate all the various causality conditions that exist to select reasonable global spacetimes. Though as we will see later, the proofs of singularity theorems mostly (at least in their most developed versions) do only require weak causality conditions. The second Hawking-Theorem for example (5.4) can be proven without any explicit causality condition and the Penrose-Hawking-Theorem (5.12) only requires the chronology condition, which prohibits closed timelike curves. This most certainly should be fulfilled for any reasonable spacetime (not the Gödel-universe[8]). For the stronger conditions it often suffices to restrict our consideration to subsets of our spacetime such that they are fulfilled. Thus we only shortly define the most important causality conditions, before narrowing down our view onto the most important but also most demanding condition : Global Hyperbolicity.

Definition 3.45. (the causal ladder) The spacetime (M, g) :

1. fulfills the chronology condition if : $\forall p \in M : p \notin I^+(p)$
2. fulfills the causality condition (is causal) if : $\forall p \in M : J^+(p) \cap J^-(p) = \{p\}$
3. fulfills the future distinguishing condition if : $\forall p \in M$ there exist arbitrarily small neighbourhoods U of p such that every fd-causal curve starting at p and ending in U is contained in U.
4. fulfills the strong causality condition if : $\forall p \in M$ there exist arbitrarily small causally convex neighbourhoods U. Here a causally convex neighbourhoods is defined as a U such that every fd-causal curve $\gamma : [a, b] \to M$ with endpoints $\gamma(a), \gamma(b) \in U$ is already fully contained in U.
5. fulfills the stable causality condition if there exists a Lorentzian metric $g' \succ g$ such that (M, g') fulfills the causality condition (if $g \in C^k$ we also demand $g' \in C^k$).
6. is causally simple if it is causal and the relation $\leq$ is closed.
7. is globally hyperbolic if it satisfies the strong causality condition and the causal diamonds $J^+(p) \cap J^-(q)$ are compact for all $p, q \in M$.

The conditions above are given such that every succeeding condition implies the previous one for $g \in C^2$. Thus global hyperbolicty is the strongest requirement, containing all the others.

Remark 3.46. Later it will be convenient to have an equivalent but seemingly weaker version of strong causality which will then give us a stronger condition on spacetime if we assume strong causality to be false. In fact by using [10] Theorem 1.35. one can deduce the equivalence of the following definition: For every $p \in M$ and neighbourhood U of p, there exists a smaller neighbourhood $V \subset U$ such that every causal curve with endpoints in V is already contained in U.

---

[8] cf. [19]

3.3.4 Global hyperbolicity (C^1)

We start this section by a further characterization of globally hyperbolicity. In fact one may have realized, that the prior definition of global hyperbolicity made use of causality defined via point sets. We already mentioned in 3.3.2 this can be obstructive in some important cases. Thus we will proceed by characterizing global hyberpolicity using causal curves as the 'atoms' of causality. For doing so, we first have to define the space of causal curves.

Definition 3.47. (space of causal curves I, cf. [16] p.1436)
We define for $p, q \in M$:

$$C_h(p;q) := \{\gamma : [0;1] \to M \mid \gamma \text{ LC, fd-causal}, \ \gamma(0) = p, \ \gamma(1) = q, \ h(\dot\gamma, \dot\gamma) = \text{constant almost everywhere}\} \quad (75)$$

Now since $C_h(p;q) \subset C([0;1]; M)$ which comes with the topology of uniform convergence (compact-open topology, T_{co}) we naturally have the subspace topology on $C_h(p;q)$. We already mentioned, that we will quite often need to use the limit-curve theorems in the following sections. Thus it only seems foresighted to already define the closure rather than $C_h(p;q)$ as the space of causal curves.

$$\tilde{C}(p;q) := \overline{C_h(p;q)}^{T_{co}} \quad (76)$$

By use of Theorem 3.39 we know that every element of $\tilde{C}(p;q)$ is a future directed causal curve from p to q defined on $[0;1]$ though not necessarily fulfilling $h(\dot\gamma, \dot\gamma)$ to be constant.

Even though this definition already looks predestined for the use of 3.39 it may seem rather unnatural. In fact a more natural definition may be given by :

Definition 3.48. (space of causal curves II, cf. [16] p.1432)
We define for $p, q \in M$:

$$C(p;q) := \{\gamma : I \to M \mid \gamma \text{ fd-causal LLC from } p \text{ to } q\}/\sim \quad (77)$$

Here $I \subset \mathbb{R}$ is a compact interval I and $\gamma \sim \gamma'$ if there exists a map $\phi : I' \to I$ such that $\gamma' = \gamma \circ \phi$ that is absolutely continuous, surjective, strictly monotonically increasing with absolute continuous inverse.
We define the topology T on $C(p;q)$ by choosing the basis to consist of the sets : $O(U) := \{\gamma \in C(p;q) \mid \gamma(I_\gamma) \subset U\}$ for $U \subset M$ open.

Remark 3.49.
1. In [16] Lemma 1.2 C.Sämann proves, that $\sim$ indeed defines an equivalence relation.
2. Also it is proven that for every $\gamma \in C(p;q)$ the parametrization with respect to h-arclength defined in 3.34 is unique and fulfills the conditions for a reparametrization given above. From here one can reparameterize by $t \mapsto L_h(\gamma)t$ to get a unique causal curve in $C_h(p;q)$. The smallest Lipschitz constant for this reparameterized curve is then given by $L_h(\gamma)$.

The above tells us therefore, that the set $C_h(p;q)$ just chooses one convenient representative of the equivalence classes of $C(p;q)$. Furthermore again in [16] Lemma 2.5/2.8 it is proven that the map $\pi : \tilde{C}(p;q) \to C(p;q)$ defined by $\pi(\gamma) = [\gamma]$ is continuous and if the strong causality[9] condition holds even proper. Thus in that case if one wants to examine the compactness of $C(p;q)$ one can equivalently analyse $\tilde{C}(p;q)$.

We need one further lemma as a preparation.

Lemma 3.50. (1) If the strong causality condition holds on $(M;g)$ for $g \in C^1$ then $(M;g)$ is non partially imprisoning. That is for any compact set $K \subset M$ and γ future (or past) inextendible causal curve which starts in K there exists an t_0 such that $\gamma(t) \notin K$ for all $t \geq t_0$.

(2) A spacetime is non totally imprisoning, that is no compact set contains a future (or past) inextendible causal curve, if and only if for all compact sets $K \subset M$ there exists $C > 0$ such that for all causal γ contained in K the h-arclengths are bounded $L_h(\gamma) < C$.

Proof. (cf. [5], 14/Lemma 13 and [16] Lemma 2.7)
(1) Let $(M;g)$ fulfill the strong causality condition, Furthermore let $\gamma : [0;b) \to M$ be a future inextendible causal curve such that $\gamma(0) \in K$. Assume there exists a sequence $(t_i)_i$ such that $t_i \to b$ and $\gamma(t_i) \in K$ for all $i \in \mathbb{N}$. Since K is compact there exists a subsequence $(t_j)_j$ such that $\gamma(t_j) \to p \in M$. Though γ is inextendible which means that it has no future endpoint. Thus we find another sequence $(s_j)_j$ such that $\gamma(s_j)$ does has no subsequence converging to p. This implies that we can find a neighbourhood $U \subset M$ of p such that no $s_j \in U$ exists. Since $\gamma(t_j) \to p$ we can find for every small neighbourhood $V \subset U$ a piece of γ which starts at an $\gamma(t_j) \in V$ then leaves U such that it reaches an $\gamma(s_l)$ with $s_l > t_j$ and then comes back again to U, ending at an $\gamma(t_k)$ with $k > j$. This contradicts strong causality.

(2) In 3.34 we have shown that inextendible curves have infinite h-arclength. Thus we only have to show that non-totally imprisonment implies the existence of a constant C as described above. As it turns out this is a direct consequence of our limit-curve theorems. To demonstrate this let $K \subset M$ be compact and assume (2) would be false. Hence there exists a sequence of causal curves $(\gamma_n)_n$ all contained in K such that $L_h(\gamma_n) \to \infty$, that is if we h-parameterize all of them their domain $[0; L_h(\gamma)]$ approach $\mathbb{R}_{\geq 0}$. Since they all start in K we can without loss of generality assume that $\gamma_n(0) \to p \in K$. Recapitulating the proof of Theorem 3.35 and 3.39 one notices, that the same procedure as in the case for all γ_n being inextendible can be carried out in this case. Thus we are able to construct an inextendible causal limit-curve contained in K. This contradicts non total imprisonment.

[9] Actually as C.Sämann points out the proofs in [16] only use the condition that there exists for every compact set K and $p;q \in K$ a $C > 0$ such that $L_h(\gamma) < C$ for all $\gamma \in C(p;q)$ contained in K. As we prove in Lemma 3.50 this follows from strong causality.

□

We are now prepared to prove the characterization of global hyperbolicity in the prior developed language of causality theory.

Theorem 3.51. (equivalent definition of global hyperbolicity) Let (M,g) a C^1 spacetime fulfill the strong causality[10] condition. Then (M,g) is globally hyperbolic if and only if for all $p,q \in M$: $C(p,q)$ is compact.

Proof. (cf. [16] Theorem 3.2)
Let (M,g) be a spacetime which fulfills the strong causality condition.

(1) $J^+(p) \cap J^-(q)$ is compact $\Longrightarrow$ $C(p,q)$ is compact
First we notice that due to [16] Theorem 2.8 (cf. 3.49), we can without loss of generality instead prove compactness of $\tilde{C}(p,q)$. This represents a crucial simplification, since now our limit-curve theorems are applicable. Since T_{co} is induced by a metric it suffices to prove sequential compactness. Thus let $(\gamma_n)_n$ be a sequence of curves in $\tilde{C}(p,q)$. From the preparatory Lemma 3.50 we know, that strong causality implies the existence of a constant $C > 0$ such that $L_h(\gamma_n) < C$. Remembering the Remark 3.49 we therefore know, that all Lipschitz constants of γ_n are bounded by C. Furthermore all γ_n start at p. Hence 3.39 is applicable and implies the existence of a causal limit-curve γ to which γ_n converges in T_{co}, that is uniformly. Since $\tilde{C}(p,q)$ is closed with respect to the uniform convergence topology, $\gamma \in \tilde{C}(p,q)$. Hence $\tilde{C}(p,q)$ is compact.

(2) $C(p,q)$ is compact $\Longrightarrow$ $J^+(p) \cap J^-(q)$ is compact
Again it suffices to prove sequential compactness. Thus let $(x_n)_n$ be a sequence contained in $J^+(p) \cap J^-(q)$. Thus for every x_n there exists an $\gamma_n \in \tilde{C}(p,q)$ such that $x_n \in \gamma_n([0,1])$. Now since $C(p,q)$ is compact, $\tilde{C}(p,q)$ is compact too. Hence there exists a subsequence of $(\gamma_n)_n$ which converges to an $\gamma \in \tilde{C}(p,q)$. The claim is now that there exists an $x \in \gamma([0,1])$ which is an accumulation point of $(x_n)_n$. Thus let $t_n \in [0,1]$ such that $\gamma_n(t_n) = x_n$. Since $[0,1]$ is compact we can without loss of generality assume $t_n \to s \in [0,1]$. We aim to prove that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \gamma_n(t_n) = \gamma(s)$. This follows from

$$\mathrm{dist}_h(\gamma(s), \gamma_n(t_n)) \leq \underbrace{\mathrm{dist}_h(\gamma(s), \gamma_n(s))}_{\to 0} + \underbrace{\mathrm{dist}_h(\gamma_n(s), \gamma_n(t_n))}_{\to 0} \qquad (78)$$

since $\gamma_n \to \gamma$ and all the γ_n are continuous. Thus by defining $x = \gamma(s)$ we have found an accumulation point and therefore, that $J^+(p) \cap J^-(q)$ is compact. □

The above characterization of global hyperbolicity may have been formulated in a different way , it's essence though stayed the same. This has the drawback that it is not really helping us to find practical examples where global hyperbolicity holds. The following discussion aims to shed some light on these issues. Through formulating an at first seemingly very different condition on spacetime, we are able to deduce an important result, which in the singularity theorems

[10] see footnote 9

often enables us to restrict our consideration to subsets of spacetime which are globally hyperbolic. The treatment will follow [16], where all of the following results are proven for C^0 metrics.

Definition 3.52. (Cauchy Hypersurface)
We call a subset $S \subset M$ a Cauchy hypersurface if every inextendible (future or past directed) causal curve meets S exactly once.

Remark 3.53. In [5] Lemma 14.29 it is proven, that for $g \in C^2$ a Cauchy hypersurface S indeed is a connected closed topological hypersurface (C^0-submanifold of dimension $n-1$). Furthermore this hypersurface is acausal which means that no two points of it can be connected by a casual curve. In [16] Prop.5.2 the result is generalized to C^0 metrics.

The claim is, that the existence of a Cauchy hypersurface already implies global hyperbolicity. We therefore have to prove that it implies strong causality and $J^+(p) \cap J^-(q)$, or as we saw in 3.51 $C(p,q)$ being compact for all $p, q \in M$. We start moderately by only proving the causality condition first.

Lemma 3.54. (Cauchy hypersurface $\Rightarrow$ Causality) The existence of a Cauchy hypersurface in a C^0 spacetime (M, g), implies the causality condition to be fulfilled.

Proof. Let (M, g) contain a Cauchy hypersurface S. Assume there would exist an $p \in M$ such that $p \in I^+(p)$. Thus there exists a timelike curve $\gamma : [0,1] \to M$ from which starts and ends at p. Thus by concatenating γ with it self infinitely many times we get $\gamma : [0, \infty) \to M$ an inextendible causal curve. Thus since S is a Cauchy hypersurface it meets S at some time t_0. Though our construction implies that $\gamma(t_0 + n) = \gamma(t_0)$ for all $n \in \mathbb{N}$. Thus γ meets S infinitely many times, which contradicts S being a Cauchy hypersurface. □

A procedure similar to the above, shows that a Cauchy hypersurfaces decomposes spacetime into three disjoint sets.

Lemma 3.55. (decomposition of spacetime, cf. [16] Lemma 5.4/5.5)
Let (M, g) be a C^0 spacetime. If it contains a Cauchy hypersurface S,
a) it decomposes into the disjoint union $M = I^+(S) \cup S \cup I^-(S) = I^+(S) \cup S \cup I^-(S)$. Hence $I^-(S) = I^-(S)$[11].
b) every inextendible causal curve has to intersect $I^+(S), I^-(S), S$.

Proof. a) (I) $M \subset I^+(S) \cup S \cup I^-(S)$
Let p be in M. Furthermore let γ be any fd-timelike path through p (which always can be constructed locally in a chart). Now in [14] Theorem 2.5.7 Piotr T. Chruściel demonstrates the existence of an inextendible timelike $\tilde{\gamma}$ extension. We shortly summarize the argument. Assume γ is extendible that is it has a future endpoint $q \in M$. Now call Γ_q the set of all fd-timelike h-parameterized

[11] as mentioned in 3.37 for C^1 metrics we have always $I^-(S) = I^-(S)$, though we have not shown a proof, which is why we prove it explicitely for this special in which it even holds for C^0 metrics

paths $: [0, L_h( )) \to M$ which start at q. As above $_q \neq \emptyset$. We can now define a partial order on $_q$ by defining $_1 \trianglelefteq {}_2$ if $_2$ is an extension of $_1$, that is $_{2}|_{[0, L_h(_1))} = {}_1$. For every chain $P \subseteq {}_q$ one can define an upper bound by $(t) := (t)$, where $\in P$ such that $t \leq L_h( )$. This defines a fd-timelike h-parameterized curve defined on the union of all the domains of curves in P. Thus we can apply Zorns-Lemma which tells us that $_p$ has a maximal element, which due to the definition of '$\trianglelefteq$' has to be inextendible, that is defined on $[0, \infty)$. Dually we can extend to the past such that the inextendible timelike curve $\tilde{}$ hast to meet S, which implies that $p \in I^+(S) \cup S \cup I^-(S)$.

(II) $S, I^+(S), I^-(S)$ are disjoint

Assume $p \in I^+(S) \cap I^-(S) \neq \emptyset$. As above this would imply the existence of an inextendible timelike curve which meets S at least twice. This contradicts S being a Cauchy hypersurface, thus $I^+(S) \cap I^-(S) = \emptyset$. Now assume that $p \in S \cap I^+(S)$. Hence the existence of an inextendible curve which meets S more than once would follow. Thus the first equality of a) is proven.

For the second equality we notice that S is also a Cauchy hypersurface for (M, g') if $g' \prec g$. The above therefore implies that $M = I^+_{g'}(S) \cup S \cup I^-_{g'}(S)$, as a disjoint union. Hence $M = I^+(S) \cup S \cup I^-(S)$ which has to be a disjoint union since $I^-(S) \subseteq I^-(S)$.

b) Assume hat $: \mathbb{R} \to M$ is an inextendible h-parameterized fd-causal curve which does not meet $I^+(S)$. We know that it has to meet S at some point and since cannot be constant or contained in S it also meets $I^-(S)$. Now let $t_1 > t_0 \in \mathbb{R}$ be such that $(t_1) \in I^-(S)$, $(t_0) \in S$. Hence there exists a fd-timelike curve from (t_1) to S. Concatenating this curve with $|_{t \geq t_1}$ we obtain a fd-causal curve which meets S twice, a contradiction. □

The preceding lemmas help us to finally prove global hyperbolicity in the presence of a Cauchy hypersurface.

Theorem 3.56. (Cauchy hypersurface $\Longrightarrow$ global hyperbolicity, cf. [16] Theorem 5.7 or [5] 14/Theorem 38 for $g \in C^2$)
Let (M, g) be a C^0-spacetime which contains a Cauchy hypersurface S. Then (M, g) is globally hyperbolic.

Proof. (I) <u>(M, g) fulfills the strong causality condition:</u>

We will follow the idea of the proof given in [5] 14/Theorem 38, though adjusted to our situation, that is to LLC curves and to the C^0-limit curve theorems (3.39). Assume (M, g) is not strongly causal, that is there exist an $p \in M$ such that for every $n \in \mathbb{N}$ there exists a fd-causal curve $_n$ which starts and ends in $B^h(p, \frac{1}{n})$ but is not contained in $B^h(p,)$, > 0. As previously we can assume that all $_n$ are proportional to h-arclength defined on $[0, 1]$. Furthermore since all $_n$ leave $B^h(p,)$ we know that $L_h(_n) \geq$. Since additionally $_n(0) \to p$ we would only need $L_h(_n)$ to be bounded from above so that we are in case (1) of 3.39. If this would indeed be true, we could obtain a closed causal limit curve which contradicts (M, g) fulfilling the causality condition (3.54). If instead $L_h(_n)$

were not bounded we can consider a subsequence consisting of causal curves parameterized by strictly increasing h-arclength (still denoted as γ_n). Thus, as we already mentioned previously we can use the same proof as in 3.39 to obtain an inextendible limit curve γ. Hence Lemma 3.55b) tells us that γ has to enter $I^+(S)$ at some time and never leave it again. We can therefore choose an $x = \gamma(t_1) \in I^+(S)$. The same procedure can be applied to the subsequence $(\tilde{\gamma}_n|_{[0,L_h(\gamma_n)-t_1]})_n$, with n large enough such that $L_h(\gamma_n) > t_1$. Here we defined $\tilde{\gamma}_n(t) := \gamma_n(L_h(\gamma_n) - t)$ as the corresponding past directed causal curves. We already assumed that $L_h(\gamma_n) \to \infty$ therefore also the h-arclengths of this new sequence of past directed curves is increasing and unbounded. This enables us to construct an inextendible past directed limit-sequence $\tilde{\gamma}$. Dually to the argument for future inextendible curves we have the existence of an $x' = \tilde{\gamma}(t_2) \in I^-(S)$. Hence for larger enough $n \in \mathbb{N}$ we would have: $\gamma_n(t_2) \in I^+(S) \cap I^-(S)$, a contradiction to S being acausal. Thus we conclude that (M, g) is strongly causal.

(II) <u>$\forall p, q \in M$: $\mathcal{C}(p, q)$ is compact:</u>

Since the topology on $\mathcal{C}(p, q)$ is induced by a metric, we only have to prove sequential compactness. Thus let $(\gamma_n)_n$ be an arbitrary sequence of causal curves in $\mathcal{C}(p, q)$. Since $L_h(\gamma_n) \geq \mathrm{dist}_h(p, q) > 0$ (we may assume $p \neq q$), the sequence $(L_h(\gamma_n))_n$ is bounded from below. Therefore if $(L_h(\gamma_n))_n$ would be bounded from above, the sequence would satisfy the conditions of Theorem 3.39 , which thus implies the existence of a limit-curve $\gamma \in \mathcal{C}(p, q)$. If instead $(L_h(\gamma_n))_n$ is unbounded from above, we can examine a subsequence (still denoted $(\gamma_n)_n$) which has strictly inreasing, diverging h-arclength. The argument now precedes as in (I). Again we would obtain an inextendible limit sequence γ by using 3.39 which thus has to enter $I^+(S)$ at some point $x = \gamma(t_1)$. Now again consider the subsequence of past directed h-parameterized curves $(\tilde{\gamma}_n|_{[0,L_h(\gamma_n)-t_1]})_n$, with $n \in \mathbb{N}$ big enough. Again they have unbounded, increasing h-arclengths which enables us to construct a past directed inextendible limit curve starting from q. Now the contradiction of the existence of an $\gamma_n(t_2) \in I^+(S) \cap I^-(S)$ for big enough $n \in \mathbb{N}$ can be derived exactly as in (I). Thus the h-arclengths of $(\gamma_n)_n$ must be bounded, which provides us with a converging subsequence as described above. □

We defined a Cauchy hypersurface as a set which is met by every inextendible causal curve in (M, g) exactly once. For a general set $A \subset M$ would it not be possible to just restrict our spacetime such that the above condition is fulfilled? This would as advertised before provide us with a subspace of our spacetime which is globally hyperbolic. The following definition will be needed to make this idea formal:

Definition 3.57. (Cauchy development) Let $A \subset M$ be an arbitrary subset. We define the future Cauchy development of A as :

$$D^+(A) := \{q \in M \mid \text{every past directed, past inextendible causal curve through } q \text{ meets } A\} \quad (79)$$

After analogously defining $D^-(A)$ we define the Cauchy development of a subset $A \subseteq M$ as : $D(S) := D^+(A) \cup D^-(A)$.

Corollary 3.58. (cf. [16] Corollary 5.8) Let $A \subseteq M$ be acausal (no two points can be connected by a casual curve), then $\operatorname{int}(D(A))$ is globally hyperbolic.

Proof. The argument is similar as in Theorem 3.56. For the C^2 case the proof is given in ([5] 14/Theorem 38).

□

Remark 3.59.
1. In fact one can also show (cf. [16] Theorem 5.9) that for C^0-spacetimes being globally hyperbolic already implies the existence of a Cauchy hypersurface. In fact if (M, g) is globally hyperbolic one always finds a homeomorphism $h : M \to \mathbb{R} \times S$ such that S is a Cauchy hypersurface. This clearly demonstrates how strong the condition of global hyperbolicity actually is.
2. A rather lengthy argument in [5] (14/Lemma 43, here $g \in C^2$) demonstrates that, if $A \subseteq M$ is not only acausal but also a topological hypersurface then $D(A)$ is open and therefore by our previous argument globally hyperbolic. This result will be needed subsequently, though since its proof would require some concepts not necessarily needed elsewhere for the theorems, we will just take it as given.

3.3.5 Maximal curves

The preceding quite lengthy discussion of global hyperbolicity and its characterizations now needs to be justified by demonstrating how it is exactly this condition which we will need for the singularity theorems. As the singularity theorems in general aim to prove geodesic incompleteness it is only natural to further study properties of their existence. In Riemannian geometry the study of minimizing curves and geodesics is closely intertwined. Thus one could hope to achieve similar insights in Lorentzian geometry. Though the twin paradox (3.27) already demonstrated that instead we should analyse maximisation properties of curves. This motivates our next definition.

Definition 3.60. (time separation) We define the time separation function $\tau : M \times M \to \mathbb{R} \cup \{\infty\}$ as :

$$\tau(p, q) := \begin{cases} \sup L_g(\gamma) & \text{if } p \leq q \\ 0 & \text{else} \end{cases} \quad (80)$$

where γ is a LLC fd-causal curve. In fact we can always assume $\gamma \in \mathcal{C}(p, q)$.

A rst property of can already be derived without any additional demands on the causality of $(M; g))$.

Lemma 3.61. The time separation function is lower-semi continuous.

Proof. The proof is based on the twin paradox and given in [5] 14/Lemma 17. Instead of repeating the argument we will draw a picture to demonstrate the idea (Fig.2).

Figure 2: (Lemma 3.27) Let (red) be a timelike curve such that $L_g( )$ $(p; q)$ for some > 0. For $U; V$ convex neighbourhoods of p respective q we nd $p^0; q^0$ as above and small enough neighbourhoods given by the grey circles such that by the Twinparadox for every $x; y$ depicted above we have $L_g( ) > (p; q) \quad 3$. Where is the concatenation of with the straight geodesics connecting y to p^0 and q^0 to x. In particular we have for all such $x; y$: $(x; y) \quad (p; q) \quad 3$.

□

The following can be viewed as a motivation to de ne global hyperbolicity in the rst place.

Lemma 3.62. Let (M, g) be globally hyperbolic C^2-spacetime. Then for any $p, q \in M$ such that $p < q$ there exists a causal geodesic γ connecting p and q such that : $L_g(\gamma) = \tau(p, q)$.

Proof. The definition of $\tau(p, q)$ as a supremum ensures the existence of a sequence $(\gamma_n)_n \subset C(p, q)$ such that $L_g(\gamma_n) \to \tau(p, q)$. If $L_h(\gamma_n)$ were unbounded there would exist a subsequence of strictly increasing, diverging h-arclength $(\gamma_{n_k})_k$. After h-parametrization we therefore obtain a sequence of curves $(\hat\gamma_{n_k})_k$ with $\hat\gamma_{n_k} : [0, L_h(\gamma_{n_k})] \to J(p, q)$. Thus again using the limit curve theorem 3.35 we would obtain an inextendible curve contained in $J(p, q)$. Since (M, g) is globally hyperbolic it is non-totally imprisoning and $J(p, q)$ is compact. Thus we arrived at a contradiction. Hence $(L_h(\gamma_n))_n$ has to be bounded. We are therefore in case (1) of 3.35 and obtain a subsequence, still denoted as $(\gamma_n)_n$ and converging uniformly to a causal limit curve $\gamma \in C(p, q)$. Though we do not know if $L_g(\gamma_n) \to L_h(\gamma)$ since we only have the convergence of curves and not of their derivatives. Up until now the argument could have been formulated equivalently in C^1-spacetimes. Though to proceed we will make explicit use of g being at least C^2. Since $\gamma([0, 1])$ is compact we can cover it with finitely many precompact convex neighbourhoods Z_i. Furthermore since (M, g) is globally hyperbolic and therefore in particular fulfills the strong causality condition we can assume Z_i to be causally convex. Now choose Z_1 such that $p_0 := p \in Z_1$. If $q \in Z_1$ define γ as the unique geodesic connecting p and q in Z_1. Since every causal geodesic leaving Z_1 cannot enter it again, γ in fact is globally maximizing by the twin paradox (3.27). If instead $q \notin Z_1$, γ and therefore infinitely many γ_n have to leave Z_1. Now define $p_1^n := \gamma_n(t_1^n)$ as the first intersection point of γ_n with ∂Z_1. Since ∂Z_1 is compact, (p_1^n) has an accumulation point : p_1. We can now choose an Z_2 which contains p_1 and therefore infinitely many p_1^n (always by instead considering the subsequence such that $p_1^n \to p_1$ and $p_1^n \in Z_2$ for all $n \in \mathbb{N}$, which we will still denote with $(\gamma_n)_n$). We can proceed similarly until reaching q. In fact q is reached after finite iterations since no γ_n can reenter Z_i after once leaving it ($\Leftarrow$ causally convex). We are now able to define for each n in the constructed subsequence a piece-wise geodesic by connecting p_i^n to p_{i+1}^n in Z_{i+1}. The same can be done for the sequence (p_i) of accumulation points to construct a piece-wise geodesic denoted as γ. Importantly the twin paradox 3.27 now tells us that :

$$L_g(\gamma_n|_{p_i^n \to p_{i+1}^n}) \leq L_g(\tilde\gamma_n|_{p_i^n \to p_{i+1}^n}) \tag{81}$$

$$\Longrightarrow \quad L_g(\gamma_n) \leq \sum_i L_g(\tilde\gamma_n|_{p_i^n \to p_{i+1}^n}) = \sum_i |\mathrm{Exp}^{-1}(p_i^n, p_{i+1}^n)| \tag{82}$$

Thus in the limit of $n \to \infty$:

$$\tau(p, q) \leq L_g(\gamma) \Longrightarrow \tau(p, q) = L_g(\gamma) \tag{83}$$

If γ would have breaks, we could use the twin paradox 3.27 to construct a longer causal curve :
Let t_0 be a break of γ and V a convex neighbourhood of $\gamma(t_0)$. Now choose

any $t_1 < t_0$ and $t_2 > t_0$ such that $_{j[t_1;t_2]}$ V . Since 2 $C^{2;pc}$ we can use the twin paradox, to obtain a strictly longer curve : $_{jt\ t_1}$ $_{jt\ t_2}$ where is the unique geodesic from (t_1) to (t_2) contained in V. Hence a maximizing curve cannot have breaks, that is is a geodesic.

□

For later purposes we will need the existence of such maximizing geodesics, not only between individual points, but also between compact sets and a point. Since continuous maps always take maximal values on compact sets, it only remains to show that is upper-semi continuous if $(M; g)$ is globally hyperbolic.

Lemma 3.63. If $(M; g)$ is globally hyperbolic, the time separation function is continuous.

Proof. (cf. [5] 14/ Lemma 21) We only sketch the proof since it is quite similar to the previous one. Assume is not upper-semi continuous, that is there exist $p; q$ 2 M a > 0 and sequences p_n ! p, q_n ! q such that : $(p_n; q_n) >$ $(p; q)+$. Now choose any p 2 I (p) and q^+ 2 $I^+(q)$. Since I is open, we have for large enough n : p_n 2 $I^+(p\ )$ and q_n 2 I (q^+). Now choose for each n 2 N an $_n$ 2 $C(p_n; q_n)$ such that $L_h(\ _n) >$ $(p_n; q_n)$ $\frac{1}{n}$. By concatenating the timelike curves from p to p_n with $_n$ and the timelike curve from q_n to q^+ we obtain causal curves, when rightly parameterized in $C(p\ ; q^+)$. Now an argument similar as in Lemma 3.62 proves the existence of piece-wise geodesic 2 $C(p; q)$ such that $L_g(\ )$ $(p; q)+$, a contradiction to the de nition of . □

We are now nally prepared to formulate the fundamental result on which the rst singularity theorem will be based.

Lemma 3.64. (cf. [5] 14/ Theorem 44) Let S M be a closed acausal topological hypersurface in the C^2-spacetime $(M; g)$. Then for every q 2 $D^+(S)$ there exists a geodesic from S to q of length $(S; q)$.

Remark 3.65. In the above lemma we used: $(S; q) := \sup_{p 2 S}(\ (p; q))$

Proof. As mentioned in 3.59 D(S) is open and globally hyperbolic. Since $(p; q) = 0$ if p 2= J (q) we only have to take the supremum on J (q) \ S. The claim is, that J (q) \ S is compact. Indeed let $(x_n)_n$ be a sequence in J (q) \ S and $_n$ respectively causal curves 2 $C(x_n; q)$ which are contained in $D^+(S)$. Now consider the past-directed causal curves $\hat{\ }(t) :=$ (1 t). If their h-arclength would be unbounded we would obtain by 3.35 an inextendible limit curve in D(S). Thus it would have to meet I (S) which is impossible since no $_n$ enters I (S). Thus the h-arclengths are bounded and we obtain a past directed causal limit curve ending at an x, which is contained in S since it is closed and x_n ! x. Thus we have proven that J (q) \ S is compact. Hence by continuity of , there exists an p 2 J (q) \ S such that : $(S; q) =$ $(p; q)$. Additionally lemma 3.62 provides the existence of a maximal geodesic such that $(S; q) =$ $(p; q) = L_g(\ )$.

□

The fundamental result underlying all of the proofs above was the twin paradox which is based on the existence of normal neighbourhoods. It seems therefore questionable if the above results could be transferred to the C^1-case. Indeed in C^1-spacetimes it is unknown whether maximal curves have to be geodesics[12]. Luckily, we will only need the mere existence of a maximal geodesic and not the fact that all such maximizing curves are geodesics already. As it turns out this weakened statement remains true for C^1- spacetimes.

Theorem 3.66. (maximal geodescs in C^1-spacetime, cf. [15] Prop.2.13)
Let $(M; g)$ be a C^1-spaectime which is globally hyperbolic. Also let $(g\ )$ be a net of smooth metrics such that $g \quad g$ and $g\ ! \quad g$ in C^1_{loc}. Then for any $p < q$ in M there exists an L_g-maximizing causal geodesic $: [0; 1] \quad M$ from p to q which is the C^1-limit curve of a sequence of $L_{g\ n}$-maximizing , $g_{\ n}$-causal geodesics $_n : [0; 1] \quad M$ for $_n\ ! \quad 0$.

Remark 3.67. Importantly both approximations, form the inside $(g\ )$ and outside $(\hat{g}\ )$, can always be explicitly constructed by convolution on manifolds (see Prop.6.15).

Proof. We assumed that $g \quad g$ and $g\ ! \quad g$ in C^1_{loc}, though we do not know whether $g \quad g_{\ 0}$ for $^0 < \quad$. A short argument (similar to [16] Lemma 1.4) shows that in our case this can without loss of generality be assumed. In fact since $([0; 1])\ 2\ J_g\ (p; q) \quad J_g(p; q)$ which is compact by global hyperbolicity, we can restrict our considerations to $J_g(p; q)$. Now choose any $_0 > 0$. Since $g_{\ 0} \quad g$ we have that $g_{\ 0}(X; X\ ) \quad 0 \implies g(X; X\ ) < 0$. Now de ne $B := f X\ 2\ TM_{J(p;q)} j\ jX j_h = 1 ; g(X; X\ ) \quad g$. The claim is, that there exists a $\ > 0$ such that $K := f X\ 2\ TM_{J_g(p;q)} j\ jX j_h = 1 ; g_{\ 0}(X; X\ ) \quad 0 g \quad B$. If this would not be the case we could construct a sequence $X_n\ 2\ K$ such that $g(X_n ; X_n) > \frac{1}{n}$. Now K is a compact subset of TM. Thus by restricting to a subsequence we can assume that $X_n\ ! \quad X\ 2\ K$. It follows that $g_{\ 0}(X; X\ ) \quad 0$ but $g(X; X\ ) \quad 0$, a contradiction to $g_{\ 0} \quad g$. Hence we can always nd such a < 0. Now since $g\ ! \quad g$ we can nd an $_1 < \quad _0$ such that $kg \quad g_{\ 1} k_{1\ ;J_g(p;q)} < \quad$. Thus if $g_{\ 0}(X; X\ ) \quad 0$ then $g(X; X\ ) \qquad$ such that $g_{\ 1}(X; X\ ) \quad g(X; X\ ) + kg \quad g_{\ 1} k_{1\ ;J_g(p;q)} < 0$. Hence $g_{\ 0} \quad g_{\ 1}$. Repeatedly using this argument we can construct a sequence $(\ _n)_n$ such that $_n\ ! \quad 0$ and $g_{\ k} \quad g_{\ l}$ if $k < l$.

(I) $p << _g q$

We already mentioned (and proved for the globally hyperbolic case see 3.55) that for C^1-spacetimes $I^+_g(p) = I^+_g(A) = \bigcup_{g^0\ g} I^+_{g^0}(A)$. Hence there exists an $_0$ such that $p << _{g\ 0}\ q$. Now construct a sequence of smooth approximation metrics $(g_{\ n})_n$ as above. Since the time cones are getting larger for higher n we also have $p << _{g\ n}\ q$. Furthermore a Cauchy hypersurface in $(M; g)$ is also one in each $(M; g_{\ n})$ since $g_{\ n} \quad g$. Hence $(M; g_{\ n})$ is globally hyperbolic and we can apply

[12] [2] 2.2/p.12

Lemma 3.62 to obtain a $L_{g_{\epsilon_n}}$-maximizing timelike geodesic $\gamma_n : [0,1] \to M$ from p to q as described when formulating the lemma.

We now aim to apply an argument similar as in Theorem 3.43. Though to do so, we need to know that we can assume $\dot\gamma_n(0) \to v \in T_pM$. Without loss of generality we can affinely reparameterize all γ_n such that $|\dot\gamma_n|_h(0) = 1$ (though in general $|\dot\gamma_n|_h(t) \neq 1$ for $t>0$). Hence we can assume $\dot\gamma_n(0) \to v \in T_pM$ for those reparametrized curves $\gamma_n : [0,t_n] \to M$ by restricting to a subsequence. Recapitulating the proof of 3.43 one observes, that inextendibility of the geodesics is only used for the fact that they eventual leave every compact set of the exhaustion K_i. Thus if we find for every K_i and $n_0 \in \mathbb{N}$ an $n \geq n_0$ such that γ_n leaves K_i we can make the same argument as in 3.43 to obtain an inextendible limit geodesic. Since all γ_n are contained in $J_g(p,q)$ the same holds true for γ. This contradicts non-total imprisonment in globally hyperbolic spacetimes. Hence there exists an $i \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $(\gamma_n)_{n \geq n_0} \subseteq K_i$. We are therefore in similar situation as in Theorem 3.42. In fact we can now reparameterize all γ_n back to be defined on $[0,1]$. Then the proof of 3.42 can be applied and provides us with a causal limit geodesic $\gamma : [0,1] \to M$ which starts at p and ends at q. It therefore only remains to prove that γ indeed is L_g-maximizing. Though as we will see this 'only' should not be taken as to literally. As it turns out we first have to prove the existence of a maximizing geodesic in a different way to come back later proving that γ in fact is maximizing too. We shortly discuss where the problems lie in proving the maximality of γ, which will then lead us to a new approach.

Preferably one would like to estimate the proper time of a g-causal geodesic c by the proper time it has for smooth approximations. The problem we are facing in doing so, is that we do not know whether a g-causal curve is $\check g_\epsilon$-causal. This would change if we instead could examine only timelike curves. Though we do not yet know if this is permissible (in fact by proving this lemma we simultaneously prove this fact too). Hence we need a different approach. The problem of causality would vanish if we instead approximate g by smooth metrics of larger lightcones. Thus let $(\hat g_\epsilon)$ be a net of smooth approximations of g such that $\hat g_\epsilon \to g$ in C^1_{loc}, and c an arbitrary g-causal curve proportional to h-parameterized on $[0,1]$. Thus it is also $\hat g_\epsilon$-causal for all $\epsilon \geq 0$. Hence for ϵ small enough such that $\|g - \hat g_\epsilon\|_\infty \leq \delta$, for $\delta > 0$, the following inequality holds:

$$L_g(c) = \int_0^1 \sqrt{-g(\dot c,\dot c)}\, dt \leq \int_0^1 \sqrt{-\hat g_\epsilon(\dot c,\dot c) + C^2\delta}\, dt \leq \tau_{\hat g_\epsilon}(p,q) + C\sqrt{\delta} \tag{84}$$

Here $C > 0$ is the bound on h-arclengths which exists due to $J_g(p,q)$ being compact and (M,g) being non-total imprisoning. If it would be possible (as it has been for $(\check g_\epsilon)$) to construct a sequence $\hat g_{\epsilon_n}$ and corresponding maximizing geodesics γ'_n which converge in C^1_{loc} to a g-geodesic γ', Equation (84) would imply :

$$L_g(c) \leq L_{\hat g_{\epsilon_n}}(\gamma'_n) + C\sqrt{\delta_n} \tag{85}$$

$$\Longrightarrow \quad \tau_g(p,q) \leq \liminf L_{\hat g_{\epsilon_n}}(\gamma'_n) = L_g(\gamma') \tag{86}$$

Where the last equation followed from uniform $C^1([0;1])$ convergence. Hence the maximality of ^{0} would have been proven. It therefore only remains to proof the existence of such ^0_n. In [16] Theorem 4.5 a rather technical argument proves the following stability result:

Theorem 3.68. (cf. [16] Theorem 4.5) For a C^0-spacetime $(M;g)$ which is globally hyperbolic there always exists a smooth g^0 such that $(M;g^0)$ is globally hyperbolic.

In the beginning of this proof we demonstrated that for g g and g ! g with g g, $J_g(p;q)$ being compact implies the existence of an $_1$ such that g g_1 g on K M compact. Essentially the same argument implies that for g g^0, $\check{g}$! g with g $\check{g}$ and K M compact, we have the existence of an $_K$ such that g $\check{g}_K$ g^0 on K M. For proving that we can choose a sequence $(g_n)_n$ such that each $(M;g_n)$ is globally hyperbolic we would need to globalize the preceding argument such that g $\check{g}_n$ g^0 on M. As shown in [20] Lemma 4.3 this in fact is possible. We shortly repeat the argument given there: Let $(K_n)_n$ be a compact exhaustion of M $(K_0 = ;)$ such that K_n int (K_{n+1}). As shown in [?] Lemma 2.7.3 there exists a smooth function : M! R such that $0 <$ (x) $_{K_n}$ if x 2 K_n int (K_{n-1}). Where $_{K_n}$ is small enough such that g $\check{g}_{K_n}$ g^0 on K_n. Now define $\check{g}_1(x) := \check{g}_{(x)}(x)$. Thus $\check{g}_1$ is a smooth metric. Most importantly it fulfills g $\check{g}_1$ g^0. For any x 2 M there exists an n 2 N such that x 2 K_n K_{n-1}. Then (x) $_{K_n}$, that is $\check{g}_{(x)}$ $g^0(x)$. Hence we can construct a sequence $(\check{g}_n)_n$ such that g $\check{g}_{n+1}$ $\check{g}_n$ g^0, which implies that all $(M;\check{g}_n)$ are globally hyperbolic. Therefore the existence of maximizing geodesics ^0_n is ensured and by restricting to a subsequence we are now able to construct by the same argument as before a g-causal limit geodesic ^{0}. Hence ^{0} is a maximal g-causal g-geodesic. Since $p <<_g q$ we furthermore have $_g(p;q) > 0$, hence ^{0} is timelike.

We can now finally come back to prove the maximality of the g-causal limit geodesic constructed by using smooth approximations from inside. Since ^{0} is timelike and C^1 there exists a constant C^0, such that $g(\dot{}^0;\dot{}^0) <$ C^0. Therefore $\dot{}^0$ is also timelike for g_n for large enough n 2 N. By the same argument as in Equation 84 we thus have :

$$_g(p;q) = L_g(\;^0) \quad _{g_n}(p;q) + C_n = L_{g_n}(\;_n) + C_n \;!\; L_g(\;) \qquad (87)$$

Hence the maximality of is proven.

(II) <u>$p <_g q$ but not $p <<_g q$</u>

First one could repeat the argument of (I) where the existence of a maximal geodesic was proven by using g $\check{g}$ approximations form outside. In fact the argument can be applied without any changes, figuratively because the $\check{g}$ can not 'differentiate' between $<_g$ and $<<_g$. Hence we obtain a maximizing limit-geodesic ^{0}. Since geodesics have a fixed causal character (see 3.41) ^{0} has to be null, that is $_g(p;q) = 0$.

We now want to construct causal g g-geodesics. To do so we have to find

points close to q which can be reached by g-causal curves in the first place. In fact we can choose those $q_n \in \partial J^+_{g_{\epsilon_n}}(p)$ such that they are connected to p by a null geodesic (see 3.32). This is possible due to an argument given in [21] Theorem 5.3: Let U_n be a descending sequence of neighbourhoods of q such that: $\cap_n U_n = \{q\}$ (for example $B^h(q, \frac{1}{n})$). In [17] Lemma 1.22 the push-up lemma we described in 3.32 for the C^2-case, is proven also for C^1-metrics. Hence the same argument as in b.3) of 3.32 can be applied to obtain that $J^+_g(p) \subseteq \overline{I^+_g(p)}$. Thus $q \in J^+_g(p) \setminus I^+_g(p) \subseteq \partial J^+_g(p)$. We therefore find for every $n \in \mathbb{N}$ a point $q^e_n \in U_n \setminus \overline{J^+_g(p)}$ and $q^i_n \in U_n \cap \operatorname{int}(J^+_g(p)) = U_n \cap I^+_g(p)$ (here e:exterior, i:interior). Furthermore since $I^+(p) = I^+_g(p)$, we can always find a small enough ϵ_n such that $q^i_n \in I_{g_{\epsilon_n}}(p)$. Without loss of generality we can also assume that $\epsilon_n < \frac{1}{n}$. Now choose an arbitrary path from q^i_n to $q^e_n \in U_n \setminus \overline{J^+_g(p)} \subseteq U_n \setminus \overline{J^+_{g_{\epsilon_n}}(p)}$ which therefore has to meet $\partial J^+_{g_{\epsilon_n}}$ at some point q_n.
As in (I) we have that (M, g_{ϵ_n}) is globally hyperbolic which provides us with the existence of maximizing g_{ϵ_n}-null geodesics $\gamma_n : [0,1] \to M$ from p to q_n. Now choose any $q^+ \in I^+_g(q)$ such that $q_n \in I^-_g(q^+)$ for large enough $n \in \mathbb{N}$. In particular, those γ_n are then contained in the compact set $J_g(p, q^+)$. Hence by the same argument as in (I) we obtain, a $C^1([0,1])$- limit g-geodesic $\gamma : [0,1] \to M$ from p to q. Since every γ_n is g_{ϵ_n}-null and $g_{\epsilon_n} \to g$ uniformly on $J_g(p, q^+)$ we have that γ is g-null. We already argued that $\tau_g(p, q) = 0$. Hence γ is maximizing. ☐

In [2] two crucial corollaries (Corollary 2.15/2.16) from the above theorem are stated.

Corollary 3.69. *Let $S \subseteq M$ be a smooth be smooth spacelike*[13] *Cauchy hypersurface contained in a C^1-spacetime (M, g). Then for any $q \in I^+(S) \subseteq D^+(S)$ there exists a maximal g-timelike, g-geodesic from S to q. In fact this g-geodesic can be obtained as a limit of g_{ϵ_n}-geodesics each maximizing g_{ϵ_n}- length from S to q.*

Remark 3.70. In Section 4 we will see that for C^2-metrics, curves which maximize length between a spacelike hypersurface and a point, not only have to be timelike geodesics, but also must be normal to the hypersurface. Hence, our obtained limit-geodesic has to be normal to S too.

Proof. Since S is a smooth, spacelike Cauchy hypersurface it is in particular a closed acausal topological hypersurface. Hence just as in Lemma 3.64 we can deduce that for every $q \in I^+_g(S)$: $J^-_g(q) \cap S$ is compact. Hence $J^+_g(J^-_g(q) \cap S) \cap J^-_g(q)$ is compact[14] Now replace in part (I) of the previous proof $J_g(p, q)$ with $J^+_g(J^-_g(q) \cap S) \cap J^-_g(q)$. To prove the existence of a maximizing geodesic we again

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[13] spacelike will be needed to apply the results of 4 , showing the failure of maximality of timelike geodesics after some finite proper time

[14] For a globally hyperbolic spacetime and K_1, K_2 compact subsets we always have $J^+(K_1) \cap J^-(K_2)$ being compact. This is shown in [16] Corollary 3.4 and follows from causal simplicity in globally hyperbolic spacetimes.

have to work with globally hyperbolic approximations from outside $g \prec \check{g}_n \prec g'$. It may seem to be a problem, that S not necessarily is a Cauchy hypersurface for $(M, \check{g}_n)$ anymore. Though we do not need a maximal curve respective to S but only of such curves starting in $J_g^-(q) \cap S$. Therefore $\check{g}_n$ being continuous already assures the existence of a maximal $\check{g}_n$-geodesic emerging form the compact set $J_g^-(q) \cap S$. For the existence of a limit geodesic which is not inextendible, we relied on the fact, that all geodesics were contained in the compact set $J_{\check{g}_0}(p, q)$. In our situation we can instead use $J^+_{\check{g}_0}(J_g^-(q) \cap S) \cap J^-_{\check{g}_0}(q)$, which is compact due to global hyperbolicity of $(M, \check{g}_0)$[15]. Hence the existence of a S-maximizing g-geodesic can be proven by following the same reasoning as in part (I) of the previous proof 3.66. Indeed also for proving that such a maximal geodesic can be obtained as the C^1-limit of maximal $\check{g}_n$-geodesics we can rely on the same argument as in 3.66. In this case it is even more direct since S is a Cauchy hypersurface for every $\check{g}_n \prec g$. □

Corollary 3.71. Let (M, g) be a globally hyperbolic C^1-spacetime. Furthermore let $N \subset M$ be a compact[16], spacelike, (n-2)-dimensional submanifold. Then for any $q \in J_g^+(N) \setminus I_g^+(N)$ there exists a g-null, g-geodesic which maximizes length to N and is obtained as the C^1-limit curve of maximizing $\check{g}_n$-null, $\check{g}_n$-geodesics. As in the preceding corollary this obtained limit geodesic, then has to be normal to N.

Proof. If $\tau_g(N, q) > 0$ then there would exist a point $p \in N$ such that $\tau(p, q) > 0$. Thus by Theorem 3.66 and (M, g) being globally hyperbolic, there would exist a maximal g-geodesic from p to q which therefore must be g-timelike. This is a contradiction to $q \in J_g^+(N) \setminus I_g^+(N)$, hence $\tau_g(N, q) = 0$ (as we will see in 4.16 in the smooth case this follows without global hyperbolicity). Similarly as in part (II) of 3.66 we can construct a sequence $q_n \in \partial J^+_{\check{g}_n}(N)$ which converges to q. Using that $(M, \check{g}_n)$ is globally hyperbolic we therefore can construct a sequence of $\check{g}_n$-null geodesics each starting at a point $p_n \in N$ and ending at q_n. As before let us choose a $q^+ \in I_g^+(q)$ and consider only $n \in \mathbb{N}$ large enough such that $q_n \in I_g^-(q^+)$. Since N is assumed to be compact there exists a subsequence $p_{n_k} \to p \in N$. Finally global hyperbolicity implies causal simplicity that is $J^-(q^+) \cap N$ is compact and therefore all $\check{g}_n$-null geodesics are contained in $J_g^+(J_g^-(q^+) \cap N) \cap J_g^-(q^+)$ which again is compact by global hyperbolicity. We can therefore construct as in part (II) of 3.66 a limit g-null, g-geodesic, from $p \in N$ to q, which therefore has to be maximal. □

4 Calculus of Variations

In the preceding section we discussed those causal properties which provide us with the existence of proper time-maximizing geodesics. The strategy underlying most of the singularity theorems is now, to similarly obtain conditions (most often formulated as initial and curvature conditions) which force geodesics to

[15] see 14

stop being maximizing. If a given spacetime therefore fulfills those causal, initial and curvature conditions, geodesics would have to be incomplete, that is only defined up to a finite parameter. Thus a 'singular' character of the spacetime under examination would have been proven.
In general there are two, from their character different approaches to find such conditions, forcing geodesics to stop being maximizing. The first one, which investigates the behavior of the so called Raychaudhuri equation and geodesic congruences, actually started the analysis of singularities in the first place (cf. [22], 1955). Even though this approach may be the more illuminating one[17], we will not discuss it much further here. The second approach is based on variational calculus. In fact, both of them show the failure of maximality by the existence of focal points. Though in the second approach it is often possible to prove their existence just by making the right guesses and therefore allows a shorter and often simpler treatment. Hence in the following we will briefly summarize the main results needed for the singularity theorems. The proofs of those results mostly consist of technical calculations and constructions and as such are interesting in their own rights, but not necessary illuminating for our further understanding of causality and singularities. Thus instead of repeating the proofs which are extensively discussed in [5] Chapter 10, we will settle for just stating the results.
As one will shortly notice, in this section a C^2-differentiability of the metric is truly necessary. Hence from now, just as in the beginning let $g \in C^1$. This may seem to be quite an obstruction to formulate C^1-singularity theorems. Though as we will see later, the idea of the C^1-singularity theorems does not rely on proving them on their own but instead proving that for close enough smooth approximations the conditions for the C^1 - theorems hold. By use of 3.43 one thereafter can deduce a singular behaviour for the C^1-spacetime.

4.1 General results

As the term 'Calculus of Variations' rightly suggests we want to do calculus with local variations of a given curve in M. That is in the end we will use similar arguments as in real analysis to analyse the existence of local maximas of the length functional L_g. Though to do so we first have to define what a variation of a given curve is:

Definition 4.1. Let P be a smooth spacelike submanifold of M and $q \in M$. That is, it is a a manifold embedded in M such that $g_{|P}$ is a Riemannian metric.

1. We define similarly as in (3.48, Space of causal curves)

$$(P;q) := \{\gamma : [0;b] \to M \mid \gamma \in C^{1;pc}\} \qquad (88)$$

Here we used $C^{1;pc}$, since for variational calculus we need enough differentiability to 'variate' in the first place. While $C^{1;pc}$ may seems to be far

[17] In [23] a introduction to general relativity is given, by taking the Raychaudhuri equation as the fundamental starting point

to much than needed, it will allow us to forget about any differentiability problems.

2. A $(P;q)$-variation of $\gamma \in \Omega(P;q)$ is defined as a continuous two-parameter map α:
$$\alpha : [0;b] \times (-\epsilon;\epsilon) \to M \tag{89}$$
such that :
(1) $\alpha(t;0) = \gamma(t)$
(2) $\forall s \in (-\epsilon;\epsilon) : \alpha(t;s) \in \Omega(P;q)$
(3) there exist $0 = t_0 < t_1 < ::: < t_N = b$ such that $\alpha|_{[t_i;t_{i+1}] \times (-\epsilon;\epsilon)}$ is C^1 for all $i = 0;:::;N-1$

3. We call $V \in \Gamma^{1;pc}(\gamma^* TM)$ a continuous piecewise C^1 section of the pull back bundle (cf. [4] 6.7 for the definition of the pull back bundle) such that $V(0) \in P$ and $V(b) = 0$ a $(P;q)$-variation vector field of $\gamma \in \Omega(P;q)$.

Remark 4.2. 1. Every $(P;q)$-variation vector field V on γ can be written as $\partial_s \alpha(t;0)$ for a $(P;q)$-variaton of γ (cf. [13] Hilfssatz 8.4). Hence $(P;q)$-variation vector fields, can be viewed as local models for a $(P;q)$-variation of a given curve $\gamma \in \Omega(P;q)$.

2. Later we will also need a more general concept of variations and variation vector fields which do not necessarily obey $\alpha(b;s) = q$ for every s. Though since we still demand all the other conditions, we are just changing the suffix $(P;q)$ to P.

The concepts introduced above now allow us to examine extremal properties of the length functional L_g by classical methods of real analysis. In fact by concatenating L_g with a given $(P;q)$-variation α we get : $L_g \circ \alpha(s) : (-\epsilon;\epsilon) \to \mathbb{R}_{\geq 0}$ which due to the leibniz-integral rule is ininitely often differentiable at $\epsilon = 0$ if $|\dot\gamma|_g > 0$. The null case (where $|\dot\gamma|_g = 0$ will be mentioned later). Indeed using the leibniz-integral rule one calculates:

Lemma 4.3. (First variation formulas) Let γ be a curve in $\Omega(P;q)$ such that $|\dot\gamma|_g > 0$ and α a $(P;q)$-variation with associated $(P;q)$-variation vector field V. Since every curve such that $|\dot\gamma|_g > 0$ can be reparameterized such that $|\dot\gamma|_g = 1$ we will assume γ to be of unit speed. Then the first variation of length is given by:
$$\delta L_g(V) := \frac{d(L_g \circ \alpha)}{ds}\Big|_{s=0} = \int_0^b g(\dot\gamma; \dot V)\,dt \tag{90}$$
Where $\epsilon := \mathrm{sign}(g(\dot\gamma;\dot\gamma))$ and $\dot V = \nabla_{\partial_t} V$.
Furthermore if we define at each break $i = 2;:::;N-1$: $\Delta\dot\gamma(t_j) := \dot\gamma(t_j^+) - \dot\gamma(t_j^-)$ we get the following equivalent formula:
$$\delta L_g(V) = -\int_0^b g(\ddot\gamma; V)\,dt - \sum_{i=2}^{N-1} g(\Delta\dot\gamma(t_j); V(t_j)) + g(\dot\gamma; V)\Big|_a^b \tag{91}$$

Corollary 4.4. (cf. [5] 14/Corollary 26)
Let $\gamma \in \Omega(P;q)$ such that $|\dot\gamma|_g > 0$. Then $dL_g(V) = 0$ for every $(P;q)$-variation vector field V, that is for every $(P;q)$-variation α, is equivalent to γ being a geodesic such that $\dot\gamma(0) \perp T_{\gamma(0)}P$. In particular since P is spacelike γ has to be timelike.

Proof. If γ is a geodesic starting normally to P, then 4.3 implies $dL_g(V) = 0$. The other direction is proven by choosing appropriate bump functions f and then evaluating $dL_g(fY)$ where Y is the parallel translation of some tangent vector y on γ. □

To find sufficient conditions on a geodesic γ (which emanates orthogonal to P) to be locally maximizing we proceed, just as in real analysis, by examining the 'Hessian' of the length functional. As we will see in the next lemma, now the Riemann curvature tensor $R \in \Gamma(T^{(1,3)}M)$ comes into play. It is defined as:

Remark 4.5. (Riemann curvature tensor))

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{92}$$

where $X, Y, Z \in \Gamma^1(TM)$ and $[.,.]$ is the Lie bracket on vector fields.
Since R is a tensor, its value at $p \in M$ only depends on $X(p), Y(p), Z(p)$. The above formular can be written in coordinates as :

$$R^m_{ijk} = \partial_j \Gamma^m_{ik} - \partial_k \Gamma^m_{ij} + \Gamma^m_{js}\Gamma^s_{ik} - \Gamma^m_{ks}\Gamma^s_{ij} \tag{93}$$

where $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ are the Christoffel symbols associated to the Levi-cevita connection. Hence the coordiante expressions makes it clear why at least $g \in C^2$ is needed for the following discussion.

Lemma 4.6. (second variation formula) Let γ be a timelike geodesic emanating orthogonal to P such that $|\dot\gamma|_g = 1$. Furthermore let α be a $(P;q)$-variation of γ with associated $(P;q)$-variation vector field V and 'transverse acceleration vector field' $A := \frac{\partial^2 \alpha}{\partial s^2}(t, 0)$. Then:

1. (Synge's formula) :

$$\left.\frac{d^2(L_g \circ \alpha)}{ds^2}\right|_{s=0} = \int_0^b \{g(V'^{\perp}, V'^{\perp}) + \tilde R(V^{\perp}, \dot\gamma, V^{\perp}, \dot\gamma)\}dt - g(\dot\gamma, A)\Big|_0^b \tag{94}$$

where $\tilde R \in \Gamma((T^{(0,2)}M))$ is the kovariant Riemann tensor defined by $\tilde R(X,Y,Z,W) := g(R(X,Y)Z,W)$. Also $V'^{\perp}$ describes the component of V' perpendicular to $\dot\gamma$. Since γ is assumed to be a geodesic the order of covariant derivative and restricting to $\dot\gamma^{\perp}$ is irrelevant.

2. To describe extrinsic curvature properties of our embedded submanifold P it is often convenient to use the second fundamental form, defined as (see [13] Def.6.12):

$$II(X;Y) := (\nabla_X d\iota(Y))^{\perp_P} \tag{95}$$

with $\iota : (P; g_{|P}) \to (M; g))$ is the isometric Immersion and $\perp_P$ the projection onto the normal bundle. In fact II is an element of $\Gamma(T^{(1;2)}M)$ and is symmetric. We are now able to write Synge's formula in a different form :

$$\frac{d^2(L_g \gamma)}{ds^2}\Big|_{s=0} = \int_0^b \{g(V'^{\perp}; V'^{\perp}) + R(V^{\perp}; \dot\gamma, V^{\perp}; \dot\gamma)\}dt + g(\dot\gamma; II(V^{\perp}(0); V^{\perp}(0)) \tag{96}$$

In multivariate real analysis the hessian describes a symmetric bilinear form of the tangent vectors at each point. If we treat $\Omega(P;q)$ as if it would be a manifold with points $\gamma \in \Omega(P;q)$, it is only natural to consider $(P;q)$-variations α as curves inside $\Omega(P;q)$ and therefore their infinitesimal model V, that is the $(P;q)$-variation vector fields as elements of the tangent space, denoted as $T_\gamma\Omega(P;q)$. Though as we already saw in 4.6 the second variation for vector fields tangent to γ is always zero. Thus in the following definition we only have to consider those $(P;q)$-variation vector fields orthogonal to γ: $T_\gamma^{\perp}\Omega(P;q)$ We are therefore motivated to make the following definiton:

Definition 4.7. (Index/hessian form) Let $\gamma \in \Omega(P;q)$ such that $|\dot\gamma|_g > 0$ and $V; W \in T_\gamma^{\perp}\Omega(P;q)$. We define the symmetric bilinear form on the $\mathbb{R}$-vector space $T_\gamma^{\perp}\Omega(P;q)$:

$$I_\gamma^{\perp}(V^{\perp}; W^{\perp}) := \int_0^b \{g(V'^{\perp}; W'^{\perp}) + R(V^{\perp}; \dot\gamma, W^{\perp}; \dot\gamma)\}dt + g(\dot\gamma; II(V^{\perp}(0); W^{\perp}(0)) \tag{97}$$

This will be the tool by which we can predict the failure of maximality of proper time for a given timelike geodesic γ. In fact if $I_\gamma^{\perp}$ is not negative semi-definite we can already deduce that γ is non (locally-) maximal. Though to achieve a better understanding of those properties which lead to the failure of being maximal , we will shortly present the most important results of studying the definiteness of $I_\gamma^{\perp}$. A natural approach is to start by determining its nullspace.

Lemma 4.8. (nullspace of $I_\gamma^{\perp}$, cf. [5] 14/Lemma 33)
Let $\gamma : [0; b] \to M$ be a timelike geodesic in $\Omega(P;q)$ emanating orthogonal from P. The nullspace of $I_\gamma^{\perp}$ is then given by those $V \in T_\gamma^{\perp}\Omega(P;q)$ which are the $(P;q)$-variation vector fields of a P-variation α of the following form:
$\forall s$:
(1) $\alpha_s(t) := \alpha(t;s)$ is a geodesic

(2) $\_{s}(0)$? $T_{s(0)} P$
(3) $V(b) = @_s(b;0) = 0$

Remark 4.9. 1. In the above lemma we do not require the variation to ful ll $: (b;s) = q$ but $(0 ;s)$ 2 P. In particular is in general no $(P;q)$-variation. Though since $V(b) = @_s(b;0) = 0$, Remark 4.2 tells us that V indeed is a $(P;q)$-variation vector eld.

2. For an arbitrary geodesic 2 $(P;q)$ we call P-variation vector elds which are associated to a P-variation ful lling (1) and (2) in the previous lemma, also P-Jacobi elds, denoted with J .

This leads us to the following de nition:

De nition 4.10. (focal point) Let 2 $(P;q)$ be a geodesic emanating normally to P. We call (t) a focal point of P along if there exists a P-Jacobi eld J 2 J such that $J(t) = 0$.

The following characterisation will be crucial to connect the existence of focal points with maximality of geodesics.

Lemma 4.11. ([5] 14/Prop.30) The following two statements are equivalent:
(1) (t) is a focal point
(2) $\mathrm{Exp} : D \setminus NP!$ M (the exponential map restricted onto the normal bundle) is singular at t $\_(0)$

As it turns out, the de nition of focal points already gave us the crucial tool we will need to analyse if a given geodesic is maximal. This can be seen by noticing that the twin paradox 3.27 could be proven analogously by de ning the position vector eld only as a vector eld on an open neighbourhood U of where $d\mathrm{Exp}_{jNP}$ is non singular. Here the gauÿ-lemma can still be applied in the same way to obtain the statements of 3.27 on U. That is for every other causal curve 2 $(P;q)$ contained in U we have $L_g( )$ $L_g( )$ and equality if is a reparametrization of . Hence would be a local maxima of the length functional (with respect to the compact open topology). In fact as Lemma 4.11 has told us, the above situation is exactly what happens if there are no focal points along . We have thus proven that the absence of focal points on a geodesic implies local-maximality. It is therefore only natural to ask if the existence of focal points also leads to the failure of being locally maximal. Since for a timelike geodesic being locally maximal implies the index form to be positive semi-de nite we are drawn to examine the connection between focal points and de niteness of $I^?$. Indeed the next theorem proves that focal points completely determine its de niteness.

Theorem 4.12. (de niteness of the $I^?$, cf. [5] 14/Lemma 13,Theorem 34) Let 2 $(P;q)$ be a timelike geodesic emanating orthogonal to P. Then the following statements hold :
(1) $I^?$ can not be positive de nite [18]

(2) $I^?$ is negative de nite if and only if there are no focal points of P along
(3) $I^?$ is negative semi-de nite if and only if (b) is the only focal point of P along .

The above theorem allows the key conclusion: If we can prove that $I^?$ for : $[0; b]$ M a timelike geodesic, is only semi-negative de nite, we also would have proven the existence of a focal point. Thus if the geodesic could be extended to larger times than b it would have a focal point strictly before its endpoint. This on the other hand would (again by the above theorem) imply $I^?$ not to be semi-1negative de nite on this larger interval, in particular to become non locally maximizing.

As in the corollary 3.71 we have to treat the null case separately. We explicitly excluded null curves from our previous consideration by demanding $j\_j_g > 0$. If this is not the case the composition L_g would in general not be di erentiable anymore which prevents us from calculating the rst or second variation. Hence we somehow need to nd a way to get rid of the $j\ j_g$ inside the length functional. As it turns out this is possible if we are willing to sacri ce some of the geometric signi cance when studying the length functional. Instead we de ne :

De nition 4.13. (the action functional) Let be a causal curve in $(P; q)$. Then we de ne the energy/action of as

$$A( ) := \frac{1}{2}\int_0^b g(\_;\ \_)dt \tag{98}$$

We can now compute the rst and second variation of the action functional. In fact the results are completely analogous, that is only di ering by signs, to the expressions which we discussed previously for the length functional.

Lemma 4.14. (rst and second variation of the action functional) Let 2 $(P; q)$ and a $(P; q)$-variation of with associated $(P; q)$-variation vector eld V. Then :

1. (rst variation)

$$\begin{aligned} d\ A(V) &:= \frac{d(A\ \ )}{ds}\Big|_{s=0} \\ &= \int_0^b g(\_;\ V\_)dt \\ &= \int_0^b g(\bullet; V)dt \quad \sum_{i=2}^{N\ 1} g(\ \_\ (t_j); V(t_j)) + g(\_; V)\Big|_a^b \end{aligned} \tag{99}$$

[18] in [5] 14/Lemma 13 this is only proven for $P = p \in M$. Though the same proof works for P as above a spacelike submanifold.

2. (second variation) If the first variation vanishes then :

$$\frac{d^2(A\circ\alpha)}{ds^2}\bigg|_{s=0} = \int_0^b \{ g(V'^{\perp}, V'^{\perp}) + \tilde{R}(V^{\perp}, \dot\gamma, V^{\perp}, \dot\gamma) \} dt - g(\dot\gamma, \Pi(V^{\perp}(0), V^{\perp}(0))) \tag{100}$$

3. (hessian form) Let $V, W \in T\Omega(P;q)$ then we define the symmetric bilinear form on the $\mathbb{R}$-vector space $T^{\perp}\Omega(P;q)$:

$$H^{\perp}(V^{\perp}, W^{\perp}) := \int_0^b \{ g(V'^{\perp}, W'^{\perp}) + \tilde{R}(V^{\perp}, \dot\gamma, W^{\perp}, \dot\gamma) \} dt - g(\dot\gamma, \Pi(V^{\perp}(0), W^{\perp}(0))) \tag{101}$$

In the discussion of the length functional we now proceeded by defining focal points as the existence of P-Jacobi fields which vanish. By the definition of P-Jacobi fields this could be interpreted as a variation through geodesics of the same causal character starting orthogonal to P and re-converging at some time up to first order. The attribute that all γ_s have the same causal character, is not so clear anymore when γ is null. Nevertheless in [5] (Corollary 40) it is shown, that a P-Jacobi field which vanishes at some time t already can be described as the variation vector field associated to a variation through null geodesics, all emanating orthogonal to P. Thus the interpretation above remains to be true. It is now our goal to find a relation between focal points along null-geodesics and the definiteness of $H^{\perp}$. This will then hopefully provide us with a similar tool to predict the failure of being (locally) maximal, as in the timelike case $I^{\perp}$ did. We have the following theorem :

Theorem 4.15. (definiteness of the $H^{\perp}$, cf. [5] 14/Prop. 41)
Let γ be null-geodesic starting orthogonal to P ($\Longrightarrow \dim(P) \leq n-2$), a spacelike submanifold of (M, g). If there are no focal points along γ then $H^{\perp}$ is positive semi-definite. Additionally if in the above case $H^{\perp}(V, V) = 0$ then V must be tangent to γ.

Therefore, also in the null case we can predict the existence of focal points just by analysing the definiteness of a bilinear form. The only thing still missing, is the connection between the existence of focal points and a failure of being maximal. This is resolved by the following theorems :

Theorem 4.16. Let P be a spacelike submanifold of (M, g) a C^1-spacetime and $\gamma \in \Omega(P;q)$ a causal curve.

1. (cf. [5] Lemma 50) If γ is a null geodesic not starting orthogonal to P than there is a timelike curve in $\sigma \in \Omega(P;q)$ arbitrarily close to γ.
2. (cf. [5] Theorem 51, [10] Theorem 6.16 a)) Let γ start orthogonal to P and assume that there is a focal point of P along γ strictly before q. Then there exists an arbitrary close timelike curve $\sigma \in \Omega(P;q)$.

In particular $\gamma \in \Omega(P; q)$ a null geodesic is not locally length maximizing unless γ starts orthogonal to P and contains no focal points of P before q.

With those methods developed we will now proceed to collect a range of general conditions on our spacetime, which if they are fulfilled, lead to the failure of maximizing up to arbitrary proper time. Thus they already hint at those conditions we will need to prove the singularity theorems later on.

4.2 Energy conditions: A source for focal points

All of the subsequent results on the existence of focal points rely on the same method, described in [1] (p.4 f.)[19]. Most importantly it allows us to formulate the conditions for the existence of focal points, as desired, in a general manner and thereby forces us to define and study the so called energy conditions. We will demonstrate this method briefly in the case of a unit-timelike geodesic $\gamma \in \Omega(P; q)$ starting orthogonal from P a spacelike submanifold of dimension $p \leq n-2$. The null case follows analogously.

The main problem we are facing when trying to prove the semi-definiteness of $I^{\perp}$ is the construction of explicit $(P; q)$-variation vector fields. As it turns out the arguably most simplest approach to this problem already suffices to get significant results. The idea of this approach lies therein to alternate the problem of constructing vector fields into the much easier problem of constructing appropriate functions on $[0; b]$. This can be realized by choosing any arbitrary tangent vector $y \in T_{\gamma(0)}P$ such that $g_{\gamma(0)}(y; y) = 1$ and parallel translating it along γ to get $Y \in \Gamma^1(\gamma^* TM)$, such that $\nabla_{\partial_t} Y = 0$. Now choose any continuous piecewise C^1- function defined $[0; b]$ such that $f(0) = 1; f(1) = 0$. Then $V(t) := fY(t) \in T^{\perp}\Omega(P; p)$ is an allowed $(P; q)$-variation vector field. We therefore arrive at :

$$I^{\perp}(V; V) = \int_0^b \{ \dot{f}^2 + f^2 R(Y; \dot{\gamma}; Y; \dot{\gamma}) \} dt + g(\dot{\gamma}; \Pi(y; y)) \tag{102}$$

The last term can be interpreted as the 'initial rate of convergence' forced upon normal geodesics by the extrinsic curvature of P. In fact this can be proven formally , that is for any P-Jacobi field Y (cf. [5] p.287) : $\frac{d(|Y|_g)}{dt}(0) = g(\dot{\gamma}; \Pi(Y(0); Y(0)))$. We can simplify our problem even further by averaging in the normal subspace $(T_{\gamma(t)}M)^{\perp}$. That is let us choose an arbitrary orthonormal basis of $T_{\gamma(0)}P$: $(e_1; \dots; e_p)$. We then proceed by defining the convergence of P as:

$$k : NP \to \mathbb{R}$$
$$k(z_p) := \frac{1}{p} \sum_{i=1}^{p} g_p(z_p; \Pi(e_i; e_i)) \tag{103}$$

[19] In [1] the index of g is chosen to be $n-1$, contrary to our convention of $\mathrm{index}(g) = 1$. Hence some sign differences will occur.

By doing the same procedure as above for y for every e_i and adding up all terms we finally arrive at:

$$\sum_{i=1}^{p} I^{\perp}(fE_i, fE_i) = \int_0^b \left(p\dot{f}^2 - f^2 \mathrm{Ric}(\dot\gamma, \dot\gamma) \right) dt + pk(\dot\gamma(0)) \tag{104}$$

where E_i are the parallel translations of e_i and $\mathrm{Ric} \in \Gamma^1(T^{(0,2)}M)$ the Ricci tensor defined as $\mathrm{Ric}_{ij} = R^m_{imj} = \mathrm{trace}(Y \to R(X, Y)Z)$ [20]. It is convenient to define the functional:

$$J[f] := \int_0^b \left(p\dot{f}^2 - f^2 \mathrm{Ric}(\dot\gamma, \dot\gamma) \right) dt \tag{105}$$

The same procedure can be repeated with the hessian of the action functional (101) which has the exact same formal expression as the index form (97), but multiplied with (-1). Importantly a parallel transported vector $y \in T_{\gamma(0)}P$ orthogonal to a null geodesic γ, stays orthogonal to $\dot\gamma$. Hence we can deduce from $H^{\perp}(fY, fY) \leq 0$ already the existence of a focal point by Theorem 4.15. We have therefore proven the following lemma on which most of our subsequent analysis of focal points will be based.

Lemma 4.17. (cf. [1] Prop.2.2) Let $\gamma : [0, b] \to M$ be a unit-timelike or a null geodesic starting orthogonal to a spacelike submanifold P of (M, g) with $\dim(P) \geq 2$. If there exists a continuous piecewise function $f : [0, b] \to \mathbb{R}$ such that $f(0) = 1$ and $f(b) = 0$ and :

$$J[f] \leq pk(\dot\gamma(0)) \tag{106}$$

then there exists a focal point of P along γ.

It is thus evident that we need conditions on the initial convergence k and the Ricci-tensor along γ to prove the existence of focal points. Those conditions should not be chosen arbitrary, but instead rely on physical arguments. Otherwise the significance of the singularity theorems themselves would be diminished. That is, a singularity in an unphysical spacetime is in general not a singularity we care much about. This approach leads us to a short discussion of the now often anticipated energy conditions, which occur in many areas of physics. In fact we should not forget, that in the end general relativity is the spacetime theory which interests us. By means of the Einstein equations :

$$G_{ab} = \mathrm{Ric}_{ab} - \frac{R}{2} g_{ab} = 8\pi T_{ab} \quad | R := \mathrm{Ric}^m_m \, ; \, c = 1 \, ; G = 1 \tag{107}$$

we therefore have a direct relationship between conditions imposed on the Ricci-curvature tensor and on the physical stress-energy tensor $T \in \Gamma(T^{(0,2)}M)$ (see Def. 4.18). It is this relationship where the connection between the singularity

[20] If $\dim(P) = n - 2$ it is not so clear why we can ignore the remaining orthogonal dimension to P when computing the Ricci-curvature. Nevertheless this follows from ([5] Lemma 8.9)

theorems, formulated on an abstract Lorentzian manifold and the rest of physics can be found. In fact the stress-energy tensor does not include gravitational energy[21] and has to be derived from the given matter fields on spacetime. In [24] (3.2/p.61) the following characterization of a stress-energy tensor is given:

Definition 4.18. (stress-energy tensor) Let $(\psi_i)_i$ a family of smooth tensor fields which model the matter content in our spacetime (M, g). We call $T \in \Gamma(T^{(0,2)}M)$ a stress-energy tensor if it only depends on the matter fields ψ_i, their covariant derivatives, the metric and furthermore obeys :
(1) For U an open subset of M : $T_{|U} = 0 \iff \psi_i = 0$ for all $i \in I$
(2) $(\nabla_{\partial_j} T)^j = 0$ ('local energy conservation')

Generally the above definition does not determine the stress-energy tensor uniquely. Additionally it does not give us concrete instructions to construct a stress-energy tensor for a given matter model. Though if the matter model on a compact n-dim region $D \subset M$ is characterized by field equations derived from the variation of an action:

$$A := \int_D L_{matter}\,[(\psi)_i; (\nabla\psi)_i; g]\, dVol_g \tag{108}$$

we can give an explicit expression for a stress-energy tensor on D (cf. [24] 3.3/p.54 f.):

$$T_{lk} = \frac{\partial(L_{matter})}{\partial g^{lk}} + g_{lk} L_{matter} \tag{109}$$

The following summary of the most important energy conditions sheds some light on the physical content of the stress-energy tensor and gives us a reasonable framework in which we can subsequently try to prove the existence of focal points. An extensive discussion on this topic, treating in detail the interpretations, problems and consequences of the energy conditions can be found in [25]. We will try to briefly skip through aspects mentioned in [25] which will be important for our further discussion. First of all it is remarkable, that up until now, regardless of the extensive use of energy conditions in a range of physical theories, there is no convention on their epistemic status. That is none of the generally discussed energy conditions can be viewed as a law or an experimental fact and neither as a strict consequence of other more fundamental axioms of our physical theories. In fact for every standard energy condition, there exists a physically reasonable example which breaks it. On the other hand they can be formulated in such a wide range of spacetime theories, that they sometimes may even appear to be more fundamental than the Einstein field equations themselves. Nevertheless it is time to finally call them by their right name, that is explicitly formulating their demands.

Definition 4.19. (Standard energy/curvature conditions)

[21] to include gravitational energy we would need a 'pseudo tensor' since the equivalence principle imposes that gravitational fields can always be transformed locally to zero in a freely falling frame of reference

1. <u>null energy condition (NEC)</u> : $\forall$ null vectors k : $Ric_{ab}k^a k^b \geq 0$
2. <u>weak energy condition (WEC)</u>: $\forall$ timelike vectors u : $G_{ab}u^a u^b \geq 0$
3. <u>dominant energy condition (DEC)</u>: $\forall$ timelike vectors u : $G_{ab}u^a u^b \geq 0$ and $G^a_b u^b$ is causal
4. <u>strong energy condition (SEC)</u>: $\forall$ timelike vectors u : $Ric_{ab}u^a u^b \geq 0$

We intentionally formulated the conditions in a geometric fashion such that up until now the Einstein field equations were not used in any way. This has the advantage that it is independent of the explicit theory we impose on spacetime. Though clearly it lacks the physical interpretation; the importance of which we have advertised repeatedly in the prior discussion. Hence let us shortly discuss the physical meaning of the above equations by means of the Einstein field equations (107). Generally the components of the stress-energy tensor are interpreted in the following way: $T^{ab} = g^{ac}g^{bd}T_{cd}$ describes the flux of the a-momentum component through a surface of constant b-coordinate. We can put this in an even simpler form by requiring that T is an energy momentum tensor of type(I) as described in [24](4.3/p.89). That is if for every point $p \in M$ there exists an orthonormal frame $(e_0, \dots, e_{n-1})$ of T_pM such that e_0 is timelike, $e_1, \dots, e_{n-1}$ are spacelike and T has a diagonal representation. Then T^{00} can be interpreted as the local energy density (ρ) and T^{ii} as the three principal pressures (p_i). Let us now analyse what the energy conditions impose on our physical system. The first one, that is the NEC can be interpreted as the fact that observers travelling along null curves measure only non-negative energy density. Though the problem with this interpretation is that no physical observer can travel along null curves. As argued in ([25],p.9) since the energy density is an observer dependent quantity this causes some serious trouble on interpreting the NEC as we did above. This problem does not occur in the case of the WEC. It similarly demands, the energy density to be measured always as positive. Though this time the observers are physical ones travelling along timelike curves. It may be worth mentioning, that the WEC by continuity always implies the NEC. Let us proceed by formulating the physical idea behind the DEC. Clearly it contains the WEC and therefore also its physical implications. Though the condition of $G^a_b u^b = T^a_b u^b$ being causal additionally forces the local energy-flow vector to be non spacelike. If T is of type (I) we can equivalently state this condition as $\rho \geq 0$ and $|p_i| \leq \rho$ for all $i = 1, 2, 3$. Interestingly there is a much more fundamental, though also more problematic interpretation of $T^a_b u^b$ being causal. Let us shortly remind ourselves where this journey into causality started from. In section 3.2 we mentioned the importance of fixing the speed of light for special relativity. In our case it led us naturally to the definition of a causal character of vectors (see Def.3.2), the basis for all concepts of causality we have subsequently constructed. The condition of $T^a_b u^b$ being causal has been generally interpreted as the prohibition of superluminal, that is non causal propagation of stress energy. In fact by definition $T^a_b u^b$ is the local energy flow as measured by an observer traveling along a timelike curve with velocity u. Hence one could argue, the DEC only formulates the already given finite speed of causality. The problem with this interpretation is, that it is formulated in an observer dependent way. That is, it always refers to a measurement of an observer. Though does it also allow a formulation of an observer independent prohibition of superluminal propagation? Indeed in [24] this is enforced by proving that for any stress energy tensors as defined in Def.4.18 and a closed achronal (no two points can be connected by a timelike curve), spacelike hypersurface A: $T_{|A} = 0 \Rightarrow T_{|D(A)} = 0$. Though if the dominant energy condition is a necessary condition for non superluminal propagation of stress energy is still debated. In [26](p.105) J.Earman argues, that an observer independent definition should be taken as the initial requirement for non superluminal propagation instead. This is done by demanding the existence of a well posed initial value problem for all matter fields in spacetime. He proceeds by demonstrating the existence of physically reasonable matter models which do not fulfill the DEC though still have a well posed initial value formulation for the stress-energy. In particular he mentions scalar fields with a negative potential which are considered by some cosmological models containing dark energy. They do not obey the DEC or WEC but still satisfy those conditions needed to prevent observer independent superluminal propagation of stress-energy as characterized by the initial value problem. Even further it should be mentioned that the DEC itself only demands stress-energy to propagate causally. What is about the fields? Should they not be restricted by the universal limit of causality too? Not to loose ourselves into subtleties we will not need explicitly later on, we proceed with the discussion of the SEC. The SEC is probably the most significant condition considering the singularity theorems concerning timelike incompleteness. In fact, some times it is even named the timelike convergence condition. That is, if written in our previously developed language to describe geodesic-variations , it is a source of focal points. By use of the Einstein equations we can reformulate the condition into:

$$T_{ab} - \frac{1}{2}\mathrm{trace}(T) g_{ab} \geq 0 \tag{110}$$

Unfortunately, there is still no evident physical interpretation of this expression. Some insight may be gained by assuming the stress-energy tensor to be of type (I). Then the SEC can be written as :

$$T^{00} + \sum_{i=1}^{n-1} T^{ii} \geq 0 \quad \wedge \quad T^{00} + T^{ii} \geq 0 \;\; \forall i = 1, \ldots, n-1 \tag{111}$$

Considering a barentropic equation of state : $p = \omega\rho$, with $\rho := T^{00}$ and assuming isotropy : $p = T^{ii}$ for all $i = 1, \ldots, n-1$ we can state the above equation equivalently as $\omega \geq -\frac{1}{3}$. Classically this is obeyed by most reasonable matter models. Though the cosmological constant has an equation of state with $\omega = -1$ and hence explicitly violates the SEC. Furthermore this is not the only case where the SEC is violated. In [27] M.Visser and C.Barelo argue, that the SEC , due to its various violations in physical reasonable situations, should not serve as a fundamental guiding principle for our physical theory. This may seem rather unsettling, since as we already mentioned, most of the classical singularity theorems rely on the SEC. It is therefore an important task to weaken the energy conditions and test if the singularity theorems still remain true. Nevertheless we proceed by rst collecting some classical existence theorems for focal points assuming the SEC (or NEC when discussing null-incompleteness). Only then we will further discuss in which way the SEC and NEC could be generalized to a wider range of physical matter models while still maintaining its convergence property needed for the singularity theorems.

Lemma 4.20. (Focussing theorem (I), cf. [5] 14/Prop.37(timelike), Prop.43(null))
Let P be a spacelike submanifold of our smooth spacetime $(M; g)$ with $\dim(P)$ 2. Furthermore let : $[0; b]$ M be a unit-timelike or null geodesic starting orthogonal from P. If :
(1) $k := k(\_(0)) > 0$
(2) $Ric(\_; \_)$ 0 8t 2 $[0; b]$
Then there is a focal point (r) of P along such that $0 < r$ $\frac{1}{k}$ provided b $\frac{1}{k}$.

Proof. De ne $f(t) := 1$ kt and assume that b $\frac{1}{k}$. Substituting this into (105) for $_{[0; \frac{1}{k}]}$ we get :

$$J[f] = pk \int_0^{\frac{1}{k}} f^2 Ric(\_; \_)dt \quad pk \tag{112}$$

Hence Lemma 4.17 proves the existence of a focal point of P along . □

In particular condition (2) is implied by the SEC on $(M; g)$. In the null case the NEC already su ces. There is a standard generalization of the above lemma. This will be helpful later when we work with approximations, since it allows a small constant deviation form the SEC or NEC.

Lemma 4.21. ((Focussing theorem (I'), cf. [2] Lemma 4.10 (timelike),Lemma 5.6 (null))
Let all conditions of Lemma 4.20 be ful lled. Though instead of but $Ric(\_; \_)$ 0 we only assume the existance of a 0 $\frac{3kp}{b}(1$ $c)$ for some $0 < c$ 1 such that $Ric(\_; \_)$. Then there exists a focal point (r) of P along such that $0 < r$ $\frac{1}{ck}$ if b $\frac{1}{ck}$.

Proof. Similarly as before we de ne $f(t) := 1$ $\frac{t}{b}$. Hence :

$$\begin{aligned} J[f] = \frac{p}{b} \int_0^b f^2 Ric(\_; \_)dt & \quad \frac{p}{b} + \int_0^b (1 \quad \frac{t}{b})^2 dt \\ & = \frac{p}{b} + \frac{b}{3} \quad \frac{p}{b} + pk(1 \quad c) \end{aligned} \tag{113}$$

If b $\frac{1}{ck}$ then :

$$J[f] \quad \frac{p}{b} + pk(1 \quad c) \quad kp \tag{114}$$

Hence the existence of a focal point is proven. □

In all of the previous discussions we always assumed geodesics to start orthogonal to some spacelike submanifold. In some case though it is desirable to instead examine geodesics in general. This has the advantage, that they allow to formulate some singularity theorems in a very general manner. The Hawking-Penrose Theorem for example can be completely formulated in abstract causality terms, that is without referring to some concrete submanifold or similar constructions. Instead the idea is to prove only afterwards that a wide range of objects in our spacetime imply these conditions and thus are covered by the general formulation of the theorem. We are therefore motivated to formulate some focussing theorems for arbitrary geodesics. This can be done in a natural way by choosing P to be a zero-dimensional submanifold $p \in M$. Focal points can then be described as the existence of a p-variation vector field associated to a variation through geodesics all starting at p. Generally focal points for $P = p$ are called conjugate points instead. In particular to prove the existence of conjugates of p along γ - an unit-timelike or null geodesic, it suffices to show $J[f] = \int_{-b}^{b} [\dot f^2 (n-1) - f^2 \mathrm{Ric}_{j}(\dot\gamma, \dot\gamma)] dt \leq 0$ for some $f \in C^{1,pc}; f(-b) = 0; f(b) = 0$. With those preparations we can now formulate the next lemma prohibiting maximizing geodesics to be defined for all times (that is on $\mathbb{R}$) in certain spacetimes.

Lemma 4.22 (Existence of conjugate points for long enough geodesics). Let γ be a causal and complete geodesic, such that $\mathrm{Ric}_{j}(\dot\gamma, \dot\gamma) \geq 0$ and $\exists s_0 \in \mathbb{R}$ where γ is defined with $\mathrm{Ric}_{j\,\gamma(s_0)}(\dot\gamma, \dot\gamma) \neq 0$. Then there are $x_0; x_1$ such that $\gamma(x_0)$ and $\gamma(x_1)$ are conjugates.

Proof. Without loss of generality we can assume $s_0 = 0$ (otherwise consider $\tilde\gamma(t) := \gamma(t + s_0)$). Assume that $x_0; x_1$ as described above do not exist. By the previous reamrk we know that $J[f] = \int_{-b}^{b} [\dot f^2 (n-1) - f^2 \mathrm{Ric}_{j}(\dot\gamma, \dot\gamma)] dt \leq 0$ implies the existence of a conjugate point along γ with $-b$ as starting point. Here $f(t)$ is a piecewise smooth function such that $f(-b) = f(b) = 0$. Therefore we can assume that for all such f: $J[f] > 0$. This can be written as:

$$\int_{-b}^{b} f^2(t) \mathrm{Ric}_{j}(\dot\gamma, \dot\gamma) dt < \int_{-b}^{b} \dot f(t)^2 dt (n-1) \tag{115}$$

Fix arbitrary $-b < b_0 < 0$ and $0 < b_1 < b$ and define:

$$f_b := \begin{cases} \frac{t+b}{b_0+b} & -b \leq t \leq b_0 \\ 1 & b_0 \leq t \leq b_1 \\ \frac{b-t}{b-b_1} & b_1 \leq t \leq b \end{cases} \tag{116}$$

As f_b is a piecewiese smooth function, it is an allowed choice for f in (115). We therefore get:

$$\int_{-b}^{b} f_b^2(t) \mathrm{Ric}_{j}(\dot\gamma, \dot\gamma) dt < \int_{-b}^{b} \dot f_b(t)^2 dt (n-1) = \frac{n-1}{b+b_0} + \frac{n-1}{b-b_1} \tag{117}$$

We now want to use the Lemma of Fatou which tells us that for $(g_m)_m$ a sequence of non-negative measurable functions into $\mathbb{R} \cup \{\infty\}$:

$$\int_{\mathbb{R}} \liminf_{m\to\infty}(g_m)\,dt \leq \liminf_{m\to\infty} \int_{\mathbb{R}} g_m\,dt \tag{118}$$

Since pointwise: $\lim_{m\to\infty} g_m(t) = \mathrm{Ric}|_{\gamma(t)}(\dot\gamma,\dot\gamma)$, for $g_m(t) := 1_{[-m,m]} f_m^2(t)\mathrm{Ric}|_{\gamma(t)}(\dot\gamma,\dot\gamma)$, which is indeed a non-negative measurable function (piecewise smooth on $\mathbb{R}$ and due to our assumption on $\mathrm{Ric}|_{\gamma(t)}(\dot\gamma,\dot\gamma)$), applying (118) to (117), we get:

$$\int_{\mathbb{R}} \mathrm{Ric}|_{\gamma(t)}(\dot\gamma,\dot\gamma)\,dt \leq \liminf_{m\to\infty} \int_{\mathbb{R}} g_m\,dt = \liminf_{m\to\infty} \int_{-m}^{m} f_m^2(t)\mathrm{Ric}|_{\gamma(t)}(\dot\gamma,\dot\gamma)\,dt$$

$$\leq \liminf_{m\to\infty} \left(\frac{n-1}{m+b_0} + \frac{n-1}{m-b_1}\right) = 0$$

Since we assumed $\mathrm{Ric}|_{\gamma(t)}(\dot\gamma,\dot\gamma) \geq 0$, it follows that $\mathrm{Ric}|_{\gamma(t)}(\dot\gamma,\dot\gamma) = 0$ almost everywhere, that is everywhere since it is a smooth function.

□

Remark 4.23. The above result remains true only assuming the SEC and the so called genericity condition along γ, that is $R(\cdot,\dot\gamma)\dot\gamma|_{\gamma(s_0)} : T_{\gamma(s_0)}M \to T_{\gamma(s_0)}M$ is not zero for some s_0 in the domain of γ. A proof can be found in [3] (Prop.2.4). Though generally those proofs rely on the Raychaudhuri equation which we explicitly tried to avoid in this thesis.

As promised, after having briefly presented the most important classical focussing theorems, we will now discuss modifications of the SEC and NEC which hope to generalize the above lemmas. Arguably the most prominent physical models which violate the SEC and NEC can be constructed in scalar field theory. Their importance comes not only from their simplicity and therefore great usability as toy models, but also from experimental grounds. In [27](p.3) one finds a range of arguments, speaking for the value of scalar fields in cosmology and astrophysics. The particle which is explicitly discussed in [1] by C.Fewster is the pion. Those are, up to certain, not too large values of momenta (measured in their rest frame), well described by scalar fields. Similarly one could examine the higgs field, which has the advantage of describing a fundamental particle and therefore not being restricted to some thresholds of momenta. Another interesting, though not yet discovered scalar field is the quintessential inflaton (introduced in [28]). It aims to describe a mechanism, leading to the fast expansion of our universe during the inflationary era and simultaneously account for dynamical dark energy today. We are therefore motivated to search for modifications of the SEC and NEC which are inspired by energy-inequalities derived in the context of scalar fields. For simplicity we will restrict our considerations to classical scalar fields. In quantum field theory all of the standard pointwise energy conditions are violated. It is therefore an interesting task , to directly derive Quantum-Energy-inequalities. In fact this is done in [29] by C.Fewster. The inequalities derived there, could in some special cases have the form we will consider subsequently for the classical scalar field. Generally the following treatment follows [30] where the inequalities for classical scalar fields were derived and [1] for the focussing theorems using the prior formulated inequalities.

4.3 Generalized energy conditions

4.3.1 The non-minimally coupled field

We start by shortly introducing the non-minimally coupled scalar field, in the context of which a modification of the SEC will be discussed.
We already described matter fields in Def.4.18 as tensor fields on M. Hence scalar fields ϕ are elements of $C^\infty(M)$, characterized in general by a field equation. In the case of the non-minimally coupled scalar field the field equation takes the form :

$$(\Box_g + \mu^2 + \xi R)\phi = 0 \tag{119}$$

, where ξ is the coupling constant, R the Ricci-scalar, μ some fixed characteristic wavelength and $\Box_g$ the d'Alembertian defined as $\Box_g := g^{ij}\nabla_i\nabla_j$. The above equation is given in natural units, where $\hbar = 1; c = 1$. We already gave an explicit formula for the stress-energy tensor as long as the field equations can be derived from a Lagrangian. In fact (119) are equivalent to a stationary action with the Lagrangian :

$$L := \frac{1}{2}[(\mathrm{grad}_g(\phi))^2 - (\mu^2 + \xi R)\phi^2] \tag{120}$$

We therefore can calculate via (109) the stress-energy tensor:

$$\begin{aligned} T_{ij} = (\nabla_{\partial_i}\phi)(\nabla_{\partial_j}\phi) + \frac{1}{2}g_{ij}(\mu^2\phi^2 - g(\mathrm{grad}(\phi);\mathrm{grad}(\phi))) \\ + \xi(g_{ij}\Box_g - \nabla_{\partial_i}\nabla_{\partial_j} + G_{ij})\phi^2 \end{aligned} \tag{121}$$

With the help of the Einstein field equations (107), we therefore arrive at an expression for the Ricci-tensor:

$$\begin{aligned} \mathrm{Ric}(X;X)(\frac{1}{8\pi} - \xi\phi^2) = (1-2\xi)X^iX^j(\nabla_{\partial_i}\phi)\nabla_{\partial_j}\phi - \frac{1-2\xi}{n-2}\mu^2\phi^2 \\ - \frac{2\xi}{n-2}\mathrm{grad}(\phi)^2 - \xi X^iX^j\nabla_{\partial_i}\nabla_{\partial_j}\phi^2 + \frac{2\xi^2}{n-2}R\phi^2 \end{aligned} \tag{122}$$

, for every timelike vector field $X \in \Gamma^\infty(TM)$. In particular for the minimal coupled field $\xi = 0$ we get:

$$\mathrm{Ric}(X;X) = 8\pi(X^iX^j(\nabla_{\partial_i}\phi)(\nabla_{\partial_j}\phi) - \frac{1}{n-2}\mu^2\phi^2) \tag{123}$$

Hence if $\mu^2\phi^2 \geq X^iX^j(\nabla_{\partial_i}\phi)(\nabla_{\partial_j}\phi)(n-2)$ the SEC is violated. On the orther hand, the NEC is still fulfilled when considering minimal-coupling . Nevertheless if $\;\; 6= 0$ one can nd physical reasonable conditions on , which violate the NEC (cf. [31]p.5). How could we possibly change the SEC and NEC such that they remain valid, not only for the minimally but also the non-minimally coupled scalar eld? While doing so one should keep in mind, that nally to prove focussing theorems it is the quantity $\int$ Ric(_; _)f ^{2}(t)dt which we have to estimate. It is therefore natural, to examine exactly those expressions in the case of a Ricci-Tensor given by (122). This has been explicitly carried out in much detail in [30]. For our purposes it will su ce to just state the result derived there and build our subsequent considerations upon it.

Lemma 4.24. (timelike worldline inequality, cf. [30] Theorem 3, Corollary 1) Let (M; g;) be a solution of the Einstein-Klein-Gordon equation, that is $G_{ij} = 8 \;T_{ij}$ with T given by (121). Furthermore assume that 2 [0; $_c$], where $_c \coloneqq \frac{n\;\;2}{4(n\;\;1)}$ is the conformal coupling constant[22]. If : I! M is a unit-timelike geodesic such that on there exist $_{max}$ and $^0_{max}$:

$$j\; j \quad _{max} < (8 \quad)^{\;1=2}; \quad jr \;\_\; j \quad ^0_{max} \tag{124}$$

Then we have the following energy inequality along:

$$\int_I \mathrm{Ric}(\_; \_)f(t)^2dt \quad k \quad f k^2 \coloneqq \quad Q(kf\!-\!k^2_{L^2(I)} + Q^2 kf k^2_{L^2(I)}) \quad 8f \;\; 2\; C^1_c\;(\mathrm{int}(I)) \tag{125}$$

, where C^1_c (int (I)) is the space of smooth functions with compact support in the interior int (I). The constants Q 0 and Q 0 are explicitly given as :

$$Q = \frac{32 \quad ^2_{max}}{1 \quad 8 \quad ^2_{max}}; \quad Q^2 = \frac{(1 \quad 2\;)\quad ^2}{4\;(n \quad 2)} + \frac{8 \quad _{max}\; ^0_{max}}{1 \quad 8 \quad ^2_{max}}\;^2 \tag{126}$$

A similar bound can be derived along null geodesics. Since T (X; X) = Ric(X; X) for every null vector, it su ces to estimate $\int_I$ T(_; _)f (t)^{2}dt for T given by (121). Once more we will just state the result:

Lemma 4.25. (null worldline inequality, cf. [32] 6/p.16)
Let (M; g;) be a solution of the Einstein-Klein-Gordon equation, that is $G_{ij} = 8\; T_{ij}$ with T given by Eq. 121. Furthermore assume that 2 [0; $_c$]. If : I! M is an a nely parameterized null geodesic such that admits the following bounds on :

$$j\; j \quad _{max} < (8 \quad)^{\;1=2}; \quad jr \;\_\; j \quad ^0_{max} \tag{127}$$

Then we have the following energy inequality along:

$$\int_I \mathrm{Ric}(\_; \_)f(t)^2dt \quad k \quad f k^2 \coloneqq \quad Q(kf\!-\!k^2_{L^2(I)} + Q^2 kf k^2_{L^2(I)}) \quad 8f \;\; 2\; C^1_c\;(\mathrm{int}(I)) \tag{128}$$

[22] at = $_c$ the action functional is invariant under conformal transformations

, where C_c^1 (int (I)) is the space of smooth functions with compact support in the interior int (I). The constants Q 0 and $\tilde{Q}$ 0 are explicitly given as :

$$Q = \frac{32 \quad {}^{2}_{max}}{1 \quad 8 \quad {}^{2}_{max}}; \qquad \tilde{Q} = \frac{8 \quad {}_{max} \quad {}^{0}_{max}}{1 \quad 8 \quad {}^{2}_{max}} \tag{129}$$

Remark 4.26. 1. As desribed in ([27], p.16) a violation of $j \; j \quad {}_{max} <$ $(8 \quad)^{1=2}$ would lead to unphysical behaviour.

2. In [33] Corollary 6.31 it is proven that for any domain of R^n of nite width (that is contained between two parallel hyperplanes) $k{:}k$ is indeed a norm on $C_c^1(\;)$ (if $Q > 0$) which is equivalent to the Sobolev norm $kf k_1 := (kf k^2_{L^2(I)} + kf{-}k^2_{L^2(I)}))^{1=2}$. Hence the above inequality also remains true for $f \; 2 \; W_0^1(I) := \overline{C_c^1 \text{ (int}(I))}^{k:k_1}$. (For Q=0 the SEC (or NEC) is ful lled along)

The above lemmas propose a formula to generalize the SEC and NEC. The question of, whether those inequalities su ce to prove foussing theorems will be discussed in the next subsection.

4.3.2 Generalization of NEC and SEC

The natural formulation of a more general energy condition inspired by scalar elds is the following.

De nition 4.27. (scalar eld inspired energy conditions)

1. We say $(M; g)$ ful lls the scalar eld inspired timelike convergence condition (:SISEC) if for every timelike geodesic $: I! \quad M$ and $8f \; 2 \; C_c^1$ (int (I)) (hence $W_0^m(I)$) an estimate of the following form holds:

$$\int_I \mathrm{Ric}(\_; \_) f(t)^2 dt \quad k \quad f k^2_m := \quad (Q_m kf^{(m)} k^2_{L^2(I)} + Q_0 kf k^2_{L^2(I)}) \tag{130}$$

2. We say $(M; g)$ ful lls the scalar eld inpired null convergence condition (:SINEC) if for every a nely parameterized null geodesic $: I! \quad M$ and $8f \; 2 \; C_c^1$ (int (I)) (hence $W_0^m(I)$) an estimate of the following form holds:

$$\int_I \mathrm{Ric}(\_; \_) f(t)^2 dt \quad k \quad f k^2_m := \quad (Q_m kf^{(m)} k^2_{L^2(I)} + Q_0 kf k^2_{L^2(I)}) \tag{131}$$

We have allowed higher derivatives of f in the norm $k{:}k_m$. In fact estimates of the above form (with m 1) do not only arise in classical scalar eld theory, but also in quantum eld theory. Though as we already mentioned we will not got further into this (to be sincere this is partly due to a lack of knowledge on the part of the author). Instead we will proceed by demonstrating two variants of focussing theorems which were proven in [1]. To do so we first need to introduce a generalization of the function (116) already used in Lemma 4.22. In fact if we would restrict ourselves to $m = 1$, (116) would already suffice to prove the following focussing theorem. Though if we also consider $m > 1$ we could not apply our newly formulated energy conditions since (116) is not of high enough regularity. Hence we somehow need to regularize (116) but maintain the desired endpoints needed to apply Lemma 4.17. As it has been done in [1] we obtain this, by using regularized incomplete Beta functions.

Definition 4.28. We define $p_m(x)$ as the unique polynomial of degree $2m-1$ such that $p_m(0) = 0$; $p_m(1) = 1$; $p_m^{(k)}(0) = 0$; $p_m^{(k)}(1) = 0$ for $1 \leq k \leq m-1$. Explicitly they are given by :

$$p_m(x) := \frac{\int_0^x y^{m-1}(1-y)^{m-1}dy}{\int_0^1 y^{m-1}(1-y)^{m-1}dy} \tag{132}$$

It is foreseeable from the form of the inequalities in 4.27 that we will need to calculate the $L^2([0;1])$ norms of p_m, its first and mth derivative . This has been done in the appendix of [1], arriving at closed expressions :

$$\begin{aligned} \|p_m\|^2_{L^2([0;1])} &= A_m := \frac{1}{2} - \frac{(2m)!^4}{4(4m)!m!^4} \\ \|p_m'\|^2_{L^2([0;1])} &= B_m := \frac{(2m-2)!^2(2m-1)!^2}{(4m-3)!(m-1)!^4} \\ \|p_m^{(m)}\|^2_{L^2([0;1])} &= C_m := \frac{(2m-2)!(2m-1)!}{(m-1)!^2} \end{aligned} \tag{133}$$

We are now prepared to formulate the often announced focussing theorems with weakened energy hypothesis.

Lemma 4.29. (Focussing theorem (III), cf. [1] Lemma 4.1/4.5)
Let P be a spacelike submanifold of our smooth spacetime $(M; g)$ and $\gamma : [0; b] \to M$ a unit-timelike (or null) geodesic emanating orthogonal from P on which the SISEC (or SINEC) holds. Furthermore assume that there exists an $b_0 \in (0; b)$ such that $\mathrm{Ric}(\dot\gamma;\dot\gamma)(t) \geq \rho_0 \geq 0$ for all $t \leq b_0$, that is initially the SEC holds on γ. We define in analogy to (116) :

$$f(t) := \begin{cases} 1 & t \in [0; b_0] \\ p_m\left(\frac{b-t}{b-b_0}\right) & t \in [b_0; b] \end{cases} \tag{134}$$

Then :

$$J[f] \qquad := (1-A_m)(-\rho_0)b_0 + \frac{Q_m C_m}{b_0^{2m-1}} + Q_0 A_m b + \frac{\sqrt{B_m}}{b-b_0} + \frac{Q_m C_m}{(b-b_0)^{2m-1}} \tag{135}$$

In particular by Lemma 4.17 if $pk(\dot\gamma(0))$ then there exists a focal point of P along γ.

Remark 4.30. The estimate of $J[f]$ would not suffice in the situation of Lemma 4.22. Only the initial curvature $k(\dot{}(0))$ can help us out to accomodate for the missing attraction on $[b_0; b]$.

Proof. (sketch, cf. [1] Lemma 4.5)
We first have to construct a function contained in $W_0^m(I)$, such that we can apply the inequality given by the energy condition. We define:

$$(t) := \begin{cases} p_m(t=b_0) & t \; 2 \; [0; b_0] \\ 1 & t \; 2 \; [b_0; b] \end{cases} \tag{136}$$

The definition of the polynomials p_m (4.28) tell us now, that f is $m \; 1$ - times continuous differentiable. Furthermore $(f \;)^{(m)}$ exists and is constant on $[0; b_0)$ and $(b_0; b]$ with a finite jump at b_0. Hence $f \; 2 \; W_0^m([0; b])$. We can rewrite f^2 as :

$$f^2 = (f \;)^2 + (1 \quad ^2)f^2 = (f \;)^2 + (1 \quad ^2) \tag{137}$$

, the last step following from $(t) = 1$ for all t such that $f(t) \neq 1$. We can therefore split the integral :

$$\int_0^b f^2 Ric(\dot{}; \dot{})dt = \int_0^b (f \;)^2 Ric(\dot{}; \dot{})dt + \int_0^{b_0} (1 \quad ^2) Ric(\dot{}; \dot{})dt \tag{138}$$

On the first summand we can apply our assumed SISEC (or SINEC) and the second summand can be estimated as usual since $Ric(\dot{}; \dot{})_{[0;b_0]} \quad 0 \quad 0$. If we plug this into $J[f]$ (105) and using (133) one derives the above estimate. □

At this point it is not clear whether the above lemma is telling us anything new. In fact if $\quad p=b_0$ then the focussing theorem 4.20 would already give us the existence of a focal point and thus making the above result unnecessary. We should therefore at least mention, that in [1] this problem is further analyzed and importantly it is demonstrated that for a fixed b_0 which fulfills $Q_0 b_0^2 \quad 1$ and $Q_m = b_0^{2(m \; 1)} \quad 1$ the above predicts under reasonable assumptions a focal point for $b \quad \sqrt{\frac{pB_m}{A_m Q_0}}$ if the initial curvature fulfills $k(\dot{}(0)) > \frac{p}{b_0} \sqrt{\frac{4A_m B_m Q_0 b_0^2}{p}} \; \frac{p}{b_0}$. In particular we have $\quad \frac{2(n \; 1)B_m}{b}$ which is of the same order as the convergence needed for the classical focussing theorem (Lemma 4.20). C.J Fewster and E.Kontou then proceed (p.24 f.) by explicitly examining the toy model of an Einstein-Klein-gordon model describing neutral pions, where the characteristic length scale is chosen to be the reduced Compton wave length $(\; = \; _c)$. They then estimate the necessary minimum timescale b_0 at which the SEC must have been obeyed such that the above conditions ($Q_0 b_0^2 \quad 1$ and $Q_m = b_0^{2(m \; 1)} \quad Q_1 \quad 1$) hold. Using the CDM -model one derives an estimate for the time at which the SEC has been fulfilled , which in fact turns out to be of two orders larger than the minimum timescale b_0 estimated before. A similar analysis of a physical toy-model is done for the null case, which again demonstrates that under physical reasonable assumptions a convergence of the

same order as in Lemma 4.20 already suffices. Hence the above theorem can in fact be employed to physical situations.
Still one could ask, if we could not somehow get along without the assumption of an initial time were the SEC or NEC is valid. In fact one could just choose any b_0 and define $_0 := \min_{j[0;b_0]} \mathrm{Ric}(\_;\_)(t)$. Though this leads to a larger convergence $k(\_)(0)$ needed for the existence of a focal point. In particular the above estimates which proved the initial convergence to be reasonable would not hold anymore. It is therefore desirable to completely do without any pointwise assumptions on the Ricci-tensor. We needed those in the above Ansatz only due to the different conditions we imposed on the test-functions f. On the one hand when formulating the SISEC : $f(0) = f(1) = 0$ and on the other hand in the focussing Lemma 4.17 : $f(0) = 1; f(1) = 0$. May there be another way to circumnavigate this problem? A natural approach to turn a function on $[0;b]$ such that $f(0) = 1; f(1) = 0$ into one which is zero at its endpoints, is just to extend the domain. That is if we extend to the past $: [b_0; b] \quad M$ we could define a function g such that $g(b_0) = 0$ and $g(0) = 1$. If this is done in the right way we can define

$$\tilde{f}(t) := \begin{cases} g(t) & t\ 2\ [b_0; 0] \\ f(t) & t\ 2\ [0; b] \end{cases} \tag{139}$$

as a function with the desired properties to apply the estimate of SISEC. This will be the approach used in the next focussing theorem.

Lemma 4.31. (Focussing theorem (IV), cf. [1] Lemma 4.3/4.7)
Let P be a spacelike submanifold of our smooth spacetime $(M; g)$ and $: [0; b] \quad M$ a unit-timelike (or null) geodesic emanating orthogonal from P. Extend to the past such that $: [b_0; b] \quad M$ is a unit-timelike (or null) geodesic on which the SISEC (or SINEC) holds. We define :

$$f_{b^0}(t) := \begin{cases} p_m(\frac{b^0\ t}{b^0}) & t\ 2\ [0; b^0] \\ 0 & t\ 2\ [b^0; b] \end{cases} \tag{140}$$

Then :

$$J[f_{b^0}] \qquad b^0 := \frac{Q_m C_m}{(b^0)^{2m\ 1}} + \frac{pB_m}{b^0} + A_m Q_0 b^0 + \inf_{b_0^0\, 2\, (0; b_0]} \frac{Q_m C_m}{(b_0^0)^{2m\ 1}} + A_m (Q_0 + \ _{max}) b_0^0 \tag{141}$$

, where $_{max} := \max_{[b_0; 0]} \mathrm{Ric}(\_;\_)(t)$. In particular by Lemma 4.17 if $(pk(\_(0))$ $:= \inf_{b^0 2 (0;b]}(\ b^0))$ then there exists a focal point of P along .

Proof. We define:

$$g(t) := \begin{cases} 0 & t\ 2\ [b_0; \ b_0^0] \\ p_m(\frac{b_0^0 + t}{b_0^0}) & t\ 2\ [b_0^0; 0] \end{cases} \tag{142}$$

Hence we can construct $\tilde f$ as in (139) which then, again by the definition of p_m is contained in $W_0^m([-b_0^0; b])$. That is we can estimate the desired integral :

$$\int_0^b f^2(t)Ric(\dot\gamma,\dot\gamma)dt = \int_{b_0}^{b} \tilde f^2(t)Ric(\dot\gamma,\dot\gamma)dt - \int_{b_0^0}^{0} \tilde f^2(t)Ric(\dot\gamma,\dot\gamma)dt$$
$$\geq -\|\tilde f\|_m^2 - \rho_{max}\int_{b_0^0}^{0} \tilde f^2(t)dt \qquad (143)$$

Again using (133) one derives the desired bounds. Importantly the constants $b_0^0 \in (0; b_0]$ and $b^0 \in (0; b]$ were chosen arbitrary, which allows us to optimize over them. Let us briefly argue, why those infima are in fact minimas. We start with infima over b_0^0. For convenience let us define $F(b_0^0) := \{\frac{Q_m C_m}{(b_0^0)^{2m-1}} + A_m(Q_0 + \rho_{max})b_0^0\}$. Then F is initially decreasing and has only one minimum at $b_0^{0min} = \left(\frac{(2m-1)Q_m C_m}{A_m(Q_0+\rho_{max})}\right)^{1/2m}$ assuming this expression is well defined. Hence if $b_0 \geq \left(\frac{(2m-1)Q_m C_m}{A_m(Q_0+\rho_{max})}\right)^{1/2m}$ the infima is given by :

$$F_{min} = \frac{2m}{2m-1}(A_m(Q_0+\rho_{max}))^{1-1/(2m)}((2m-1)C_m Q_m)^{1/(2m)} \qquad (144)$$

If $b_0 \leq b_0^{0min}$ the infima is given by $F(b_0)$.
The second optimization is done over $b^0 \in (0; b]$. Unfortunately no closed expression for the minima can be given for a general h. Nevertheless by taking the derivative of $G(b^0) := \{\frac{Q_m C_m}{(b^0)^{2m-1}} + \frac{\sqrt{B_m}}{b^0} + A_m Q_0 b^0\}$ one concludes, that G initially decreases, then at some point reaches a minimum and then only increases from there on.

□

This time we are not obliged to prove, that the above result is not already covered by the classical focussing theorems : We did not assume the SEC (or NEC) at any time, the classical results are therefore powerless. Still if turns out to be in general unreasonable larger than the contraction required in the classical focussing theorems, the above result would lose most of its physical significance. Hence it is an important result in [1] , that under physical reasonable assumptions : $\frac{\sqrt{B_m}+1}{b}$ which for small values of m is close to the order of the convergence needed to prove the classical theorems. Finally it is interesting to notice that, a large SEC violation at the beginning ($\rho_{max} < 0$) implies a smaller initial convergence to obtain the existence of focal points. This may seem rather unituitive at first, but generally follows due to the SISEC being an averaged energy condition: An initial violation has to be followed by closer fulfillment later on.

Before finally getting into proving the singularity theorems it may be appropriate to shortly remind ourselves what we have done so far and where we are at now. We began by constructing an arena, giving us the basic structures in which we could formulate causality (Sec.2) . Thereby we have always been guided by the intuition of basic causal relations introduced by special relativity (Sec.3.1, Sec.3.2). The twin paradox (Lemma 3.27) has been a fundamental result on this path, not only concerning the intuition for lorentzian causality but also being the crucial ingredient to connect the concept of focal points with those of maximizing properties of geodesic (Sec.4.1). After generalizing these causal concepts globally onto our spacetime (Sec.3.3), we have seen that only a few properties could be preserved in this process. Hence we needed to impose causality conditions from outside. Considering causal curves as the fundamental building blocks of causality (Sec.3.3.2) lead us quite directly to the concept of global hyperbolicity. As we have seen in (Sec.3.3.4) it is exactly this property of spacetime which is needed to fully unleash the power of the limit curve theorems. It allowed us to prove crucial existence results for maximizing geodesics (Sec.3.3.5). Importantly those results could be generalized to C^1-spacetimes (Theorem 3.66) which therefore gives us hope to later establish the singularity theorems also in this regularity regime. We then proceeded with the opposite goal: We searched for conditions one could impose on our spacetime to predict the failure of maximality of a geodesic. We emphasized the importance of the physical reasonableness those condition must have and therefore proceeded by giving a brief overview of the standard energy conditions and their physical interpretations (Sec.4.2). Finally we have formulated generalized energy conditions which as argued in (Sec.4.3.2), apply to a wider range of physical models but still secure the required focussing of geodesics. Thus it only remains to combine those two counteracting forces which on the one hand provide the existence of maximizing geodesics and on the other hand predict their failure after some nite a ne parameter.

5 Classical Results

The above treatement of causality and maximality of curves is already predestined to a certain notion of what a singularity is supposed to be (namely based on geodesic incompleteness). Though we should, at least brie y, see how our 'intuition' of singularities corresponds to this formal concept. The intuitive picture of a singularity is some point at which geometrical (curvature) or physical invariants become in nite. The problem with this notion is, that those points cannot be included in spacetime which is choosen to be regular. We therefore have to ask: How could we possibly locate such points?
The natural objects to point at things in spacetime are causal curves. If they are pointing at a 'singularity' they would approach it, though never reaching what they are aiming at. Let us assume those curves happen to be geodesics, that is representing freely falling observers. If those geodesics then would be complete, that is de ned up to arbitrary large a ne parameters, nothing unreasonable would have been explicitly detected in our spacetime. If on the other hand the geodesic ends at some nite a ne parameter, the observer would suddenly vanish from its existence. Clearly something unphysical happened. This leads us to a new idea. Independently of curvature problems, would we not call a spacetime

with a freely falling observer abruptly disappearing, singular? Where did the observer went, if he was not devoured by a singularity? We are thus driven to base our definition of a singular spacetime on the existence of incomplete geodesics instead. It is not a coincidence, that in the preceding chapters we did nothing, but to develop the right tools to prove geodesic incompleteness. Still if we would leave it with that, we would have forgotten those observers which are influenced by external forces; those on a rocket ship for example. The problem one faces now is that we do not have an affine parameter for general causal curves. How then to define completeness? In ([24], p.259) this problem is tackled by defining a generalized affine parameter, suitable for any causal curve. Using this generalized affine parameter (which in fact is an affine parameter for geodesics) they define the concept of b-incompleteness, on which their subsequent definition of a singular spacetime is based. Even though this leads to a precise concept of a singular spacetime, the confusion in connection with singularities does not end there. Generally there is no clear connection between curvature singularities and those based on incompleteness described above[23]. Furthermore even if a spacetime contains incomplete curves the possibility of an extension of spacetime itself is still open. If we are for example considering $(M, g) := (\mathbb{R}^4 \setminus \{0\}, \eta)$ with η the minkowski metric, we would have infinitly many incomplete geodesics. Though it would be hard to argue that (M, g) is truly singular. Hence one generally only considers 'inextendible' spacetimes when speaking of singular spacetimes (cf. [3] Def.3.3). That is for every isometric embedding of (M, g) into a larger manifold $(\tilde{M}, \tilde{g})$, $\tilde{g}|_{\partial M}$ is of lower regularity than C^1 (possibly not even well defined). This seemingly satisfactory view of singularities nevertheless comes with new problems. What if a given spacetime with incomplete causal curves is extendible? Generally there are numerous different extensions, not seldom with rather different physical interpretations and consequences. The above discussion tried to briefly demonstrate how problematic it can be to define singularities in a rigorous way. The classical singularity theorems though, generally only consider geodesic incompleteness.

Definition 5.1. (geodesic completeness) A C^k-spacetime (M, g) with $k \geq 1$ is geodesically complete if all inextendible geodesics are complete that is defined on $\mathbb{R}$. Similarly one defines causal geodesic completeness by only concerning causal geodesics above.

As we have seen this definitely restricts the consequences derivable from the theorems. Explicitly they do not necessarily predict any curvature problems. Nonetheless it indicates that something goes wrong and allows due to its minimalistic nature to be proven by rather general conditions on spacetime. We will proceed, contrary to the historical development but in line with our treatment, by proving the singularity theorem first presented 1966 by S.W.Hawking [34].

[23] A result which tries to eluminate this relation can be found in ([24],Prop.8.5.2). Here it is proven that for generic spacetimes and an so called imprisoned b-incomplete curve γ also some kind of curvature singularity along γ appears.

Theorem 5.2. (Hawking I (1966), cf. [34] Theorem 1 or [5] Theorem 55a)
Let (M, g) be a C^1 spacetime. If:

(1) M contains a spacelike Cauchy-hypersurface S

(2) Along every future directed unit-timelike geodesic $\gamma : [0, b] \to M$ emanating orthogonal from S there exists a focal point of S

Then every future directed timelike curve starting in S has a length bounded by b. In particular (M, g) is geodesically incomplete.

Proof. To speak the truth, in its essence we have already proven the theorem in the previous chapters. It only remains to put the right theorems in the right order:
Since M contains a Cauchy hypersurface it is globally hyperbolic (Theorem 3.56). Hence let $q \in I^+(S)$ be arbitrary and $\sigma : [0, c] \to M$ any future directed timelike curve starting in S and ending at q. By Lemma 3.64 we know there exists a maximal unit-timelike geodesic γ from S to q. In particular $L_g(\sigma) \leq L_g(\gamma)$. We have assumed the existence of an focal point along γ after an affine parameter greater or equal b. Now the key conclusion of Theorem 4.12 implies thus a failure of maximality after b. It therefore follows that $L_g(\sigma) \leq L_g(\gamma) \leq b$. □

In general condition (2) is instead given as :

(2.1) (M, g) fulfills the SEC ($Ric(X, X) \geq 0$ for all timelike X)

(2.2) for every future directed unit-timelike normal vector:

$$n \in (TS)^{\perp} : k(n) \geq c > 0$$

In fact, using the first focussing theorem 4.20 we deduce from (2.1) and (2.2) condition (2) setting $b = 1/c$. The above formulation has the advantage though that it does not explicitly relies on the energy conditions. Hence we can use the results of 4.2 and 4.3.2 to derive (2) using weakened energy conditions:

Lemma 5.3. (Hawking I with weakened energy conditions, cf. [1] Theorem 4.2/4.4)
Assume condition (1) of 5.2 and either of :
<u>Version (a)</u>

(2.1) (M, g) fulfills the SISEC (4.27) for all future directed unit-timelike geodesics emanating orthogonal to S:

$\gamma : [0, b] \to M$ with Q_m, Q independent of γ

(2.2) there exists an $b_0 \in (0, b)$ such that for all such geodesics :

$Ric(\dot\gamma, \dot\gamma)(t) \geq \kappa_0 \geq 0$ for all $t \leq b_0$

(2.3) for every future directed unit-timelike normal vector:

$$n \in (TS)^{\perp} : (n-1)k(n) \geq \min\left(\frac{n-1}{b_0}, \ldots\right) \qquad \text{(see (135))}$$

Version (b)

(2:1) there exists an $b_0 > 0$ such that all future directed uni-timelike geodesics emanating orthogonal to S can be extended as an a ne geodesic to :

$: [\; b_0; b] \; M$

(2:2) $(M; g)$ ful lls the SISEC for all such extended geodesics as in (2.1) with $Q_m; Q$ independent of and there exists a nite upper bound

${}_{max}$ such that: $Ric(\_; \_)_{j[\; b_0; b]}$ ${}_{max}$

(2:3) for every future directed unit-timelike normal vector:

$n \, 2 \, (TS)^? : (n \quad 1) k(n)$ (see (141))

Then condition (2) of 5.2 is ful lled and therefore $L_g(\;)$ b for all future directed timelike geodesics emananting orthogonal to S.

Proof. The above implications are exactly what Lemma 4.29 and 4.31 show. □

Event though the above theorem may be applied to Robertson-Walker models of spacetime which satisfy the right energy conditions, it has the drawback of requiring the existence of a Cauchy hypersurface (and therefore global hyperbolicity). As we have seen previously, global hyperbolicity is a rather strong requirement imposed on spacetime. If we would for example delete any closed subset of $I^+(S)$ from our spacetime, S would cease to be a Cauchy hypersurface and the above theorem would loose its ground. Aiming for a more stable singularity theorem, with weaker conditions on causality; the following theorem has been proven and formulated rst by S.W.Hawking. It assumes weaker conditions though also proves less.

Theorem 5.4. (Hawking II (1967), cf. [35] Theorem 1 or [5] Theorem 55b) Let $(M; g)$ be a C^1 -spacetime. If :

(1) $(M; g)$ contains a compact spacelike acausal hypersurface A
(2) Along every future directed unit-timelike geodesic $: [0; b] \; M$ emanating orthogonal from A there exists a focal point of A

Then $(M; g)$ is future geodesically incomplete.

Proof. This time we need some further preparations. The basic idea is, to restrict our spacetime to a globally hyperbolic subset in which A is in fact a Cauchy hypersurface. Luckily this is exactly what 3.59 assures us : Choose the new spacetime as $(D(A); g_{jD(A)})$. Thus for $(D(A); g_{jD(A)})$ all of the conditions of 5.2 hold which implies that : $D^+(A) = D(A) \setminus (I^+(A) [\; A) \quad f \; p \, 2 \, M \, j \quad {}_g(S; p) \quad bg$. In fact every causal curve from A to $p \, 2 \, D^+(A)$ must be contained in $D^+(A)$. If not, let $: [0; c]! \quad M$ be a future directed causal curve from A to p and $q^0 \, 2 \; M \quad D^+(A)$. Then construct a inextendible past directed causal curve curve starting at q^0. By concatenating $(c \quad t)$ with we get a past directed inextendible causal curve which does not meet S and starts at p, a contradiction.

Therefore the previously stated inclusion is true. If $\overline{D^+(A)} \setminus D^+(A) = \emptyset$, then $I^+(M) \subset D^+(A)$ and hence our result would follow from 5.2. Thus assume there exists an $q \in \overline{D^+(A)} \setminus D^+(A)$. Conclusively we want to prove the existence of an incomplete future directed unit-timelike geodesic emanating orthogonal form A and of proper time smaller or equal to b. Let us assume the contrary and derive a contradiction in 5 steps.

(1) The first step in deriving a contradiction is showing that under the above assumptions $\overline{D^+(A)} \subset \{p \in M \mid \tau_g(A,p) \le b\}$. This done by choosing any sequence $(p_n)_n \subset D^+(A)$ such that $p_n \to q \in \overline{D^+(A)} \setminus D^+(A)$. Since, as argued above, the yet to be proven inclusion holds for $D^+(A)$ we know that $\tau_g(S,p_n) \le b$. Now let $K := \{v \in NA \mid |v|_g \le b, v \text{ future directed}\}$, which is compact. In particular we have $D^+(K) \subset \mathrm{Exp}(K)$ which is well defined on K by our assumption that every future directed unit-normal geodesic emanating orthogonal from A is at least defined up to an affine parameter of b. We proceed by choosing $v_n \in \mathrm{Exp}_{|K}^{-1}(p_n)$, that is constructing a sequence $(v_n)_n \subset K$. Since K is compact we can assume $v_n \to v \in K$. Using that $\mathrm{Exp}_{|K}$ is continuous we find $\mathrm{Exp}(v) = q$. In lemma 3.61 we noticed that the time separation function τ_g is lower semi-continuous. In particular we can find for every $k \in \mathbb{N}$ an n_k such that $|v_n|_g = \tau_g(S,p_n) \ge \tau_g(S,q) - 1/k \to |v|_g \ge \tau_g(S,q)$. In particular we have $\tau_g(S,q) \le b$, that is $\overline{D^+(A)} \subset \{p \in M \mid \tau_g(S,p) \le b\} \subset \mathrm{Exp}(K)$. We therefore conclude that $\overline{D^+(A)}$ is compact.

(2) Let again $q \in \overline{D^+(A)} \setminus D^+(A)$. In particular there is an inextendible causal curve $\gamma : [0,c) \to M$ starting at q, never reaching A. The next step consists in proving that γ in fact is a conjugate free null geodesic. If γ were timelike, choose any $q_\varepsilon = \gamma(t_\varepsilon)$ with $t_\varepsilon > 0$. Then $I^+(q_\varepsilon)$ is an open neighbourhood of q. In particular $I^+(q_\varepsilon) \cap D^+(A) \neq \emptyset$ which allows us to choose $q' \in I^+(q_\varepsilon) \cap D^+(A)$. Construct a curve by connecting q' to q_ε by a past directed timelike curve and then concatenating it with $\gamma_{|t \ge t_\varepsilon}$. Hence we have constructed a past directed inextendible timelike curve from $q' \in D^+(A)$ which never meets A; a contradiction. From here we can not only deduce that γ is not timelike but using Theorem 4.16 we obtain that indeed γ must be a conjugate free null geodesic. In fact if not, let $(c_k)_k$ be a sequence converging strictly increasing to c such that $(\gamma(c_k))_k$ does not converge and $c_0 < c$ being large enough such that $\gamma_{|[0,c_0]}$ is not a conjugate free null geodesic. By Theorem 4.16 we have a past directed timelike curve from q to $\gamma(c_0)$. This curve may be arbitrary close to γ and therefore can be chosen to avoid A. Let us construct a past inextendible causal curve γ_0 by concatenating this newly obtained timelike curve at $\gamma(c_0)$ with γ. Since for large enough $n_0 \in \mathbb{N}$ $\gamma_{0|[c_0 - 1/n_0, c_1]}$ is not a conjugate free null geodesic, we can apply 4.16 to obtain a timelike curve from $\gamma_0(c_0 - 1/n_0)$ to $\gamma_0(c_1) = \gamma(c_1)$ (still avoiding A). From here we can construct a past directed causal curve by concatenating $\gamma_{0[0,c_0 - 1/n_0]}$ with the new timelike curve up to $\gamma(c_1)$ and then with γ. Call this curve γ_1. If we repeat this procedure recursively (with strictly increasing n_k) we can define a past directed

timelike curve :

$$(t) := \quad _k(t) \text{ if } t \quad c_k \quad 1{=}n_k \qquad (145)$$

If were past extendible we would have $\lim_{k!1}$ $(c_k) =$ $\lim_{k!1}$ $(c_k \quad 1{=}n_k) =$ $\lim_{k!1}$ $(c_k \quad 1{=}n_k) =$ $p\ 2\ M$. Contrary to the choice for $(c_k)_k$ as a sequence on which does not converge. Hence we would be in the same situation as previously when considering to be timelike. Hence is a conjugate free null geodesic.

(3) There are two additional properties we will need for . The rst one is the fact that it can not leave $\overline{D^+(A)} \quad D^+(A)$. Assume this is false, that is there exists a t_0 such that $(t_0)\ 2\!\!\!= \overline{D^+(A)} \quad D^+(A)$. If (t_0) would enter $D^+(A)$ it would meet A at some time which we explicitly demanded not to be the case. Hence (t_0) is not contained in $\overline{D^+(A)}$. In particular there exists a past directed inextendible timelike curve from (t_0) which does not meet A. It is important for the subsequent argument, that in fact can be chosen to be timelike. If not every past directed inextendible timelike curve starting at (t_0) would have to meet A. Let us assume this would be true and choose to be an arbitrary such curve. Then $q^0 = \quad (t^0)$ for $t^0 > 0$ is contained in $D^+(A)$ and therefore $(t_0)\ 2\ \overline{D^+(A)}$. This can be seen by noticing, that every past directed causal curve which starts at q^0 and never meets A allows us to construct (just as we have done when proving that is a conjugate free null geodesic) an inextendible timelike curve from q which also does not meet A. Again we are facing a contradiction, which therefore allows us to assume that in fact is timelike. We proceed, de ning a new past directed curve as the concatenation of until (t_0) with . As we have done in the prior arguments we can construct an inextendible timelike curve which starts at q but does not meet A. Just like in the second step, this is a contradiction to $q\ 2\ \overline{D^+(A)}$. Hence we are concluding that is a conjugate free null geodesic which is determined to stay inside the boundary $\overline{D^+(A)} \quad D^+(A)$ [24].

(4) As remarked above, there is still one crucial property of which we will need to prove: The function $_g(S; \ (t))$ is strictly decreasing. The proof follows the same line as the preceding steps. Concretely let $t < s$ be in the domain of . Since $(s)\ 2\ \overline{D^+(A)}$ we know by the beginning of the proof that there is a timelike geodesic from S to (s) of length $L_g(\ ) = \quad (S; \ (s)) \quad 1{=}b$. Concatenating with $_{j[s;t]}$ we have constructed a future directed causal curve from S to (t) with the same length as (here we have used that is a null geodesic). Though since this concatenated curve for sure is not a geodesic (break at (s)) it can not be maximal (4.3). Hence $_g(S; \ (t)) > \ _g(S; \ (s))$.

(5) As we have proven all preparatory results it only remains to conclude the contradiction. In fact by the compactness of $\overline{D^+(A)} \quad D^+(A)$ the function $_g(S; \ )$ must take a de nite minima at a $q_{min}\ 2\ \overline{D^+(A)} \quad D^+(A)$. This contradicts our previous result, by choosing $q := q_{min}$ and traveling along the null geodesic along which $_g(S; \ )$ only decreases.

[24] Such geodesics are generally called 'null -generators'

Finally a contradiction has been found which therefore forces us to conclude the existence of an inextendible future directed unit-timelike geodesic emanating orthogonal to A and most importantly of length smaller or equal to b. □

Remark 5.5. 1. It is possible to prove the above theorem without the assumption of A being acausal, thus without any explicit causality condition. This is due to Prop.48 in [5] which tells us that for any connected component A^0 of A we are able to to define a semi Riemannian covering map $k : (\tilde{M}, \tilde{g}) \to (M, g)$ which sends a closed spacelike acausal hypersurface $\tilde{A} \subset \tilde{M}$ isometric to A^0. If condition (2) of the above theorem is fulfilled in (M, g) with respect to A and therefore A^0 so it is in $(\tilde{M}, \tilde{g})$ with respect to $\tilde{A}$ (lift geodesic variations). In particular we can apply 5.4 to $(\tilde{M}, \tilde{g})$. Using Satz 4.69 of [13] we conclude the existence of an inextendible unit-timelike geodesic emanating orthogonal to A and of proper time bounded by $1/b$. In particular it follows that (M, g) is geodesically incomplete.

2. Just as in Theorem 5.2 we purposely gave condition (2) possibly in a less intuitive but easier to generalize way as in standard references. In fact it is the same condition as in Theorem 5.2 and therefore can be formulated in the exact same way as we did in Lemma 5.3 (Version (a) and Version (b)) only assuming the SISEC and certain initial conditions.

In Sec.4 and all the subsequent discussions based upon it, we always treated the null case separately. In particular we introduced the action functional only to be able to generalize the study of focal points to null geodesics. Hence it is time to justify this detour we have taken. In fact in all places of the last theorem where we have used 4.16 we did not explicitly need the concept of focal points along null geodesics. It would have been enough to only prove that is a null geodesic contained in $\overline{D^+(A)} \setminus D^+(A)$. Nevertheless we will subsequently see, that this changes when trying to prove a singularity theorem concerning the geometry found in the context of Black holes. Here the full formulation of 4.16 and therefore the concept of focal points along null geodesics will be crucial. In general when modelling (non rotating) Black holes one considers spacetimes on which $SO(3)$ acts via isometries (spherical symmetry) and can be described as $Q \times S^2$ with Q a Lorentz surface. It is therefore natural to examine the consequences of S^2, or to keep it general any spacelike $(n-2)$-dimensional submanifold P, to have a positive initial convergence. In particular by formulating such general conditions and proving them resulting in singularities, would set us free from the explicit symmetry normally imposed on Black hole models. This is exactly what Penrose's singularity Theorem (1965, the first modern singularity theorem!) does. The problem which holds us back from applying the same reasoning as in Hawkings Theorems (5.2, 5.4) is that we are not able to restrict our spacetime, making P to some kind of Cauchy hypersurface anymore. The crucial idea from here is to examine the compactness of the whole causal future and hope to derive similarly as before, when considering $\overline{D^+(A)}$ a contradiction to (M, g) being geodesically complete. This motivates the following definition introduced by Penrose in [36]:

Definition 5.6. We call a non-empty achronal set $B \subset M$ a future trapped set if $E^+(B) = J^+(B) \setminus I^+(B)$ is compact and non-empty.

This fundamental concept allows us to formulate the Penrose singularity theorem.

Theorem 5.7. (Penrose Theorem (1965), cf. [36] or [5] Theorem 61, Cor.A) Let $(M; g)$ be a C^1-spacetime. The following two statements cannot hold simultaneously :

(1) $(M; g)$ contains a non compact Cauchy hypersurface S

(2) $(M; g)$ contains a compact trapped set B

Remark 5.8. In the above formulation the connection to compact spacelike submanifolds of dimension $(n - 2)$ with positive initial convergence is hidden behind condition (2). This will be explained in more detail later on.

Proof. Assume (1) and (2) were true. The following (3) steps will derive a contradiction.

(1) $E^+(B) = \partial J^+(B)$.
We start by noticing that $(M; g)$ is globally hyperbolic since it contains a Cauchy hypersurface (3.56). If $(q_n)_n$ is a sequence contained in $J^+(B)$ such that $q_n \to q \in M$, choose $p_n \in B$ such that $q_n \in J^+(p_n)$. Since B is compact we can assume $p_n \to p \in K$. In particular causal simplicity due to global hyperbolicity implies $p \leq q$ and therefore $q \in J^+(B)$. Hence $J^+(B)$ is closed and $E^+(B) = \partial J^+(B)$.

(2) $\partial J^+(B)$ and therefore $E^+(B)$ is a topological hypersurface.
Since $I^+(J^+(B)) \subset J^+(B)$, we have that $J^+(B)$ is a future set (which is defined by this exact property). Hence a classical result (cf. [24] Prop.6.3.1 or [5] Cor.27) concerning future sets can be applied, telling us that the boundary $\partial J^+(B)$ in fact is an achronal $C^{0;1}$ (lipschitz continuous)-hypersurface.

(3) There is an embedding: $\psi : E^+(B) \to S$.
Since our spacetime $(M; g)$ is time-orientable we have the existence of a global timelike vector field $u \in \Gamma(TM)$. A standard result of differential geometry ([4] Satz 8.3) is the existence of a smooth flow $\Phi_u : U_u \to M$ with U_u an open subset of $\mathbb{R} \times M$ and inextendible integral curves of X given by $\gamma(\cdot; p)$. Thus for every point $p \in E^+(B)$ we have $\gamma(\cdot; p)$ an inextendible timelike curve which has to meet S at some time t_p. This allows us to define $\psi(p) := \gamma(t_p; p)$. If $\psi(p) = \psi(p') = q \in S$, we would have that p and p' would lie on the timelike curve $\gamma(\cdot; q)$ which contradicts achronality of $E^+(B)$. Hence ψ is injective. To

prove that ψ in fact is continuous we define the restriction $\varphi := \psi|_{(\mathbb{R}\times S)\cap U_u}$. We already noticed that every integral curve has to meet S. Hence φ is surjective. Furthermore since S is achronal we can similar as previously for $E^+(B)$ and deduce that φ is injective. Using that S is a topological hypersurface and U_u is open we conclude that $(\mathbb{R}\times S)\cap U_u$ is a manifold of dimension n. We have therefore constructed a continuous bijective map between two manifolds which by the invariance of domain theorem has to be a homeomorphism. Now choose an arbitrary $q \in S$ and $t \in \mathbb{R}$ such that $\varphi(t;q) \in E^+(B)$. Then $\rho(\varphi(t;q)) = q = \pi_2(t;q)$ where π_2 is the projection onto the second factor. Finally we have arrived at the result that $\rho = \pi_2 \circ \varphi^{-1}$, which in fact is continuous. Once more we can apply the invariance of domain theorem by which we obtain that ρ is a continuous embedding.

It only remains to conclude the contradiction: Since $E^+(B)$ is compact $\rho(E^+(B)) \neq S$ is an open and closed subset which contradicts S being connected.

$\square$

Remark 5.9. In [10] a future trapped set B is not assumed to be achronal or $E^+(B)$ to be non-empty. Instead it is proven (cf.Theorem 6.23), that in globally hyperbolic spacetimes and B being a non-empty compact set such that $\overline{E^+(B)}$ is compact we can define $A := E^+(B) \cap B$ a compact, achronal and non-empty set such that $E^+(A) = E^+(B) \neq \emptyset$ is compact. Hence the above theorem already follows if B is only assumed to be a non-empty compact set such that $\overline{E^+(B)}$ is compact.

The question which is now of great importance to answer, is that after a more concrete description of future trapped sets. Up to this point they were only described in an abstract hard to test manner and therefore need to be characterized by giving some examples. Most importantly we have not seen yet the connection between geodesic incompleteness and therefore singularities and the above theorem. Luckily such a description has been given already in the original formulation by Penrose. We will again formulate the following lemma in such a way, that our focal point results are easy to be applied.

Lemma 5.10. (future converging surfaces, cf. [5] Prop.60 or [10] Theorem 6.24)
Let $(M;g)$ be a C^1-spacetime. Furthermore let P be an achronal compact spacelike $(n-2)$ dimensional submanifold of M. Assume that along every future directed null geodesic $\gamma : [0;b] \to M$ emanating orthogonal from P there exists a focal point of P. If $(M;g)$ is future null geodesic complete then $E^+(P)$ is compact and non-empty, that is P is a trapped set.

Proof. The proof is analogous to the one given in 5.4 where it has been shown that $\overline{D^+(S)} \subseteq \mathrm{Exp}(K)$ with K a compact subset of the normal bundle. Thus let us define $K^0 := \{v \in NP \mid g(v;v) = 0\}$ that is the set of all normal null vectors in NP. In particular we can define $\pi : K^0 \to P$ as $\pi(v_p) := p$. Since for every $p \in P$ the subspace $(T_pP)^\perp$ is timelike we have by Lemma 3.4 the existence of two linear independent null normal vectors at p. By ([13] Satz 3.21) we conclude the existence of an orthonormal frame $(e_0, e_1, \dots, e_{n-1})$ of the Pullback bundle $\iota^* TM$ of $(\iota : P \hookrightarrow M)$ on an open neighbourhood U of $p \in P$ such that e_0 is timelike, e_1 spacelike and they together a basis for $NP_{|U}$. Since $e_0 + e_1$ and $e_0 - e_1$ describe two linearly independent null vectors orthogonal to P we can define $U_1 := \{v_q \in K' \mid q \in U, v_q \in \mathbb{R}(e_0 + e_1)\}$ and $U_2 := \{v_q \in K' \mid q \in U, v_q \in \mathbb{R}(e_0 - e_1)\}$ two open subsets of K' such that $\pi_{|U_i} : U_i \to U$ are homeomorphisms for $i = 1, 2$. In particular π is a double covering of P and thus compact. We proceed, using Theorem 4.16 which tells us that for $q \in J^+(P) \setminus I^+(P)$ any causal curve γ connecting P to q must be a normal null geodesic without any focal points (or the constant curve). It thus follows from our assumption $\gamma : \gamma(s) = q$ for $s < b$. Hence $E^+(P) \subset Exp(\tilde{K})$ with $\tilde{K} := \{tv \in NP \mid v \in K', 0 \leq t \leq b\}$ being compact and well defined since (M, g) is null geodesically complete. For any sequence $(q_n)_n$ in $E^+(P)$ we have the existence of an accumulation point q in $Exp(\tilde{K})$. Though since $I^+(P)$ is open and all $q_n \notin I^+(P)$ we conclude that $q \in E^+(P)$. We have proven that $E^+(P)$ is a closed set contained in the compact set $Exp(\tilde{K})$ which therefore by itself is compact. If $E^+(P) = \emptyset$ we would have $P \subset I^+(P)$ contrary to P being achronal. □

In particular if we combine this lemma with Theorem 5.7 we get:

Corollary 5.11. Let (M, g) be a C^1-spacetime. The following statements cannot hold simultaneously:

(1) (M, g) contains a non compact Cauchy hypersurface S

(2) (M, g) contains an achronal compact spacelike $(n-2)$- dimensional submanifold P such that along every future directed null geodesic $\gamma : [0, b) \to M$ emanating orthogonal from P there exists a focal point of P.

(3) (M, g) is future null geodesic complete

Hence we can exactly as in Lemma 5.3 use our different focal point theorems of Sec.4.2 to conclude the fulfillment of the above condition (2) only using the SINEC and appropriate initial conditions. If in a given spacetime model one could additionally demonstrate the existence of a non compact Cauchy hypersurface we would have proven (M, g) to be null geodesic-incomplete. Though this last requirement of a non-compact Cauchy hypersurface is even more demanding as the mere existence of a Cauchy hypersurface in Theorem 5.2. Just as we did in 5.4 we could instead assume the existence of an acausal topological hypersurface S and restrict our spacetime to $(D(S), g_{|D(S)})$. Then the preceding results would imply geodesic incompleteness inside $D(S)$, but would not necessarily assure incompleteness of (M, g) if $\overline{D^+(S)} \setminus D^+(S) \neq \emptyset$. This may still be interpreted as the failure to predict the future from S. In fact for every $q \in \overline{D^+(S)} \setminus D^+(S) \neq \emptyset$ we have $I^+(q) \cap D^+(S) = \emptyset$. Otherwise we could choose $q' \in I^+(q) \cap D^+(S)$ and therefore construct a past directed causal curve by concatenating a timelike curve from q' to q with a past inextendible null generator from q. Since achronality of S implies that the beginning timelike segment can not meet S we have constructed a past inextendible causal curve from q' which does not meet S. A contradiction to $q' \in D^+(S)$. The Penrose singularity theorem therefore (as argued in ([24],p.265)) may be interpreted as demonstrating the failure of predictability rather than the unalterable conclusion of a singularity. It is nevertheless desirable to formulate a theorem which similarly concerns the formation of trapped sets but does not rely on the existence of a non-compact Cauchy hypersurface. The search for such a theorem culminated 1970 in the Hawking and Penrose Theorem a version of which can be fromulated as:

Theorem 5.12. (Hawking and Penrose cf. [37] (1970) or [3] Lemma 5.1)
Let (M, g) be a C^1-spacetime. The following thee statements cannot hold simultaneously:

(1) every inextendible causal geodesic has a conjugate point
(2) (M, g) satisfies the chronology condition
(3) there exists a future (or past) trapped set B

We will not include the proof of this theorem. After all we want to examine the connection between the generalized energy conditions and low regularity. The above proof though requires quit some work, rather unrelated to this question. A thorough treatment and discussion is given in ([10],6.6.3). We will instead only shortly try to place the above theorem among the other singularity theorems. As we remarked in Lemma 4.22 assuming the SEC and the genericity condition along every causal geodesic already suffice to assure condition (1). The question of how general the genericity condition truly is, is a topic for itself. While for non-totally imprisoning spacetimes there seems to be a consensus on it being a reasonable condition, there is none when only assuming the chronology condition (cf. [18] Remark 6.22). Interestingly the above theorem also generalizes to a certain extent Hawking's Theorem 5.4. In fact any compact acausal spacelike hypersurface is also a future (and past) trapped set.

Lemma 5.13. Let (M, g) be a spacetime and A a compact acausal spacelike hypersurface. Then $E^+(A) = E^-(A) = A$. In particular A is a trapped set.

Proof. .
(1) $A \subseteq E^+(A)$
We have $A \setminus E^+(A) \subseteq I^+(A) \cap A = \emptyset$, where the last equality follows from A being acausal.
(2) $E^+(A) \subseteq A$
If $q \in E^+(A) \setminus A$ then there exists a null geodesic γ which maximizes length to A. In particular we have by 4.16 that γ has to start orthogonal to A which contradicts A being a spacelike hypersurface. □

Remark 5.14. This result may be generalized to any compact achronal topological hypersurface (cf. [3] Ex.4.3).

Even though the above theorem shines with its generality it has the clear disadvantage that it does not give us any concrete hint if we are facing a future or past singularity. In [10] it is argued, that by admitting the past-singularity (Big Bang) on our spacetime model advocated by Hawkings theorem (5.4), the Hawking and Penrose Theorem is not giving us any new information. The Penrose theorem instead could still predict the existence of a future singularity (Black Hole). Nevertheless since concrete models of Black holes (Kruskal,Reissner-Nordström...) generally forget about the rest of the universe, the Hawking and Penrose theorem may still be applied to break loose from their strict symmetry conditions.

We have quite often now repeated the statement that singularity theorems show the stability of singularities under small deviations from the strict symmetry assumptions of a concrete model. To end this chapter we should therefore briefly come back to this promise in a more concrete way. In all of the preceding discussions we always assumed to have a spacetime model (M, g) with certain properties given and then analysed the consequences following from these assumptions. To demonstrate stability as described above we have to take the Einstein field equations (107) into account which determine the spacetime model in the first place. In general one starts with some hypersurface S on which certain initial conditions are imposed. The aim of solving the so called Cauchy problem is then to find a (up to diffeomorphism unique) spacetime (M, g) fulfilling (107) such that S is a Cauchy hypersurface. A crucial result, studying the Cauchy problem is that solutions of it are Cauchy stable (cf. [24] Chapter 7.5 on the Cauchy problem). This is crudely speaking that the solution of (107) restricted to a compact subspace of M depends continuously on the initial data given. Hence if one chooses a compact region of spacetime which contains a trapped set given as a $(n-2)$ dimensional submanifold as in Lemma 5.10 or a hypersurface as in Lemma 5.13, small enough deviations from the symmetry conditions in the original initial data still lead to the formation of those trapped sets. In particular the singularity theorems are still applicable which thus in fact demonstrates their importance in predicting the existence of singularities in physically realistic spacetime models. Speaking of a spacetime model being physical reasonable, we shall now come back to the problem we avoided throughout this section : What is the regularity of the metric needed to cover the physically reasonable models?

6 The question of continuity

It should be mentioned first, that all of our preceding results remain true without major changes up until a regularity of $g \in C^2$. Why is this regularity regime not considered as satisfactory? In fact already in ([24],1973) the problem of regularity concerning the singularity theorems has been mentioned and discussed as one which should be taken seriously. One reason brought up there is the existence of various physical examples of lower regularity :

1. Oppenheimer-Snyder model of a collapsing star ([38],1939) $\Rightarrow$ $g \in C^{1,1}$ (lipschitz continuous derivatives)
2. in general matched spacetimes ([39],1993) $\Rightarrow$ $g \in C^{1,1}$
3. hydrodynamic shock waves $\Rightarrow$ $g \in C^{1,1}$
4. gravitational shock waves $\Rightarrow$ $g \in C^{0,1}$ (lipschitz continuous)
5. thin mass shells ([40],1966) $\Rightarrow$ $g \in C^{0,1}$

Though as argued in [24], the above examples with $g \in C^{0,1}$ are only mathematical idealizations of C^2-solutions and therefore may not endanger the singularity theorems as much as those with $g \in C^{1,1}$. Instead of studying concrete examples, one could also try a more general approach by examining properties of solutions for the Cauchy problem. In fact, classical existence results of the Cauchy problem consider even lower regularities than C^1. There is also a more indirect but not less important reasoning. Assume our C^2-spacetime (M, g) fulfills the conditions demanded by the singularity theorems. Accepting those conditions as physical reasonable would lead us to the conclusion that any reasonable (that is such that the conditions are still satisfied) extension of spacetime which hopes to remedy geodesic incompleteness must be of lower regularity than C^2. Though as we saw above, $C^{1,1}$-spacetimes generally correspond to finite matter discontinuities which therefore do not represent physical unreasonable situations. Even if $g \in C^{0,1}$ it would have (by Radmacher's Theorems) square integrable first derivatives which therefore allows volume integrals of curvature components over compact regions of spacetime to be defined and finite. Hence an extension of the above form with distributional curvature, may still be considered as physical and not containing a curvature singularity. If the singularity theorems would thus fail at low regularities, this could be a generic way to avoid their consequences. When trying to assess the physical relevance of the theorems we therefore conclude that it is an important task to probe the needed regularity. Since singularity theorems in general concern geodesic incompleteness it seems reasonable to start with such regularities where geodesics still exist. If g is only assumed to be $C^{0,1}$ there is no canonical way to define geodesics (though it is possible using Filippov solutions [41] Cor.3.3). On the contrary if $g \in C^1$ not only the existence is assured by Peano's existence theorem but also the crucial properties mentioned in 3.41 hold. It thus seems to be so that C^1 is the lowest regularity where the formulation of the singularity theorems themselves is not too much affected. Even though it may be arguable how reasonable it is to abstain from the uniqueness of geodesics, we shall therefore proceed by choosing C^1 as the regularity in which we want to further discuss the Singularity theorems. Luckily the most important results of global causality in Sec.3.3.2, Sec.3.3.4 and Sec.3.3.5 , which are needed to prove the Singularity theorems have been already proven there for the C^1 case. It is therefore mostly the fundamentally local concept of focal points and in general Sec.4 which bring us trouble when trying to obtain Singularity theorems for C^1 spacetimes. Here Lemma 3.43 will help us out: If there are for close enough smooth approximations (M, g_ε), g_ε-geodesics all starting in a compact set and only defined up to a finite affine parameter smaller than some $N > 0$, then (M, g) is geodesically incomplete. It is therefore sufficient to show that imposing appropriate conditions on our C^1-spacetime (M, g) imply those needed for the Singularity theorems on close enough smooth approximations. This will be the main goal pursued in the remaining of this thesis. In general we will thereby follow the treatment given by M.Graf in [2]. In fact it has been proven there that in C^1-spacetimes Hawking's Theorems (5.2,5.4) and the Penrose Theorem (5.7) remain true under the assumption of a distributional SEC or NEC.

We will thus start by briefly introducing Distributions on manifolds which will allow us to formulate such distributional energy conditions and therefore account for the fact that in C^1-spacetimes the Ricci-curvature is not well defined as a tensor field anymore.

6.1 Some distributional geometry

We already saw in Sec.4.3 that pointwise energy conditions do not suffice if we want to describe scalar fields. That is if we aim to formulate generalized energy conditions obeyed by scalar fields we have to consider conditions fulfilled on worldlines rather than points. This need becomes even more relevant when considering actual quantum fields. Taking this approach one step further, that is considering worldvolume inequalities, we would have already arrived at formulating energy conditions of distributional character. It thus seems (independent of the regularity) natural to introduce distributions on manifolds in the language of which those generalized energy conditions can be formulated. As we have mentioned already, in the case of a C^1-spacetime this approach becomes necessary rather than natural.

In the classical theory of distributions on $\mathbb{R}^n$ there is a canonical way to embed smooth functions $f \in C^\infty(\mathbb{R}^n)$ into the space of distributions by integration :

$$f[\varphi] := \int_{\mathbb{R}^n} f(x)\varphi(x)\,d\mu \tag{146}$$

, where $\varphi \in C_c^\infty(\mathbb{R}^n$ (smooth with compact support) is a test function and μ is the Lebesgue measure on $\mathbb{R}^n$. Though to have such an embedding on general manifolds we need a concept of integration over the whole spacetime in the first place. In fact, we implicitly assumed this to exist when speaking of worldvolume inequalities. Before introducing distributions we therefore have to define test objects which allow to be integrated. Similar as above in the $\mathbb{R}^n$-case it is then possible to embed smooth tensor fields into the topological dual of those test objects, later defined as tensor distributions. This will lead us to a natural generalization of the Ricci-curvature as a distribution.

6.1.1 Densities on manifolds

Since the characteristic property of manifolds is to be locally homeomorphic to R^n we are motivated to base the concept of integration on manifolds on the one already given in R^n. Let us assume we would have already de ned our test objects which allow integration and would be one of them. In the case of $M = R^n$ those objects would be given by sections of the one-dimensional vector bundle (line-bundle) : M R of compact support. That is just as in (146) elements of $C^1_c(M)$. It is therefore convenient to describe those new test objects also as sections with compact support of a yet to de ne line-bundle. Locally, that is for any coordinate chart (V ;) of M, integration of should coincide with the one induced by the normal Lebesgue integral over (V), that is:

$$\int_V = \int_{(V)} (\ ^1(x))\,dx \tag{147}$$

, where is the coordinate expression of . Since the above should be true for any chart, to be consistent we would need the following equality to hold:

$$\begin{aligned} \int_{V \cap V} &= \int_{(V \cap V)} (\ ^1(x))\,dx \\ &= \int_{(V \cap V)} (\ ^1(y))\,|\det(D(\ ^1))|(y)\,dy \\ &\overset{!}{=} \int_{(V \cap V)} (\ ^1(y))\,dy \end{aligned} \tag{148}$$

In the second equality the transformation formula of the Lebesgue integral has been used. Using the vector space structure given by the line bundle for each $p \in V \cap V$ we can choose an arbitrary $f \in C^1_c((V \cap V))$ to de ne a new test object by : $\tilde{} := (f \ )$. Then the (148) becomes:

$$\int_{(V \cap V)} f(y) \left\{ (\ ^1(y))\,|\det(D(\ ^1))|(y) \quad (\ ^1(y)) \right\} dy = 0 \tag{149}$$

Hence by the fundamental lemma of calculus of variations we conclude the transformation rule :

$$^1|\det(D(\ ^1))| \overset{!}{=} \ ^1 \tag{150}$$

Another crucial property of a second-countable manifold is the existence of partitions of unity subordinate to any open cover $\{U\}_{\in A}$ of M, that is a family $R = \{ : M \to R\}_{\in A}$ of smooth functions such that :

1. $\{\mathrm{supp}( )\}_{\in A}$ is a locally nite cover of M and $\mathrm{supp}( ) \quad U$
2. $0 \quad 1 \ \forall \in A$ and furthermore $\sum_{\in A} (p) = 1 \ \forall p \in M$ (well de ned due to local niteness (1.))

In particular if the integral of ω as one should demand is linear we would have for every Atlas $\mathcal{A} = \{(V_\alpha, \varphi_\alpha) : \alpha \in A\}$ and compact subset K of M:

$$\begin{aligned} \int_K \omega = \int_K \Big(\sum_{\alpha \in A'} \rho_\alpha \omega\Big) &= \sum_{\alpha \in A'} \int_K \rho_\alpha \omega \\ &= \sum_{\alpha \in A'} \int_{\varphi_\alpha(V_\alpha)} \rho_\alpha(\varphi_\alpha^{-1}(x))\, \omega_\alpha(\varphi_\alpha^{-1}(x))\, dx \end{aligned} \tag{151}$$

, where $A' := \{\alpha \in A : \mathrm{supp}(\rho_\alpha) \cap K \neq \emptyset\}$ is a finite set. In fact using the transformation rule (150) one can demonstrate, that the last expression given above is independent of our choice of Atlas and subordinate partition of unity and therefore suits to be the definition in the first place. It thus only remains to construct a line-bundle fulfilling the transformation behaviour given by (150). Fortunately the vector (or in general fiber)-bundle construction theorem assures us the existence of such:

Theorem. Let $(U_\alpha)_{\alpha \in A}$ be an open cover of M a smooth manifold: For any family $\{A_{\alpha\beta} : V_\alpha \cap V_\beta \to GL_k(\mathbb{R})\}$ of smooth functions such that the cocycle condition is fulfilled :

1. $A_{\alpha\beta} A_{\beta\gamma} = A_{\alpha\gamma}$ on $V_\alpha \cap V_\beta \cap V_\gamma$
2. $A_{\alpha\alpha} = \mathrm{id}_{\mathbb{R}^k}$ on V_α

there exists an up to isomorphism unique vector bundle (E, M, π) which has exactly those $A_{\alpha\beta}$ as transition functions.

In fact (1.) and (2.) both are fulfilled for the transition functions demanded by (150). We are therefore driven to make the following definition.

Definition 6.1. (volume bundle)
Let $(V_\alpha, \varphi_\alpha)$ be an Atlas of a smooth manifold M. We define the real line bundle $Vol(M)$ as the vector bundle given by the cocycle:

$$\begin{aligned} A_{\alpha\beta} &: V_\alpha \cap V_\beta \to \mathbb{R} \setminus \{0\} = GL_1(\mathbb{R}) \\ A_{\alpha\beta}(p) &= |\det(D(\varphi_\beta \circ \varphi_\alpha^{-1}))|^{-1}(\varphi_\alpha(p)) \end{aligned} \tag{152}$$

Remark 6.2. There exists a vector bundle isomorphism $\Phi : M \times \mathbb{R} \to Vol(M)$ (a smooth diffeomorphism which is an isomorphism on each fiber). That is $Vol(M)$ is trivial and a globally non-vanishing section of the volume bundle can be defined by $\sigma(p) := \Phi(p, v)$ for some fixed $v \in \mathbb{R} \setminus \{0\}$. Nevertheless the above isomorphism is not a canonical one but relies on choosing an Atlas and subordinate partition of unity for the vector bundle (cf. [42] Prop.1.1.3).

We therefore finally know what ω is an element of:

Definition 6.3. (Densities)

1. We call $\Gamma^k(M; Vol(M))$ the C^k sections of $Vol(M)$ the space of C^k-densities on M. The C^k-sections of compact support will be denoted as $\Gamma_c^k(Vol(M))$. If $k = \infty$ we will as usual write $\Gamma_c(Vol(M))$.

2. If $\lambda \in \Gamma^k(M; Vol(M))$ and $(\Phi : \pi^{-1}(V) \to V \times \mathbb{R})$ a vector bundle chart we denote the coordinate expression of λ as: $\lambda_\Phi := \pi_2 \circ \Phi \circ \lambda|_V \in C^k(V)$, where $\pi_2 : V \times \mathbb{R} \to \mathbb{R}$ is the projection onto the second component.

Conclusively we have thus defined with (151) a reasonable concept to integrate on compact subsets of our manifolds (which will suffice for our purposes). We now aim to find a topology on the space of densities which will allow us to define distributions as their topological dual. After that we will proceed by stating some for our further discussion fundamental properties concerning the theory of distributions. The basis of this short exposition will be the much more detailed discussions in ([43],Chapter 3) and ([42],Chapter 1).

The following definition of convergence of nets on any general $\Gamma^k(M; E)$, with (E, M, π) a vector bundle generalizes the one already given in Def.2.11 for tensor fields. As usual we base our definition on the property of manifolds to be locally euclidean. That is for each chart (V, φ) of M we define a topology on $\Gamma^k(V; E)$ such that $u \mapsto u \circ \varphi^{-1} \in C^\infty(\varphi(V); \mathbb{R}^k)$ is a homeomorphism. Here the topology on $C^\infty(\varphi(V); \mathbb{R}^k)$ is chosen to be given by uniform convergence of all derivatives up to order k on compact subsets. This leads us to a natural concept of convergence in the whole space $\Gamma^k(M; E)$:

Definition 6.4. (topology/convergence in $\Gamma(M; E)$)
Let (E, M, π) be a vector bundle of rank k. For any net $(u_i)_{i \in I} \subset \Gamma^k(M; E)$, we say $u_i \to u \in \Gamma^k(M; E)$ if for all charts (V, φ) of M we have: $u_i|_V \to u|_V$ in $\Gamma^k(V; E)$.

Remark. With the above topology the space of densities $\Gamma^k(M; E)$ is a Fréchet-space, that is a Hausdorff, locally convex and complete vector space with a countable neighbourhood basis of zero (cf. [42],1.1.5).

Nevertheless we have already mentioned that our test-objects should have compact support to allow integration as we defined above. It may be desirable to maintain completeness when restricting our consideration to the subspace $\Gamma_c^k(M; E)$. Though since $\Gamma_c^k(M; E)$ is dense in $\Gamma^k(M; E)$, we need a finer topology than just the subspace topology induced by $\Gamma^k(M; E)$. Fortunately there is an elegant way to construct such a topology:

Definition 6.5. (Topology on $\Gamma_c^k(M; E)$)
Let $(K_m)_m$ be an exhaustive sequence of compact subsets of M, that is $K_m \subset int(K_{m+1})$ and $M = \bigcup_{m \in \mathbb{N}} K_m$. We define the topology on $\Gamma_c^k(M; E)$ as the inductive limit topology of the inclusions

$$\Gamma_{K_m}^k(X; E) := \{u \in \Gamma^k(M; E) : supp(u) \subset K_m\} \hookrightarrow \Gamma_c^k(M; E) \qquad (153)$$

, where $\Gamma^k_{K_m}(M;E)$ is endowed with the subspace topology. In particular, a net $(u_i)_{i\in I} \subset \Gamma^k_c(M;E)$ converges to $u \in \Gamma^k_c(M;E)$ if and only if there is an $m \in \mathbb{N}$ such that :

$$\bigcup_{i\in I} \mathrm{supp}(u_i) \cup \mathrm{supp}(u) \subseteq K_m \text{ and } u_i \to u \text{ in } \Gamma^k(M;E) \tag{154}$$

Finally it should be remarked, that the above construction does not depend on the chosen exhaustion $(K_m)_m$ and provides us with a locally convex complete (though not Fréchet-) topological vector space $\Gamma_c(M;E)$.

Without further ado we now define distributions.

6.1.2 Distributions on manifolds

Definition 6.6. (Distribution)
Let M be a smooth manifold. We define the space of distributions (of order k) on M as :

$$\mathcal{D}'^{(k)}(M) := \Gamma^k_c(M;\mathrm{Vol}(M))' \tag{155}$$

That is the space of continuous linear functionals from $\Gamma^k_c(M;\mathrm{Vol}(M))$ into $\mathbb{R}$ (topological dual).

Though there has been a reason to instead of treating just the special case of $E = \mathrm{Vol}(M)$, define a topology on $\Gamma^k_c(M;E)$ for an arbitrary vector bundle :

Definition 6.7. (E-valued distributions)
Let (E,M,π) be a vector bundle. we define the space of E-valued distributions (of order k) as:

$$\mathcal{D}'^{(k)}(M;E) := \Gamma^k_c(M;E^*\otimes \mathrm{Vol}(M))' \tag{156}$$

Here E^* denotes the dual bundle corresponding to E (cf. [4] Sec.6.9.1).

After all we aim to define the Ricci-curvature in a distributional way, which suggests $E = T^{(0,2)}M$ such that Ric becomes a so called tensor distribution. Let us proceed as promised by collecting some fundamental properties of E-valued distributions. In hope to clarify the concept of distributions we will begin by presenting two different but equivalent ways to describe them. In particular the last one is of great practical use since it allows us to treat distributions on manifolds as families of the simpler case of distribution on open subsets of $\mathbb{R}^n$.

Facts 6.8. (two further characterizations of E-valued distributions)

1. There exists an isomorphism of $C^\infty(M)$-modules:

$$\Xi : \mathcal{D}'(M) \otimes \Gamma(M;E) \xrightarrow{\cong} \mathcal{D}'(M;E) \tag{157}$$

   Explicitly Ξ is induced by the bilinear map :

$$\begin{aligned}
&\tilde{\Xi} : \mathcal{D}'(M) \times \Gamma(M;E) \to \mathcal{D}'(M;E)\\
&\tilde{\Xi}(T,z)[\omega] := T[(\mathrm{tr}_E \otimes \mathrm{id}_{\mathrm{Vol}(M)})(z\otimes\omega)] = T[\omega(z)]\\
&z \in \Gamma(M;E),\ \omega \in \Gamma_c(M;E^*\otimes \mathrm{Vol}(M))
\end{aligned} \tag{158}$$

where we have always considered the tensor products of $C^\infty(M)$ modules and used $tr_E(z(p) \otimes w(p)) := \omega(z)(p)$ for $p \in M$ to describe the trace operation. In particular we can treat tensor distributions as elements of $D'(M) \otimes \Gamma(T^{(r,s)}M)$.

2. There is a very general and aesthetically pleasing way to describe the 'locality' of structures as $D'^{(k)}(M;E)$ by the concept of sheaves. Without further introducing this approach we just state the important results for our concrete example of $D'^{(k)}(M;E)$.

Lemma 6.9. $D'^{(k)}(M;E)$ is a fine sheave of $C^\infty(M)$-modules. Translated to our example this implies that for any $(U_i)_{i\in I}$, an open covering of M we have :

(1) if $T, J \in D'^{(k)}(M;E)$ such that $T_{|U_i} = J_{|U_i}$ $\forall i \in I$ then $T = J$

(2) for a family $(T_i)_{i\in I}$ of E-valued distributions $T_i \in D'^{(k)}(U_i;E)$ such that:

$T_{i\,|U_i\cap U_j} = T_{j\,|U_i\cap U_j}$ $\forall i,j \in I$ with $U_i \cap U_j \neq \emptyset$

there exists an $T \in D'^{(k)}(M;E)$ with $T_{|U_i} = T_i$ for all $i \in I$

(3) there exists a partition of unity subordinate to $(U_i)_{i\in I}$ that is $C^\infty(M)$-morphisms : $(\chi_i : D'^{(k)}(M;E) \to D'^{(k)}(M;E))$ such that for $T \in D'^{(k)}(M;E)$: $(\mathrm{supp}(\chi_i(T)) \subset U_i)$ is a locally finite cover of M and $\sum_{i\in I} \chi_i(T) = T$

In the above the restriction of a $T \in D'^{(k)}(M;E)$ to an open neighbourhood V, is just defined as $T_{|V}[u] := T[u]$ for all $u \in \Gamma^k_c(V;E \otimes Vol(M))$. Furthermore the support of a distribution is defined as: $x \in \mathrm{supp}(T) \Leftrightarrow$ for all open neighbourhoods V of x there exists an $u \in \Gamma^k_c(V;E \otimes Vol(M))$ and $T[u] \neq 0$.
In particular property (2) will be of importance for us, which is why we briefly sketch the proof:

Proof. (2) Choose any partition of unity subordinate to $(U_i)_{i\in I}$ denoted with $(\psi_i)_{i\in I}$. For any $u \in \Gamma^k_c(M;E \otimes Vol(M))$ we define $u_i := \psi_i u \in \Gamma^k_c(U_i;E \otimes Vol(M))$ so that $u = \sum_{i\in I} u_i$. Assume we would have constructed $T \in D'^{(k)}(M;E)$. By linearity we would have: $T[u] = \sum_{i\in I} T[u_i] = \sum_{i\in I} T_i[u_i]$. Furthermore if ψ'_j is any other other subordinate partition of unity we have

$$: \sum_{i\in I} T_i[u_i] = \sum_{i\in I}\sum_{j\in I} T_i[\psi_i \psi'_j u] = \sum_{i\in I}\sum_{j\in I} T_j[\psi_i \psi'_j u] = \sum_{j\in I} T_j[u_j] \quad (159)$$

. In the second equality we used our assumption $T_{i\,|U_i\cap U_j} = T_{j\,|U_i\cap U_j}$. We are therefore motivated to make the definition $T[u] := \sum_{i\in I} T_i[\psi_i u]$. In fact we again using $T_i|_{U_i \cap U_j} = T_j|_{U_i \cap U_j}$ we conclude for $u \in \Gamma^k_c(U_i; E \otimes Vol(M))$:

$$T|_{U_i}[u] = \sum_{j \in I} T_j[u_j] = \sum_{j \in I} T_j|_{U_i \cap U_j}[u_j] = \sum_{j \in I} T_i|_{U_i \cap U_j}[u_j] = T_i[u] \quad (160)$$

It only remains to show that T in fact is continuous. Thus assume the net $(u_\iota) \subset \Gamma^k_c(M; E \otimes Vol(M))$ converges to $0 \in \Gamma^k_c(M; E \otimes Vol(M))$. Hence there is a compact set K such that $supp(u_\iota) \subseteq K$ and all coordinate expressions (as their derivatives up to order k) converge locally uniformly to zero. Since $supp(\chi_i u_\iota) \subseteq supp(u_\iota)$ we also conclude that $\chi_i u_\iota \to 0$ for an arbitrary fixed $i \in I$. Since T_i is an element of $D'^{(k)}(U_i; E)$ and therefore in particular continuous we have $T_i[\chi_i u_\iota] \to 0$ for every fixed $i \in I$. Hence by the definition of T also $T[u_\iota] \to 0$, that is T is continuous. □

Hence we have demonstrated the possibility to describe any distribution as a certain family of distributions which are only defined locally. If we choose the open sets to be charts of M, that is $(U_\alpha; \varphi_\alpha)$ we have a topological isomorphism: $(\psi_\alpha)_*: \Gamma^k_c(U_\alpha; E \otimes Vol(M)) \to \Gamma^k_c(\varphi_\alpha(U_\alpha); \mathbb{R}^k)$ defined by $(\psi_\alpha)_* u := \psi_\alpha \circ u \circ \varphi_\alpha^{-1}$ where $\psi_\alpha: E \otimes Vol(M)_{U_\alpha} \to \mathbb{R}^k$ with $k = \dim(E) = \dim(E \otimes Vol(M))$ is a trivialization composed with the projection $pr_2: U_i \times \mathbb{R}^k \to \mathbb{R}^k$. This allows us to define the adjoint map:

$$\begin{aligned}(\psi_\alpha)'_*: D'^{(k)}(\varphi_\alpha(U_\alpha); \mathbb{R}^k) &\to D'^{(k)}(U_\alpha; E)\\ ((\psi_\alpha)'_* T)[u] &:= T[(\psi_\alpha)_* u]\end{aligned} \quad (161)$$

The above construction provides the following result:

Theorem 6.10. (cf. [43] Theorem 3.1.9)
The distributional space $D'^{(k)}(M; E)$ (with $(E; M; \pi)$ a vector bundle of rank k) can be identified with families (T_α) of distributions $T_\alpha \in D'^{(k)}(\varphi_\alpha(U_\alpha); \mathbb{R}^k)$ satisfying:

$$(\psi_\alpha)'_* T_\alpha = (\psi_\beta)'_* T_\beta \quad \text{on } U_\alpha \cap U_\beta \quad (162)$$

We will only sketch that a family as described above in fact leads to a well defined E-valued distribution:

Proof. Define $\tilde{T}_\alpha := ((\psi_\alpha)'_* T_\alpha)$. Our assumption on the family of local distributions then can be written as: $\tilde{T}_\alpha|_{U_\alpha \cap U_\beta} = \tilde{T}_\beta|_{U_\alpha \cap U_\beta}$. This is exactly the condition demanded in Lemma 6.9 hence it provides us with the existence of an $\tilde{T}$ in $D'^{(k)}(M; E)$ such that $((\psi_\alpha^{-1})'_*)\tilde{T}|_{U_\alpha} = T_\alpha$. □

One principle idea which in the end motivated the definition of densities and thus the whole construction examined above, has been the demand for natural embedding of the space $\Gamma(M; E)$ into $D'^{(k)}(M; E)$. Hence in closing the preceding discussion we shall define it explicitly:

Definition 6.11. For any $v \in \Gamma^0(M, E)$ we define $\Psi(v) \in \mathcal{D}'^{(0)}(M, E)$ as :

$$\Psi(v)[\omega] := \int_M \omega(v) \tag{163}$$

where $\omega \in \Gamma^0_c(M, E^* \otimes Vol(M))$. The above integral is well defined, since $\operatorname{supp}(\omega)$ is compact.

6.1.3 Distributional curvature

We are now prepared to define the often announced Ricci-curvature distribution on our spacetime (M, g). Let (U, φ) be a coordinate chart of M. Then $(\varphi^{-1})^* g \equiv (g_{ij})$ defined as $((\varphi^{-1})^* g)_x(v, w) := g_{\varphi^{-1}(x)}(d\varphi^{-1} v, d\varphi^{-1} w)$ is a C^1-Lorentz metric on $\varphi(U)$. Hence the Christoffel symbols are given by the continuous expressions : $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$. In Remark 4.5 we already gave the coordinate expression of the Riemann-curvature tensor if $g \in C^2$:

$$R^m_{ijk} = \partial_j \Gamma^m_{ik} - \partial_k \Gamma^m_{ij} + \Gamma^m_{js}\Gamma^s_{ik} - \Gamma^m_{ks}\Gamma^s_{ij} \tag{164}$$

This expression seems to be undefined if the Γ^k_{ij} are only continuous. Though if we consider the above derivatives as distributional ones, that is $\partial_m \Gamma^k_{ij}[\chi] := -\int_{\varphi(U)} \Gamma^k_{ij} \partial_m \chi$ for every $\chi \in \Gamma^1_c(\varphi(U), Vol(M)) = \mathcal{D}^1_c(\varphi(U))$ we can use the exact same expression as above to define $(R^m_{ijk}) \in \mathcal{D}'^{(1)}(\varphi(U), (\mathbb{R}^{n^4}))$. Hence if we could proof (162) to be fulfilled, Theorem 6.10 would provide us with a global tensor distribution whose local coordinate expressions are exactly given by (164). We begin by rewriting (162):

$$\begin{aligned} T_\varphi &= (\psi \circ \varphi^{-1})'^{0} T_\psi = (\varphi^{-1})'^{0}(\psi)'^{0} T_\psi \\ &= ((\psi) \circ (\varphi^{-1}))'^{0} T_\psi = (\psi \circ \varphi^{-1})'^{0} T_\psi \end{aligned} \tag{165}$$

Let $\omega \in \Gamma^1_c(U_\varphi \cap U_\psi, T^{(3,1)}M)$; $\chi \in \Gamma^1_c(U_\varphi \cap U_\psi, Vol(M))$ with coordinate expressions $\omega_\varphi, \chi_\varphi$ in the chart $(\varphi(U_\varphi) \cap \varphi(U_\psi), \varphi)$ and analogous for $(\psi(U_\varphi) \cap \psi(U_\psi), \psi)$. Furthermore let us denote with

$$\{(\varphi R)^m_{ijk}\}, \{(\psi R)^m_{ijk}\} \in \mathcal{D}'^{(k)}(\varphi(U), (\mathbb{R}^{n^4})) \tag{166}$$

the respective expressions of (164) in each coordinate system. Using this notation we can write the above relationship as:

$$\{(\varphi R)^m_{ijk}\}[(\omega_\varphi) \circ \varphi^{-1}] \stackrel{!}{=} \{(\psi R)^m_{ijk}\}[(\psi \circ \varphi^{-1}) \circ ((\omega_\psi) \circ \varphi^{-1})] \tag{167}$$

If we are choosing the trivializations $\delta: T^{(3,1)}M \otimes Vol(M) \to \mathbb{R}^{n^4}$ as $(\omega_\varphi) := \omega(dx^i, dx^j, dx^k, \partial_m)$ and define $\delta \circ \psi \circ \varphi^{-1}$ the Jacobian of which we denote as $J^j_i := D(\psi \circ \varphi^{-1})^j_i = \partial_i((\psi \circ \varphi^{-1})^j)$ then :

$$\begin{aligned} &(\psi \circ \varphi^{-1}) \circ ((\omega_\psi) \circ \delta^{-1})^{ijk}_{m} \\ &= J^i_l J^j_r J^k_n (J^{-1})^s_m |\det(J)| (\omega_\psi \circ \delta^{-1})^{lrn}_s (\psi \circ \varphi^{-1}) \\ &= J^i_l J^j_r J^k_n (J^{-1})^s_m (\omega_\psi \circ \delta^{-1})^{lrn}_s (\psi \circ \varphi^{-1}) \end{aligned} \tag{168}$$

where the transformation rule for the volume bundle (150) has been used in the second equality. A similar proof as in the smooth case shows that for any $2 \ _c(U \ \backslash \ U \ ; Vol(M))$:

$$(\ R)^{s}_{lrn} [\qquad {}^{1}] = (\ R)^{m}_{ijk} J^{i}_{l} J^{j}_{r} J^{k}_{n} (J \ \ {}^{1})^{s}_{m} [\qquad {}^{1}] \qquad (169)$$

where the multiplication of a smooth function a with a distribution T is de ned as $aT[\ ] := T[a \ ]$. Finally putting all of the above results together we obtain:

$$\begin{aligned} & f (\ R)^{m}_{ijk} \ g \ [(\qquad {}^{1}) \ ((\ ! \ ) \qquad {}^{1})] \\ &= (\ R)^{m}_{ijk} [J^{i}_{l} J^{j}_{r} J^{k}_{n} (J \ \ {}^{1})^{s}_{m} (\ ! \qquad {}^{1})^{lrn}_{s} (\qquad {}^{1})] \\ &= (\ R)^{m}_{ijk} J^{i}_{l} J^{j}_{r} J^{k}_{n} (J \ \ {}^{1})^{s}_{m} [((\ ! \ )^{lrn}_{s} \quad) \qquad {}^{1})] \qquad (170) \\ &= (\ R)^{s}_{lrn} [(\ ! \ )^{lrn}_{s} \qquad {}^{1}] \\ &= \ f (\ R)^{m}_{ijk} \ g \ [(\ ! \ ) \qquad {}^{1}] \end{aligned}$$

Thus we have proven the existence of a tensor distribution (R) of order 1, which due to its local form (164) coincides if $g \ 2 \ C^{2}$ with the embedded Riemann tensor (R) (see Def.6.11). Nevertheless our energy conditions are formulated, using the Ricci tensor not R. Fortunately we can just like in the smooth case contract the distributional Riemann-tensor to obtain the distributional Ricci tensor which is locally given by :

$$Ric_{ij} \ = \ R^{m}_{imj} \ = \ @_{m} \ {}^{m}_{ij} \quad @_{j} \ {}^{m}_{im} + \ {}^{m}_{ms} \ {}^{s}_{ij} \quad {}^{m}_{js} \ {}^{s}_{im} \quad 2 \ D^{0(1)} (\quad (U \ )) \qquad (171)$$

6.2 Regularization techniques

After having de ned a reasonable structure for the Ricci-curvature in general C^{k}-spacetimes ($k \quad 1$), we now proceed by introducing standard techniques to approximate tensor-distributions through smooth tensor elds. As we will see, the same procedure may be applied to smoothen out tensor elds of low regularity (as for example the metric in C^{1}-spacetimes).

As usual, let us start in a chart (V ;). Choose a subset U such that $B^{R^{n}}(\quad (U \ ); \ ) \qquad (V \ )$ (that is for all $p \ 2 \quad (U \ ) : B^{R^{n}}(p; \ ) \qquad (V \ ))$. Now let $T \ 2 \ D^{0}(\quad (V \ ))$ be a distribution. Furthermore choose any standard molli er , that is $2 \ C^{1}_{c} \ (B^{R^{n}}(0; 1))$ and $(x) := \ {}^{n} \ (\frac{x}{})$ such that supp( ) $B^{R^{n}}(0; \ )$. the above conditions allow us to de ne:

$$T \ ? \quad (x) := T \ [\quad (x \quad :)] \quad j x \ 2 \quad (U \ ) \qquad (172)$$

The following is a standard result of functional-analysis the proof of which is given in ([44],3.19) and will be shown in a more general setting later on:

Proposition 6.12. In the situation described above we have:
(1) $T \ ? \quad 2 \ C^{1} (\quad (U \ ))$
(2) For every $2 \ C^{1}_{c} (\quad (U \ ))$:

$$(T \ ? \quad)[\ ] ! \quad T \ [\ ] \quad \text{for} \quad ! \quad 0 \qquad (173)$$

We now aim to generalize this idea onto tensor-distributions de ned on the whole manifold:

1. Let $T \in D'(M; T^{(r,s)}M)$ and $f(V ; \ )g_{\in N}$, $f(U ; \ _{|U})g_{\in N}$ two countable and locally nite atlas of M such that $\overline{U}$ is a compact subset of V (exists due to [4] Hilfssatz 3.25 and Satz 3.65). We therefore have for each $\in N$ a tensor distribution $T := (\ ^{1})' T_{|V} \in D'(\ (V); R^{n^{r+s}}) = D'(\ (V))^{n^{r+s}}$. Now choose such that $B^{R^n}(\ (U); \ ) \ (V)$. Hence by the above proposition $: T \ ? \ \in (\ (U); R^{n^{r+s}})$, where the convolution is done component wise. In particular we have $(\ ) (T \ ? \ ) \in (U ; T^{n^{r+s}} M)$. Here we used $(\ )$ - the pullback of a tensor eld by a di eomorphism. We could have equivalently associated $(T \ ? \ )$ via the embedding with a distribution and write $(\ )'(T \ ? \ )$

2. To construct a global section of the tensor bundle we need to cut o the tensor distribution constructed above on U. Thus let $\in C_c^1(V)$ such that $j \ j \ 1$ and $_{|K} \ 1$ for some compact subset $K \ U$. We are now able to de ne a global smooth tensor eld: $(\ ) (T \ ? \ )$ which is zero outside of K.

3. We now have to add all of those local expressions in a well de ned way and such that the total sum approximates the original distribution T. The above properties already strongly propose to use a partition of unity. Hence let $f \ g_{\in N}$ a partition of unity subordinate to $(U)_{\in N}$ and $^{0} := \ ^{1}$. It makes sense to choose $K := \mathrm{supp}(\ ) \ U$. Finally we de ne:

$$T ?_M := \sum_{\in N} (\ ) ((\ ^{0} T) ? \ ) \tag{174}$$

   Even though on the left hand side the chosen Atlas and subordinate partition of unity do not appear, it should be remarked that the de nition does depend on the choice taken for them. Each summand is well de ned since $\mathrm{supp}(\ ^{0})$ is a compact subset of U. Also the sum is well de ned since (U) is a locally nite cover of M and $\mathrm{supp}((\ ) ((\ ^{0} T) ? \ ))$ U.

The whole construction would be rather uninteresting if we could not also prove some connection between $T ?_M$ and T. We therefore aim to demonstrate point (2) of the previous Proposition on the whole manifold.

Theorem 6.13. (Convergence of $(T ?_M \ )$)
For $T \in D'(M; T^{(r,s)}M)$ the construction (174) of $T ?_M$ obeys :

$$(T ?_m \ )[u] \xrightarrow{!\ 0} T[u] \quad \forall \in \ _c(T^{(s,r)}M \quad Vol(M)) \tag{175}$$

Proof. To simplify the notation let us assume : T 2 D^0(M) and 2 $_c$(V ol(M)). The tensorial case follows similarly component wise. From Def.6.11 we have:

$$(T ?_M \quad)[\ ] = \int_M (T ?_M \quad) \qquad (176)$$

By the de nition of the integral we need a partition of unity subordinate to (U) $_{2N}$. Let us just choose the same as in the de nition of T $?_M$, that is f g $_{2N}$. Since supp( ) is compact we only need a nite number of : Let A N be the nite subset such that 8 2 A : supp( ) \ U 6= ;. The above integral then becomes:

$$\int_M (T ?_M \quad) = \sum_{2A} \int_{(U\ )} (\quad ^1(x))(T ?_M \quad)(\quad ^1(x)) \quad (\quad ^1(x))\,dx \quad (177)$$

If we are now using the de nition of (T $?_M$):

$$\int_M (T ?_M \quad) = \sum_{2A} \int_{(U\ )} (\quad ^1(x)) \Big\{ \sum_{2N} (\quad) ((\ ^0T\ ) ? \quad)(\quad ^1(x)) \Big\} (\quad ^1(x))\,dx \qquad (178)$$

Again since supp(( ) ((^{0}T) ?)) U we only have to consider those 2 N such that U \ supp( ) 6= ;. That is :

$$\int_M (T ?_M \quad) = \sum_{2A} \sum_{2A} \int_{(U\ )} (\quad ^1(x)) \Big\{ (\quad) ((\ ^0T\ ) ? \quad)(\quad ^1(x)) \Big\} (\quad ^1(x))\,dx \qquad (179)$$

Thus let us concentrate on each of these summands :

$$\begin{aligned} &\int_{(U\ )} (\quad ^1(x)) \Big\{ (\quad) ((\ ^0T\ ) ? \quad)(\quad ^1(x)) \Big\} (\quad ^1(x))\,dx \\ &= \int_{(U \ \backslash\ U\ )} (\quad ^1(x)) \Big\{ (\quad ^1(x)) ((\ ^0T\ ) ? \quad)(\qquad ^1(x)) \Big\} (\quad ^1(x))\,dx \\ &= \int_{(U \ \backslash\ U\ )} (\quad ^1(x)) \Big\{ (\quad ^1(x)) (\ ^0T\ )[\ ((\qquad ^1(x)) \quad z)] \Big\} (\quad ^1(x))\,dx \end{aligned} \qquad (180)$$

Here we have used z- as a dummy variable and in the second equality applied de nition (172) of ? on open neighbourhoods of R^n. By linearity of (^{0}T) we obtain:

$$= \int_{(U \cap U)} ({}^{0}T) \quad (\quad {}^{1}(x)) \quad (\quad {}^{1}(x)) \quad ((\quad {}^{1}(x)) \quad z) \quad (\quad {}^{1}(x)) \; dx$$

$$= \int_{(U \cap U)} (T) \quad {}^{0}(z) \quad (\quad {}^{1}(x)) \quad (\quad {}^{1}(x)) \quad ((\quad {}^{1}(x)) \quad z) \quad (\quad {}^{1}(x)) \; dx \qquad (181)$$

For the subsequent argument let us define

$$(x; z) := {}^{0}(z) \quad (\quad {}^{1}(x)) \quad (\quad {}^{1}(x)) \quad ((\quad {}^{1}(x)) \quad z) \quad (\quad {}^{1}(x) \qquad (182)$$

which is a smooth function of compact support on $(U) \quad (U)$. The following argument follows the idea in ([44],Satz 3.11): Let us construct a sequence of simple functions with respect to x: $({}_{k}(x; z))_{k \in N}$ which are functions of compact support and converge uniformly on $\overline{(U)} \quad \overline{(U)}$ to $(x; z)$. We can assume them to have the form:

$$_{k}(x; z) = \sum_{i=1}^{n_k} (x_{k_i}; z) 1_{E_{k_i}}(x) \qquad (183)$$

for E_{k_i} compact measurable sets contained in $\overline{(U)}$, $x_{k_i} \in E_{k_i}$ and $1_{E_{k_i}}$ the indicator function. In particular $_{k}(x; z)$ is a smooth function of z with compact support in (U). We thus obtain:

$$T [\int_{(U \cap U)} {}_{k}(x; z) dx] = T [\sum_{i=1}^{n_k} (E_{k_i}) \; (x_{k_i}; z)]$$

$$= \sum_{i=1}^{n_k} (E_{k_i}) T [\; (x_{k_i}; z)] \qquad (184)$$

where is the Lebesgue measure. Since

$$\int_{(U \cap U)} {}_{k}(x; z) dx \to \int_{(U \cap U)} (x; z) dx \qquad (185)$$

in $C^{1}_{c}(\overline{(U)})$ we conclude using continuity of T that the left hand side converges to $T [\int_{(U \cap U)} (x; z)] dx$. Furthermore using ([44] 3.19 (2)) it follows that $T [\; (x; z)]$ is a smooth function for $x \in (U)$ which thus assures

$$\sum_{i=1}^{n_k} T [\; (x_{k_i}; z)] 1_{E_{k_i}}(x) \to T [\; (x; z)] \qquad (186)$$

. Hence the right hand side of (184) converges to $\int_{(U \cap U)} T [\; (x; z)] dx$. Finally using the definition of (182) we conclude:

$$\int_{(U \cap U)} T \quad {}^{0}(z) \quad (\quad {}^{1}(x)) \quad (\quad {}^{1}(x)) \quad ((\quad {}^{1}(x)) \quad z) \quad (\quad {}^{1}(x)) \; dx$$

$$= T [\int_{(U \cap U)} {}^{0}(z) \quad (\quad {}^{1}(x)) \quad (\quad {}^{1}(x)) \quad ((\quad {}^{1}(x)) \quad z) \quad (\quad {}^{1}(x)) dx] \qquad (187)$$

Let us examine the argument of T a little further. First we use the transformation formula for the Lebesgue integral to change our integration variable with $(x = \quad ^{1}(y))$:

$$\int_{(U \cap U\ )} {}^{0}(z) \quad (\quad ^{1}(x)) \quad (\quad ^{1}(x)) \quad ((\quad ^{1}(x)) \quad z) \quad (\quad ^{1}(x))\, dx$$

$$= \int_{(U \cap U\ )} {}^{0}(z) \quad (\quad ^{1}(y)) \quad (\quad ^{1}(y)) \quad (y \quad z) \quad (\quad ^{1}(y))\, j \det(D (\quad ^{1}))j(y)\, dy$$

$$= \int_{(U \cap U\ )} {}^{0}(z) \quad (\quad ^{1}(y)) \quad (\quad ^{1}(y)) \quad (y \quad z) \quad (\quad ^{1}(y))\, dy$$

$$= \ ^{0}(z) \ f (\quad) \quad ^{1} g ? \quad (z) \tag{188}$$

In the second step we used the transformation rule for the Volume bundle (150) and the last one follows for small enough and the de nition (z) := (z) . Since ( ) ^{1} is a smooth function we have in particular:

$$f (\quad) \quad ^{1} g ? \quad \overset{!}{!}\ ^{0} (\quad) \quad ^{1} \tag{189}$$

uniformly in $\overline{\quad (U\ )}$. Hence again by continuity of T :

$$(\ ^{0}T\ )[\ f (\quad) \quad ^{1} g ? \quad]\ !\ \ (\ ^{0}T\ )[(\quad) \quad ^{1}] \tag{190}$$

Let us put this result back into the sum of (179):

$$\begin{aligned} &\int_M (T ?_M \quad) \\ &= \sum_{2A} \sum_{2A} (\ ^{0}T\ )[\ f (\quad) \quad ^{1} g ? \quad] \\ &\overset{!}{!}\ ^{0} \sum_{2A} \sum_{2A} (\ ^{0}T\ )[(\quad) \quad ^{1}] \end{aligned} \tag{191}$$

But this can be evaluated to :

$$\begin{aligned} \sum_{2A} \sum_{2A} (\ ^{0}T\ )[(\quad) \quad ^{1}] &= \sum_{2A} (T\ )[(\quad) \quad ^{1}] \\ &= \sum_{2A} T_{jU}\ [(\quad _{jU}\ )] \\ &= T[\sum_{2A} (\quad _{jU}\ )] = T[\ ] \end{aligned} \tag{192}$$

where we used in the last equality that 1 on supp( ). All in all we have proven :

$$(T ?_m \quad)[u] \overset{!}{!}\ ^{0}\ T[u] \tag{193}$$

for an arbitrary $\quad 2 \quad _c(Vol(M))$. After all of the above calculations it may became understandable why we did not handle the case of T being a general tensor distribution but restricted our consideration to $D^0(M)$. Nevertheless as we already mentioned the general result follows component wise. □

As a special case the exact same construction can be used to smooth tensor elds of low-regularity $T \; 2 \quad ^k(T^{(r;s)}M)$. Importantly the additional regularity present in comparison to tensor distributions allow stronger convergence properties:

Proposition 6.14. (cf. [42] Prop.3.5)
If $T \; 2 \quad ^k(T^{(r;s)}M)$ then $T \; ?_M \quad ! \quad T$ in C^k_{loc}. If $k \quad 1$ and $K \quad M$ is an arbitrary compact subset then there exists xed $c_K > 0$ and $_0(K)$ such that:

$$kT \quad T \, ?_M \quad k_{1 \; ;K} \quad c_K \qquad 8 < \; _0(K) \tag{194}$$

.

This result is crucial for the smooth approximating metrics as we have already used for example in in Theorem 3.66. Nevertheless the above constructed smooth metric $g \, ?_M$ does not su ce as there is no clear causal relationship to g.

Proposition 6.15. (smooth approximations (cf. [2] Lemma 4.2))
Let $(M; g)$ be a C^1-spacetime. It is possible to construct for > 0 arbitrarily close smooth approximations of g with smaller (g) or larger lightcones $(\hat{g})$ and time-orientable by the same vector eld. Explicitly we have:

1. $g \quad g \quad \hat{g} \; 8$
2. $g \quad g \, ?_m \quad ! \quad 0$ in C^1_{loc} and for any compact subset $K \quad M$ there are $c_K > 0; \; _0(K) > 0$ such that:

   $$kg \quad g \, ?_M \quad k_{1 \; ;K} \quad c_K \qquad 8 < \; _0(K) \tag{195}$$

   Analogous statements hold for $\hat{g}$.
3. $\hat{g} \; ! \quad g$ and $g \; ! \quad g$ in C^1_{loc}

Proof. A complete proof can be found in [2] Lemma 4.2. We will only shortly sketch the idea to construct g as those approximations will be needed later once more. As we are used to it, one starts locally in a chart $(U \; ; \quad _{jU})$ which is pre-compact in a larger chart $(V \; ; \quad)$. Using the smooth g-timelike vector eld which exists by time-orientability we can construct a corresponding timelike one-form $(: \, !)$ using the canonical isomorphism induced by g. Hence we can construct a frame of the dual tangent bundle on (V) of the form: $f(\quad)(!);(\quad)_1; ::::;(\quad)_{n \; 1} g$ where $(\quad)_i$ are g-spacelike one-forms. The idea is now to shrink the width of the lightcones of the smooth approximations $g \, ?_M$ using the spacelike one-forms. That is we de ne on (U):

$$(\; g) := (\; g) + \quad (\;)(\quad) \tag{196}$$

where $(\ g\ ) := ((\quad)\ (\quad g))\ ?$ with $f\quad g$ a partition of unity subordinate to $(U\ )$, $(\quad) = \sum_{i=1}^{n}{}^{1}(\quad)_i\quad(\quad)_i$ and $(\ )$ a function yet to be determined. In fact using $(\ ) := C$ with some large enough constant C suffices to assure that: $(\ \check{g}\ )\quad(\quad)\ g$ on $(U\ )$. We have thus altered the local expressions of our initial smooth approximation metric $g\ ?_m$ to fulfill our required causality relation. Hence we proceed just like in the construction of the convolution to patch those local expressions to a global metric:

$$\check{g} := \sum_{2N} (\quad)\ ((\ \check{g}\ )) \tag{197}$$

where $\quad 1$ on $\mathrm{supp}(\quad)$ as before. One can demonstrate now that for every compact subset of M we can find an $_K$ such that for all $\ <\ _K$ all of our desired conditions are fulfilled. Hence by the globalization lemma ([20] Lemma 4.3) as already shortly mentioned in the proof of 3.66 we find a map $u : (\ ;p) \mapsto \mathbb{R}_0$ such that $g := \check{g}_{u(\ ;p)}$ fulfills globally all our demands. □

6.3 Distributional energy conditions

As we have already mentioned, the general idea of proving C^1-singularity theorems relies on demonstrating only indirectly geodesic incompleteness for the C^1-spacetime $(M;g)$. That is one aims to prove causal geodesic incompleteness of $(M;g)$ by demonstrating causal geodesic incompleteness for all close enough smooth approximations. This can be done by proving that certain conditions imposed on $(M;g)$ imply conditions on such close smooth approximations $(M;g\ )$ which then lead to the emergence of focal points in $(M;g\ )$. As we have seen in Sec. 4.2 the archetype for such conditions are energy conditions. The main part of the C^1-singularity theorems thus consists in proving that certain distributional energy conditions imposed on $(M;g)$ imply those focal points generating conditions discussed in Sec. 4.3.2. Let us proceed by introducing some distributional energy conditions which seem promising.

1. <u>The distributional SEC</u>

   Definition 6.16. We say a C^1-spacetimes $(M;g)$ fulfills the distributional strong energy conditions (:DSEC) if for all timelike vector fields $X \in (TM)$ the distribution $\mathrm{Ric}(X;X) \in \mathcal{D}'(M)$ is positive. That is for all $\ \in\ _c(M;\mathrm{Vol}(M))$ such that $(\quad 0)$[25] we have $\mathrm{Ric}(X;X)[\ ]\quad 0$.

   This seems to be the most natural candidate to generalize the SEC to distributional curvatures. In fact if g is at least C^2 the above condition is equivalent to the classical SEC: If $X \in T_pM$ we can construct a smooth timelike vector field X such that $X(p) = X$ (in the null case this becomes problematic). Then $(\mathrm{Ric}(X;X))$ is positive if and only if $\mathrm{Ric}(X;X)$ is

[25] By the transformation rule for densities this is a well defined coordinate independent condition.

positive at every point in particular at $p \in M$. On the other hand if $\mathrm{Ric}(X,X) \geq 0$ for all tangent vectors X we also have $(\mathrm{Ric}(X,X)) \geq 0$ in the distributional sense. Finally we should remark that using local description of distributions discussed in the previous section, we can equivalently state the DSEC as : For all local vector fields $X \in \Gamma(U;M)$ the local expressions $(\mathrm{Ric})_{ij} X^i X^j \in \mathcal{D}'^0(U)$ are positive distributions.

2. <u>The distributional NEC</u>

   In the formulation of the SEC on can rely on the fact that being timelike is an open condition. That is not only can we extend every timelike vector to a local timelike vector field, but even stronger every extension on a small enough domain is timelike. In the null case on the other hand it already becomes problematic to define (in a general C^1-spacetime) local extensions. Furthermore even if we can construct such extensions (which then in general are only of low regularity) we are loosing the second property: The causal character of extensions of a null vector is in general not under our control. Unfortunately this property will turn out crucial for us later. How to circumnavigate this problem? One solution is given in [15] (Def.5.1):

   Definition 6.17. We say a C^1-spacetimes fulfills the distributional null energy condition if for any compact $K \subseteq M$ and any $\delta > 0$ there exists $\varepsilon(\delta;K)$ such that $\mathrm{Ric}(X;X) > -\delta$ for any local smoth vector field $X \in \Gamma(U;TM)$ with $U \subseteq K$ and $\|X\|_h = 1$ which satisfies $|g(X;X)| < \varepsilon(\delta;K)$ on U.

   That is if X is almost a null vector field then Ric almost obeys the 'naive' distributional null energy condition. If this could be shown to imply an almost NEC for smooth approximations then Lemma 4.21 could be used to show the existence of focal points. In [2] further equivalent definitions of the above distributional null energy condition are given which also demonstrate the equivalence to the classical NEC in C^2-spacetimes. Nevertheless we will subsequently focus on timelike energy conditions as their distributional treatment seems more natural and for now sufficient. On behalf of this spirit we will in the following only try to define a distributional SISEC.

3. <u>A distributional SISEC</u> Let us recall the definition of the scalar field inspired strong energy condition given in Def.4.27:

   Definition. We say $(M;g)$ fulfills the scalar field inspired timelike convergence condition (:SISEC) if for every timelike geodesic $\gamma : I \to M$ and $\forall f \in C^1_c(\mathrm{int}(I))$ (hence $W^m_0(I)$) an estimate of the following form holds:

$$\int_I \mathrm{Ric}(\dot\gamma;\dot\gamma) f(t)^2 dt \geq -\|f\|_m^2 := -(Q_m \|f^{(m)}\|^2_{L^2(I)} + Q_0 \|f\|^2_{L^2(I)}) \quad (198)$$

   At first this may seem to be predestined for a distributional formulation, as it is already defined against a space of test functions. In fact it is a distributional inequality, though interpreting the Ricci-cuvature as a distribution on the geodesic and not on spacetime as we did previously when considering C^1-spacetimes. Let us start by formulating the above condition in a di erent way which will then motivate our generalization to C^1-spacetimes. To do so we will need the concept of q-densities (q 2 R). Those are de ned analogously to (Def.6.1,Def.6.3) where we introduced 1-densities.

De nition 6.18. (q-densities)
Let (V ;) be an Atlas of a smooth manifold M. We de ne the real line bundle $V ol^q(M)$ (q 2 R) as the vector bundle given by the cocycle:

$$\begin{aligned} A \quad &: V \setminus V ! \quad R \ f \ 0g = GL_1(R) \\ A \quad (p) &= j\det(D(\quad ^1))j \ ^q(\quad (p)) \end{aligned} \tag{199}$$

We thus de ne C^k-q-densities as the as the C^k-sections of $V ol^q(M)$.

Now let : I! M be a timelike geodesic on which the SISEC holds. Let us assume (I) is a one dimensional submanifold with boundary of M (this is true if for example the strong energy condition is ful lled ([5] 14/Ex.11)). Now de ne the 1=2-densitity (:) on (I) such that for every parameterization ^{1} : I! (I) that is a chart its coordinate expression is given by

$$(\quad ^1(\ )) := f(\quad ^1(\ )) \ \frac{d(\quad ^1 \quad ^1)}{d}(\ )^{1=2} \tag{200}$$

with f 2 C^1_c (int((I))). This naturally induces a 1-density given by :

$$(\ ^2) \ (\quad ^1(\ )) := f^2(\quad ^1(\ )) \ \frac{d(\quad ^1 \quad ^1)}{d}(\ ) \tag{201}$$

Hence we can write the left hand side of the SISEC as:

$$\begin{aligned} Ric[\_ \quad \_ \quad ^2] &= \int_{(I)} Ric(\_; \_) \ ^2 \\ &= \int_{(I)} Ric(\_; \_)(\quad ^1(\ ))(\ ^2) \ (\quad ^1(\ ))d \end{aligned} \tag{202}$$

where we have regarded $\_ \quad \_ \quad ^1$ 2 ^1_c(T((I))) and thus Ric 2 $D^{0(1)}$((I); $T^{(0;2)}$((I))). To handle the right hand side we need to generalize the derivative used there into the language of manifolds. In general there are two natural concepts which may provide this: The covariant and the Lie-derivative. Since the derivative in the original formulation of the SISEC is completly independent of the geometry of spacetime it seems inadequate to use the covariant derivative. Hence let us shortly de ne the

Lie-derivative of q-densities. Let $\varphi : M \to N$ be a diffeomorphism and $\in {}_c(N; Vol^q(M))$. We define the q-density $(\varphi \ )$ on M as:

$$(\varphi \ )_p(v_0 \wedge ... \wedge v_{n-1}) := {}_{\varphi(p)}(\varphi \ v_0 \wedge ... \wedge \varphi \ v_{n-1}) \tag{203}$$

If $(U_M; \ _M)$ and $(V_N; \ _N)$ are charts of M and respectively N such that $\varphi(U_M) = V_N$ we can give a concrete coordinate expression of the pullback q-density:

$$(\varphi \ ) \ ^M = |\det(D(\ _N \ \varphi \ _M^{-1})|^q \ {}^N_{\varphi(\ )} \tag{204}$$

Now let $\varphi \ (\varphi_t : M \to M)$ be the one parameter group of diffeomorphisms induced by a vector field $X \in (TM)$. Furthermore choose $(V; \ )$ a chart with $U \ V$ such that $\varphi_t(U) \ V$ for $|t| < \ $. The above expression then can be written as:

$$(\varphi_t \ ) \ = |\det(D(\ \varphi_t \ ^{-1})|^q \ {}_{\varphi_t(\ )} \tag{205}$$

We are now prepared to define the Lie-derivative of q-densities. It measures the change along the flow of a vector field when connecting the different tangent spaces such that the vector field itself stays constant.

Definition 6.19. (Lie-derivative of q-densities)
Let $X \in {}^1(TM)$ and (φ_t) the corresponding one parameter group of C^1-diffeomorphisms. For any q-density $\in {}^k(Vol^q(M))$ $(k \ 1)$ we define the Lie-derivative in direction X as:

$$L_X \ := \lim_{t \to 0} \frac{\varphi_t \ - \ }{t} \in {}^{k-1}(Vol^q(M)) \tag{206}$$

which can be written in coordinates as:

$$(L_X \ ) \ = \sum_{i=0}^{n-1} \frac{\partial}{\partial x^i} X^i + q(\frac{\partial X^i}{\partial x^i}) \tag{207}$$

In particular in our case when $X = \_ \ ^{-1}$ and the $1=2$-density on $M = \ (I)$ we get:

$$(L_{\_} \ ) \ ^{-1} = (f \ )^0(t) \tag{208}$$

If is C^k we have $\in {}^k_c(Vol^{\frac{1}{2}}(\ (I)))$ and hence can formulate the SISEC for $m \ k$ in an invariant way:

$$Ric[\_ \ \_ \ ^2] \ (Q_m \|L_{\_}^{(m)} \ \|^2_{L^2(\ (I))} + Q_0 \| \ \|^2_{L^2(\ (I))}) \tag{209}$$

where we have used that we can define for all $\in {}^0_c(M; Vol^{\frac{1}{p}}(M))$ the L^p-norm on M as:

$$\| \ \|_{L^p(M)} := \int_M | \ |^p \tag{210}$$

The expression above allows as promised a direct generalization to a worldvolume inequality. The idea is to extend the above formulation (209) on a small neighbourhood U of (I):

De nition 6.20. (Distributional SISEC)
We say $(M; g)$ ful lls the distributional scalar eld inspired timelike convergence condition (:DSISEC) if for every timelike geodesic $: I! \quad M$ there exists a relatively compact neighbourhood U of (I) such that for all causal $X\ 2 \quad ^k(U\ ; TM)$ and $2 \quad ^k_c(U\ ; Vol^{\frac{1}{2}}(M))$ an estimate of the following form holds $(1 \quad m \quad k)$:

$$Ric[X \quad X \qquad ^2] = Ric(X; X)[\ ^2] \qquad (Q_m kL^{(m)}_X \quad k^2_{L^2(U\ )} + Q_0 k \quad k^2_{L^2(U\ )}) \tag{211}$$

To provide a physical example we may examine the most simple scalar eld : the minimally coupled scalar eld. In fact a worldvolume inequality proven in ([30],Theorem 2) can be equivalently rewritten as:

$$\int_M Ric(X; X) \ ^2 \qquad \int_M \frac{^2 m^2}{n \quad 2} \ ^2 \tag{212}$$

In particular for an relatively compact neighbourhood U of (I) we we can de ne $_{max}$ as the maximum of on $\overline{U}$ and thus arrive at an inequality of the form (242) with $\tilde{Q} = 0$ and $Q = \frac{^2_{max} \ m^2}{n \quad 2}$. Finally it is important to show (otherwise our strategy of proving C^1-singularity theorems could not be applied anymore) that if $g\ 2\ C^2$ the above inequality (242) already implies the worldline SISEC. This can be seen using the following lemma:

Lemma 6.21. ([3],Lemma 2.3) For any unit timelike curve $: I! \quad M$ which starts in a spacelike submanifold $P \quad M$ (or at a point $p\ 2\ M$) there is a synchronous coordinate system $(V\ ; \ )$ such that V is a neighbourhood of up to the rst focal (or conjugate) point to P (or p).

Here a synchronous coordinate system is meant to describe coordinates $= (x^0; ::::; x^{n \quad 1})$ such that the expression for the metric becomes :

$$(\ ) \ g(x) = \quad dx^0 \quad dx^0 + g_{ij}(x) dx^i \quad dx^j \qquad i; j = 1; ::::; n \quad 1 \tag{213}$$

In particular either we already have the existence of a focal (/conjugate)-point or we can use a synchronous coordinate system to de ne a $1=2$-density on V by: $\quad ^1\ 2\ C^1_c(V\ )$ which is yet to be de ned explicitly. We can assume that $\overline{V} \quad U$ which therefore implies that we can view as $1=2$-density on U which is zero outside $supp(f\ ) \quad V$. Using the fact that the coordinate expression of in a synchronous coordinate system is given by $(\ (t)) = (\ t; ::::; 0)$ we are motivated to de ne for any $f^2\ 2\ C^1_c(int(I))$ the function $\quad 2\ C^1_c(V\ )$ as:

$$^1(x^0; ::::; x^{n \quad 1}) := f(x^0)(\ _1(\ x^1) \ ::: \quad _{n \quad 1}(\ x^{n \quad 1}))^{\frac{1}{2}} \tag{214}$$

where is a standard molli er [26] and $_i$ small enough such that the above expressions are well de ned. In fact we can assume $(V\ ) = int(I)$

$I_1 \; ::: \; I_{n-1}$ for small enough $_i$, $I_i = (\; _i ; \; _i)$ and $_i < \; _i$ for all $i = 1; :::; n \; 1$. Furthermore let us define the a vector field on U by :

$$X_{|V} := \; @_0 \tag{215}$$

where $\; 2 \; _c(U)$ such that $\; 1$ on V. As we have assumed the DSISEC to hold along we have the inequality :

$$\int_U \mathrm{Ric}(X;X) \; ^2 \quad Q_m \int_U (L_X^{(m)} \;)^2 \quad Q \int_U \; ^2 \tag{216}$$

We can calculate L_X in synchronous coordinates to be:

$$\begin{aligned}(L_X \;) = & \sum_{i=0}^{n-1} @_i (f(x^0)(\; _1(\; x^1) \; ::: \; _{n-1}(\; x^{n-1}))^{\frac{1}{2}})X^i \\ & + \frac{1}{2}(f(x^0)(\; _1(\; x^1) \; ::: \; _{n-1}(\; x^{n-1}))^{\frac{1}{2}}) \sum_{i=0}^{n-1} @_i (X^i)\end{aligned} \tag{217}$$

Though since supp(g) $\; V$ and here $X = @_0$ the above expression can be simplified to:

$$(L_X^{(m)} \;) = @_0^{(m)}(f(x^0))(\; _1(\; x^1) \; ::: \; _1(\; x^{n-1}))^{\frac{1}{2}}) \tag{218}$$

Thus we can write the above inequality in a more concrete way:

$$\begin{aligned} & \int_{I_1} ::: \int_{I_{n-1}} \int_I \mathrm{Ric}(@_0;@_0) f^2(x^0) \; _1(\; x^1) \; ::: \; _{n-1}(\; x^{n-1}) dx^0 ::: dx^{n-1} \\ & Q_m \int_{I_1} ::: \int_{I_{n-1}} \int_I (@_0^{(m)}(f(x^0)))^2 \; _1(\; x^1) \; ::: \; _{n-1}(\; x^{n-1}) dx^0 ::: dx^{n-1} \\ & Q \int_{I_1} ::: \int_{I_{n-1}} \int_I f^2(x^0) \; _1(\; x^1) \; ::: \; _{n-1}(\; x^{n-1}) dx^0 ::: dx^{n-1} \\ = & \; Q_m \int_I (@_0^{(m)}(f(x^0)))^2 dx^0 \quad Q \int_I f^2(x^0) dx^0 \end{aligned} \tag{219}$$

where we used the normalization of all $_i$ in the last equation. Since $\int_{I_i} \mathrm{Ric}(@_0;@_0) \; _i(\; x^i)dx^i = \mathrm{Ric}(@_0;@_0)? \; _i(x^0; :::; 0; :::; x^{n-1})$ where $x^i = 0$ we conclude in the limit $_i \; ! \; 0$ for all $i = 1; :::; n \; 1$:

$$\int_I \mathrm{Ric}(\_; \_) f^2(x^0) dx^0 \quad Q_m \int_I (@_0^{(m)}(f(x^0)))^2 dx^0 \quad Q \int_I f^2(x^0) dx^0 \tag{220}$$

which is exactly the originally introduced SISEC.

---

[26] We can for example choose $(x) := \begin{cases} \exp(\; 1=(1 \; x^2))=I_n & |x| < 1 \\ 0 & |x| \; 1 \end{cases}$

with I_n a normalization factor. In particular we have that $(\;)^{\frac{1}{2}} \; 2 \; C_c^1(\mathbb{R})$.

As we have provided some examples of distributional energy conditions well defined even on C^1-spacetimes, it is now time to aim at proving C^1-singularity theorems.

7 C^1-singularity theorems

As mentioned earlier the strategy is based on proving causal geodesic incompleteness for smooth approximations instead of the C^1-spacetime itself. Though we are still missing some explicit connection between the distributional energy conditions introduced in the previous section and conditions imposed on the Ricci-curvature of smooth approximations to apply this strategy. Not to loose our self in rather technical and involved calculations we will only state the following main results concerning this matter, presented in [2] by M.Graf.

Facts 7.1. (Some facts concerning the Ricci-curvature of smooth approximations) In the following we will always denote the Ricci-curvature corresponding to a metric g' as $\mathrm{Ric}_{g'}$. Furthermore we will in general consider smooth approximations of the from (g_ε) as constructed in Prop.6.15 .

1. $\mathrm{Ric}_{g_\varepsilon} - \mathrm{Ric}_{g \star_M \rho_\varepsilon} \to 0$ locally uniformly (see [2] Lemma 4.5).

   Essentially this follows from the convergence results given Prop.6.15 and the fact that for every function $f \in C^0(\mathbb{R}^n)$ and compact subset $K \subset M$ we have $\|\partial_m (f \star \rho_\varepsilon)\|_{K,\infty} \to 0$ (see [15] Lemma 4.7).

2. $\mathrm{Ric}_{g \star_M \rho_\varepsilon} - \mathrm{Ric}_g \star_M \rho_\varepsilon \to 0$ locally uniformly (see [2] Lemma 4.6).

   Here a version (see [2] Lemma 4.6) of Friedrichs Lemma is crucial. From the above facts we can deduce $\mathrm{Ric}_{g_\varepsilon} - \mathrm{Ric}_g \star_M \rho_\varepsilon \to 0$ locally uniformly.

3. Using the preceeding convergence properties one can deduce the following fundamental lemma for proving the C^1-singularity theorems assuming the DSEC:

   Lemma 7.2. (see [2] Lemma 4.6) Let (M, g) be a C^1-spacetime and $K \subset TM$ a compact subset. Assume the DSEC is fulfilled ($\mathrm{Ric}_g(X, X) \geq 0$ for every g-timelike smooth vector field). Then:

$$\forall \delta > 0\ \exists \varepsilon_0 > 0\ \forall \varepsilon < \varepsilon_0\ \forall X \in K \text{ with } g_\varepsilon(X, X) = -1 : \ \mathrm{Ric}_{g_\varepsilon}(X, X) > -\delta \tag{221}$$

Finally our odyssey comes to an end : We are prepared to reproduce the C^1-versions of Hawkings singularity theorems proven by M.Graf in [2].

Theorem 7.3. (Hawking I, C^1- version), see [2] Theorem 4.13)
Let $(M;g)$ be a C^1-spacetime. If :

(1) $(M;g)$ contains a a smooth spacelike Cauchy-hypersurface S
(2) $(M;g)$ fulfills the DSEC
(3) for every future directed unit-timelike normal vector:

$n \in (TS)^{\perp} : k(n) \geq k_0 > 0$

then $\tau_g(S;p) \leq \frac{1}{k_0}$ for all $p \in I^+_g(S)$.

Proof. Keeping the tradition, just like in proof of the classical Hawking (5.2) theorem, it only remains to put the right theorems in the right place. Assume there exists a $p \in I^+_g(S)$ such that $l := \tau_g(S;p) > 1/k_0$. We have proven in Cor.3.69 the existence of a g-timelike, g-geodesic from S to p which is obtained as a $C^1([0;1])$ limit of g_{ϵ_n}-geodesics (γ_n) which themselves maximize g_{ϵ_n}-length : $\tau_{g_{\epsilon_n}}(S;p) = L_{g_{\epsilon_n}}(\gamma_n) =: l_n \geq l$. As our focal point theorems are formulated for unit-timelike geodesics, we may reparameterize γ and all γ_n to unit-geodesics. We now aim to apply Lemma 7.2 and therefore need an appropriate compact subset $K \subseteq TM$. In the end we are hoping to conclude some kind of SEC on close enough approximations γ_n, the images $\mathrm{im}(\dot\gamma_n)$ thus should be certainly contained in K. Fortunately $K := \bigcup_{n \in \mathbb{N}} \mathrm{im}(\dot\gamma_n) \cup \mathrm{im}(\dot\gamma)$ is due to the C^1 convergence of the approximating geodesics in fact compact: If $(q_k)_{k \in \mathbb{N}}$ is a sequence contained K there are two possibilities; either we find some $n \in \mathbb{N}_0$ such that there are infinitely many $q_k \in \mathrm{im}(\dot\gamma_n)$ ($\dot\gamma \equiv \dot\gamma_0$) or we find an $k_n \in \mathbb{N}$ for every $n \in \mathbb{N}$ such that $q_{k_n} \in \mathrm{im}(\dot\gamma_{m_n})$ with $m_n \geq n$. In the first case we find a convergent subsequence by compactness of $\mathrm{im}(\dot\gamma_n)$. For the second case we always find an $t_{k_n} \in [0;l]$ such that $\dot\gamma_{m_n}(t_{k_n}) = q_{k_n}$. By compactness of $[0;l]$ we can assume that $t_{k_n} \to t_0 \in [0;l]$. We therefore conclude $\mathrm{dist}_h(\dot\gamma(t_0); q_{k_n}) \leq \mathrm{dist}_h(\dot\gamma(t_0); \dot\gamma(t_{k_n})) + \mathrm{dist}_h(\dot\gamma(t_{k_n}); \dot\gamma_{m_n}(t_{k_n}))$ which both converge to zero and thus proves the existence of a convergent subsequence in all cases, that is compactness. Hence we can apply Lemma 7.2 to K which tells us that:

$$\forall \delta > 0 \; \exists \epsilon_0 > 0 \; \forall \epsilon < \epsilon_0 \; \forall X \in K \text{ with } g(X;X) = -1 : \; \mathrm{Ric}_{g_\epsilon}(X;X) > -\delta \tag{222}$$

Which δ may be sufficient? Let us choose a small enough neighbourhood $U \subseteq S$ of $\gamma(0)$ and large enough $N_0 \in \mathbb{N}$ such that for every future directed g_{ϵ_n}- unit-timelike normal vector: $n_{g_{\epsilon_n}} \in (TU)^{\perp_{g_{\epsilon_n}}} : k_{g_{\epsilon_n}}(n_{g_{\epsilon_n}}) > k_0 > 0$ if $n \geq N_0$. The existence of such U and N_0 is assured by C^1_{loc} convergence of g_{ϵ_n} to g. Recalling Lemma 4.21 : 'If $0 \leq \delta \leq \frac{3k_0(n-1)}{l_n}(1-c)$ for some $0 < c \leq 1$ such that $\mathrm{Ric}(\dot\gamma;\dot\gamma) \geq -\delta$. Then there exists a focal point $\gamma(r)$ of S along γ such that $0 < r \leq \frac{1}{ck_0}$ if $l_n \geq \frac{1}{ck_0}$', we only have to choose $0 < c < 1$ such that $l > \frac{1}{ck_0} > \frac{1}{k_0}$ and set $\delta := \frac{3k_0(n-1)}{l}(1-c)$. Hence for all $n \geq N_0$ such that $\gamma_n(0) \in U$ the existence of a focal point is proven for $l_n > \frac{1}{ck_0}$ which due to the maximality of all γ_n implies : $l_n \leq \frac{1}{ck_0} < l$. This contradicts $l_k \to l$. Hence our assumption has to be wrong which thus implies $\tau_g(S;p) \leq \frac{1}{k_0}$ for all $p \in I^+_g(S)$.

□

We directly proceed by proving Hawkings second theorem for C^1-spacetimes. While in the classical case (5.4) this turned out to be the much more complicated theorem, the C^1 generalization requires less work than Hawkings rst theorem, as proven above. This is due to the fact, that as before we do not aim to prove the singularity theorems once more but only transfer the properties needed for the classical theorems onto our smooth approximations which then should lead to a contradiction to $(M; g)$ being complete. As mentioned in ([2], Remark 4.14) assuming that all g-geodesics are de ned for large proper times implies much easier (by Theorem 3.43 instead of 3.69) the same for close enough approximations as merely assuming the existence of one geodesic with large proper time.

Theorem 7.4. (Hawking II, C^1- version), see [2] Theorem 4.11)
Let $(M; g)$ be a C^1.spacetime. If :

(1) $(M; g)$ contains a compact spacelike hypersurface A

(2) $(M; g)$ ful lls the DSEC

(3) for every future directed unit-timelike normal vector:

$n \, 2 \, (TS)^{?} : k(n) > 0$

then there exists at least one inextendible future directed timelike geodesic starting at A with proper time less than $\frac{1}{}$ for any $< \min_A (k(n))$.

Proof. As usual let us assume the above to be wrong, that is there exists an $0 < \; < \min_A (k(n)) =: k_0$ such that all future directed timelike unit-geodesics emanating from A are de ned up to an a ne parameter larger than $\frac{1}{}$. Let us choose some xed ; such that $0 < \; < \; < \; < k_0$. As previously we can use C^1_{loc}-convergence of the smooth approximations g to nd an ^{0} such that for every future directed g - unit-timelike normal vector: $n_g \, 2 \, (TA)^{?g} : k_g (n_g) > \; > 0$ if ^{0}. In fact A is still a spacelike hypersurface measured by g since $g (X; X) \; 0 =) \; g(X; X) < 0$ by construction of $g (X; X)$. Once more we want to nd a compact subset of F TM which contains all curves of interest that is all g - unit-geodesics starting orthogonally from A for small enough to be de ned up to an a ne parameter $a := \frac{1}{}$. In fact since A is assumed to be compact we can de ne:

$$K := \bigcup_{0< \; 1} f n \; 2 \; (TA)^{?g} ; g (n ; n) = \; 1; \text{ future directed } g \quad (223)$$
$$[f \; n \, 2 \, (TA)^{?g} ; g(n; n) = \; 1; \text{ future directed } g$$

which is a compact subset of TM. Since every g-geodesic starting in K is de ned up to an a ne parameter larger then $\frac{1}{} > \frac{1}{} = a$ we can use Theorem 3.43 and Cor.3.44 to de ne F as the set containing the curves: $F := F_{0;K;a}$ (see (73)). Again we apply Lemma 7.2 to $F_{0;K;a}$ which tells us that:

$$8 > 0 \, 9_1 > 0 \, 8 < \; _1 8X \; 2 \; F_{0;K;a} \text{ with } g (X; X) = \; 1 : \; \mathrm{Ric}_g (X; X) > \quad (224)$$

and again we have to ask our selves which may be su cient? Let us choose $0 < c < 1$ such that $\frac{1}{c} < \frac{1}{} = a$ and $= \frac{3\ (n\ \ 1)}{a}(1\ \ c)$. Then $\mathrm{Ric}_g\ (X; X\ ) >$ on $F\ \ _{0;K;a}$ for all $< \min f\ _1;\ _0;\ ^0 g$ implies the existence of an focal point (again by Lemma 4.21) along every such g -unit-normal future directed geodesic starting in A since $a = \frac{1}{} > \frac{1}{c}$. Hence for such small enough we conclude that every such g - geodesic stops to be maximizing for a ne parameters larger than a. Hence we are in the Situation of Theorem 5.4. We therefore may proceed exactly as in the classical case to derive a contradiction, proving our desired theorem. □

Apparently our e ort to introduce curvature as a distribution on a manifold has been rewarded. The above theorems give a clear framework to analyse singularities even when facing spacetime models of C^1-regularity. Most importantly they demonstrate that if one interprets the demanded assumptions as physical reasonable, extending a singular spacetime to a spacetime of lower regularity does in general not solve the problem. In fact any extension of order C^1 which still maintains those reasonable conditions, must be singular (as shown in the above theorems). Nevertheless as we have seen explicitly, physical models which do not obey the strong energy conditions appear quite numerously and therefore possibly still allow a physical reasonable non-singular extension of C^1-regularity. It is thus desirable to set the above theorems free from the strong energy condition and only demand a weaker, hopefully physically more reasonable energy condition. In the last act of this thesis, we shall try to examine this question a bit further. As there is no better way to 'examine' this question than trying to prove a concrete theorem, we present one possible version which may be interpreted as a C^1-version of Lemma 5.3 Version (a).

Theorem 7.5. (Hawking I with weakened energy conditions (a) , C^1-version) Let $(M; g)$ be a C^1-spacetime. If :

(1) M contains a spacelike Cauchy-hypersurface S

(2:1) $(M; g)$ ful lls the DSISEC (242) for all future directed unit-timelike geodesics emanating orthogonal to S:
: $[0; b] \quad M$ with $1 \quad m \quad k$ and $Q_m; Q$ independent of

(2:2) there exists an b_0 2 $(0; b)$ such that for all such geodesics as in (2.1) there exists an open neighbourhood $U_{\sim}$ of $\sim :=\ _{j[0;b_0]}$ such that $(\mathrm{Ric}_g(X; X)\quad _0)\quad 0$ (distributional) for all C^2-timelike vector elds X on $U_{\sim}$ for some $_0 > 0$

(2:3) for every future directed unit-timelike normal vector:

$$n\ 2\ (TS)^{?} : (n\quad 1)k(n) > \min\ \frac{n\quad 1}{b_0};\qquad \text{(see (135))}$$

then $_g(S; p)\quad b$ for all $p\ 2\ I^+_g(S)$.

Proof. We begin exactly as in Theorem 7.3: Assume there exists a $p\ 2\ I^+(S)$ such that $l :=\ _g(S; p) > b$. We have proven in Cor.3.69 the existence of a g-timelike, g-geodesic from S to p which is obtained as a $C^1([0;1])$ limit of g_n-geodesics $_k : [0;1]$ M which themselves maximize g_k-length : $_{g_k}(S;p) = L_{g_k}(\ _k) =: l_k$ l . As our focal point theorems are formulated for unit-timelike geodesics, we may reparameterize and all $_k$ to unit-geodesics.

Similarly as in the previous theorems, the idea we will pursue is trying to deduce the original SISEC for close enough approximations $_k$. Previously we could use the fact that $\mathrm{Ric}_g(X;X)$ 0 immediately implies $\mathrm{Ric}_g(X;X) ?_M$ 0. In the case of the DSISEC such a conclusion requires some work .

(1) An inequality for $\mathrm{Ric}_g(X;X) ?_M$

Luckily we have already derived an expression for the action of a distribution $T ?_M$ in Theorem 6.13. Just before taking the limit ! 0 we have arrived at the following equation (191):

$$\begin{aligned} &\int_M (T ?_M \quad) \\ &= \sum_{2A} \sum_{2A} (\ ^0T\ )[\ f(\qquad)\quad ^1g?\quad] \\ &= \sum_{2A} (\ ^0T\ )[\ f(\quad)\quad ^1g?\quad] \end{aligned} \tag{225}$$

for any $T\ 2\ D^0(M)$ and $2\ _c^k(M;Vol(M))$. We will now t the notation in the above equation to suit out current situation. To begin with in our case $2\ _c^k(U\ ;Vol^{\frac{1}{2}}(M))$ in the above expression we thus have to replace by ^{2}. It will later turn out convenient if we also write ^{2}. In fact we can de ne a new partition of unity subordinate to (U) by:

$$\tilde{\ } := \frac{^2}{P\ ^2} \tag{226}$$

Since the expression $P\ ^2$ is always larger than zero we have $\sqrt{\tilde{\ }}\ 2\ C_c^1(U\ )$ is well de ned and $\tilde{\ }$ a partition of unity where each function may be written as a square. If we are just using our old notation for the new partition of unity $\sqrt{\tilde{\ }}$ we may rewrite the above expression as:

$$\begin{aligned} &\int_M (T ?_M \quad)\ ^2 \\ &= \sum_{2A} (\ ^{20}T\ )[\ f(\quad (\ ^2)\ )\quad ^1g?\quad] \\ &= \sum_{2A} (\ ^{20}T\ )[\int_{(U\ )} (\quad (\ ^2)\ )\quad ^1((\ )\quad y)\quad (y)dy\ ] \end{aligned} \tag{227}$$

In the proof of Theorem 6.13 we demonstrated, the possibility to pull a local distribution outside of an integral. Similarly we can now take the distribution inside the integral. Though this time it acts on (^{2}(^{2})) ^{1}(z y) as we have used the symmetry of the convolution of smooth functions to switch their roles. We thus arrive at:

$$\int_M (T \,?_M\ )\ ^2 = \sum_{2A} \int_{(U\ )} (T\ )[\ ^{20}(\ )(\ \ (\ ^2)\ )\ \ ^1((\ )\ \ y)]\ \ (y)dy \tag{228}$$

In particular we can omit from now on as 1 on an open neighbourhood (: Z) of supp( ) and may be chosen small enough such that B ^{h}(supp( );) Z for all in the nite covering of $\overline{U}$. If one nally puts T = Ric$_g$(X ; X) :

$$\begin{aligned}\int_M (\mathrm{Ric}_g(X;X)\,?_M\ )\ ^2 &= \sum_{2A} \int_{(U\ )} (\mathrm{Ric}_g(X;X))\ [\ ^{20}(\ )(\ ^2)\ \ ^1((\ )\ \ y)]\ \ (y)dy \\ &= \sum_{2A} \int_{(U\ )} \mathrm{Ric}_g(X;X)_{jU}\ [\ ^2\ (\ ^2)\ \ ^1(\ \ (\ )\ \ y)\ \ j\wedge d\ \ j\ ]\ \ (y)dy\end{aligned} \tag{229}$$

where we have used j^ d j to describe the unique 1-density on U determined by j^ d j(@$_0$ ^ ::: ^ @$_{n-1}$) = 1 . We can now apply the assumed DSISEC for the 1=2-density : := ^{1}(( ) y) j^ d j$^{\frac{1}{2}}$. Here j^ d j$^{\frac{1}{2}}$ analogously is de ned as the unique 1=2-density such that j^ d j$^{\frac{1}{2}}$((@$_0$ ^ ::: ^ @$_{n-1}$) = 1 . Finally we get a promising expression:

$$\int_M (\mathrm{Ric}_g(X;X)\,?_M\ )\ ^2 \sum_{2A} \int_{(U\ )} (Q_m k L_X^{(m)}\ \ ^2 k^2_{L^2(U\ )} + Q_0 k\ \ k^2_{L^2(U\ )})\ \ (y)dy \tag{230}$$

As we will see the above inequalty will already be su cient for our special choice of and X .

(2) The special case

In the end we want to derive the SISEC along close enough approximations $_n$. Thus we should concentrate on those vector elds and densities which could be used to derive the original SISEC from the DSISEC. In fact we already have found such when we the de ned the DSISEC in the rst place :

$$\begin{aligned} X_k &:= \ _k\ @_{x^0_k} \\ \ _k{}^{k}\ \ _k{}^1(x^0_k;:::;x^{n-1}_k) &:= f\,(x^0_k)(\ \ _1(\ \ x^1_k)\ \ :::\ \ \ _{n-1}(\ \ x^{n-1}_k))^{\frac{1}{2}} \end{aligned} \tag{231}$$

where $_k = (x^0_k, \ldots, x^{n-1}_k) : V_{n} \to {}_k(V_{n})$ is a synchronous coordinate system along $_k$. If there exists some $k_0 \in \mathbb{N}$ such that all $_n$ contain a focal point before or at b we would have already derived a contradiction to $l := {}_g(S, p) > b$ and $l_k \to l$. By choosing an appropriate subsequence we can therefore assume the existence of an synchronous coordinate system up to an affine parameter b for all $k \in \mathbb{N}$. Furthermore by the convergence of $_k \to$ in $C^1([0,1])$ we can define any fixed $\in C^\infty_c(U)$ such that $\equiv 1$ on Z an open neighbourhood of (I) and use it as $_k$ for large enough k in the above definition of X_k. In particular we will restrict the domain of the synchronous coordinate systems for large enough k such that $_k(I) \subset Z$ to $\tilde{V}_n := V_n \cap Z$. Thus let us explicitly calculate (230) for the above choice of X_k and ^{k}.

$$^k \circ {}_k^{-1} = f(x^0_k - y^0)(_1(y^1 - x^1_k) \ldots _{n-1}(y^{n-1} - x^{n-1}_k))^{\frac{1}{2}} \tag{232}$$

where we have used that we do not need a partition of unity as the support of $_k$ is contained in one chart and for a small enough we have

$$supp(^k \circ {}_k^{-1}({}_k(.) - y) \subset \tilde{V}_n \quad y \in supp( ) \tag{233}$$

. The right hand side of Equation (230) thus can be written as:

$$\begin{aligned}&\int_{(U)} \int_{(U)} Q_m(\partial^{(m)}_{x^0_k}(f(x^0 - y^0)))^2 \, _1(y^1 - x^1_k) \ldots _1(y^{n-1} - x^{n-1}_k)) \\ &Q_0 f^2(x^0_k - y^0) \, _1(y^1 - x^1_k) \ldots _{n-1}(y^{n-1} - x^{n-1}_k) \, (y)dxdy \\ &\int_{(U)} Q_m(\partial^{(m)}_{x^0_k}(f(x^0)))^2 \, _1(- x^1_k) \ldots _1(- x^{n-1}_k)) \\ &Q_0 f^2(x^0_k) \, _1(- x^1_k) \ldots _{n-1}(- x^{n-1}_k) \, dx\end{aligned} \tag{234}$$

Where we have used Young's convolution inequality in the last step.

(3) From $Ric_g(X, X)$ to $Ric_g(X, X)_{\perp M}$

A crucial result for the already presented C^1-singularity theorems has been the locally uniform convergence $Ric_g - Ric_g{}_{\perp M} \to 0$. In the following we will use this result to estimate $Ric_g(X_k, X_k)[^2_k] - Ric_g{}_{\perp M}(X_k, X_k))[^2_k]$.

$$\begin{aligned}&|Ric_g(X_k, X_k)[^2] - Ric_g{}_{\perp M}(X_k, X_k))[^2_k]| \\ &= |\int_{(U)} Ric_g(\partial_0{}^k, \partial_0{}^k) - Ric_g{}_{\perp M}(\partial_0{}^k, \partial_0{}^k)) (^2_k) \, ^k \circ {}_k^{-1}(x)dx| \\ &\int_{(U)} |(Ric_g)^k_{00} - (Ric_g{}_{\perp M})^k_{00}|(^2_k) \, ^k \circ {}_k^{-1}(x)dx \\ &\| _k\|^2_{L^2(U)}\end{aligned} \tag{235}$$

where δ is given by the uniform convergence of $|Ric_g \star_M - Ric_g|_{\overline{U}} \to 0$ and is thus only dependent on and $\overline{U}$. The next step in our estimating chain consists of comparing $Ric_g \star_M (X_k, X_k))[\,^2_k]$ with $Ric_g(X_k, X_k)) \star_M [\,^2_k]$. To prove Lemma 7.2 this is done by extending a tangent vector $X \in T_pM$ to a locally timelike vector field given by the constant vector field in a chart multiplied with an appropriate cutoff function. Luckily we can use a similar argument only that we explicitly needed a synchronous coordinate system for the construction of X_k.

$$
\begin{aligned}
Ric_g \star_M (X_k, X_k)[\,^2_k] &= Ric_g \star_M [X_k \quad X_k \quad ^2_k] \\
&= (Ric_g \star_M)_{ij}{}^k [X_k^i X_k^j \,^2] = (Ric_g \star_M)_{00}^k [\,^2_k] \\
&= (Ric_g)_{00}^k \star_M [\,^2_k] = Ric_g(X_k, X_k) \star_M [\,^2_k]
\end{aligned} \tag{236}
$$

Thus we have finally completed the chain from $Ric_g (X_k, X_k)[\,^2_k]$ to $Ric_g(X_k, X_k) \star_M [\,^2_k]$. All in all we have therefore derived the following estimate:

$$
\begin{aligned}
Ric_g (X_k, X_k)[\,^2_k] \quad (Ric_g \star_M)(X_k, X_k)[\,^2_k] \quad & k \quad _k k^2_{L^2(U)} \\
= (Ric_g(X_k, X_k) \star_M)[\,^2_k] \quad & k \quad _k k^2_{L^2(U)}
\end{aligned} \tag{237}
$$

In particular using the result of part (2) that is (234) we conclude:

$$
Ric_{g_k}(X_k, X_k)[\,^2_k] \qquad (Q_m k L^{(m)}_{X_k} \; k^2_{L^2(U)} + (Q_0 + \;_k) k \; k^2_{L^2(U)}) \tag{238}
$$

Which as we already have shown before implies the SISEC along $_k$ with $Q_0^k := (Q_0 \quad _k)$.

<u>(4) initial SEC for close enough approximations</u>

Let us parameterize all $_k$ back on $[0, 1]$ and let $t_0 \in (0, 1)$ be the corresponding parameter to b_0. We thus have a sequence $\tilde{}_k := \;_k|_{[0,t_0]}$ of g_k-unit timelike geodesics which maximize g_k-length from S to p and converge in $C^1([0, t_0]$ to $\tilde{}$. Let us furthermore restrict our subsequence to large enough $k \in \mathbb{N}$ such that $\tilde{}_k([0, t_0]) \quad U_{\tilde{}}$ and parameterize the curves back to unit-geodesics. We are thus finding ourselves in a situation like in the beginning of Theorem 7.3. That is if we define $K := \bigcup_{k \in \mathbb{N}} im(\dot{\tilde{}}) \cup im(\dot{\tilde{}}) \quad TM$ we have by the exact same proof as for Lemma 7.2:

$$
\forall \varepsilon \quad _0 \, \exists \;_0 > 0 \; \forall \varepsilon \quad _0 \forall X \in K \text{ with } g(X, X) = -1: \; Ric_g (X, X) > \quad > 0 \tag{239}
$$

That is we find for all $b_0^0 < b_0$ large enough $k \in \mathbb{N}$ such that $Ric_g (\dot{}_k, \dot{}_k) >$ for all $t \quad b_0^0$. This allows us to define an ^{k} as in (135) with $b_0^0,$ and Q_0^k instead of $b_0, \;_0$ and Q_0. Nevertheless as all three parameters can be chosen arbitrary close to the original values for large enough $k \in \mathbb{N}$ we have $^k \to$ as $k \to \infty$. Finally just like in the other C^1-singularity theorems we can assume due to the

C^1_{loc}-convergence of $g \;\; ! \;\; g$ that for large enough $k\,2\,\mathbb{N}$ the convergence fulfills $(n \quad 1)k_{g \;\; k}(\_k(0)) > \min \;\; \frac{n \;\; 1}{b_0};$ (not the same k!). That is if $k_{g\;\;k}(\_k(0)) > \frac{1}{b_0}$ we deduce by Lemma 4.20 the existence of a focal point along $\;_k$ after an affine parameter $\quad b_0$ which implies that $l_k \quad b_0 < b$; a contradiction. On the other hand if

$$(n \quad 1)k_{g\;\;k}(\_k(0)) > \qquad = (1 \quad A_m)(\quad _0)b_0 + \frac{Q_m C_m}{b_0^{2m \quad 1}} + Q_0 A_m b + \frac{pB_m}{b \quad b_0} + \frac{Q_m C_m}{(b \quad b_0)^{2m \quad 1}} \tag{240}$$

we find since since $\;^k \;\; !$ for large enough $k\,2\,\mathbb{N}$ that:

$$(n \quad 1)k_{g\;\;k}(\_k(0)) > \quad ^k \tag{241}$$

Hence all of the conditions of Lemma 4.29 are fulfilled which implies the existence of focal points along $\;_k$ for an affine parameter $t \quad b$. In particular it follows that $l_k \quad b < l$; a contradiction. □

Even though it was shown and used in the prior arguments that the $DSISEC$ implies the $SISEC$ in C^2 spacetimes, it yet remains to demonstrate a converse directed connection : Does the $SISEC$ in C^2-spacetimes imply the $DSISEC$? Such a connection is not needed when trying to prove C^1-singularity theorems, but it would help to justify the reasonableness of the $DSISEC$. The problem one faces is that the $SISEC$ generally only talks about the Ricci-tensor applied to geodesics while the $DSISEC$ also restricts the Ricci-tensor on any general vector field defined on a small neighbourhood of a geodesic. Hence to have any hope in this regard we need to assume the $SISEC$ along other curves than only g-geodesics as well. We may further realize that in the proof of Theorem 7.5 only one kind of vector field, $1{=}2$-density pair has been used, such that assuming a condition on all vector fields and $1{=}2$- densities (as in the formulation of the $DSISEC$) seems a bit strong. It is thus possible to weaken the $DSISEC$ while maintaining the result of Theorem 7.5 .

Definition 7.6. *(stable $DSISEC$)*
We say $(M;g)$ fulfills the stable $DSISEC$ if for all timelike g-geodesics there exists a pre-compact neighbourhood U and a metric $g^0 \quad g$ such that for all $g^0 \quad g \quad g$ and g - geodesics $\; : I \;\; ! \;\; M$ which are contained in a g -synchronous coordinate system $(U \;\; ; \;' \;)$ with $U \qquad U$ an estimate of the following form holds $(1 \quad m \quad k)$

$$Ric[@_0^{'} \quad @_0^{'} \qquad ^2] = Ric(@_0^{'}\;;@_0^{'}\;)[\;\;^2] \qquad (Q_m k L^{(m)}_{@_0^{'}} \;\; k^2_{L^2(U \;\;)} + Q_0 k \;\; k^2_{L^2(U \;\;)}) \tag{242}$$

where $\quad 2 \quad ^k_c(U \;\; ; Vol^{\frac{1}{2}}(M))$

In particular if a C^2-spacetime $(M;g)$ fulfills the $SISEC$ along $\;$ with fixed Q_m and Q_0 also for g -geodesics close to $\;$ and $\;$ small enough, then :

$$Ric[@_0^{'} \qquad @_0^{'} \qquad ^2] \;\; = \;\; \int_{'\,(U\;\;)} Ric(@_0^{'}\;;@_0^{'}\;) \;\; ^2 \quad ' \;\; ^1 dx^0 ::: dx^{n \;\; 1} \tag{243}$$

That is if we apply the $SISEC$ along the coordinate curves $\{x^i = const\}_{i=1,\dots,n-1}$ the stable $DSISEC$ along follows. One may interpret this result as the fact, that a $SISEC$ which also holds for almost g-geodesics (that is g -geodesics approximating a g-geodesic) is equivalent (in C^2-spacetimes) to the distributional energy condition needed in the C^1-singularity theorem (7.5). As the strategy of proving C^1-singularity theorems relies exactly on this property, that energy conditions remain to be true on close enough approximations, one probably should expect the distributional version of the $SISEC$ to have this form.

From here it may be interesting to explicitly examine physical models (apart from the minimal coupled scalar field) and try to find conditions on the matter fields such that an energy inequality of the stable DSISEC-form is fulfilled.

In conclusion one may say that even violations of the SEC together with low regularity seem to be incapable to save us from the rather unsettling consequences of the singularity theorems. Shall we just accept them? Try to generalize them even more? Search for counterexamples? As it is part of human nature to ask for the past and reasons of our present existence, I am pleased knowing that all of the above questions will be further investigated and possibly lead to further such ingenious and rich results as the first formulation of a modern singularity theorem by R.Penrose (1965).

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