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Blaschke Conjecture and Hopf Rigidity

Bachelor's Thesis in Mathematics

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Abstract

Using classical surface geometry, in this thesis the Blaschke conjecture for wiedersehen surfaces and a theorem of E. Hopf about surfaces without conjugate points are proven.

Mittels klassischer Flächentheorie werden in dieser Abschlussarbeit die Blaschke Vermutung für Wiedersehensflächen und ein Theorem von E. Hopf über Flächen ohne konjugierte Punkte bewiesen.

Contents

Chapter 1

Introduction

A wiedersehen surface is a connected two-dimensional Riemannian manifold such that every point has exactly one conjugate point. The round sphere S^2 is a wiedersehen surface, for every point has exactly one conjugate point, namely the antipodal point. Imagine two unit-speed geodesics starting at a point p. If we follow these two geodesics, they will meet again at the antipodal point of p at time π.

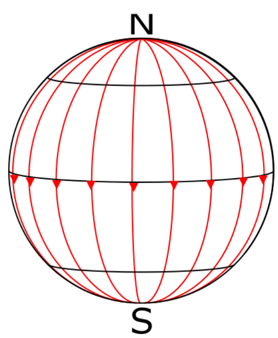

Figure 1.1: The red lines are the geodesics starting at the north pole N. They meet again at the south pole S.

The same geometric property holds for wiedersehen surfaces: There exists a time $a > 0$ such that any two unit speed geodesics starting at a common point p will meet again after time a at the conjugate point of p. This behaviour explains the name "wiedersehen". It is astonishing that the time a is independent of the chosen starting point p and the two geodesics. This property is by no means obvious from the definition. A priori the following could happen:

- There are two unit speed geodesics (the red and black one in figure 2.1) starting at the point p and meeting again at the conjugate point $Con(p)$ of p after different times.

- There exists a geodesic (the blue one in figure 2.1) starting at p that doesn't even hit $Con(p)$.

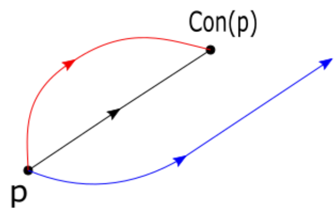

Figure 1.2: Behaviour on a wiedersehen manifold that could happen a priori but doesn't. The red, black and blue line illustrate unit speed geodesics starting at the point p. The red and black geodesics hit the point $Con(p)$ after different times. The blue geodesic doesn't hit $Con(p)$.

It is then a natural question to ask how many different wiedersehen surfaces there are up to isometry. In 1921 Blaschke conjectured in the first edition of his "Vorlesungen über Differentialgeometrie" that every wiedersehen surface in $\mathbb{R}^3$ is isometric to the round sphere. If we look at the conjecture in the abstract setting of Riemannian surfaces, we see that the real projective plane, equipped with the canonical metric is a wiedersehen surface as well. On this surface a point is only conjugate to itself and every unit speed geodesic starting at a point comes back after time π.

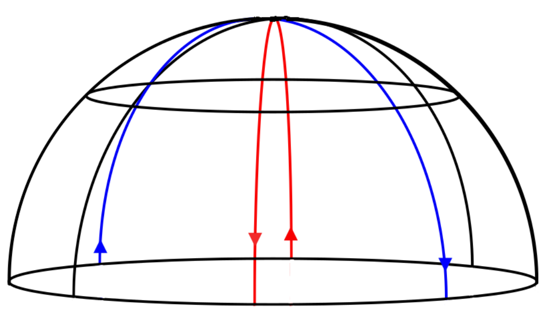

Figure 1.3: We consider the real projective plane as the quotient of the sphere under the antipodal map. The red and blue lines are geodesics starting at the north pole N. After time π they meet there again.

More than 40 years after Blaschke published this conjecture, Green proved that the sphere and real projective plane are in fact the only wiedersehen surfaces, i.e.

Blaschke Conjecture 1.1. *A wiedersehen surface is isometric to the round sphere or the real projective plane (with positive multiples of the canonical Riemannian metric).*

In order to do this, Green integrated over the unit tangent bundle, a technique used earlier by E. Hopf to prove the following

Theorem 1.2. *The total curvature of a Riemannian surface without conjugate points is non-positive. If it equals zero, then the Gaussian curvature vanishes.*

In this thesis we will give proofs of the Blaschke conjecture and the theorem of Hopf. They both have the common idea of integrating some equality (or inequality) over the unit tangent bundle. Hence chapter 2 is devoted to the geometry of the tangent bundle. We consider the concept of densities on a manifold to be able to integrate over nonorientable manifolds and provide tools from integration theory. We introduce a natural metric, called the Sasaki metric, on the total space of the tangent bundle and the unit tangent bundle and show that the geodesic flow is volume preserving.

In chapter 3 we present geometric properties of wiedersehen manifolds. We will see that every two unit speed geodesics starting at the same point will meet again after some time a, which is independent of the starting point and the geodesics. The main result of this chapter is that simply connected wiedersehen manifolds are diffeomorphic to the sphere, the injectivity radius and diameter are both equal to a and all geodesics are periodic with period $2a$.

In chapter 4 we prove the Blaschke conjecture for wiedersehen surfaces. Firstly, we use convering space theory to reduce to the simply connected case. The key for the Blaschke conjecture is then to prove the following theorem.

Theorem 1.3. *Let (M, g) be a closed Riemannian surface and let there be a time $a > 0$ such that along all unit speed geodesics no conjugate point appears before time a. Then*

$$vol(M) \geq \frac{2a^2}{\pi}\chi(M)$$

and equality holds if and only if the sectional curvature is constant $K = \frac{\pi^2}{a^2}$. Here $\chi(M)$ denotes the Euler characteristic of M.

Motivated by this characterization for a Riemannian surface having constant Gaussian curvature, we are then interested in the volume of simply connected wiedersehen surfaces. Using a formula of Santalo we will then show that for a wiedersehen surface the above inequality is indeed an equality.

Chapter 2 enables us to derive theorem 1.2 in chapter 5.

Chapter 2

Geometry of Tangent and Cotangent Bundle

In this chapter we mainly follow Ballmann [3]. Here we provide the technical tools needed for the proofs of the Blaschke conjecture and the theorem of Hopf.

2.1 Splitting of the Double Tangent Bundle

Let M be a smooth m-dimensional manifold and ∇ a connection on M. We have the projection $\pi : TM \to M$. Let $F : N \to M$ be a smooth map between smooth manifolds. We denote by ${}^{F}\nabla$ the pullback connection of ∇ under F. We write $\tau^{{}^F\nabla}$ for the torsion of ${}^{F}\nabla$. For definitions see [4] and [5]. The differential of F will be denoted by F_*.

Definition 2.1. *Define the* ***connection map*** $C : TTM \to TM$ *as follows: Let* $v \in TM$ *and* $Z \in T_vTM$*. Now take any smooth curve* $V : (-\epsilon, \epsilon) \to TM$ *in the tangent bundle such that* $V(0) = v$ *and* $\frac{dV(t)}{dt}_{|t=0} = Z$*. We define*

$$C(Z) := {}^{\pi\circ V}\nabla_{\partial_t} V(0).$$

For C to be well defined one has to show that it is independent of the choice of such a V.

Proposition 2.2. *Let* (x, U) *be local coordinates on* M *and* (x, y) *the induced local coordinates on* TM*. These coordinates itself induce local coordinates* (x, y, z, w) *on* TTM*. In these coordinates we have:*

$$\pi_*(x, y, z, w) = (x, z) \text{ and } C(x, y, z, w) = (x, w^i + z^j y^k \omega^i_{jk}(x))$$

where ω *denotes the matrix of connection forms. In particular* C *is well defined.*

Proof. Let V denote a curve in TM such that $\dot{V}(0) = (x, y, z, w)$. In coordinates we have $V(s) = (x(s), y(s))$ and $\dot{V} = (x, y, \dot{x}, \dot{y})$. We obtain

$$\pi_*(x,y,z,w) = \pi_* \circ \dot{V} = \frac{d(\pi \circ V(t))}{dt}_{|t=0} = (x(0), \dot{x}(0)) = (x,z).$$

For C we calculate

$$\begin{aligned} {}^x\nabla_{\partial_t} V &= {}^x\nabla_{\partial_t} y^i \partial_i = \dot{y}^i \partial_i + y^i ({}^x\nabla_{\partial_t}(\partial_i \circ x)) = \dot{y}^i \partial_i + y^k \nabla_{\dot{x}} \partial_k \\ &= \dot{y}^i \partial_i + y^k \dot{x}^j \nabla_{\partial_j} \partial_k = (\dot{y}^i + y^k \dot{x}^j \omega^i_{jk}(x)) \partial_i. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} C(x,y,z,w) &= C(\dot{V}(0)) = {}^{\pi \circ V}\nabla_{\partial_t} V(0) \\ &= (x, \dot{y}^i + y^k \dot{x}^j \omega^i_{jk}(x))_{|t=0} = (x, w^i + y^k z^j \omega^i_{jk}(x)). \end{aligned}$$

Since the coordinate expression of C is independent of the chosen curve V in TM, C is well defined. C is smooth because it is smooth in coordinates. $\square$

Corollary 2.3. *We have the following properties.*

1. *$\mathcal{H} := kerC$ and $\mathcal{V} := ker\pi_*$ are m-dimensional subbundles of $TTM \to TM$.*
2. *$TTM = \mathcal{H} \bigoplus \mathcal{V}$.*
3. *For each $v \in TM, \pi_* : \mathcal{H}_v \to T_{\pi(v)}M$ and $C : \mathcal{V}_v \to T_{\pi(v)}M$ are isomorphims.*

Proof. 1. We apply the local frame criterion for subbundles of [1, Thm. 10.32]. We choose coordinates around $x \in M$ as in proposition 2.2 and obtain for $(x,y) \in TM$

$$\begin{aligned} &(ker\pi_*)_{(x,y)} = \{(x,y,0,w) | w \in \mathbb{R}^n\} \\ &and \\ &(kerC)_{(x,y)} = \{(x,y,z,-z^j y^k \omega^i_{jk}(x)) | z \in \mathbb{R}^n\} \end{aligned}$$

In these coordinates the smooth local sections

$$(x,y,0,e_i),\ (x,y,z,-y^k \omega^l_{ik}(x)),\ i \in \{1,...,m\}$$

form a basis of $\mathcal{V}_{(x,y)}$, $\mathcal{H}_{(x,y)}$, respectively. Here e_i denotes the i-th unit vector in $\mathbb{R}^m$.
2. Since $dim\mathcal{H}_{(x,y)} = dim\mathcal{V}_{(x,y)} = m$ it suffices to show

$$\mathcal{H}_{(x,y)} \cap \mathcal{V}_{(x,y)} = \{(x,y,0,0)\}.$$

For $(x,y,z,w) \in \mathcal{H}_{(x,y)} \cap \mathcal{V}_{(x,y)}$ we have z=0 and thus w=0.
3. C and π_* are linear by proposition 2.2 and isomorphisms since

$$\mathcal{H}_{(x,y)} \cap \mathcal{V}_{(x,y)} = \{(x,y,0,0)\}.$$

$\square$

Remark 2.4. $\mathcal{H}$ and $\mathcal{V}$ are called horizontal and vertical distribution. Since the double tangent bundle TTM splits we write for each $Z \in TTM$

$$Z = (X, Y),$$

where $X = \pi_*(Z)$ and $Y = C(Z)$.

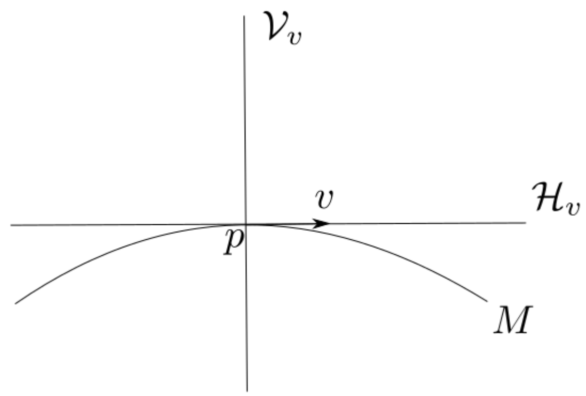

Figure 2.1: The splitting of the double tangent bundle.

Corollary 2.5. *Let $v \in TM$ and $p = \pi(v)$.*

1. *A parallel vector field V along a curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = X$ satisfies $\dot{V}(0) = (X, 0)$.*
2. *The curve $V : (-\epsilon, \epsilon) \to T_pM, V(s) = v + sY$ satisfies $\dot{V}(0) = (0, Y)$ where $Y \in T_pM$.*

Proof. We compute

$$\pi_*\dot{V}(0) = \dot{\gamma}(0) = X \text{ and } C(\dot{V}(0)) = {}^{\gamma}\nabla_{\partial_t} V(0) = 0$$

for 1. and

$$\pi_*\dot{V}(0) = \frac{d(\pi \circ V(t))}{dt}_{|t=0} = \frac{dp}{dt}_{|t=0} = 0 \text{ and } C(\dot{V}(0)) = {}^{\gamma}\nabla_{\partial_t}(v + tY)_{t=0} = Y$$

for 2. $\square$

Notation 2.6. Let M be a smooth manifold, ∇ a connection on M, $p \in M$ and $v \in T_pM$. We denote by $\gamma_v : I_v \to M$ the unique maximal geodesic such that $0 \in I_v, \gamma_v(0) = p$ and $\dot{\gamma_v}(0) = v$. We write $\Phi : \mathcal{G} \to TM$ for the geodesic flow. Here $\mathcal{G}$ denotes the subset of $\mathbb{R} \times TM$, where Φ is defined. The vector field on TM associated to the geodesic flow will be denoted by G.

Proposition 2.7. *Let $v \in TM$. Then $G(v) = (v, 0)$.*

Proof. We compute

$$\pi_* \circ G(v) = \pi_* \circ G \circ \Phi^0(v) = \pi_* \frac{d\Phi^t(v)}{dt}_{|t=0} = \frac{d\pi \circ \Phi^t(v)}{dt}_{|t=0} = \dot{\gamma}_v(0) = v \text{ and}$$

$$C \circ G(v) = C \circ G \circ \Phi^0(v) = C(\frac{d\Phi^t(v)}{dt}_{|t=0}) = {}^{\gamma_v}\nabla_{\partial_t}\dot{\gamma_v} = 0.$$

□

Lemma 2.8. *Let $F : N \to M$ be smooth. Then $F^*(\tau^\nabla) = \tau^{^F\nabla}$.*

Proof. See [5, Satz 4.12] for more details. □

Proposition 2.9. *Let $V : (-\epsilon, \epsilon) \to TM$ be a smooth curve in TM with $V(0) = v$, $\dot{V}(0) = (X, Y)$ and $t \in I_v$. We consider the variation $\Gamma(s,t) = \gamma_{V(s)}(t)$ with Jacobi field $J(t) = \partial_s\Gamma(0,t)$. Then the differential of the geodesic flow is given by*

$$\Phi^t_*(X, Y) = (J(t), {}^{\gamma_v}\nabla_{\partial_t}J(t) + \tau^\nabla(J(t), \dot{\gamma}_v(t))).$$

Proof. We compute

$$\pi_*\Phi^t_*\dot{V}(0) = \frac{d}{ds}_{|s=0}(\pi\Phi^t V(s)) = \partial_s\Gamma(0,t) = J(t)$$

and using lemma 2.8 we obtain

$$\begin{aligned}
C\Phi^t_*\dot{V}(0) &= C(\frac{d}{ds}_{|s=0}(\Phi^t V(s))) = C(\frac{d}{ds}_{|s=0}\partial_t\Gamma(s,t)) = ({}^\Gamma\nabla_{\partial_s}\partial_t\Gamma)(0,t) \\
&= ({}^\Gamma\nabla_{\partial_t}\partial_s\Gamma)(0,t) + \tau^{^\Gamma\nabla}(\partial_s, \partial_t)(0,t) = {}^{\gamma_v}\nabla_{\partial_t}J(t) + \Gamma^*(\tau^\nabla)(\partial_s, \partial_t) \\
&= {}^{\gamma_v}\nabla_{\partial_t}J(t) + \tau^\nabla(J(t), \dot{\gamma}_v(0)).
\end{aligned}$$

□

2.2 Symplectic Geometry

In this section we assemble some basic facts about symplectic manifolds following [1].

Definition 2.10. *Let V be a finite-dimensional vector space and ω an alternating covariant 2-tensor on V. ω is called* ***nondegenerate*** *or* ***symplectic*** *if for each nonzero $v \in V$, there exists $w \in V$ such that $\omega(v, w) \neq 0$.*

Definition 2.11. *Let M be a smooth manifold. A nondegenerate 2-form ω on M is a 2-form such that ω_p is nondegenerate for each $p \in M$. A* ***symplectic form*** *on M is a closed nondegenerate 2-form. A tuple (M, ω) is called a* ***symplectic manifold*** *if ω is a symplectic form on M.*

On the total space of the cotangent bundle $\bar{\pi} : T^*M \to M$ there is a canonical symplectic form ω. Let (q, ϕ) denote a point in T^*M. The pointwise pullback of $\bar{\pi}$ is denoted by $\bar{\pi}^*_{(q,\phi)} : T^*_q M \to T^*_{(q,\phi)} T^*M$. The **tautological 1-form** $\lambda \in \Omega^1(T^*M)$ is defined by $\lambda_{(q,\phi)} = \bar{\pi}^*_{(q,\phi)}\phi$. That is $\lambda_{(q,\phi)}(v) = \phi(\bar{\pi}_{*(q,\phi)}(v))$ for $v \in T_{(q,\phi)}T^*M$.

Proposition 2.12. *Let M a smooth manifold. Then the 2-form $\omega \in \Omega^2(T^*M)$ defined by $\omega = -d\lambda$ is symplectic. ω is called the **canonical symplectic form on** T^*M.*

Proof. See [1, Thm 22.11]. □

Definition 2.13. *Let (M, ω) be a symplectic manifold. For any smooth function $f \in C^\infty(M)$ we define the **Hamiltonian vector field of** f to be the unique vector field X_f on M that satisfies $\iota_{X_f}\omega = df$.*
*A smooth vector field X on M is called **symplectic** if ω is invariant under the flow of X, i.e. $\Phi^*_X\omega = \omega$. X is called **Hamiltonian** if there exists a function $f \in C^\infty(M)$ such that $X = X_f$. It is called **locally Hamiltonian** if each $p \in M$ has a neighborhood on which X is Hamiltonian.*

Proposition 2.14. *Let (M, ω) be a symplectic manifold. A smooth vector field on M is symplectic if and only if it is locally Hamiltonian.*

Proof. See [1, Prop.22.17]. □

2.3 Legendre-Transformation

In this section we assume (M, g) to be a semi-Riemannian manifold.

Definition 2.15. *The bundle-isomorphism $\mathcal{L} : TM \to T^*M$ defined by $\mathcal{L}(v)(w) = g(v, w)$ is called **Legendre-Transformation**.*

We now use the Legendre-Transformation to transport the canonical symplectic structure of the cotangent bundle to the tangent bundle.

Definition 2.16. *We denote by $\mathcal{L}^* : T^*T^*M \to T^*TM$ the pullback of $\mathcal{L}$ and define $\lambda_g : TM \to T^*TM$ by $\lambda_g(v) = \mathcal{L}^*\lambda\mathcal{L}(v)$.*

Proposition 2.17. *$\lambda_g \in \Omega^1(TM)$. For each $v \in TM$ and $Z \in T_vTM$ we have*

$$(\lambda_g)_v(Z) = g(v, \pi_* Z).$$

Proof. We calculate

$$(\lambda_g)_v = \mathcal{L}^*\lambda\mathcal{L}(v) = \mathcal{L}^* \circ \bar{\pi}^*_{\mathcal{L}(v)}(\mathcal{L}(v)) = (\bar{\pi} \circ \mathcal{L})^*(\mathcal{L}(v)) = \pi^*(\mathcal{L}(v)) = g(v, \pi_* \cdot).$$

Inserting Z we obtain the assertion. □

From now on we assume ∇ to be a metric connection on M.

Lemma 2.18. *Let M be a smooth manifold, $p \in M, Z_1, Z_2 \in T_pM$. Then there is a $V : (-\epsilon, \epsilon)^2 \to M$ such that*

$$V(0,0) = p, \partial_s V(0,0) = Z_1 \text{ and } \partial_t V(0,0) = Z_2.$$

For any 1-form η on M we have

$$d\eta(\partial_s V, \partial_t V) = \partial_s \eta(\partial_t V) - \partial_t \eta(\partial_s V).$$

Proof. This is a local question. Choose coordinates (U, ϕ) around p and define $V(s,t) := \phi^{-1}(s\phi_{*,p}Z_1 + t\phi_{*,p}Z_2)$. The second statement can be checked by a computation in coordinates. □

Proposition 2.19. *The 2-form $\omega_g := -d\lambda_g$ on TM satisfies*

$$\omega_g(Z_1, Z_2) = g(X_1, Y_2) - g(Y_1, X_2) - g(v, \tau^\nabla(X_1, X_2))$$

for each $v \in TM$ and $Z_1 = (X_1, Y_1), Z_2 = (X_2, Y_2) \in T_vTM$.

Proof. Replace in lemma 2.18 M, p, η by TM, v, λ_g, respectively. We obtain a $V : (-\epsilon, \epsilon)^2 \to TM$ such that $V(0,0) = v, \partial_s V(0,0) = Z_1$ and $\partial_t V(0,0) = Z_2$ and writing $\Gamma = \pi \circ V$ we get:

$$\begin{aligned}
&d\lambda_g(\partial_s V, \partial_t V) \\
&= \partial_s \lambda_g(\partial_t V) - \partial_t \lambda_g(\partial_s V) \\
&= \partial_s g(V, \partial_t(\pi \circ V)) - \partial_t g(V, \partial_s(\pi \circ V)) && (2.17) \\
&= g({}^\Gamma\nabla_{\partial_s} V, \partial_t \Gamma) + g(V, {}^\Gamma\nabla_{\partial_s} \partial_t \Gamma) - g({}^\Gamma\nabla_{\partial_t} V, \partial_s \Gamma) - g(V, {}^\Gamma\nabla_{\partial_t} \partial_s \Gamma) \\
&= g({}^\Gamma\nabla_{\partial_s} V, \partial_t \Gamma) - g({}^\Gamma\nabla_{\partial_t} V, \partial_s \Gamma) + g(V, {}^\Gamma\nabla_{\partial_s} \partial_t \Gamma - {}^\Gamma\nabla_{\partial_t} \partial_s \Gamma) \\
&= g({}^\Gamma\nabla_{\partial_s} V, \partial_t \Gamma) - g({}^\Gamma\nabla_{\partial_t} V, \partial_s \Gamma) + g(V, \tau^{{}^\Gamma\nabla}(\partial_s, \partial_t)) \\
&= g({}^\Gamma\nabla_{\partial_s} V, \partial_t \Gamma) - g({}^\Gamma\nabla_{\partial_t} V, \partial_s \Gamma) + g(V, \tau^\nabla(\partial_s \Gamma, \partial_t \Gamma). && (2.8)
\end{aligned}$$

Evaluating at $(s,t) = (0,0)$ we obtain the formula for ω_g. □

Corollary 2.20. *Let ∇ be a metric connection. Then ω_g is symplectic.*

Proof. We need to show the nondegeneracy of $(\omega_g)_v$ for each $v \in TM$. Write $Z_1 = (X_1, Y_1)$ and assume $\omega_g(Z_1, Z_2) = 0$ for all $Z_2 = (X_2, Y_2) \in T_vTM$. This implies

$$g(X_1, Y_2) - g(Y_1, X_2) - g(v, \tau^\nabla(X_1, X_2)) = 0$$

for all $X_2, Y_2 \in T_{\pi(v)}M$. If we choose $X_2 = 0$ we obtain

$$g(X_1, Y_2) = 0, \forall Y_2 \in T_{\pi(v)}M$$

and since g is nondegenerate we infer $X_1 = 0$. Hence

$$-g(Y_1, X_2) = 0, \forall X_2 \in T_{\pi(v)}M$$

and again the nondegeneracy of g implies $Y_1 = 0$. □

2.4 Densities and Integration on Riemannian Manifolds

In the next chapters we want to integrate on Riemannian manifolds that are not necessarily orientable. To do this, we introduce the concept of densities on a manifold. Firstly we define densities on a vector space to be the objects that transform in the right way. In our discussion we follow [1].

Definition 2.21. *Let V be an m-dimensional vector space. A function $\mu : \underbrace{V \times \cdots \times V}_{m \text{ times}} \to \mathbb{R}$ is called a **density** if for every linear map $T : V \to V$ and $v_1, \ldots, v_m \in V$ we have*

$$\mu(Tv_1, \ldots, Tv_m) = |detT|\mu(v_1, \ldots, v_m).$$

The next proposition summarizes some basic properties of densities.

Proposition 2.22. *Let V be an m-dimensional vector space. Then*

1. *The set $\mathcal{D}(V)$ of densities on V is a vector space.*
2. *Two densities on V agreeing on a basis are equal.*
3. *If $\omega \in \Lambda^m(V^*)$ is an alternating covariant tensor, then the map*

   $$|\omega| : V \times \cdots \times V \to \mathbb{R}, |\omega|(v_1, \ldots, v_m) = |\omega(v_1, \ldots, v_m)|$$

   is a density.
4. *$\mathcal{D}(V)$ is 1-dimensional.*

Proof. See [1, Proposition 16.35]. □

We now turn to the case of manifolds.

Definition 2.23. *If M is a smooth m-dimensional manifold, then a map*

$$\mu : M \to \mathcal{D}(M) := \coprod_{p \in M} \mathcal{D}(T_pM) \text{ with } \mu_p \in \mathcal{D}(T_pM)$$

is called a density on M. $\mathcal{D}(M)$ is called the density bundle of M.

Proposition 2.24. *If M is a smooth manifold, its density bundle is a smooth line bundle.*

Proof. See [1, Proposition 16.36]. □

As for differential forms we can define the pullback of a smooth map.

Definition 2.25. *Let $F : M \to N$ be a smooth map between m-dimensional manifolds. We define the pullback by*

$$F^* : \mathcal{D}(N) \to \mathcal{D}(M), (F^*\mu)_p(v_1, ..., v_n) := \mu_{F(p)}(F_{*p}v_1, ..., F_{*p}v_n).$$

Proposition 2.26. *Let $G : P \to M$ and $F : M \to N$ be smooth maps between m-dimensional manifolds and μ a density on N. Then*

1. *For functions $f : N \to \mathbb{R}$, $F^*(f\mu) = (f \circ F)F^*\mu$.*
2. *If ω is an n-form on N, then $F^*|\omega| = |F^*\omega|$.*
3. *If μ is smooth, then $F^*\mu$ is a smooth density on M.*
4. *$(F \circ G)^*\mu = G^* \circ F^*\mu$.*

Let us turn toward integration of densities.

Definition 2.27. *Let U be an open subset of $\mathbb{R}^n$ and μ a density on U. We call μ integrable if the unique function $f : U \to \mathbb{R}$ with $\mu = f|dx^1 \wedge \cdots \wedge dx^m|$ is integrable. We call a compactly supported density μ on a manifold integrable if for all charts (U, ϕ) on M the density $(\phi^{-1})^*\mu$ is integrable.*

Proposition 2.28. *Let $F : M \to N$ be a smooth map between n-manifolds, $(U; (x^i)), (V, (y^i))$ charts on M, N respectively. Then for each function $f : V \to \mathbb{R}$*

$$F^*(f|dy^1 \wedge \cdots \wedge dy^m|) = f \circ F|detDF||dx^1 \wedge \cdots \wedge dx^m| \text{ on } U \cap F^{-1}(V),$$

where DF is the matrix of partial derivatives of F in these coordinates.

Proof. See [1, Proposition 16.40]. □

Here we see quite nicely why densities are the right objects to integrate on a manifold. The transformation of densities under a change of coordinates involves the absolute value of the Jacobian determinant, exactly as the transformation rule for integration. If $U \subset \mathbb{R}^n$ is an open set and μ a compactly supported density on U with $\mu = f|dx^1 \wedge \cdots \wedge dx^n|$ such that f is Lebesque integrable on U we define the integral of μ over U by

$$\int_U \mu := \int_U f dx,$$

where the right side is the Lebesque integral in $\mathbb{R}^n$.

Proposition 2.29. *Let U, f, μ be as above, $V \subset \mathbb{R}^n$ open and $F : V \to U$ a diffeomorphism. Then*

$$\int_U \mu = \int_V F^*\mu.$$

Proof. By the preceding proposition and the transformation rule we obtain

$$\begin{aligned}\int_V F^*\mu &= \int_V (f \circ F)F^*(|dx^1 \wedge \cdots \wedge dx^m|)\\ &= \int_V (f \circ F)|detDF||dx^1 \wedge \cdots \wedge dx^m|\\ &= \int_U f|dx^1 \wedge \cdots \wedge dx^m| = \int_U \mu.\end{aligned}$$

□

Now let μ be a Lebesque integrable density on M with compact support in some chart (U, ϕ) . We define the integral of μ over M by

$$\int_M \mu := \int_{\phi(U)} (\phi^{-1})^* \mu.$$

This is well defined by the preceding proposition. Now we turn to the general case, where *supp*μ is not necessarily contained in a single chart. Let μ be a Lebesque integrable density on M with compact support. Let $\{U_i\}$ be finitely many charts covering *supp*μ and $\{\psi_i\}$ a subordinate smooth partition of unity. We define the integral of μ over M by

$$\int_M \mu := \sum_i \int_M \psi_i \mu.$$

This is well defined. To see this we choose a different finite cover $\{V_j\}$ and subordinate smooth partition of unity $\{\rho_j\}$. Then

$$\begin{aligned} \sum_i \int_M \psi_i \mu &= \sum_i \int_M \sum_j \rho_j \psi_i \mu = \sum_j \sum_i \int_M \rho_j \psi_i \mu \\ &= \sum_j \int_M \sum_i \rho_j \psi_i \mu = \sum_j \int_M \rho_j \mu. \end{aligned}$$

Proposition 2.30. *Let M, N be smooth n-manifolds and μ, η compactly supported integrable densities on M.*

1. *For all $a, b \in \mathbb{R}$:*
$$\int_M a\mu + b\eta = a \int_M \mu + b \int_M \eta.$$
2. *If $F : N \to M$ is a diffeomorphism, then $\int_M \mu = \int_N F^* \mu$.*

Proposition 2.31. *Let (M, g) be a Riemannian manifold. There is a unique smooth positive density μ_g (or sometimes we denote it by μ_M), called the Riemannian density, on M such that*

$$\mu_g(E_1, ..., E_m) = 1$$

for each orthonormal frame $(E_1, ..., E_m)$.

Proof. See [1, Proposition 16.45]. $\square$

Proposition 2.32. *Let (M, g) be a Riemannian manifold and $(U; (x^i))$ a chart. Then the Riemannian density μ_g is given by*

$$\mu_g = \sqrt{det(g_{ij})} |dx^1 \wedge \cdots \wedge dx^m|.$$

Proof. Since μ_g is positive, there exists a positive function $f \in C^\infty(M)$ with $\mu_g = f|dx^1 \wedge \cdots \wedge dx^m|$. Let $p \in M$, and (E_i) be a smooth orthonormal frame around p. Denote by (E_i^*) its dual coframe. If we write $\frac{\partial}{\partial x^i} = A_i^j E_j$, then

$$f = \mu_g(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}) = |E_1^* \wedge \cdots \wedge E_m^*|(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}) = |det(A)|.$$

If we compute the metric in the coordinate frame, we get

$$g_{ij} = g(A_i^k E_k, A_j^l E_l) = \sum_k A_i^k A_j^k$$

and hence

$$det(g_{ij}) = det(A^T A) = det(A)^2.$$

Thus $f = |det(A)| = \sqrt{det(g_{ij})}$. □

Let (M, g) be a Riemannian manifold. We call a function $f : M \to \mathbb{R}$ integrable if $f\mu_g$ is integrable. If such a f is compactly supported, we define the integral of f to be

$$\int_M f\mu_g.$$

Proposition 2.33. *Let (M, g) be an oriented Riemannian manifold and $f : M \to \mathbb{R}$ a continuous compactly supported function. Then*

$$\mu_g = |vol_g| \text{ and } \int_M f\mu_g = \int_M f vol_g,$$

where vol_g denotes the Riemannian volume form on (M, g).

Proof. Let (E_i) be a orthonormal basis of T_pM. Then

$$\mu_g(E_1, \dots, E_m) = 1 = |vol_g(E_1, \dots, E_m)|.$$

Since densities that agree on some basis are equal, $\mu_g = |vol_g|$.

Without loss of generality we can assume that the support of f is contained in a positively oriented chart (U, ϕ). Then

$$\int_U f\mu_g = \int_{\phi(U)} (\phi^{-1})^*(f\mu_g) = \int_{\phi(U)} (f \circ \phi^{-1})\sqrt{det(g_{ij})}dx = \int_U f vol_g.$$

□

Because of the last proposition it doesn't matter in the orientable case, whether we integrate a function against the Riemannian density or against the Riemannian volume form. Now we are able to provide the theorems of integration theory, specialized to Riemannian manifolds.

Proposition 2.34. *(Fubini) Let $F : (M, g) \to (N, h)$ be a Riemannian submersion and $f : M \to \mathbb{R}$ an integrable function with compact support in M. Then the function*

$$\bar{f} : N \to \mathbb{R}, q \mapsto \int_{F^{-1}(q)} f \mu_{F^{-1}(q)}$$

is integrable and

$$\int_M f \mu_g = \int_N \int_{F^{-1}(q)} f \mu_{F^{-1}(q)} \mu_h(q).$$

Proof. See [15, Theorem 5.6]. □

Proposition 2.35. *Let g_{euk} denote the Euclidean metric on $\mathbb{R}^n$. If $A : \mathbb{R}^n \to \mathbb{R}^n$ is an endomorphism, then*

$$\frac{1}{n} tr(A) = \frac{1}{vol(S^{n-1})} \int_{S^{n-1}} g_{euk}(Av, v) vol_{S^{n-1}}.$$

Proof. Let $\iota : S^{n-1} \to \mathbb{R}^n$ denote the inclusion, $A = (a_{ij})$. For $i \neq j$ we consider the map

$$\tilde{\phi}_{ij} : \mathbb{R}^n \to \mathbb{R}^n, (x^1, ..., x^i, ..., x^j, ..., x^n) \mapsto (x^1, ..., x^j, ..., -x^i, ..., x^n),$$

which restricts to a map

$$\phi_{ij} := \tilde{\phi}_{ij} \circ \iota : S^{n-1} \to S^{n-1}.$$

The latter is an isometry since

$$\phi_{ij}^* g_{S^{n-1}} = \phi_{ij}^* \iota^* g_{euk} = \iota^* \tilde{\phi}_{i,j}^* g_{euk} = \iota^* g_{euk} = g_{S^{n-1}}.$$

Thus

$$\int_{S^{n-1}} x^i x^j vol_{S^{n-1}} = \int_{S^{n-1}} \phi_{ij}^* (x^i x^j vol_{S^{n-1}}) = - \int_{S^{n-1}} x^i x^j vol_{S^{n-1}} = 0.$$

On the other hand for all $i = 1, ..., n$

$$n \int_{S^{n-1}} (x^i)^2 vol_{S^{n-1}} = \sum_{i=1}^n \int_{S^{n-1}} (x^i)^2 vol_{S^{n-1}} = vol(S^{n-1}).$$

Using the last two equalities we compute

$$\begin{aligned}
\int_{S^{n-1}} g_{euk}(Ax, x) vol_{S^{n-1}} &= \sum_{i,j=1}^n \int_{S^{n-1}} a_{ij} x^i x^j vol_{S^{n-1}} \\
&= \sum_{i=1}^n \int_{S^{n-1}} a_{ii} (x^i)^2 vol_{S^{n-1}} \\
&= \sum_{i=1}^n a_{ii} \frac{1}{n} vol(S^{n-1}) = \frac{1}{n} tr(A) vol(S^{n-1}).
\end{aligned}$$

□

In the transformation rule for integrals of functions in $\mathbb{R}^m$ the norm of the Jacobian determinant is involved. On the other hand, if we consider a smooth map $F : M \to N$ between m-dimensional manifolds, there is no meaningful way to define the determinant of $F_* : T_pM \to T_{F(p)}N$. But if we require M and N to be Riemannian manifolds, we can make the following definition.

Definition 2.36. *Let (M, g) and (N, h) be Riemannian m-manifolds and $\psi : M \to N$ a smooth map. We define the **Jacobian** of ψ as the smooth function $|\psi|_* : M \to [0, \infty)$ that satisfies*

$$\psi^* \mu_h = |\psi|_* \mu_g.$$

Proposition 2.37. *(transformation rule) Let (M, g) and (N, h) be Riemannian manifolds and $\psi : M \to N$ a diffeomorphism. If $f : N \to \mathbb{R}$ is integrable and compactly supported in N, then*

$$\int_{\psi(M)} f \mu_h = \int_M (f \circ \psi) |\psi|_* \mu_g.$$

Proof. This follows from

$$\psi^*(f \mu_h) = (f \circ \psi) \psi^* \mu_h = (f \circ \psi) |\psi|_* \mu_g$$

and Proposition 2.30. □

The following proposition shows how the Jacobian of a map ψ can be calculated if the differential ψ_* is known.

Proposition 2.38. *Let (M, g), (N, h) be m-dimensional Riemannian manifolds and $\psi : M \to N$ a smooth map. Let $p \in M$ and $(b_1, ..., b_m)$ be a basis of T_pM. Then we have*

$$|\psi_*|(p) = \frac{|\psi_{*p} b_1 \wedge ... \wedge \psi_{*p} b_m|}{|b_1 \wedge ... \wedge b_m|},$$

where $|b_1 \wedge ... \wedge b_m| := |det(g(b_i, b_j))|^{\frac{1}{2}}$.

Proof. By definition

$$\mu_h(\psi_{*p} b_1, \ldots, \psi_{*p} b_m) = |\psi|_*(p) \mu_g(b_1, \ldots, b_m).$$

If ψ_{*p} is no isomorphism, then the left hand side of the above equation is equal to zero. Since $\mu_g(b_1, \ldots, b_m) \neq 0$, we must have $|\psi|_*(p) = 0$. On the other hand, $(\psi_{*p} b_i)$ is linearly dependent, thus $|\psi_{*p} b_1 \wedge ... \wedge \psi_{*p} b_m| = 0$.
If ψ_{*p} is an isomorphism, then the same argument as in proposition 2.32 shows

$$\mu_g(b_1, \ldots, b_m) = \sqrt{det(g(b_i, b_j))}, \mu_h(\psi_{*p} b_1, \ldots, \psi_{*p} b_m) = \sqrt{det(g(\psi_{*p} b_i, \psi_{*p} b_j))}.$$

□

Proposition 2.39. *Let $\pi : \tilde{M} \to M$ be a smooth k-sheeted covering map and μ an integrable density on M. Then*

$$\int_{\tilde{M}} \pi^*\mu = k \cdot \int_M \mu.$$

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be a finite cover of charts of *supp*μ, where the U_α are evenly covered neighborhoods with

$$\coprod_{i=1}^{k} U_{\alpha,i} = \pi^{-1}(U_\alpha).$$

Let $\{\psi_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$. Then $\{\pi^*_{|U_{\alpha,i}}\psi_\alpha\}$ is a smooth partition of unity subordinate to $\{U_{\alpha,i}\}$ and $\{(U_{\alpha,i}, \pi^*_{|U_{\alpha,i}}\phi_\alpha)\}$ is a finite cover of charts of *supp*$(\pi^*\mu)$. Therefore

$$\begin{aligned}\int_{\tilde{M}} \pi^*\mu &= \sum_{\alpha,i} \int_{U_{\alpha,i}} \pi^*_{|U_{\alpha,i}}\psi_\alpha \pi^*\mu = \sum_{\alpha,i} \int_{U_{\alpha,i}} \pi^*_{|U_{\alpha,i}}(\psi_\alpha \mu) \\ &= \sum_{\alpha,i} \int_{U_\alpha} (\psi_\alpha \mu) = k \cdot \int_M \mu.\end{aligned}$$

□

Proposition 2.40. *Let $F : (M, g) \to (N, h)$ be a local isometry between Riemannian manifolds. Then*

$$F^*\mu_h = \mu_g.$$

Proof. Let (E_i) be an orthonormal frame around $p \in M$. Then (F_*E_i) is an orthonormal frame around $F(p) \in N$. Hence

$$F^*\mu_h(E_1, \cdots, E_m) = \mu_h(F_*E_1, \cdots, F_*E_m) = 1 = \mu_g(E_1, \cdots, E_m).$$

By uniqueness of the Riemannian density, $F^*\mu_h = \mu_g$. □

Now let (M, g) be a non-orientable Riemannian manifold. We want to show that the Gauss-Bonnet theorem holds in the non-orientable case as well. To see this, we use the orientation covering $\pi : \tilde{M} \to M$ of M. Properties of the orientation covering can be found in [1]. We just need the following two facts:

- π is a smooth 2-sheeted covering map and
- $\tilde{M}$ is orientable.

Proposition 2.41. *Let (M, g) be a closed (not necessary orientable) Riemannian manifold. Then*

$$\int_M K\mu_g = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

Proof. We only have to consider the non-orientable case. We use the orientation covering $\pi : \tilde{M} \to M$ to pull the metric g back to a Riemannian metric π^*g on $\tilde{M}$. Then the Gauss-Bonnet theorem for the closed oriented Riemannian manifold $(\tilde{M}, \pi^*g)$ implies

$$\int_{\tilde{M}} \tilde{K} vol_{\pi^*g} = 2\pi\chi(\tilde{M}) = 4\pi\chi(M).$$

By propositions 2.39 and 2.40

$$\int_{\tilde{M}} \tilde{K} vol_{\pi^*g} = \int_{\tilde{M}} \pi^* K \mu_{\pi^*g} = \int_{\tilde{M}} \pi^* K \pi^* \mu_g = \int_{\tilde{M}} \pi^*(K\mu_g) = 2\int_M K\mu_g,$$

which proves the asserted equality. □

2.5 The Sasaki Metric

Let (M, g) be a semi-Riemannian manifold and ∇ the Levi-Civita connection. In this section we endow the total space of the tangent bundle with a natural metric. In section 2.1 we saw that for $v \in TM$ the fiber T_vTM is isomorphic to the direct sum of two copies of $T_{\pi(v)}M$. This enables us to use the metric g to define a metric on TM.

Definition 2.42. *The* ***Sasaki metric*** *is the semi-Riemannian metric g^S on TM defined by*

$$g^S_v(Z_1, Z_2) = g_{\pi(v)}(X_1, X_2) + g_{\pi(v)}(Y_1, Y_2),$$

for $Z_1 = (X_1, Y_1), Z_2 = (X_2, Y_2) \in T_vTM$.

Proposition 2.43. *Let $v \in T_pM$.*

1. *The Sasaki metric is a semi-Riemannian metric.*
2. *$\mathcal{H} \perp \mathcal{V}$.*
3. *$\pi_*, C : \mathcal{H}_v \to T_pM$ are orthogonal transformations.*
4. *$\pi : (TM, g^S) \to (M, g)$ is a semi-Riemannian submersion.*

Proof. 2),3) are clear from the definition. For 4) we note that π is a submersion, for each $p \in M$ the fiber $\pi^{-1}(p) = T_pM$ is a semi-Riemannian submanifold of TM and for each $v \in T_pM, Z_1 = (X_1, 0), Z_2 = (X_2, 0) \in \mathcal{H}_v$ we have $g^S_v(Z_1, Z_2) = g_p(\pi_* Z_1, \pi_* Z_2) + 0$. □

We now want to investigate how volume on the tangent bundle changes under the geodesic flow. We will show that the geodesic flow preserves volume.

Proposition 2.44. *Let (M, g) be a Riemannian manifold.*

1. *The symplectic form ω_g and the density μ_{g^S} on TM are related by*

$$|\underbrace{\omega_g \wedge \cdots \wedge \omega_g}_{m \ times}| = m!\mu_{g^S}.$$

2. *The geodesic flow Φ is Hamiltonian (i.e. $\Phi = \Phi_{X_h}$) with Hamiltonian*

$$h : TM \to \mathbb{R}, h(v) = \frac{1}{2}g_{\pi(v)}(v, v).$$

3. *The geodesic flow is symplectic, i.e. $(\Phi^t)^*\omega_g = \omega_g$.*

4. *The geodesic flow preserves μ_{g^S}.*

Proof. 1. Let $v \in T_pM$ for some $p \in M$. Choose an orthonormal basis $(e_1, \cdots, e_m)$ of (T_pM, g_p). Define $Z_i = (e_i, 0)$ and $Z_{i+m} = (0, e_i)$ for $i = 1, \dots, m$. $(Z_1, \dots, Z_{2m})$ is an orthonormal basis of (T_vTM, g^S_v). For $j \in \{1, \dots, m\}$ proposition 2.19 implies

$$\omega_g(Z_j, Z_k) = \begin{cases} 0, & \text{if } k \neq j + m \\ 1, & k = j + m. \end{cases}$$

Thus, when we denote by (Z_i^*) the dual basis of (Z_i) we have

$$\omega_g = \sum_{i=1}^{m} Z_i^* \wedge Z_{i+m}^*$$

and therefore

$$\begin{aligned} \omega_g^m &= m! \sum_{1 \leq i_1 \cdots \leq i_m \leq m} (Z_{i_1}^* \wedge Z_{i_1+m}^*) \wedge \cdots \wedge (Z_{i_m}^* \wedge Z_{i_m+m}^*) \\ &= m!(Z_1^* \wedge Z_{1+m}^*) \wedge \cdots \wedge (Z_m^* \wedge Z_{m+m}^*). \end{aligned}$$

Hence $|\omega_g^m| = \mu_g$.
2. Using proposition 2.7 it remains to show that $X_h(v) = (v, 0)$. First, we compute $grad(h)$. Let $V : (-\epsilon, \epsilon) \to TM$ such that $V(0) = v, \dot{V}(0) = Z \in T_vTM$.

$$\begin{aligned} &g(Cgrad(h(v)), CZ) + g(\pi_* grad(h(v)), \pi_* Z) = g^S(grad(h(v)), \dot{V}(0)) \\ &= d_v h \dot{V}(0) = \frac{d}{dt}_{|t=0}(h \circ V(t)) = \frac{d}{dt}_{|t=0} \frac{1}{2} g(V(t), V(t)) \\ &= g({}^{\pi \circ V}\nabla_{\partial_t} V(0), V(0)) = g(CZ, v) = g(v, CZ). \end{aligned}$$

First choosing $Z = (X, 0)$ the nondegeneracy of g implies $\pi_* grad(h(v)) = 0$ and inserting $Z = (0, Y)$ we obtain $Cgrad(h(v)) = v$. Together we get

$$grad(h(v)) = (0, v).$$

Finally proposition 2.19 implies

$$g(\pi_* X_h, CZ) - g(CX_h, \pi_* Z) = \omega_g(X_h, Z) = d_v h Z = g(v, CZ).$$

Choosing Z as above and again using the nondegeneracy of g we obtain $X_h(v) = (v, 0)$.
3. Hamiltonian flows are symplectic, see [1, 22.17].
4. By 1. and 3. □

Definition 2.45. *Let (M, g) be a Riemannian manifold. We define the* ***unit tangent bundle*** *to be the subset $SM \subset TM$ given by*

$$SM = \{(p, v) \in TM | g_p(v, v) = 1\}.$$

We denote by $\iota : SM \to TM$ the inclusion.

Proposition 2.46. *Let (M, g) be a Riemannian manifold. Then*

1. *SM is a smooth codimension-1 submanifold of TM.*
2. *M is connected iff SM is connected.*
3. *M is compact iff SM is compact.*

Proof. See [2, Prop 2.9]. □

We can thus pull back the Sasaki metric via $\iota : SM \to TM$. Then $g_{SM} := \iota^* g^S$ is a Riemannian metric on the unit tangent bundle SM. Since every geodesic has constant speed the geodesic flow $\Phi^t : TM \to TM$ restricts to a smooth map $\Phi^t : SM \to SM$(where it is defined). The geodesic flow is also volume preserving if it is restricted to the unit tangent bundle. This is the content of the following lemma.

Lemma 2.47. *Let (M, g) be a Riemannian manifold.*

1. *For each $p \in M, v \in S_pM$ the tangent space of the unit tangent bundle at v is given by: $T_vSM = \{(X, Y) | X, Y \in T_pM \, and \, Y \perp v\}$.*
2. *We have $|\iota^*\lambda_g \wedge (\iota^*\omega_g)^{m-1}| = (m-1)!\mu_{\iota^* g^S}$. In particular $\iota^*\lambda_g$ is a contact form on SM and SM is orientable.*
3. *The geodesic flow on SM preserves $\iota^*\lambda_g$ and $\iota^*\omega_g$.*
4. *The geodesic flow on SM preserves μ_{SM}.*

Proof. 1. The tangent space of SM at $v \in SM$ is given by

$$T_vSM = \{\dot{V}(0) \mid V : (-\epsilon, \epsilon) \to SM \text{ and } V(0) = v\}.$$

Take such a V, then

$$g(C\dot{V}(0), v) = g({}^{\pi \circ V}\nabla_{\partial_t} V(0), V(0)) = \frac{1}{2} \frac{d}{dt}_{|t=0} \underbrace{g(V(t), V(t))}_{1} = 0.$$

Thus, we have shown the inclusion $\subset$. Equality follows from dimensional reasons. 2. We will show that $\iota^*\lambda_g \wedge (\iota^*\omega_g)^{m-1}$ is a nowhere vanishing form of top degree on SM. This will imply orientability of SM. Let $v \in S_pM$ and using Gram-Schmidt we obtain a orthonormal basis (e_i) of (T_pM, g_p) such that $e_m = v$. Then $(Z_1, \dots, Z_{2m-1})$ is a orthonormal basis of $(T_vSM, (g_{SM})_v)$, where $Z_i = (e_i, 0)$ for $i \le m$ and $Z_i = (0, e_i)$ for $m+1 \le i \le 2m-1$. By proposition 2.17 we compute

$$\iota^*\lambda_g(Z_i) = g(v, \pi_* Z_i) = \begin{cases} 0, & \text{if } i \neq m \\ 1, & i = m. \end{cases}$$

Therefore $\iota^*\lambda_g = Z_m$.By proposition 2.19 we compute for $j \le m$

$$\iota^*\omega_g(Z_j, Z_k) = \begin{cases} 0, & \text{if } k \neq j+m \\ 1, & k = j+m. \end{cases}$$

Thus, we have $\iota^*\omega_g = \sum_{i=1}^{m-1} Z_i^* \wedge Z_{i+m}^*$ and

$$(\iota^*\omega_g)^{m-1} = \pm(m-1)!(Z_1^* \wedge Z_{1+m}^*) \wedge \cdots \wedge (Z_{m-1}^* \wedge Z_{m-1+m}^*).$$

Finally, we get

$$\iota^*\lambda_g \wedge (\iota^*\omega_g)^{m-1} = \pm(m-1)! Z_m^* \wedge (Z_1^* \wedge Z_{1+m}^*) \wedge \cdots \wedge (Z_{m-1}^* \wedge Z_{m-1+m}^*),$$

which proves both orientability of SM and the asserted equality.
3. We only need to show that $\iota^*\lambda_g$ is preserved by the geodesic flow, for then

$$\Phi^{t*}\iota^*\omega_g = \Phi^{t*}\iota^* - d\lambda_g = -d\Phi^{t*}\iota^*\lambda_g = -d\iota^*\lambda_g = \iota^*\omega_g.$$

Let $V : (-\epsilon, \epsilon) \to SM, V(0) = v \in S_pM$ and $Z = \dot{V}(0) \in T_vSM$. If we write J for the Jacobi field as in proposition 2.9, we have

$$(\Phi^{t*}\iota^*\lambda_g)(Z) = (\lambda_g)_{\Phi^t(v)}(\Phi^t_* Z) \stackrel{2.17}{=} g(\Phi^t(v), \pi_*\Phi^t_* Z) \stackrel{2.9}{=} g(\Phi^t(v), J(t)).$$

Now we show that $(\Phi^{t*}\iota^*\lambda_g)(Z)$ is independent of t.

$$\begin{aligned} \frac{d}{dt}(\Phi^{t*}\iota^*\lambda_g)(Z) &= \frac{d}{dt} g(\Phi^t(v), J(t)) = g(\Phi^t(v), {}^{\gamma_v}\nabla_{\partial_t} J(t))) \\ &\stackrel{2.9}{=} g(\Phi^t(v), C\Phi^t_{*,v} Z) = 0, \end{aligned}$$

where the last equality follows from 1. since $\Phi^t_{*,v} Z \in T_{\Phi^t(v)}SM$.
4. This follows directly from 2. and 3. $\square$

Corollary 2.48. *Let (M, g) be a complete Riemannian manifold. For all $f \in C^\infty(SM)$ with compact support and $t \in \mathbb{R}$ we have*

$$\int_{SM} f \circ \Phi^t \mu_{SM} = \int_{SM} f \mu_{SM}.$$

Proof. By the preceding Lemma the Jacobian $|\Phi^t|_*$ of $\Phi^t : SM \to SM$ is identically equal to 1. The geodesic flow $\Phi^t : SM \to SM$ is a diffeomorphism for all $t \in \mathbb{R}$ since $\Phi^{-t} \circ \Phi^t = id_{SM}$ for all $t \in \mathbb{R}$. Now apply the transformation rule 2.37. $\square$

In the next chapters we will see that it is useful to consider unit speed geodesics on M as integral curves of the geodesic flow on SM, because these fill up the whole unit tangent bundle and do not intersect.

Chapter 3

Wiedersehen Manifolds

In this chapter we mainly follow [3] and always assume (M, g) to be a complete and connected Riemannian manifold.

Definition 3.1. *Let $v \in SM$. We define $con(v) \in (0, \infty]$ to be the first positive time t such that $\gamma_v(0)$ is conjugate to $\gamma_v(t)$ along γ_v. If no such time exists, we set $con(v) = \infty$. For $p \in M$ we define the **first conjugate locus of p** by $Con(p) := \{\gamma_v(con(v)) \mid v \in S_pM \text{ and } con(v) < \infty\}$.*

Notation 3.2. Let γ be a geodesic in M and V a vector field along γ. We sometimes write $\dot{V}$ for ${}^\gamma\nabla_{\partial_t}V$.

Proposition 3.3. *For all $v \in SM$ with $con(v)$ finite we have*

$$con(-\dot{\gamma}_v(con(v)) = con(v).$$

Proof. Let $v \in SM$ such that $con(v)$ is finite and set $w := -\dot{\gamma}_v(con(v))$.
First, we show $con(w) \leq con(v)$. Since $\gamma_v(con(v))$ is conjugate to $\gamma_v(0)$ along γ_v, there is a nontrivial Jacobi field J along γ_v with $J(0) = 0$ and $J(con(v)) = 0$. We now define a vector field $J^-(t) := J(con(v) - t)$ along $\gamma_w(t) = \gamma_v(con(v) - t)$ and compute

$$\begin{aligned} &\ddot{J^-}(t) + R(J^-(t), \dot{\gamma}_w(t))\gamma_w(t) \\ =&\ddot{J}(con(v) - t) + R(J(con(v) - t), \dot{\gamma}_v(con(v) - t))\dot{\gamma}_v(con(v) - t) = 0. \end{aligned}$$

Here R denotes the Riemannian curvature endomorphism of M. Thus J^- is a nontrivial Jacobi field along γ_w vanishing at $\gamma_w(0)$ and $\gamma_w(con(v))$.
Now we show that $con(v) \leq con(w)$. Assume not, then $con(v) > con(w)$. But then $\gamma_{w|[0,con(v)]} : [0, con(v)] \to M$ has an interior conjugate point at time $con(w)$. By [2, Thm 10.26] there is a proper normal vector field X along $\gamma_{w|[0,con(v)]}$ such that the index form $I(X, X)$ is strictly negative. Then the vector field $X^-(t) = X(con(v) - t)$ along γ_v is proper, normal and satisfies $I(X^-, X^-) = I(X, X) < 0$. But on the other hand $\gamma_v : [0, con(v)] \to M$ has no interior conjugate points and therefore [2, Thm 10.28] implies $0 \leq I(X^-, X^-)$, which is a contradiction. $\square$

Lemma 3.4. *$con : SM \to (0, \infty]$ is continuous.*

Proof. Let $v \in SM$ such that $con(v)$ is finite and $\epsilon > 0$. In this proof we sometimes write (p, v) instead of v, where $p = \pi(v)$. Without loss of generality we assume $\epsilon < con(v)$. We consider the smooth map

$$F : TM \to M \times M, F(v) := (\pi(v), exp_{\pi(v)}(v))$$

and compute its differential in coordinates

$$d_{(p,v)}F = \begin{pmatrix} 1 & 0 \\ * & d_v exp_p \end{pmatrix}.$$

We therefore have $det(d_{(p,v)}F) = det(d_v exp_p)$.
For each $t \leq con(v) - \epsilon$ we have $det(d_{(p,tv)}F) = det(d_{tv} exp_p) \neq 0$. Therefore we can find, for each such t, an open neighborhood $U_t \subset TM$ of (p, tv) such that for each $(p', v') \in U_t$

$$det(d_{(p',v')}F) = det(d_{v'} exp_{p'}) \neq 0.$$

The set $U := \cup_{0 \leq t \leq con(v) - \epsilon} U_t$ is thus an open neighborhood of

$$\{(p, tv) | t \in [0, con(v) - \epsilon]\}.$$

We can thus find an open neighborhood V of (p, v) in SM such that $(p', tv') \in U$ for all $(p', v') \in V, t \in [0, con(v) - \epsilon]$. Hence con is greater or equal than $con(v) - \epsilon$ on V. (If $con(v) = \infty$, the same argument shows that con is continuous at v.)
To finish the proof we have to show that there is a neighborhood of (p, v) in SM on which con is smaller than $con(v) + \epsilon$. By [2, Theorem 10.26] there is a proper normal vector field X along $\gamma_{v|[0,con(v)+\epsilon]}$ such that the index form is negative, i.e. $I(X, X) < 0$. The idea is then to transport this vector field X to a proper normal vector field X' along the geodesic $\gamma_{v'|[0,con(v)+\epsilon]}$ with vanishing index form, where (p', v') is close to (p, v). Then due to [2, Theorem 10.28], $\gamma_{v'|[0,con(v)+\epsilon]}$ must have an interior conjugate point and we are done. Let's do this in detail.
We choose $\delta < 2inj_{(p,v)}SM$, let $S \subset T_{(p,v)}SM$ denote the sphere of radius 1 and define the smooth map

$$\Gamma : [0, con(v) + \epsilon] \times [0, \delta] \times S \to M, \Gamma(t, s, w) := exp^M_{\pi \circ exp^{SM}_{(p,v)}(sw)}(t exp^{SM}_{(p,v)}(sw)),$$

where $\pi : SM \to M$ is the projection of the unit tangent bundle. For fixed w and t we can parallel transport $X(t)$ along $\Gamma(t, \cdot, w)$. By the differentiable dependence theorem of parameters for initial value problems one infers that

$$X : [0, con(v) + \epsilon] \times [0, \delta] \times S \to TM, X(t, s, w) := P^{\Gamma(t,\cdot,w)}_{0,s} X(t)$$

is smooth. Hence $X(\cdot, s, w)$ is a smooth, proper vector field along $\gamma_{exp^{SM}_{(p,v)}(sw)}$. Now the index form $I(X(\cdot, s, w), X(\cdot, s, w))$ depends continuously on s and w. We can therefore decrease δ such that

$$I(X(\cdot, s, w), X(\cdot, s, w)) < 0$$

for all $s \in [0,\delta], w \in S$. The vector fields $X(\cdot, s, w)$ are not necessarily normal to the corresponding geodesic, but we can just take the normal part $X(\cdot, s, w)^{\perp}$ and obtain

$$I(X(\cdot, s, w)^{\perp}, X(\cdot, s, w)^{\perp}) \leq I(X(\cdot, s, w), X(\cdot, s, w)) < 0$$

for all $s \in [0,\delta], w \in S$. Let $B_{\delta}^{SM}((p,v)) \subset SM$ denote the geodesic ball of radius δ around (p,v). We have shown that for all $(p', v') \in B_{\delta}^{SM}((p,v))$ there is a smooth, proper, normal vector field along $\gamma_{v'|[0,con(v)+\epsilon]}$ with negative index form, hence $con(v') < con(v) + \epsilon$. □

Definition 3.5. *(M, g) is called* ***wiedersehen manifold*** *if $Con(p)$ consists of exactly one point for all $p \in M$. A wiedersehen manifold (M, g) is called* ***wiedersehen surface*** *if in addition $dim M = 2$.*

In the following we will first develop some geometric properties of wiedersehen manifolds and then prove the Blaschke conjecture for wiedersehen surfaces. Firstly we show that *con* is constant and finite on SM. This is the content of the following two lemmas.

Lemma 3.6. *Let (M, g) be a wiedersehen manifold. For each $p \in M$ the restriction of con to S_pM is constant and finite. Therefore the function*

$$a : M \to (0, \infty), \qquad a(p) = con(v), \qquad \text{where } v \in S_pM$$

is well defined.

Proof. Let $p \in M$. The wiedersehen property says that p has a conjugate point, hence there exists a $v \in S_pM$ such that $con(v)$ is finite. By lemma 3.4 we can find an open neighborhood B of v in S_pM and $\epsilon > 0$ such that $con(w)$ is finite and bigger than ϵ for all $w \in B$.
Now we show that *con* is smooth in a neighborhood of v in S_pM. For that we set $q = Con(p)$ and take the above ϵ smaller than inj_qM. Let S_ϵ be the geodesic sphere of radius ϵ around q. Due to the choice of ϵ for all $w \in B$ we have $\gamma_w(con(w) - \epsilon) \in S_\epsilon$. We now apply the Gauss lemma to show that this intersection is perpendicular. We set $\tilde{v} = exp_q^{-1}(\gamma_v(con(v) - \epsilon))$. For each $u \in T_{\tilde{v}}S_{\epsilon}^{T_qM}$ the Gauss lemma implies

$$0 = g_{\tilde{v}}^q(u, \tilde{v}) = g_{exp_q(\tilde{v})}(d_{\tilde{v}}exp_q u, d_{\tilde{v}}exp_q\tilde{v}). \tag{3.1}$$

For the second argument we compute

$$exp_q(t\tilde{v}) = \gamma_{\tilde{v}}(t) = \gamma_{\frac{\tilde{v}}{|\tilde{v}|}}(t|\tilde{v}|) = \gamma_v(con(v) - t|\tilde{v}|).$$

And thus

$$d_{\tilde{v}}exp_q\tilde{v} = \frac{d}{dt}_{|t=1} exp_q(t\tilde{v}) = -\dot{\gamma}_v(con(v) - |\tilde{v}|) = -\dot{\gamma}_v(con(v) - \epsilon).$$

Since the left entry in (3.1) covers $T_{\gamma_v(con(v)-\epsilon)}S_\epsilon$ as u varies, the nondegeneracy of g implies

$$\dot{\gamma}_v(con(v)-\epsilon) \perp T_{\gamma_v(con(v)-\epsilon)}S_\epsilon.$$

Since S_ϵ is a codimension 1 submanifold of M there is some open neighborhood U of $\gamma_v(con(v)-\epsilon)$ and smooth function smooth $f : U \to \mathbb{R}$ such that

$$U \cap S_\epsilon = \{p' \in U | f(p') = 0\}.$$

By the continuity of the geodesic flow and shrinking of B we can find a $\delta > 0$ such that

$$\pi \circ \Phi(t, w) \in U \text{ for all } (t, w) \in (con(v) - \epsilon - \delta, con(v) - \epsilon + \delta) \times B =: V.$$

We apply the implicit function theorem to the function

$$F : V \to \mathbb{R}, F(t, w) := f(\pi \circ \Phi(t, w)).$$

Since γ_v intersects S_ϵ transversally at time $con(v) - \epsilon$, we have

$$\partial_t F_{|(con(v)-\epsilon,v)} = f_{*\gamma_v(con(v)-\epsilon)}\dot{\gamma}_v(con(v)-\epsilon) = g(gradf, \dot{\gamma}_v(con(v)-\epsilon)) \neq 0,$$
$$F(con(v)-\epsilon, v) = 0.$$

The implicit function theorem implies that there is some open neighborhood A of v in B and a unique smooth function $g : A \to \mathbb{R}$ such that $F(g(w), w) = 0$ for each $w \in A$. For all $w \in A$ we have $F(con(w) - \epsilon, w)$, hence $g(w) = con(w) - \epsilon$. Thus con is smooth on A.

Now we show that con is constant on A. For that we consider smooth curves $v : (-\epsilon, \epsilon) \to A$ and the associated variations $\Gamma : (-\epsilon, \epsilon) \times \mathbb{R} \to M, \Gamma(s, t) := exp_p(tv(s))$. We obtain a family of Jacobi fields $J(s, t) = \partial_s \Gamma(s, t)$ with

$$J(s,0) = \Gamma(s,0)(t\frac{dv}{ds})_{|t=0} = 0 \textit{ and}$$
$$({}^\Gamma\nabla_{\partial_t} J)(s,0) \perp \partial_t\Gamma(s,0), \text{ since}$$
$$g(({}^\Gamma\nabla_{\partial_t} J)(s,0), \partial_t\Gamma(s,0)) = \frac{1}{2}\frac{d}{ds}g(\partial_t\Gamma(s,0), \partial_t\Gamma(s,0)) = \frac{1}{2}\frac{d}{ds}1 = 0.$$

This implies that every Jacobi field $J(s, \cdot)$ is normal, i.e. $J(s, t) \perp \partial_t\Gamma(s, t)$ for all (s, t). Due to the definition of con and the wiedersehen property of M we get

$$q = \Gamma(s, con(v(s))$$

and thus

$$0 = \frac{d}{ds}\Gamma(s, con(v(s)) = J(s, con(v(s))) + \partial_t\Gamma(s, con(v(s)))\frac{d}{ds}con(v(s)).$$

Due to the perpendicularity of the two summands this equation yields

$$\frac{d}{ds}con(v(s)) = 0.$$

Therefore, con is locally constant at points $v \in S_pM$ where $con(v)$ is finite. Then con must be finite on S_pM. Otherwise there would be a smooth curve $\gamma : [0,1) \to S_pM$ with $con \circ \gamma_{|[0,1)} < \infty$ and $con(\gamma(1)) = \infty$. Hence $con \circ \gamma_{|[0,1)}$ would be constant and finite by the preceding discussion, contradicting the continuity of con. Consequently, con is locally constant on S_pM. Since S_pM is connected and con is continuous, con is constant on S_pM. $\square$

Lemma 3.7. *Let (M, g) be a wiedersehen manifold. The function con is constant and finite on SM. Let a be the value of con. All geodesics on M are periodic with common period $2a$.*

Proof. Let $p \in M$ and γ be a unit speed geodesic with $\gamma(0) = p$. Since con is continuous, the function a from the above lemma is continuous. Therefore there is some $\delta < \frac{inj_pM}{2}$ such that for each $p' \in M$ with $d(p, p') < \delta$

$$|2a(p') - 2a(p)| < \frac{inj_pM}{2}.$$

We now show that γ is periodic and $a(\gamma(\epsilon)) = a(p)$ for all $|\epsilon| < \delta$. Let ϵ be fixed and set $p' = \gamma(\epsilon)$.
Case 1.1: $\epsilon > 0$ and $2a(p) < 2a(p') + \epsilon$. Then we have $d(p, p') = \epsilon < \delta$. Using the triangle inequality we obtain

$$|\epsilon + 2a(p') - 2a(p)| < inj_pM.$$

We know that

$$Con(p') = \gamma(\epsilon + a(p'))$$

and from proposition 3.3 we infer

$$p' = Con(Con(p')) = \gamma(\epsilon + 2a(p')) \text{ and } \gamma(2a(p)) = p.$$

We therefore conclude that $\gamma_{|[2a(p),\epsilon+2a(p')]}$ is the unique minimizing geodesic from p to p'. But this is also true for $\gamma_{|[0,\epsilon]}$. Thus, we have $\dot\gamma(2a(p)) = \dot\gamma(0)$ and γ closes smoothly at p. Therefore, γ is periodic with period $2a(p)$.
Now we show that a is constant along γ. We know that

$$\epsilon + 2a(p') - 2a(p) = L_g(\gamma_{|[2a(p),\epsilon+2a(p')]}) = L_g(\gamma_{[0,\epsilon]}) = \epsilon,$$

which implies $a(p') = a(p)$.
Case 1.2: $\epsilon > 0$, $2a(p) \geq 2a(p') + \epsilon$. Then $2a(p') < 2a(p) + \epsilon$. Using the same argument as in case 1.1 for the geodesic $\gamma(\epsilon - \cdot)$ and changed roles of p and p' we see that $a(p) = a(p')$ and $\gamma(\epsilon - \cdot)$ closes smootly at p' and is hence periodic. Thus, γ is periodic as well.
Case 2: $\epsilon < 0$. As in cases 1.1 and 1.2 with $\gamma(-\cdot)$ instead of γ.

Thus a is locally constant along γ. By continuity a is constant along γ. To prove the assertion that a is constant on the whole manifold M we note that completeness implies that every two points on M can be connected by a unit-speed geodesic. $\square$

Lemma 3.8. *Let (M, g) be a wiedersehen manifold. The map $Con : M \to M$ is an isometry.*

Proof. We use the Myers-Steenrod theorem, see [5, 3.31]. Con is a bijection since it is involutive, that is $Con \circ Con = id_M$.
Now let $p, q \in M$ be arbitrary and $\gamma : [0, d(p,q)] \to M$ be a minimizing unit-speed geodesic with $\gamma(0) = p$ and $\gamma(d(p,q)) = q$. By the preceding lemma we have $Con(\gamma(t)) = \gamma(a+t)$. Setting $t = 0$ and $t = d(p,q)$ we get $Con(p) = \gamma(a)$ and $Con(q) = \gamma(a + d(p,q))$, respectively. We conclude that

$$d(Con(p), Con(q)) \leq L(\gamma_{[a,a+d(p,q)]}) = d(p,q)$$

for arbitrary $p, q \in M$. Since $Con^2 = id_M$ we get

$$d(p,q) = d(Con^2(p), Con^2(q)) \leq d(Con(p), Con(q)).$$

These two inequalities imply

$$d(p,q) = d(Con(p), Con(q)).$$

□

Lemma 3.9. *Let (M, g) be a wiedersehen manifold. For $p \in M$ we consider the Euclidean sphere $S_p^m(a/\pi)$ as the quotient of the closed Ball $\bar{B}(0_p, a)$ of radius a in T_pM, where the boundary $S(0_p, a)$ is identified to the south pole S_p. Let $q_p : \bar{B}(0_p, a) \longrightarrow S_p^m(a/\pi)$ denote the quotient map. Then there is a smooth covering map $F_p : S_p^m(a/\pi) \longrightarrow M$ such that $F_p \circ q = exp_p$.*

Proof. The map F_p is well defined, since for $v \in \partial B(0_p, a)$ we have $con(\frac{v}{a}) = a$ and thus

$$exp_p(v) = exp_p(a\frac{v}{a}) = Con(p).$$

Due to [1, 4.46] we only have to show that that F_p is a local diffeomorphism. On $S_p^m(a/\pi) \setminus \{S_p\}$ the map F_p is given by exp_p and hence smooth. A vector $v \in B(0_p, a)$ can't be a critical point of exp_p since conjugate points to p occur not before time a. Thus F_p is a local diffeomorphism on $S_p^m(a/\pi) \setminus \{S_p\}$. It remains to check that F_p is smooth and a local diffeomorphism at the collapsed point S_p. Since Con is an isometry we have for all $v \in S_pM$ and $t \geq 0$

$$exp_p(tv) = Con(exp_p((t-a)v)) = exp_{Con(p)}((t-a)Con_*v).$$

Let $\alpha : S_p^m(a/\pi) \to S_p^m(a/\pi)$ denote the antipodal map. For $u \in \bar{B}(0_p, a) \setminus \{0\}$ we set $v = \frac{u}{|u|}$ and $t = |u|$ and the above equality implies

$$\begin{aligned} F_p(q_p(u)) &= F_{Con(p)} q_{Con(p)} Con_*(|u| - a)\frac{u}{|u|} \\ &= F_{Con(p)} \circ q_{Con(p)} \circ Con_* \circ (q_{p|S_p^m(a/\pi) - S_p})^{-1} \circ \alpha(q_p(u)). \end{aligned}$$

Thus, F_p is on $S_p^m(a/\pi) \setminus N_p$ (N_p is the north pole) given by

$$S_p^m(a/\pi) \setminus \{N_p\} \xrightarrow{\alpha} S_p^m(a/\pi) \setminus \{S_p\} \xrightarrow{q_p^{-1}} B(0_p, a) \xrightarrow{Con_*} B(0_{Con(p)}, a)$$
$$\xrightarrow{q_{Con(p)}} S_{Con(p)}(a/\pi) \setminus \{S_{Con(p)}\} \xrightarrow{F_{Con(p)}} M.$$

All involved maps are local diffeomorphisms. This concludes the proof.

□

Theorem 3.10. *Let (M, g) be a simply connected wiedersehen manifold. Then*

1. *M is diffeomorphic to S^m.*
2. *$injM = diamM = a$.*
3. *For all $p \in M$ and $v \in S_pM$ we have $\gamma_v(a) = Con(p)$.*
4. *All unit speed geodesics in M are periodic with (least) period $2a$.*
5. *Con is an involutive isometry with d(p,Con(p))=a.*

Proof. By the universal property of the universal cover the map $F : S^m(a/\pi) \to M$ of lemma 3.9 is a diffeomorphism. Therefore exp_p restricted to $B(0_p, a)$ is injective and $exp_p = Con(p)$ on $S(0_p, a)$ for every $p \in M$. From this everything follows. □

We now want to give an equivalent condition for the wiedersehen property. To do that it is convenient to introduce Allamigeon-Warner manifolds.

Definition 3.11. *Let $k \in \{1, ..., m-1\}, l > 0$. A complete and connected Riemannian manifold (M, g) is called* ***Allamigeon-Warner*** *manifold of type (k, l) if $con(v) = l$ for all $v \in SM$ and if the multiplicity of $\gamma_v(l)$ as a conjugate point of $\gamma_v(0)$ along γ_v is equal to k.*

Proposition 3.12. *An m-dimensional Riemannian manifold (M, g) is Allamigeon Warner of type $(m-1, l)$ if and only if (M, g) is wiedersehen.*

Proof. Let (M, g) be wiedersehen and take $v \in S_pM$. Set $l = con(v)$. We have to show that the multiplicity of $\gamma_v(l)$ along γ_v is equal to $m-1$. Since the multiplicity of a conjugate point never exceeds $m-1$, it suffices to show that there are $m-1$ linearly independent Jacobi fields along γ_v vanishing at 0 and l. Let $v : (-\epsilon, \epsilon) \to S_pM$ be a smooth curve with $v(0) = v$. Consider the variation $\Gamma : (-\epsilon, \epsilon) \times \mathbb{R} \to M, \Gamma(s, t) := \exp_p(tv(s))$. Then the associated Jacobi field $J(t) = \partial_s\Gamma(0, t)$ satisfies $J(0) = 0$. Since $\Gamma(s, l) = 0$ we have $J(l) = 0$. Moreover

$$\dot{J}(0) = {}^{\gamma_v}\nabla_{\partial_t}\partial_s\Gamma|_{(s,t)=(0,0)} = {}^{\Gamma}\nabla_{\partial_t}\partial_s\Gamma|_{(s,t)=(0,0)} = {}^{\Gamma}\nabla_{\partial_s}\partial_t\Gamma|_{(s,t)=(0,0)}$$
$$= {}^{\pi\circ v}\nabla_{\partial_s}v(0) = \frac{d}{ds}_{|s=0}v(s).$$

In the last two equalities we used $\partial_t \Gamma(s,0) = v(s)$ and $\pi \circ v \equiv p$, respectively. Since the dimension of S_pM is $m-1$, we are done.
Conversely let (M,g) be Allamigeon Warner of type $(m-1,l)$. For each $p \in M$ we have to show that $Con(p)$ consists of a single point, i.e. $exp_p(lv) = exp_p(lw)$ for all $v,w \in S_pM$. Take a smooth curve $v : [0,1] \to S_pM$ with $v(0) = v, v(1) = w$. The Jacobi fields $J(s,t) = \partial_s \Gamma(s,t)$ associated to the variation $\Gamma : [0,1] \times \mathbb{R} \to M, \Gamma(s,t) := \exp_p(tv(s))$ satisfy $J(s,0) = 0$ and $\dot{J}(s,0) \perp \partial_t \Gamma(s,0)$. Since M is Allamigeon Warner of type $(m-1,l)$, we must have $0 = J(s,l) = \partial_s \Gamma(s,l)$. Hence $exp_p(lv) = \Gamma(0,l) = \Gamma(1,l) = exp_p(lw)$. $\square$

Corollary 3.13. *A Riemannian surface (M,g) is wiedersehen if and only if con is finite and constant.*

Lemma 3.14. *Let $\pi : (\tilde{M},\tilde{g}) \to (M,g)$ be a Riemannian covering. Then*

$$\begin{aligned} &(\tilde{M},\tilde{g}) \text{ is Allamigeon Warner of type } (k,l) \\ \Leftrightarrow\ &(M,g) \text{ is Allamigeon Warner of type } (k,l). \end{aligned}$$

Proof. Since π is a Riemannian covering, g is complete if and only if $\tilde{g}$ is complete. For each unit speed geodesic $\tilde{\gamma}$ in $\tilde{M}$, set $\gamma = \pi \circ \tilde{\gamma}$. Let J be a Jacobi field along γ. Denote by $\tilde{J}$ the unique vector field along $\tilde{\gamma}$ with $\pi_* \tilde{J} = J$. [5, 6.17] implies that for every vector field V along $\tilde{\gamma}$

$$\pi_* {}^{\tilde{\gamma}}\nabla^{\tilde{M}}_{\partial_t} \tilde{V} = {}^{\gamma}\nabla^{M}_{\partial_t} \pi_* \tilde{V}.$$

Applying this twice we obtain

$$\pi_* {}^{\tilde{\gamma}}\nabla^{\tilde{M}}_{\partial_t} {}^{\tilde{\gamma}}\nabla^{\tilde{M}}_{\partial_t} \tilde{J} = {}^{\gamma}\nabla^{M}_{\partial_t} \pi_* {}^{\tilde{\gamma}}\nabla^{\tilde{M}}_{\partial_t} \tilde{J} = {}^{\gamma}\nabla^{M}_{\partial_t} {}^{\gamma}\nabla^{M}_{\partial_t} J.$$

Consequently

$$\begin{aligned} \tilde{g}({}^{\tilde{\gamma}}\nabla^{\tilde{M}}_{\partial_t} {}^{\tilde{\gamma}}\nabla^{\tilde{M}}_{\partial_t} \tilde{J} + \tilde{R}(\tilde{J},\dot{\tilde{\gamma}})\dot{\tilde{\gamma}}, \cdot) &= g(\pi_* {}^{\tilde{\gamma}}\nabla^{\tilde{M}}_{\partial_t} {}^{\tilde{\gamma}}\nabla^{\tilde{M}}_{\partial_t} \tilde{J} + \pi_* \tilde{R}(\tilde{J},\dot{\tilde{\gamma}})\dot{\tilde{\gamma}}, \pi_* \cdot) \\ &= g({}^{\gamma}\nabla^{M}_{\partial_t} {}^{\gamma}\nabla^{M}_{\partial_t} J + R(J,\dot{\gamma})\dot{\gamma}, \pi_* \cdot). \end{aligned}$$

Thus J is a nontrivial Jacobi field vanishing at $t = 0, l$ if and only if $\tilde{J}$ is a nontrivial Jacobi field vanishing at $t = 0, l$. Hence the assertion holds. $\square$

Corollary 3.15. *Let $\pi : (\tilde{M},\tilde{g}) \to (M,g)$ be a Riemannian covering. Then*

$$(\tilde{M},\tilde{g}) \text{ is wiedersehen } \Leftrightarrow (M,g) \text{ is wiedersehen.}$$

Chapter 4

Blaschke Conjecture

In this chapter we follow [3] and present Green's proof of the Blaschke conjecture. The key for the proof is the following theorem, which uses a volume inequality to characterize whether a Riemannian manifold has constant positive sectional curvature.

Theorem 4.1. *Let (M, g) be a closed Riemannian manifold and $a > 0$ such that every $v \in SM$ satisfies $con(v) \geq a$. Then*

$$\frac{a^2}{\pi^2} \int_M scal \mu_g \leq m(m-1) vol_g(M).$$

And equality holds if and only if M has constant sectional curvature $\frac{\pi^2}{a^2}$. Here $scal : M \to \mathbb{R}$ denotes the scalar curvature.

Proof. First let $\gamma : [0, a] \to M$ be a unit speed geodesic and $E \in {}^{\gamma}TM$ a parallel normal unit vector field along γ . Then the vector field $V(t) = sin(\frac{\pi t}{a})E(t)$ is a proper normal vector field along γ. Therefore [2, Thm 10.28] implies

$$0 \leq \int_0^a g({}^{\gamma}\nabla_{\partial_t} V, {}^{\gamma}\nabla_{\partial_t} V) - R(V, \dot{\gamma}, \dot{\gamma}, V) dt, \tag{4.1}$$

with equality if and only if V is a Jacobi field. With

$${}^{\gamma}\nabla_{\partial_t} V = {}^{\gamma}\nabla_{\partial_t} sin(\frac{\pi t}{a})E(t) = \frac{\pi}{a} cos(\frac{\pi t}{a})E(t)$$

we can calculate the first term of the above inequality

$$\int_0^a g({}^{\gamma}\nabla_{\partial_t} V, {}^{\gamma}\nabla_{\partial_t} V) dt = \int_0^a \frac{\pi^2}{a^2} cos^2(\frac{\pi t}{a}) dt = \frac{\pi^2}{a^2} \frac{a}{2} = \frac{\pi^2}{2a}.$$

Thus, we have

$$\int_0^a sin^2(\frac{\pi t}{a}) R(E, \dot{\gamma}, \dot{\gamma}, E) dt \leq \frac{\pi^2}{2a}.$$

V is a Jacobi field if and only if it solves the Jacobi equation, which is in this case given by

$$-\frac{\pi^2}{a^2}sin(\frac{\pi t}{a})E(t)+sin(\frac{\pi t}{a})R(E,\dot\gamma)\dot\gamma=0.$$

This holds if and only if $K(E,\dot\gamma)=\frac{\pi^2}{a^2}$ and $R(E,\dot\gamma)\dot\gamma, E$ are colinear. Now we choose a parallel orthonormal frame $(E_1,...,E_m)$ along γ such that $E_1=\dot\gamma$. Then

$$\begin{aligned}\int_0^a sin^2(\frac{\pi t}{a})Ric(\dot\gamma)dt &= \sum_{i=1}^m\int_0^a sin^2(\frac{\pi t}{a})R(E_i,\dot\gamma,\dot\gamma,E_i)dt\\ &= \sum_{i=2}^m\int_0^a sin^2(\frac{\pi t}{a})R(E_i,\dot\gamma,\dot\gamma,E_i)dt \le (m-1)\frac{\pi^2}{2a},\end{aligned}\tag{4.2}$$

with equality if and only if $K(E_i,\dot\gamma)=\frac{\pi^2}{a^2}$ and $R(E_i,\dot\gamma)\dot\gamma, E_i$ are colinear for $i=2,...,m$. Since the equation is independent of the chosen frame, we get that equality holds iff $K(E,\dot\gamma)=\frac{\pi^2}{a^2}$ and $R(E,\dot\gamma)\dot\gamma, E$ are colinear for every normal parallel unit vector field E along γ. Since M is compact and therefore SM is compact, we can integrate (4.2) over the unit tangent bundle SM. For the right side we compute

$$\begin{aligned}\int_{SM}\mu_{SM} &\overset{2.34}{=} \int_M\int_{S_pM}\mu_{S_pM}\mu_M\\ &= \int_M vol(S^{m-1})\mu_M = vol(M)vol(S^{m-1}).\end{aligned}$$

For the left side we compute

$$\begin{aligned}\int_{SM}\int_0^a sin^2(\frac{\pi t}{a})Ric(\dot\gamma_v)dt\mu_{SM} &= \int_{SM}\int_0^a sin^2(\frac{\pi t}{a})Ric(\Phi^t(v))dt\mu_{SM}\\ &\overset{2.34}{=} \int_0^a sin^2(\frac{\pi t}{a})\int_{SM}Ric(\Phi^t(v))\mu_{SM}dt\\ &\overset{2.48}{=} \int_0^a sin^2(\frac{\pi t}{a})\int_{SM}Ric(v)\mu_{SM}dt\\ &= \frac{a}{2}\int_{SM}Ric(v)\mu_{SM}\\ &\overset{2.34}{=} \frac{a}{2}\int_M\int_{S_pM}Ric(v)\mu_{S_pM}\mu_M\\ &\overset{2.35}{=} \frac{a}{2}\int_M\frac{1}{m}vol(S^{m-1})scal\mu_M\\ &= \frac{a}{2}\frac{1}{m}vol(S^{m-1})\int_M scal\mu_M.\end{aligned}$$

Together we obtain the asserted inequality. Now equality holds iff for every unit speed geodesic γ and normal parallel unit vector field E along γ the sectional curvature satisfies $K(E,\dot\gamma) = \frac{\pi^2}{a^2}$ and $R(E,\dot\gamma)\dot\gamma, E$ are colinear. But this is equivalent to M having constant sectional curvature $\frac{\pi^2}{a^2}$ by [5, Satz 6.10]. $\square$

If for a closed Riemannian manifold no conjugate points exist, let a go to $+\infty$ to obtain the following corollary.

Corollary 4.2. *Let (M,g) be a closed Riemannian manifold without conjugate points. Then*

$$\int_M scal\mu_M \leq 0.$$

Since we did this limit process, we don't know what equality implies for the sectional curvature. One would expect that in the case of equality the sectional curvature vanishes identically. This is in fact true and is proved in [10]. Later we will show a theorem due to Eberhard Hopf, which states this result for Riemannian surfaces.
Let us now specialize the above theorem to the case of surfaces.

Corollary 4.3. *Let (M,g) be a closed Riemannian surface and $a > 0$ such that for all $v \in SM, con(v) \geq a$. Then*

$$vol(M) \geq \frac{2a^2}{\pi}\chi(M).$$

And equality holds iff the sectional curvature is constant $K = \frac{\pi^2}{a^2}$. Here $\chi(M)$ denotes the Euler characteristic of M.

Proof. By the Gauss-Bonnet theorem

$$\int_M K\mu_M = 2\pi\chi(M).$$

On a surface we have $scal = 2K$, thus by the preceding theorem

$$\begin{aligned} vol(M) &\geq \frac{a^2}{2\pi^2}\int_M scal\mu_M \\ &= \frac{a^2}{\pi^2}\int_M K\mu_M = \frac{2a^2}{\pi}\chi(M). \end{aligned}$$

$\square$

We already know that a simply connected wiedersehen surface M is diffeomorphic to S^2. Hence to prove the Blaschke conjecture in the simply connected case, the preceding corollary shows that it is sufficient to show that the volume of M is equal to $\frac{4a^2}{\pi}$. We will compute the volume of SM, because there the integral curves of the geodesic flow $\Phi^t : SM \to SM$ fill up SM and do not intersect each other. This will allow us to parametrize SM by unit speed geodesics starting at a hypersurface of M, namely one of the closed geodesics on M. The technical tool for this is Santalo's formula, which we shall prove in the following.

Let Σ be a non-degenerate hypersurface in the Riemannian manifold (N, g_N) and X a vector field on N with complete flow $\Phi_X : \mathbb{R} \times N \to N$. We define a smooth map

$$F : \mathbb{R} \times \Sigma \to N, F(t, x) = \Phi_X^t(x).$$

We endow Σ with the induced Riemannian metric, $\mathbb{R}$ with the Euclidean metric and $\mathbb{R} \times \Sigma$ with the product metric.

Lemma 4.4. *If $(\Phi_X^t)^* \mu_N = \mu_N$ for every $t \in \mathbb{R}$, then the Jacobian $|F|_*$ is independent of t and*

$$|F|_*(t, x) = |g_N(X^\perp(x), X^\perp(x)))|^{\frac{1}{2}},$$

where $X^\perp$ is the component of X perpendicular to Σ.

Proof. We define the map $\psi_s : \mathbb{R} \times \Sigma \to \mathbb{R} \times \Sigma, \psi(t, x) := (t + s, x)$. Then

$$\psi_s^* \mu_{\mathbb{R}\times\Sigma} = \mu_{\mathbb{R}\times\Sigma}$$

and $F \circ \psi_s = \Phi_X^s \circ F$ by the flow property. We can now compute

$$\begin{aligned} |F|_* \mu_{\mathbb{R}\times\Sigma} = F^* \mu_N &= F^*(\Phi_X^s)^* \mu_N \\ &= (\Phi_X^s \circ F)^* \mu_N \\ &= \psi_s^* F^* \mu_N \\ &= \psi_s^*(|F|_* \mu_{\mathbb{R}\times\Sigma}) \\ &= |F|_* \circ \psi_s \psi_s^* \mu_{\mathbb{R}\times\Sigma} \\ &= |F|_* \circ \psi_s \mu_{\mathbb{R}\times\Sigma}. \end{aligned}$$

But from this we obtain $|F|_*(t, x) = |F|_*(t + s, x)$ for all $t, s \in \mathbb{R}, x \in \Sigma$. This proves the independence of t. Let (U, ϕ) be a Σ-adapted chart on N with coordinate functions $\phi = (x^1, ..., x^n)$ such that the inclusion $U \cap \Sigma \to U$ is in coordinates given by $(x^2, ..., x^n) \mapsto (0, x^2, ..., x^n)$. For $\mathbb{R} \times \Sigma$ we have the chart $(y^1, ..., y^n) = id_{\mathbb{R}} \times \phi_{|\Sigma}$. For all $x \in U \cap \Sigma$

- $d_{(0,x)} F \partial_{y^1} = (\partial_t)_{|t=0} \Phi_X^t(x) = X(x)$
- $d_{(0,x)} F \partial_{y^i} = (\partial_{y^i})_{|(0,x)} \Phi_X^t(x) = \partial_{x^i}$ for $i \geq 2$, thus $d_{(0,x)} F_{|T_x\Sigma} = id_{T_x\Sigma}$.

Now let $(v_1, ..., v_n)$ be an orthonormal basis of $T_{(0,x)}\mathbb{R} \times \Sigma$ with $v_1 = \partial_t$. Then by proposition 2.38 we get

$$\begin{aligned} |F|_*(0, x) &= |det(g_N(d_{(0,x)} F v_i, d_{(0,x)} F v_j))|^{\frac{1}{2}} \\ &= |det(\begin{pmatrix} g_N(X(x), X(x)) & g_N(X(x), v_2) & \cdots & \cdots & g_N(X(x), v_n) \\ g_N(X(x), v_2) & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ g_N(X(x), v_n) & 0 & \cdots & 0 & 1 \end{pmatrix})|^{\frac{1}{2}} \\ &= (g_N(X, X) - \sum_{i=2}^n g_N(X, v_i)^2)^{\frac{1}{2}}. \end{aligned}$$

Using this we obtain the assertion, since

$$g_N(X^\perp, X^\perp) = g_N(X - \sum_{i=2}^n g_N(X, v_i)v_i, X - \sum_{j=2}^n g_N(X, v_j)v_j)$$
$$= g_N(X, X) - 2\sum_{i=2}^n g_N(X, v_i)^2 + \sum_{i=2}^n g_N(X, v_i)^2 = |F|_*(0, x)^2.$$

□

Let us apply this lemma to the geodesic flow.

Theorem 4.5. *(Santalo's formula) Let H be a hypersurface in the complete Riemannian manifold (M, g) and $\Sigma = SM_H = \{(p, v) \in SM \mid p \in H\}$. Let G be the vector field of the geodesic flow $\Phi^t : SM \to SM$. With F defined as above*

$$|F|_*(t, v) = sin(\theta(v)),$$

where $\theta(v) \in [0, \pi/2]$ is the angle between $T_{\pi(v)}H$ and v.

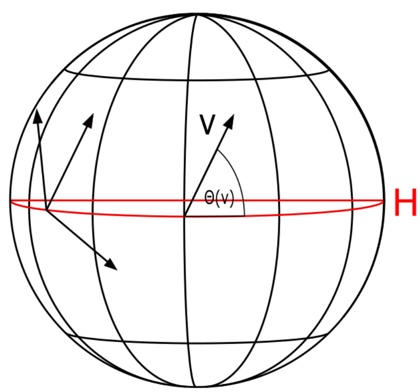

Figure 4.1: Santalo's formula: The red line is the hypersurface H, $v \in SM_H$. The arrows illustrate the directions taken by the geodesic flow.

Proof. By lemma 2.47 the geodesic flow preserves μ_{SM}, so we can apply the preceding lemma with $(N, g_N) = (SM, \iota^* g^S)$ and obtain

$$|F|_*(t, v) = |g^S(G^\perp(v), G^\perp(v))|^{\frac{1}{2}}.$$

Again, by lemma 2.47 we have $T_vSM = T_pM \bigoplus v^\perp$, where $p = \pi(v)$. As in proposition 2.7 one shows that $G(v) = (v, 0)$. If $v \in \Sigma$, then $T_v\Sigma = T_pH \bigoplus v^\perp$, which follows directly from corollary 2.5. Let $\nu_H \in T_pM$ be the outer unit normal of H in M at p. Then $\nu_\Sigma := (\nu_H, 0)$ is a unit normal of Σ in SM at v, since $g^S(\nu_\Sigma, \nu_\Sigma) = 1$ and for each $Z = (X, Y) \in T_v\Sigma$

$$g^S(\nu_\Sigma, Z) = g(\nu_H, X) + g(0, Y) = 0.$$

Thus

$$G^\perp(v) = g^S(G(v), \nu_\Sigma)\nu_\Sigma = (g(v, \nu_H) + 0)\nu_\Sigma$$

and finally

$$g(G^\perp(v), G^\perp(v)) = \frac{g(v, \nu_H)^2}{|v|_g^2 |\nu_H|_g^2} = cos^2(\frac{\pi}{2} - \theta(v)) = sin^2\theta(v).$$

□

We are now able to prove the Blaschke conjecture. Firstly, we treat the case where M is simply connected and then use the theory of covering spaces to obtain the general result.

Theorem 4.6. *(Blaschke conjecture for simply connected wiedersehen surface) Every simply connected wiedersehen surface (M, g) has constant positive sectional curvature.*

Proof. By theorem 3.10 M is diffeomorphic to the sphere S^2. Hence the Euler characteristic $\chi(M)$ is equal to 2. The idea is to show

$$vol(M) = \frac{4a^2}{\pi}.$$

Then corollary 4.3 implies that K is equal to π^2/a^2. The formula for the volume of M can be derived from Santalo's formula as follows. Let $H \subset M$ be one of the closed geodesics of length $2a$ (where a is from theorem 3.10). Define F, Σ as in Santalo's formula. We will show that

$$F : [0, a) \times (\Sigma - SH) \to SM - SH$$

is a diffeomorphism and therefore

$$\begin{aligned} vol(SM) = vol(SM - SH) &= \int_{F([0,a)\times(\Sigma - SH))} vol_{SM} \\ &= \int_{[0,a)\times(\Sigma - SH)} sin\theta(v) vol_{\mathbb{R}\times\Sigma} \\ &= \int_{[0,a)\times\Sigma} sin\theta(v) vol_{\mathbb{R}\times\Sigma} \\ &= \int_{[0,a)} vol_{\mathbb{R}} \int_\Sigma sin\theta(v) vol_\Sigma \\ &= a \int_\Sigma sin\theta(v) vol_\Sigma. \end{aligned}$$

By Fubini we have

$$\begin{aligned}\int_\Sigma sin\theta(v)vol_\Sigma &= \int_H \int_{S_pM} sin\theta(v)vol_{S_pM}(v)vol_H(p)\\ &= \int_0^{2\pi} |sin\theta| d\theta \int_H vol_H = 4\cdot 2a.\end{aligned}$$

Therefore the volume of the unit tangent bundle is $8a^2$ and

$$8a^2 = vol(SM) = \int_M \int_{S_pM} vol_{S_pM} vol_M = 2\pi vol(M),$$

which proves the formula for the volume of M.
We now show that F from above is a diffeomorphism. To see that F is surjective, let $(x,v) \in SM - SH$.
Case 1: $(x,v) \in SM_{|H}.F(0,(x,v)) = (x,v)$.
Case 2: $x \in M - H$. Take any $p \in H$ and a minimizing unit speed geodesic γ that connects p and x such that $\gamma(0) = p$. By the Jordan curve theorem H separates M into two disjoint components N and S. Without loss of generality we assume $x \in N$. Since $injM = a$, γ won't hit H in the time interval $(0,a)$. Moreover $\gamma(a) = Con(p) \in H$. Because $x \notin H, \gamma$ intersects H transversally in $Con(p)$ and thus $\gamma(a, 2a) \subset S$. Since $Con(p)$ lies on H and x doesn't, we have $Con(p) \neq x$. Therefore, there must be some positive real number $t_x < a$ with $\gamma(t_x) = x$. Consequently $Con(x) = \gamma(t_x + a) \in S$. Let γ_{-v} be the geodesic such that $\gamma_{-v}(0) = x, \dot\gamma_{-v}(0) = -v$. Then $\gamma_{-v}(a) = Con(x) \in S$ and thus there must be a first positive time $t_0 < a$ such that $\gamma_{-v}(t_0) \in H$. Then $-\dot\gamma_{-v}(t_0) \in (\Sigma - SH)$ and $F(t_0, -\dot\gamma_{-v}(t_0)) = -\dot\gamma_{-v}(0) = (x,v)$.
For injectivity we take $(t,(x,v)),(s,(y,w)) \in [0,a) \times (\Sigma - SH)$ with $s \geq t$ such that $F(t,(x,v)) = F(s,(y,w))$, i.e. $\Phi^t(v) = \Phi^s(w)$. Thus $\Phi^{s-t}(w) = v$, but $\pi \circ \Phi^{\cdot}(w)$ can only hit H after times that are multiples of a. Hence $s - t = 0$, which proves $(x,v) = (y,w)$.
Let $(t,v) \in [0,a) \times (\Sigma - SH)$, and $(X_1, ..., X_{2m-1})$ a positively oriented basis of $T_{(t,v)}\mathbb{R} \times \Sigma$, then

$$vol_{SM}(dFX_1, ..., dFX_{2m-1}) = F^* vol_{SM}(X_1, ..., X_{2m-1}) = sin\theta(v) \neq 0$$

implies that $(dFX_1, ..., dFX_{2m-1})$ is a basis of $T_{F(t,v)}SM$. Thus, F is a local diffeomorphism and we are done. □

Corollary 4.7. *Let (M,g) be a wiedersehen surface. Then (M,g) has constant positive sectional curvature.*

Proof. Let $\pi : \tilde M \to M$ be the universal cover of M. By [1, 4.40] there is a unique smooth structure on $\tilde M$ such that p is a smooth covering map. We endow $\tilde M$ with the Riemannian metric $\tilde g := \pi^* g$. By corollary 3.15, $(\tilde M, \tilde g)$ is a simply connected wiedersehen surface and has therefore constant positive sectional curvature $\tilde K$. Since π is a local isometry, (M,g) has constant positive sectional curvature. □

An even dimensional complete connected Riemannian manifold (M^m, g) with constant positive curvature is isometric to the sphere S^m or the real projective space $\mathbb{RP}^m$, both equipped with the canonical metric times some positive constant (see [8, Chapter 8, Proposition 4.4]). Thus, the only wiedersehen surfaces are S^2 and $\mathbb{RP}^2$ with positive multiples of the canonical metrics.

In the proof of the Blaschke conjecture the fact that the wiedersehen manifold is a surface is crucial, both in applying corollary 4.3 and Santalo's formula. In the first case we used the Gauss-Bonnet theorem and in the second we used that a closed geodesic in a two-dimensional manifold is a hypersurface. Nevertheless, Berger was able to prove the following.

Theorem 4.8. *Let g be a Riemannian metric on S^m with*

$$inj(S^m, g) = diam(S^m, g).$$

Then (S^m, g) has constant positive sectional curvature.

For details see [6]. By our discussion of wiedersehen manifolds we know that simply connected wiedersehen manifolds are exactly of this type, namely they are diffeomorphic to the sphere and the injectivity radius equals the diameter. Hence simply connected wiedersehen manifolds have constant positive sectional curvature. By chapter 3 we obtain

Corollary 4.9. *Every wiedersehen manifold has constant positive sectional curvature.*

But this isn't the end of the story. One can even go farther and consider Blaschke manifolds, i.e. Riemannian manifolds (M, g) with the property $injM = diamM$. The generalized Blaschke conjecture is then: Up to isometry the only Blaschke manifolds are the compact symmetric spaces of rank 1. This problem is still not solved. For a summary of what is known see [13].

Chapter 5

Closed Surfaces without Conjugate Points

In this chapter we will prove the following theorem of E. Hopf [12].

Theorem 5.1. *Let (M, g) be a closed Riemannian surface. If on M no conjugate points exist, then*

$$\int_M K\mu_M \leq 0$$

and equality holds if and only if the Gaussian curvature K is identically zero.

In this chapter we always assume (M, g) to satisfy the assumptions of this theorem. First, we investigate how the non-existence of conjugate points affects Jacobi fields on M.

Proposition 5.2. *Let $v \in S_pM$ and E be a parallel vector field along γ_v such that $(\dot{\gamma}_v, E)$ is an orthonormal frame along γ_v. Then*

1. *J is a normal Jacobi field along γ_v if and only if $\ddot{J} + K \circ \gamma_v J = 0$ and $\dot{J}(0), J(0) \perp \dot{\gamma}_v(0)$.*

2. *Writing $J = yE$ for a function y, the Jacobi equation from 1. becomes*

$$\ddot{y} + K \circ \gamma_v y = 0. \tag{5.1}$$

3. *The solutions of (5.1) are defined on all of $\mathbb{R}$.*

4. *Any nontrivial solution of (5.1) has at most one zero. Two non-identical solutions intersect at most once. (intersection property)*

5. *For all $a, b \in \mathbb{R}$ there is a unique solution $y_v(\cdot; a, b)$ of (5.1) such that*

$$y_v(a; a, b) = 1, y_v(b; a, b) = 0. \tag{5.2}$$

Proof. 1. By [5, 6.10] and the fact that M is a surface, we have

$$R(J,\dot\gamma_v)\dot\gamma_v = K(\gamma_v)(g(\dot\gamma_v,\dot\gamma_v)J - g(J,\dot\gamma_v)\dot\gamma_v) = K(\gamma_v)(J - g(J,\dot\gamma_v)\dot\gamma_v).$$

If J is a normal Jacobi field along γ_v, then

$$0 = \ddot J + R(J,\dot\gamma_v)\dot\gamma_v = \ddot J + K(\gamma_v)J.$$

Together with

$$0 = \frac{d}{dt}g(J,\dot\gamma_v) = g(\dot J,\dot\gamma_v)$$

this shows the first implication.
Conversely assume $\ddot J + K(\gamma_v)J = 0$ and $J(0), \dot J(0) \perp \dot\gamma_v(0)$. Then the function $g(J,\dot\gamma_v)$ and its derivative $g(\dot J,\dot\gamma_v)$ vanish at $t = 0$. Moreover

$$\frac{d^2}{dt^2}g(J,\dot\gamma_v) = g(\ddot J,\dot\gamma_v) = -K(\gamma_v)g(J,\dot\gamma_v),$$

and by the uniqueness of such an initial value problem $g(J,\dot\gamma_v) = 0$. Then J is a normal Jacobi field since

$$0 = \ddot J + K(\gamma_v)J = \ddot J + R(J,\dot\gamma_v)\dot\gamma_v.$$

2. Follows easily since E is parallel.
3. This follows from the theory of ordinary differential equations, see [11, 21.3].
4. If a nontrivial solution had more than one zero, there would exist conjugate points on M. If there were two nonidentical solutions intersecting at two points, we could subtract them and obtain a nontrivial solution with more than one zero.
5. See [8, Proposition 3.9, Chapter 5]. □

With the notation of the preceding proposition we are now going to prove the theorem.

Proof. For $\alpha, \beta, a, b \in \mathbb{R}$ with $\alpha \neq \beta, a \neq b$ the identity

$$y_v(s;a,b) = y_v(\alpha;a,b)y_v(s;\alpha,\beta) + y_v(\beta;a,b)y_v(s;\beta,\alpha) \tag{5.3}$$

holds, since by linearity of (5.1) both sides are solutions of the Jacobi equation and they are equal at $s = \alpha, \beta$. If $\alpha = a', \beta = b$ this becomes

$$y_v(s;a,b) = y_v(a';a,b)y_v(s;a',b). \tag{5.4}$$

We have

$$y_v(s;a,b') > 0 \text{ for } s < b' \text{ and } a < b', \tag{5.5}$$

because otherwise there would exists some $s < b'$ such that $y_v(s;a,b') \leq 0$. By the intermediate value theorem there would exist $s_0 < b'$ with $y_v(s_0;a,b') = 0 = y_v(b';a,b')$. By the intersection property $y_v(\cdot;a,b') = 0$, which contradicts $y_v(a,a,b') = 1$.
For $a < b < b'$ we get $y_v(b;a,b) = 0 < y_v(b;a,b')$. Hence the solutions

$y_v(s;a,b), y_v(s;a,b')$ are different and by the intersection property they intersect if and only if $s = a$. Therefore the function $f(s) := y_v(s;a,b) - y_v(s;a,b')$ solves (5.1), $f(a) = 0, f_{|(a,\infty)} > 0$ and $f'(a) \neq 0$, since otherwise $f \equiv 0$ by uniqueness of the solutions of (5.1). Thus $f_{|(-\infty,a)} < 0$ or equivalently

$$y_v(s;a,b) \geq y_v(s;a,b') \text{ for } s \leq a < b < b'. \tag{5.6}$$

By (5.5) and (5.6) for all $s \leq a$ the limit

$$y_v(s;a) := \lim_{b \to +\infty} y_v(s;a,b) \tag{5.7}$$

exists. If we choose $\alpha, \beta \leq a$ in (5.3), we see that the limit in (5.7) exists for all $s \in \mathbb{R}$ and is a solution of (5.1) since it is a linear combination of solutions. Furthermore, this limit commutes with differentiation by s

$$y_v'(s;a) = \lim_{b \to +\infty} y_v'(s;a,b) \text{ for } s \in \mathbb{R}. \tag{5.8}$$

This can be shown as follows. Choose $\alpha, \beta \leq a$, then

$$\begin{aligned}
y_v'(s;a) &= \frac{d}{ds}\left[\lim_{b \to +\infty} y_v(\alpha;a,b) y_v(s;\alpha,\beta) + \lim_{b \to +\infty} y_v(\beta;a,b) y_v(s;\beta,\alpha)\right] \\
&= \lim_{b \to +\infty} y_v(\alpha;a,b) y_v'(s;\alpha,\beta) + \lim_{b \to +\infty} y_v(\beta;a,b) y_v'(s;\beta,\alpha) \\
&= \lim_{b \to +\infty} [y_v(\alpha;a,b) y_v'(s;\alpha,\beta) + y_v(\beta;a,b) y_v'(s;\beta,\alpha)] \\
&= \lim_{b \to +\infty} y_v'(s;a,b).
\end{aligned}$$

From (5.2) and (5.5) we see that

$$y_v(a;a) = 1, y(s;a) \geq 0 \text{ for all } s \in \mathbb{R}.$$

Assume $y_v(s_0;a) = 0$. Then $y_v(\cdot,a)$ has a minimum at $y_v(s_0;a)$, hence $y_v'(s_0;a) = 0$. And since $y_v(s;a)$ solves the Jacobi equation (5.1), $y_v(s;a) = 0$ for all $s \in \mathbb{R}$. This contradicts $y_v(a;a) = 1$. Therefore $y_v(s;a) > 0$ for all $s \in \mathbb{R}$ and we can define a function $u_v : \mathbb{R} \to \mathbb{R}$ by

$$u_v(s) := \frac{y_v'(s;a)}{y_v(s;a)}.$$

As the notation suggests, this definition is independent of a. Let $a, a' \in \mathbb{R}$. Then by (5.4)

$$y_v(s;a) = \lim_{b \to +\infty} y_v(s;a,b) = \lim_{b \to +\infty} y_v(a';a,b) y_v(s;a',b) = y_v(a';a) y_v(s;a')$$

and differentiating by s we obtain

$$y_v'(s;a) = y_v(a';a) y_v'(s;a').$$

Dividing these two equations implies that u_v is independent of a. If we differentiate u_v and use that $y_v(s;a)$ solves the Jacobi equation we obtain

$$u_v'(s) = \frac{y_v''(s;a)y_v(s;a) - y_v'(s;a)^2}{y_v^2(s;a)} = -\frac{K(\gamma_v(s)) \cdot y_v(s;a)^2}{y_v(s;a)^2} - u^2(s)$$

and hence u_v is a solution of the Riccati equation

$$u_v'(s) + u_v^2(s) + K(\gamma_v(s)). \tag{5.9}$$

Let $Y_v(s;b)$ denote the solution of the Jacobi equation (5.1) such that $Y_v(b;b) = 0, Y_v'(b;b) = 1$. We compute

$$\begin{aligned}
&\frac{d}{ds}[y_v(s;a,b)Y_v'(s;b) - y_v'(s;a,b)Y_v(s;b)] = 0 \text{ and} \\
&y_v(b;a,b)Y_v'(b;b) - y_v'(b;a,b)Y_v(b;b) = 0.
\end{aligned}$$

Thus, for all $s \neq b$ we get

$$\frac{y_v'(s;a,b)}{y_v(s;a,b)} = \frac{Y_v'(s;b)}{Y_v(s;b)}. \tag{5.10}$$

Going to the limit we can thus write u_v in terms of Y_v

$$u_v(s) = \lim_{n \to +\infty} \frac{y_v'(s;a,s+n)}{y_v(s;a,s+n)} = \lim_{n \to +\infty} \frac{Y_v'(s;s+n)}{Y_v(s;s+n)}. \tag{5.11}$$

We now show that there exists a constant $A > 0$ such that

$$|u_v| \leq A \text{ for all } v \in SM. \tag{5.12}$$

Since M is compact and the Gaussian curvature K is continuous, there exists $A > 0$ such that $K > -A^2$ on M. Let $a \neq b$. The function

$$z(s;a,b) = \frac{sinh[A(b-s)]}{sinh[A(b-a)]}$$

is the solution of the initial value problem

$$z''(s) - A^2 z(s) = 0, \qquad z(a;a,b) = 1, \qquad z(b;a,b) = 0.$$

If we write $y_v(s) = y_v(s;a,b), z(s) = z(s;a,b)$, then

$$\begin{aligned}
(zy_v' - y_v z')' &= zy_v'' + z'y_v' - y_v'z' - y_v z'' = zy_v(-K(\gamma_v)) - y_v z A^2 \\
&= -(K(\gamma_v) + A^2) y_v z.
\end{aligned}$$

Now let $a < b$. If $s < b$, then $y_v(s), z(s) > 0$ by (5.5) and hence $(zy_v' - y_v z')' < 0$. Therefore, the function $(zy_v' - y_v z')$ is strictly decreasing on $s < b$ and is equal to 0 when $s = b$. Thus $(zy_v' - y_v z') > 0$ for $s < b$. Going to the limit we get

$$u_v(s) = \lim_{b \to \infty} \frac{y_v'(s;a,b)}{y_v(s;a,b)} \geq \lim_{b \to \infty} \frac{z'(s;a,b)}{z(s;a,b)} = \lim_{b \to \infty} \frac{-Acosh(A(b-s))}{sinh(A(b-a))} = -A.$$

Now let $a > b$. Then the functions $y_v(s) = y_v(s;a)$ and $z(s) = z(s;a,b)$ are positive for $s > b$. The same reasoning as above shows $(zy_v' - y_v z')' < 0$ for $s > b$ and hence we get

$$(zy_v' - y_v z')(s) < (zy_v' - y_v z')(b) = -y_v(b)z'(b) < 0 \text{ if } s > b.$$

Hence, we obtain

$$u_v(s) = \frac{y_v'(s;a)}{y_v(s;a)} < \frac{z'(s;a,b)}{z(s;a,b)}$$

for $s > b$ and going to the limit $b \to -\infty$ implies

$$u_v(s) \leq A$$

for all $s \in \mathbb{R}$.
We now consider u as a function $u : SM \times \mathbb{R} \to \mathbb{R}$. We have

$$u(\Phi^t(v), s) = u(v, s+t) \text{ for all } s, t \in \mathbb{R}, v \in SM. \tag{5.13}$$

To see this we consider the function $f : \mathbb{R} \to \mathbb{R}, f(s) := y_v(s+t;a,b)$. Then

$$f''(s) = -K(\pi \circ \Phi^s \circ \Phi^t(v))f(s), \qquad f(a-t) = 1, \qquad f(b-t) = 0.$$

By the intersection property we obtain $y_v(s+t;a,b) = y_{\Phi^t(v)}(s;a-t,b-t)$ and hence

$$u(\Phi^t(v), s) = \lim_{b\to\infty} \frac{y'_{\Phi^t(v)}(s;a-t,b-t)}{y_{\Phi^t(v)}(s;a-t,b-t)} = \lim_{b\to\infty} \frac{y'_{(v)}(s+t;a,b)}{y_v(s+t;a,b)} = u(v, s+t).$$

If we set $s = 0$ in (5.13) we get $u(\Phi^t(v), 0) = u(v,t)$ and inserting this into (5.9) we conclude

$$\frac{d}{dt}u(\Phi^t(v), 0) + u^2(\Phi^t(v), 0) + K(\pi \circ \Phi^t(v)) = 0. \tag{5.14}$$

We now show that the function $u(\cdot, 0)$ is measurable. Using (5.10) we obtain

$$u(v, 0) = \lim_{n\to\infty} \frac{Y_v'(0;n)}{Y_v(0;n)},$$

where $Y_v(s;n)$ is a solution of the Jacobi equation with $Y_v(n;n) = 0, Y_v'(n;n) = 1$. From the theory of ordinary differential equations one infers that for $n \in \mathbb{N}$ fixed, $Y_v(0;n)$ and $Y_v'(0;n)$ depend continuously on $v \in SM$.(See [14, 4.1.2]) Thus $\frac{Y_v'(0;n)}{Y_v(0;n)}$ is continuous in $v \in SM$ and hence $u(\cdot, 0)$ is measurable as the limit of continuous functions. We have already shown that $u(\cdot, 0)$ is bounded on SM, together we conclude that it is integrable over SM.
Integrating (5.14) with respect to t implies

$$u(\Phi^1(v), 0) - u(v, 0) + \int_0^1 K(\pi \circ \Phi^t(v))dt = -\int_0^1 u^2(\Phi^t(v), 0)dt. \tag{5.15}$$

By the preceding discussion, we can integrate the left side of this equality over SM. Using corollary 2.48 we obtain

$$\int_{SM} u(\Phi^1(v), 0) - u(v, 0)\mu_{SM} = 0.$$

Moreover

$$\begin{aligned}
\int_{SM}\int_0^1 K(\pi\circ\Phi^t(v))dt\mu_{SM} &\overset{2.34}{=} \int_0^1\int_{SM} K(\pi\circ\Phi^t(v))\mu_{SM}dt \\
&\overset{2.48}{=} \int_{SM} K(\pi(v))\mu_{SM} \\
&\overset{2.34}{=} \int_M\int_{S_pM} K(\pi(v))\mu_{S_pM}(v)\mu_M(p) \\
&= \int_M K(p)vol(S_pM)\mu_M(p) \\
&= 2\pi\int_M K\mu_M.
\end{aligned}$$

Together we conclude

$$2\pi\int_M K\mu_M = -\int_{SM}\int_0^1 u^2(\Phi^t(v), 0)dt\mu_{SM}.$$

Therefore, the total curvature is non positive. If the total curvature is equal to zero, then

$$\int_{SM}\int_0^1 u^2(\Phi^t(v), 0)dt\mu_{SM} = 0.$$

Thus, for almost all $v \in SM, \int_0^1 u^2(\Phi^t(v), 0)dt = 0$. For such a v, since $u(\Phi^t(v), 0) = u_v(t)$ is continuous in t, one infers that $u(\Phi^t(v), 0) = 0$ for all $t \in [0, 1]$. By (5.14) we get $K(\pi(v)) = 0$. Thus $K\circ\pi$ is a continuous function on SM that vanishes almost everywhere, hence it is equal to zero on SM. Therefore $K \equiv 0$. □

Corollary 5.3. *Let M be the two-dimensional torus or the Klein bottle. Let g be a Riemannian metric on M without conjugate points. Then (M, g) is flat.*

Proof. The Gauss-Bonnet theorem implies

$$\int_M K\mu_M = 0,$$

and the preceding theorem implies that $K \equiv 0$. Hence (M, g) is flat. □

One can then ask whether the same conclusion holds for higher dimensional tori. This was an almost 50 years standing conjecture, which was finally solved by Burago and Ivanov [7]. They showed that higher dimensional tori without conjugate points are flat.

Bibliography

[1] John M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013.

[2] John M. Lee, *Introduction to Riemannian Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 176, Springer, International Publishing, 2018.

[3] Werner Ballmann, *Lectures on the Blaschke conjecture*, 2016.

[4] Gabriele Benedetti, *Differentialgeometrie 1*, 2019.

[5] Gabriele Benedetti, *Differentialgeometrie 2*, 2020.

[6] A. L. Besse, *Manifolds all of whose geodesics are closed*, With appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bèrard-Bergery, M. Berger, and J. L.Kazdan. Ergebnisse der Mathematik und ihrer Grenzgebiete, 93. SpringerVerlag, 1978. ix+262 pp.

[7] D. Burago, S. Ivanov, *Riemannian tori without conjugate points are flat*, Geometric and Functional Analysis. 4. 259-269, 1994.

[8] Manfredo P. do Carmo *Riemannian Geometry*, Mathematics: Theory and Applications, Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[9] L. W. Green, *Auf Wiedersehensflächen*, Ann. of Math. (2) 78, 1963, 289299.

[10] L. W. Green, *A theorem of E. Hopf*, Michigan Math. J., 5, 1958, 31-34.

[11] Harro Heuser, *Gewöhnliche Differentialgleichungen: Einführung in Lehre und Gebrauch*, zweite Auflage, Aus dem Programm Mathematik, Vieweg+Teubner Verlag, 1991.

[12] E. Hopf, *Closed surfaces without conjugate points*, Proc. Nat. Acad. Sci. U.S.A., 34, 47-51, 1948.

[13] Benjamin McKay *Summary of the Blaschke conjecture*, 2013.

[14] Jan W. Prüss, Mathias Wilke, *Gewöhnliche Differentialgleichungen und dynamnische Systeme*, Grundstudium Mathematik, Birkhäuser, 2010.

[15] Takashi Sakai, *Riemannian Geometry*, American Mathematical Society, Mathematical Monographs 149, 1996.

Notation Index

$\pi : TM \to M$,	tangent bundle
${}^F\nabla$,	pullback of the connection under map
F_*,	differential of a smooth map
τ^∇,	torsion of a connection
$C : TTM \to TM$,	connection map
γ_v,	geodesic with initial velocity v
$\Phi : \mathcal{G} \to TM$,	geodesic flow
G,	vector field of the geodesic flow
$\bar{\pi} : T^*M \to M$,	cotangent bundle
$\iota_X \omega$,	interior product
$\mathcal{L}$,	Legendre-Transformation
λ_g,	tautological one-form on tangent bundle
ω_g,	canonical symplectic form on tangent bundle
$\mathcal{D}(M)$,	density bundle
μ_g, μ_M,	Riemannian density
vol_g, vol_M,	Riemannian volume form
$\lvert\psi\rvert_*$,	Jacobian of smooth map ψ
K,	sectional curvature
g^S,	Sasaki metric on TM
SM,	unit tangent bundle
$g_{SM}, \iota^* g^S$,	Sasaki metric on SM
$C^\infty(M)$,	smooth real valued functions on M
$con(v)$,	first conjugate time along γ_v
$Con(p)$,	first conjugate locus of p
$I(\cdot,\cdot)$,	index form
R,	Riemannian curvature endomorphism
$inj_p M$,	injectivity radius of p in M
$diam(M)$,	diameter of M
$scal$,	scalar curvature
${}^\gamma TM$,	set of vector fields along the curve γ
$\chi(M)$,	Euler characteristic of M

Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit eigenständig und ohne fremde Hilfe angefertigt habe. Textpassagen, die wörtlich oder dem Sinn nach auf Publikationen oder Vorträgen anderer Autoren beruhen, sind als solche kenntlich gemacht. Die Arbeit wurde bisher keiner anderen Prüfungsbehrde vorgelegt und auch noch nicht veröffentlicht.

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