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The Saddle Connection Graph of the Golden L

Bachelor Thesis in Mathematics
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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die hier vorliegende Bachelorarbeit selbstständig angefertigt und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe. Das gilt auch für die in der Arbeit enthaltenen Zeichnungen, Skizzen und graphischen Darstellungen.

Heidelberg, am 17.04.2023 .... Isabel Giray ....

(Isabel Giray)

Abstract

The saddle connection graph of a translation surface is a graph with saddle connections as vertices and sets of two disjoint saddle connections as edges. It characterizes translation surfaces up to affine equivalence. The graph of slopes summarizes saddle connections with the same slope in equivalence classes.
With the help of an algorithm that generates all directions of saddle connections and some findings that reduce the number of pairwise comparisons to find the edges of the graph, we construct a finite approximation of the graph of slopes of a finite translation surface, which is called the golden L.

Zusammenfassung

Der Sattelverbindungsgraph einer Translationsfläche ist ein Graph mit Sattelverbindungen als Ecken und Mengen von je zwei disjunkten Sattelverbindungen als Kanten. Er charakterisiert Translationsflächen bis auf affine Äquivalenz. Der Graph der Richtungen fasst Sattelverbindungen mit der gleichen Richtung in Äquivalenzklassen zusammen.
Mithilfe eines Algorithmus, der alle Richtungen von Sattelverbindungen generiert, und einigen Erkenntnissen, die die Anzahl der paarweisen Vergleiche reduziert, um die Kanten des Graphen zu bestimmen, konstruieren wir eine endliche Annäherung des Graphen der Richtungen einer endlichen Translationsfläche, die das goldene L genannt wird.

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1 Introduction

The goal for this bachelor thesis is to understand the saddle connection graph of a special translation surface, the so called "golden L". What makes it special is that it consists of one 1×1-square, one $1 \times \frac{1}{\phi}$- and one $\frac{1}{\phi} \times 1$-rectangle, with ϕ being the golden ratio. The golden ratio is the positive solution of the equation $x^2 - x - 1 = 0$. The saddle connection graph of a translation surface is a graph whose vertices are saddle connections and whose edges are sets of two disjoint saddle connections. It is a tool to study equivalence classes of translation surfaces. This is because its combinatorial structure completely describes the translation surface up to affine equivalence. The graph of slopes is $(2, 1)-$quasi-isometric to the saddle connection graph and its vertices are equivalence classes of saddle connections that have the same slope. There is an edge between two equivalence classes if there exists at least one edge between two saddle connections from each class.
The once-punctured torus carries the structure of a translation surface, that is built of the unit square. Here the saddle connection graph can be constructed with the help of two general criteria that determine vertices and edges of the graph. This graph can be visualized in the unit circle, as pictured below, and is called the Farey graph.

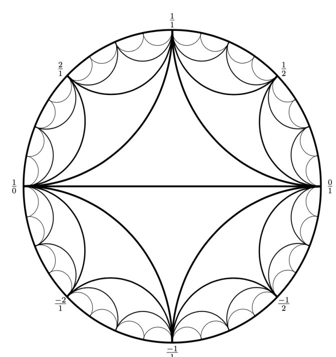


Figure 1: The Farey graph [7]

Apparently for the golden L such general criteria are not known. In [2], an algorithm is introduced that generates all saddle connections. From this we get all directions of saddle connections on the golden L and construct the graph of slopes. Since quasi-isometries respect large-scale geometry, the structure of the saddle connections graph is comparable to the structure of the graph of slopes.
The mentioned terms and other background is introduced in the first few pages. Subsequently we study the relation of the golden L to the double pentagon translation surface and find their Veech groups. The Veech group is a subgroup of $SL(2, \mathbb{R})$ and is

defined as the stabilizer of $GL(2,\mathbb{R})$ acting on the translation surface. We introduce the algorithm [2] that generates all long saddle connections with positive slope and extend it to the saddle connections with negative slope. These are all directions that admit a saddle connection and form the vertices of the graph. Since the computation of all pairwise comparisons for the edges is laborious, we only construct the graph with the first 16 saddle connections that the algorithm generates, to get an idea of how the graph looks like.

2 Background

2.1 Translation surfaces

This section is based on [8] (chapter 1.2).

Definition 2.1 (Surface)**.** *A* ***surface*** *is a compact Hausdorff space where every point has an open neighbourhood which is homeomorphic to an open subset of* $\mathbb{R}^2$*.*

Definition 2.2 (Simple closed curve)**.** *Let M be a surface. A* ***simple closed curve*** *in M is a continuous injective map $S^1 \to M$.*

Definition 2.3 (Genus)**.** *Let M be a connected surface. The* ***genus*** *g of M is defined to be the maximum number of disjoint and simple closed curves in M such that M without the images of those curves is still connected.*
Intuitively the genus can be understood as the number of "holes" of a surface.

Definition 2.4 (Chart)**.** *Let M be a surface.*
A ***chart*** *on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to V$ is a homeomorphism on an open subset V of $\mathbb{R}^2$.*

Definition 2.5 (Translation atlas)**.** *Let M be a surface.*
A ***translation atlas*** *of M is a set of charts $\{(U_i, \varphi_i)\}_{i\in I}$ with $M = \bigcup_{i\in I} U_i$ and transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ that are translations in $\mathbb{R}^2$ for all charts (U_i, φ_i), (U_j, φ_j) with $U_i \cap U_j \neq \emptyset$.*

Definition 2.6 (Translation structure)**.** *A* ***translation structure*** *ω is the equivalence class of a translation atlas, where two atlases are said to be equivalent if there are transition maps between all charts of the two atlases and these transition maps are translations.*

There are quite a few equivalent definitions of translation surfaces, but for our purposes the following definition will be sufficient.

Definition 2.7 (Finite translation surface)**.** *Let $P_1, ..., P_n$ be finitely many disjoint polygons in the plane $\mathbb{R}^2$, P_i^* each of the polygons without its vertices and D the union of all edges of the polygons without the vertices. Furthermore choose an orientation of the plane. Let T be an involution (a self-inverse map) on D such that T restricted to the interior of any edge is a translation to an oppositely oriented edge.*
If the surface $X := \left(\bigcup_{i=1}^n P_i^* \right) / T$ is connected and oriented and ω is the translation structure given by the edge identifications via T, we call (X, ω) a ***finite translation surface****.*

Remark 2.8. *In the following, let $X := (X, \omega)$ be a translation surface.*

Remark 2.9. *Visually, one can imagine a finite translation surface as a finite union of polygons in the plane that are glued together pairwise on parallel edges with the same length via translations.*

Definition 2.10 (Isomorphism of translation surfaces)**.** *Two translation surfaces (X, ω) and $(Y, \bar{\omega})$ are said to be isomorphic $(X, \omega) \cong (Y, \bar{\omega})$ if there is a bijective translation between them.*

Remark 2.11. *The Euclidian metric on $\mathbb{R}^2$ induces a* ***flat metric*** *on X:*

$$d_X(x, y) := \inf\{d_{\mathbb{R}^2}(x, x_1) + \sum_{i=1}^{m-1} (d_{\mathbb{R}^2}(T(x_i), x_{i+1})) + d_{\mathbb{R}^2}(T(x_m), y), m \in \mathbb{N} : \\ x_1, ..., x_m \text{ on the boundary of the polygons}\}$$

In general a translation surface X is not complete regarding the flat metric and not compact. We denote the metric completion of X by $\overline{X}$.

Definition 2.12 (Singularities)**.** *The elements of $\overline{X} \setminus X$ are called* ***singularities****.*

Remark 2.13. *In the above definition of a translation surface singularities only arise in the corners of the polygons. The total interior angle around a singularity is always a conical angle of $k2\pi, k \in \mathbb{N}$. We call k the* ***multiplicity*** *of the singularity and if $k = 1$ we call the singularity removable.*

Definition 2.14 (Saddle connection)**.** *A* ***saddle connection*** *on a translation surface X is a geodesic segment between two not necessarily different singularities that does not contain any other singularities. We denote the set of saddle connections of X by $\mathcal{SC}(X)$.*

Remark 2.15. *If a translation surface has s singularities with angles*

$$(1 + k_1)2\pi, (1 + k_2)2\pi, ..., (1 + k_s)2\pi,$$

then the ***genus*** *g satisfies $2g - 2 = \sum k_i$.[10]*

One particular interest when studying saddle connections is their direction. Two saddle connections have the same direction if their holonomy vector has the same direction.

Definition 2.16 (Holonomy vector)**.** *Let X be a translation surface and $\gamma \in \mathcal{SC}(X)$. Now choose points $..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...$ on γ such that there are charts (U_i, φ_i) with $x_i, x_{i+1} \in U_i$, the union of all U_i covers γ and each $\gamma \cap U_i$ is connected.*

*Transform the $\varphi_i(U_i)$ in $\mathbb{R}^2$ by transition maps such that $\varphi_{i-1}(x_i)$ and $\varphi_i(x_i)$ coincide. The resulting image of γ is an open geodesic in $\mathbb{R}^2$. The difference vector between starting and ending point of the geodesic completion is called the **holonomy vector** of γ.*

Denote by $\mathcal{V}_{\mathcal{SC}}(X)$ the set of all holonomy vectors of saddle connections.

Remark 2.17. *The set $\mathcal{V}_{\mathcal{SC}}(X)$ is discrete. See [6] (chapter 1.3) for more explanation.*

Definition 2.18 (Cylinder)**.** *A **cylinder** in X of circumference $w > 0$ and height $h > 0$ is a maximal open subset of X which is isometric to an Euclidean cylinder $\mathbb{R}/w\mathbb{Z} \times (0, h)$.*
*The **modulus** of a cylinder is the ratio of its circumference to its height.*
*A **cylinder decomposition** of X is a collection of cylinders in X so that the closures of these cylinders cover X and so that each two cylinders are disjoint.*

Remark 2.19. *The **boundary** of a cylinder consists of a union of saddle connections and singularities.*

The following example of a translation surface was already mentioned in the introduction:

Example 2.20 (The torus with one marked point)**.** *The torus with one marked point carries the structure of a translation surface. It can be built of the unit square with opposite edges identified by translations. Since the angles in the corners add up to 2π, the singularity is removable. By marking one point on the torus, which will correspond to the corners of the square surface, we study saddle connections with this singularity leading to the Farey graph.*

2.2 The golden L

Our object of interest is a particular translation surface which is called "**The golden L**". Its name originates from the golden ratio ϕ, which is defined to be the positive solution to the equation $x^2 - x - 1 = 0$. This also implies the interesting property

$\phi^2 = \phi + 1$. The golden L consists of three rectangles of edge lengths $\frac{1}{\phi} \times 1, 1 \times \frac{1}{\phi}$ and 1×1. The edge identifications are pictured in Figure 2. Note that there is only one singularity with multiplicity 3 and with the equation from 2.15 we determine the genus to be 2.

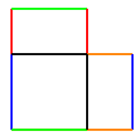
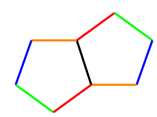


Figure 2: The golden L with edge identifications

2.3 The double pentagon

There is a close relation between the golden L and the **regular double pentagon**, which is discussed in chapter 3. This translation surface consists of two regular pentagons glued together in a certain way:

Identify two arbitrary edges of the two pentagons. The result is an 8-gon and each edge has a unique parallel edge. We glue these edges pairwise and obtain a compact orientable surface of genus 2, the double pentagon translation surface. Just like on the golden L, there is one singularity with conical angle $3 \cdot 2\pi$.



Remark 2.21. *Let $n \in \mathbb{N}$ be odd. In general for two n-gons one gets a $2(n-1)$-gon. The translation surface has genus $g = \frac{n-1}{2}$ and one singularity with conical angle $(n-2) \cdot 2\pi$. Regular double-n-gons including the statements above are treated in [5] (originally in [9]).*

2.4 The Veech group

In this chapter let X be a translation surface.

Definition 2.22 (Veech group)**.**
The group $GL^+(2, \mathbb{R}) := \{A \in GL(2, \mathbb{R}) | \det(A) > 0\}$ acts on X by linear transformations. For $B \in GL^+(2, \mathbb{R})$ this means that $B(X) = \left(\bigcup_{i=1}^{n} B(P_i^*)\right)/B \circ T \circ B^{-1}$. The stabilizer of this action is denoted by $SL(X)$ and is called the ***Veech Group****.*

Remark 2.23. *This action changes the translation structure ω in the following way: Let (U, φ) be a chart in ω. Then $B \cdot \varphi : U \to B \cdot V$, $z \mapsto B \cdot \varphi(z)$ is the corresponding chart in $B \cdot \omega$.*

Remark 2.24. *For $A \in SL(X)$ holds $(X, A \cdot \omega) \cong (X, \omega)$.*

Example 2.25 (Veech group of a torus)**.** *We continue with a generalization of Example 2.20, a translation surface that is topologically a torus, i.e., has genus 1. We can imagine X as a parallelogram (instead of the unit square) with opposite and parallel edges identified. The Veech group of X is given by $SL(X) := B \cdot SL(2, \mathbb{Z}) \cdot B^{-1}$. The map $B \in GL(2, \mathbb{R})$ is given by the linear independent vectors $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$ that span the parallelogram. ([8], Ex.4.9)*

The next lemma shows that saddle connections are sent to saddle connections by elements of the Veech group. More generally, this holds for elements of $GL^+(2, \mathbb{R})$:

Lemma 2.26. *The $GL^+(2, \mathbb{R})$ action on X induces an action on the set of saddle connections, i.e., every $A \in GL^+(2, \mathbb{R})$ induces a bijection*
$A_* : \mathcal{SC}(X) \to \mathcal{SC}(A(X))$.

Proof. The holonomy vector of a saddle connection $\gamma \in \mathcal{SC}(X)$ is a straight line segment in $\mathbb{R}^2$, then $A(\gamma)$ for $A \in GL^+(2, \mathbb{R})$ is still a straight line segment and hence a geodesic. Singularities are defined to be the elements of $\overline{X} \setminus X$. When $A \in GL^+(2, \mathbb{R})$ acts on $\gamma \in \mathcal{SC}(X)$, the linear transformation only applies on the points of the saddle connection, that lie in X, i.e., in particular not on the singularities. Hence the resulting geodesic $A_*(\gamma)$ is still a saddle connection. □

Proposition 2.27 (Properties of the Veech group)**.** *$SL(X)$ has the following properties:*

(i) $SL(X)$ is a subgroup of $SL(2, \mathbb{R})$.

(ii) $SL(X)$ is discrete.

(iii) $SL(X)$ is never cocompact.

Proof. (i) Let $A \in SL(X)$. The area of $A(X)$ equals to $\det(A) \cdot \mathcal{A}(X) = \mathcal{A}(X)$. This implies $\det(A) = 1$ and therefore $A \in SL(2, \mathbb{R})$.

(ii) This proof follows [8]. By Example 2.25 the Veech group of translation surfaces of genus 1 is discrete.
For a translation surface X of higher genus, there has to be at least one non-removable singularity (see the equation in Remark 2.15). Furthermore X is not simply connected, therefore there must exist non-trivial geodesics in $\overline{X}$ that connect this singularity with itself. These consist of finitely many saddle connections. Because we set the genus to be > 1 there are at least two disjoint and simple closed curves in X and therefore at least two directions admitting saddle connections.

By Lemma 2.26, saddle connections are sent to saddle connections by $A \in SL(X)$ as subgroup of $GL^+(2, \mathbb{R})$.

Let $\{A_n\}_{n\in\mathbb{N}} \subset SL(X)$ be a sequence approaching the identity $I \in SL(2, \mathbb{R})$. Furthermore, let $u, v \in \mathcal{V}_{\mathcal{SC}}(X)$ be linearly independent. $A_n u \to u$ and $A_n v \to v$ for $n \to \infty$. Since $\mathcal{V}_{\mathcal{SC}}(X)$ is discrete, for $n \in \mathbb{N}$ sufficiently large on gets $A_n u = u$ and $A_n v = v$.
The saddle connections u and v are linearly independent, so they form a basis for $\mathbb{R}^2$.
$\Rightarrow \forall N \geq n, N \in \mathbb{N}$ we have that $A_N = I$. $\Rightarrow SL(X)$ is discrete.

(iii) [6].

□

2.5 The saddle connection graph and the graph of slopes

In this section, let X be a translation surface. In [4] it is shown that the combinatorial structure of the saddle connection graph completely describes the translation surface up to affine equivalence. Hence we can characterize affine equivalence classes of translation surfaces by their saddle connection graph. In this thesis, we study the saddle connection graph, respectively the graph of slopes, which will be introduced in this section.

Definition 2.28 (Graph)**.** *A* ***graph*** *consists of a set of vertices and a set of edges together with a map that assigns every edge exactly two not necessarily different vertices. A sequence of vertices $\sigma_0, ..., \sigma_n$ is called a* ***path*** *if σ_i and σ_{i+1} form an edge that is contained in the graph $\forall\, i \in \{0, ..., n-1\}$.*
The ***distance*** *between two vertices σ and σ' is defined to be the number of edges of the shortest path from σ to σ'.*

Definition 2.29 (The saddle connection graph)**.** *The* ***saddle connection graph*** *$\mathcal{A}(X)$ is a graph, whose vertices are saddle connections on X and edges are pairs of disjoint saddle connections.*

Example 2.30 (The Farey graph)**.** *We continue Example 2.20. As already mentioned in the introduction, the saddle connection graph of this translation surface, that is a unit square with opposite edges identified, is the* ***Farey Graph****.*
One can prove a short criterion to determine whether a vector in $\mathbb{R}^2$ is a saddle connection on the translation surface.

$$\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \text{ is a saddle connection } \Leftrightarrow \tfrac{a}{b} \text{ is a rational number in lowest terms or } \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \quad (1)$$

Furthermore there is a criterion to check whether two saddle connections form an edge in the graph:

$$\left\{ \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\} \in G \Leftrightarrow \det\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right) = \pm 1 \qquad (2)$$

We present the Farey graph with vertices on the unit circle. Saddle connections with positive slope are in the upper half circle and those with negative slope in the bottom half. The saddle connection $\binom{a}{b}$ *is the same as* $\binom{-a}{-b}$ *and* $\binom{-a}{b}$ *is the same as* $\binom{a}{-b}$*, so they are represented by one vertex. By normalizing all the vectors and drawing the edges not as straight lines but as circle segments, we get to Figure 1.*

Definition 2.31 (The graph of slopes (Def. 4.1 in [3])**.** *The* ***graph of slopes*** $\mathcal{G}(X)$ *associated to a translation surface* X *has as vertices slopes* $\theta \in \mathbb{R}P^1$ *(i.e., directions in* $\mathbb{R}^2$*) admitting a saddle connection on* X*, with two slopes declared adjacent (there is an edge between the vertices) if they can be realised by a pair of disjoint saddle connections.*

Definition 2.32 ((C, K)-quasi-isometry)**.** *Let* $f : X \to Y$ *be a map between metric spaces* (X, d_X) *and* (Y, d_Y)*. The map* f *is called a* (C, K)***-quasi-isometry*** *if*

$$\frac{1}{C} \cdot d_X(a, b) - K \leq d_Y(f(a), f(b)) \leq C \cdot d_X(a, b) + K \quad \forall a, b \in X$$

and there is a constant $M \geq 0$ *such that* $\forall\, y \in Y \,\exists\, x \in X \,:\, d_Y(y, f(x)) \leq M$.

In this case X *and* Y *are said to be* (C, K)***-quasi-isometric****.*

Quasi-isometries respect large-scale geometry and ignore local details. With the following lemma we can consider the graph of slopes instead of the saddle connection graph for our purposes.

Lemma 2.33. *The quotient map* $\Theta : \mathcal{A}(X) \to \mathcal{G}(X)$ *sending a saddle connection to its slope is a* $(2, 1)$*-quasi-isometry.*

Proof. Let $u, v \in \mathcal{A}(X)$ be two saddle connections. Recall that distance between two vertices a and b of a graph is defined to be the number of edges of the shortest path from a to b.
In case $u = v$, one gets $d_{\mathcal{A}(X)}(u, v) = 0$ and $d_{\mathcal{G}(X)}(u, v) = 0$ because there is no edge between u and v in $\mathcal{A}(X)$ and they are in the same equivalence class in $\mathcal{G}(X)$. Hence the inequality from Definition 2.32 is satisfied.
Let now $u \neq v$. We distinguish two cases:

1. There is an edge between u and v, i.e., the saddle connections do not intersect. Then $d_{\mathcal{A}(X)}(u,v) = 1$.
   If u and v have the same slope, they are in the same equivalence class in $\mathcal{G}(x)$. Hence $d_{\mathcal{G}(X)}(u,v) = 0$.
   If u and v do not have the same slope, their slopes are adjacent because u and v are disjoint. Then $d_{\mathcal{G}(X)}(u,v) = 1$.
   In both cases the inequality from Definition 2.32 is satisfied for $C = 2$ and $K = 1$.

2. The saddle connections u and v intersect.
   In particular this means that $d_{\mathcal{A}(X)}(u,v) \geq 2$ and u and v have different slopes because $u \neq v$.
   Let $u, r_1, ...r_n, v$ be a shortest path between u and v. The distance that r_i has to the first r_j in the adjacent equivalence class is at most 2. This is because a path between two vertices of the same equivalence class has length 1 and there is only one more edge to connect r_i ($\Rightarrow d_{\mathcal{A}(X)}(r_i, r_j) = 1$) or the to r_i parallel saddle connection r_i' ($\Rightarrow d_{\mathcal{A}(X)}(r_i', r_j) = 2$) with r_j, because the equivalence classes are adjacent. The distance between $\Theta(r_i)$ and $\Theta(r_j)$ remains always 1.
   This implies $\frac{1}{2} \cdot d_{\mathcal{A}(X)}(u, r_n) \leq d_{\mathcal{G}(X)}(\Theta(u), \Theta(r_n)) \leq 2 \cdot d_{\mathcal{A}(X)}(u, r_n)$ (1)
   The saddle connection v was excluded before because we have to distinguish two cases.
   If $\Theta(r_n) \neq \Theta(v)$, (1) applies equally. If $\Theta(r_n) = \Theta(v)$, we subtract/add 1 to make the inequality valid.
   $\Rightarrow \frac{1}{2} \cdot d_{\mathcal{A}(X)}(u, r_n) - 1 \leq d_{\mathcal{G}(X)}(\Theta(u), \Theta(r_n)) \leq 2 \cdot d_{\mathcal{A}(X)}(u, r_n) + 1$.

$\Rightarrow \Theta$ is a $(2, 1)$-quasi-isometry. □

3 The Veech group of the golden L

The Veech Group preserves the area of the translation surface it acts on. We will use this property for the construction of the saddle connection graph of the golden L later.
We have an affine equivalence between the golden L and the double pentagon which is given by

$$P = \begin{pmatrix} 1 & \cos\frac{\pi}{5} \\ 0 & \sin\frac{\pi}{5} \end{pmatrix}.$$

Lemma 3.1. *The matrix P is a shear on the golden L composed with a vertical scaling. It takes the golden L to the double pentagon and its inverse vice versa.*

Proof. Visually this becomes clear in Figure 3.
The matrix P preserves the bottom line (green and orange) and shifts all line segments, that are parallel to it, by a factor of $\cos(\frac{\pi}{5})$, depending on the distance from the bottom line. The vertical line segments get rotated by an angle of $\frac{\pi}{5}$. Then the length l of the red and orange line segments and the shorter diagonal of the small rectangles equals to $l = \sqrt{1^2 + \phi^2 - 2\phi\cos(\frac{\pi}{5})} = \phi - 1$ and corresponds to the edge lengths of the double pentagon. The matrix P doesn't preserve the area and scales the height of the parallelograms the golden L consists of. Cutting off triangles and gluing them like in Figure 3 shows the statement. □

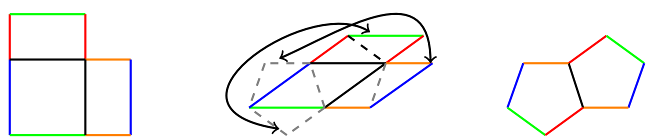


Figure 3: Visualization of the transformation from the golden L to the double pentagon

Let Γ be the Veech group of the golden L and Γ' the Veech group of the double pentagon. Γ' is generated by

$$R = \begin{pmatrix} \cos(\frac{\pi}{5}) & -\sin(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 2\cot(\frac{\pi}{5}) \\ 0 & 1 \end{pmatrix}. \quad [5]$$

Define

$$\sigma_0 := \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}, \sigma_1 := \begin{pmatrix} \phi & \phi \\ 1 & \phi \end{pmatrix}, \sigma_2 := \begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix}, \sigma_3 := \begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}. \quad [2]$$

Proposition 3.2. *The σ_i, $i \in \{0, 1, 2, 3\}$ generate the Veech group Γ of the golden L.*

Proof. We use the shear matrix P from Lemma 3.1. First we define τ as

$$\tau = \sigma_0 \cdot \sigma_3^{-1} \cdot \sigma_0 \cdot \sigma_3^{-1} \cdot \sigma_0 \cdot \sigma_0 = \begin{pmatrix} 0 & -1 \\ 1 & \phi \end{pmatrix} \text{ with } \sigma_3^{-1} = \begin{pmatrix} 1 & 0 \\ -\phi & 1 \end{pmatrix}.$$

Conjugating σ_0, respectively τ with P i.e.,

$$P \cdot \sigma_0 \cdot P^{-1} = T \text{ and } P \cdot \tau \cdot P^{-1} = R.$$

delivers the generators of Γ'. Γ and Γ' are conjugated groups[1], because by Proposition 2.27 the Veech Group is a subgroup of $GL(2, \mathbb{R})$, $P \in GL(2, \mathbb{R})$ and $P\Gamma P^{-1} = \Gamma'$.

[1]Two subgroups U, V of G are said to be conjugated, if there is $g \in G$, such that $gUg^{-1} = V$.

Therefore, σ_0 and τ respectively

$$\sigma_0 \text{ and } \rho = \tau \cdot \sigma_0^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

generate the Veech group Γ of the golden L. Furthermore, $\sigma_i \in \Gamma, i \in \{1, 2, 3\}$, because

$$\begin{aligned} \sigma_1 &= \sigma_0 \cdot \rho \cdot \sigma_0 \\ \sigma_2 &= \sigma_0 \cdot \rho \cdot \sigma_0 \cdot \rho \cdot \sigma_0 \\ \sigma_3 &= \sigma_0 \cdot \rho \cdot \sigma_0 \cdot \rho \cdot \sigma_0 \cdot \rho \cdot \sigma_0 \end{aligned}$$

You can find these calculations in more detail in the Appendix A. $\square$

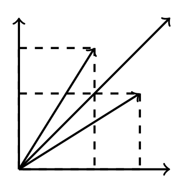
4 The tree of saddle connections

In [2] (chapter 2.2) an algorithm to construct all directions of saddle connections of the golden L is presented, which shall be explained in detail in this section.
We start with the matrices from last section

$$\sigma_0 = \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} \phi & \phi \\ 1 & \phi \end{pmatrix}, \sigma_2 = \begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}.$$

They divide the first quadrant Σ in $\mathbb{R}^2$ into four parts:

$\sigma_0\Sigma = \Sigma_0 := \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : 0 \leq y < \frac{1}{\phi}x\}$
$\sigma_1\Sigma = \Sigma_1 := \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : \frac{1}{\phi}x \leq y < x\}$
$\sigma_2\Sigma = \Sigma_2 := \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : x \leq y < \phi x\}$
$\sigma_3\Sigma = \Sigma_3 := \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : \phi x \leq y\}$



Remark 4.1. *A saddle connection on the golden L has its slope in* $\mathbb{Q}[\sqrt{5}]$ *(this is covered in [1]).*

In each direction that admits a saddle connection there exists one short and one long cylinder which are at a circumference ratio of ϕ. They form a cylinder decomposition and have the same modulus.
Furthermore there are always one short and two long saddle connections. The easiest

direction to see this is the horizontal direction. One long saddle connection connects the left bottom corner with the right bottom corner of the 1×1-square. The second long saddle connection connects the two upper corners of the 1×1-square. The short saddle connection connects the two bottom corners of the right small rectangle (respectively the two upper corners, this is the same saddle connection). The long and the short saddle connection are also at a ratio of ϕ.

Lemma 4.2 (2.4. in [2])**.** *A vector v is a saddle connection on the double pentagon $\Leftrightarrow P^{-1}v$ is a saddle connection on the golden L.*

Proof. Let v be a saddle connection on the double pentagon. By Lemma 2.26, $P^{-1}v$ is still a saddle connection on the golden L because $P \in GL^+(2, \mathbb{R})$. See the mapping in Figure 3. $\square$

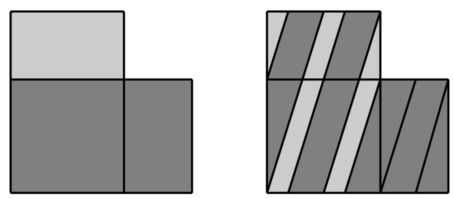


Figure 4: Two examples of a periodic direction with the long cylinder (dark grey) and the short cylinder (light grey).

Now let Λ (resp. Λ_i) be the intersection of the set of long saddle connections with Σ (resp. Σ_i). Let moreover be $A = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ and A^* be the set of all finite words in A including the empty word. For $\sigma \in A^*, \sigma = \sigma_{k_1}, \sigma_{k_2}, ..., \sigma_{k_n}, k_i \in \{1, 2, 3, 4\}$ we set $m(\sigma) = \sigma_{k_n} \cdot \sigma_{k_{n-1}} \cdot ... \cdot \sigma_{k_1}$ as the matrix product.

Theorem 4.3 (Tree Theorem [2])**.** *The set of all vectors $m(\sigma)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ for $\sigma \in A^*$ gives the entire set Λ. Furthermore, leaving out the words that start with σ_0, we get a bijection between the set $M := \{m(\sigma)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) : \sigma \in A^*\}$ and Λ.*

Proof. The shortest short saddle connection lies in the horizontal direction and the long and the short saddle connections are always at a ratio of ϕ. Then the shortest long saddle connection also has to be $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.
σ_0 sends $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ to itself and $\sigma_0, \sigma_1, \sigma_2$ and σ_3 send any other vector in Σ to a longer vector in Σ. In particular $\sigma_i,\ i \in \{1, 2, 3, 4\}$ sends any vector in Σ into Λ_i. Thus the σ_i^{-1} send any vector in Σ_i to a shorter vector in Σ.
By Lemma 2.26 saddle connections are sent to saddle connections by σ_i or σ_i^{-1} and Λ is discrete. If we start with any long saddle connection in Λ_i and apply σ_i^{-1}, we get a shorter long saddle connection in Σ. Eventually we'll get to the shortest long saddle connection $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. If we start the other way around with $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and apply the σ_i

this generates the entire set Λ. Since σ_0 sends $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ to itself, leaving out all the words that start with σ_0, we get the last statement. $\square$

For every direction that admits a saddle connection on the golden L, this algorithm delivers one of the two long saddle connections. These can be presented in a tree structure. We start with $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ as root and apply σ_i, $i \in \{1,2,3\}$, because $\sigma_0\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. From there on we apply all σ_i, hence from every saddle connection we get four saddle connections in the next level of the tree. The first eight saddle connections in this tree are pictured in Figure 5.

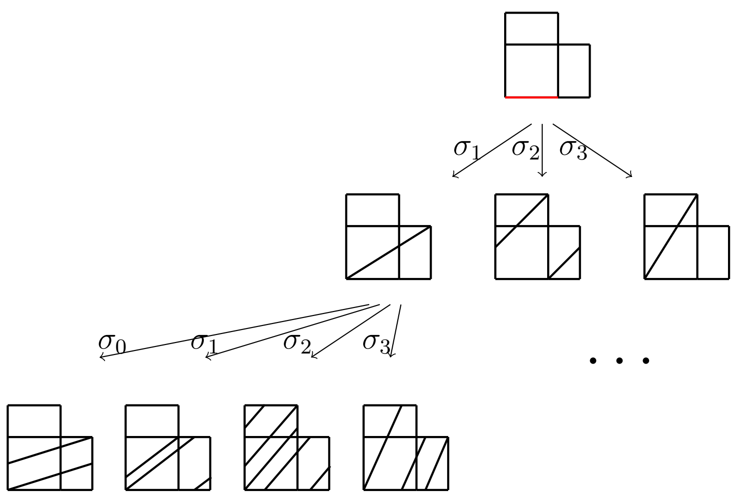


Figure 5: The tree of long saddle connections

Remark 4.4. *We could also start with the saddle connection $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, which delivers the same saddle connections and from the third level on results in the same tree. That is because*

$$\sigma_3\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \sigma_0\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}\phi\\1\end{smallmatrix}\right), \sigma_1\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}\phi\\\phi\end{smallmatrix}\right), \sigma_2\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\\phi\end{smallmatrix}\right).$$

Then the first node is a saddle connection in the direction $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and further instead of σ_1, σ_2 and σ_3, we apply σ_0, σ_1, and σ_2. From the third level on, the tree looks the same as in Figure 5.

4.1 The expanded tree of saddle connections

The above tree only delivers vectors with positive slope in the first quadrant. We can expand the above tree by using the inverse matrices σ_i^{-1}, which divide $\Sigma^{-1} := \{(x,y) \in R^2 : x > 0 \text{ and } y \le 0\}$ (the fourth quadrant) into four parts:

$\sigma_0^{-1}\Sigma = \Sigma_0^{-1} := \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : -\frac{1}{\phi}x < y \leq 0\}$

$\sigma_1^{-1}\Sigma = \Sigma_1^{-1} := \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : -x < y \leq -\frac{1}{\phi}x\}$

$\sigma_2^{-1}\Sigma = \Sigma_2^{-1} := \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : -\phi x < y \leq -x\}$

$\sigma_3^{-1}\Sigma = \Sigma_3^{-1} := \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : y \leq -\phi x\}$

We proceed like before in order to expand the tree by the negative slopes in the fourth quadrant. Let $\bar{A} = \{\sigma_0^{-1}, \sigma_1^{-1}, \sigma_2^{-1}, \sigma_3^{-1}\}$ and $\bar{A}^*$ be the set of all finite words in $\bar{A}$ including the empty word. In an analogous way, denote by $m(\sigma)$ the matrix product for a word $\sigma \in \bar{A}^*$. Furthermore, let Λ' be the intersection of the long saddle connections with Σ^{-1}.

Theorem 4.5 (Expanded tree)**.** *The set of all vectors $m(\sigma)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ for $\sigma \in \bar{A}^*$ gives the entire set Λ'. Leaving out the words, that start with σ_0^{-1}, we get a bijection between the set $\overline{M} := \{m(\sigma)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) : \sigma \in \bar{A}^*\}$ and Λ'.*

Proof. The proof follows the same argumentation as in 4.3. □

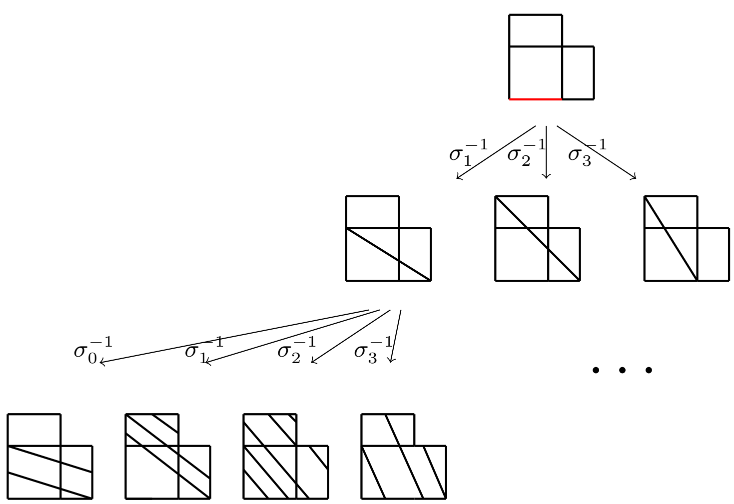


Figure 6: The tree of long saddle connections with negative slope

Remark 4.6. *The saddle connections in Figure 6 differ from the ones in Figure 5, but their holonomy vectors in $\mathbb{R}^2$ are symmetric to the x-axis.*

Remark 4.7. *There is a way to generate the saddle connections in the second and third quadrant, too.*
*For the **third quadrant**, put $\rho^2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ defined in the proof of Proposition 3.2 in front of every word and apply Theorem 4.3. The matrix ρ^2 is*

a rotation by the angle π. This mirrors all vectors from the first quadrant point-symmetrically around the origin into the third quadrant. We get

$$\rho^2\sigma_0 \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \begin{pmatrix}-1\\0\end{pmatrix}, \ \rho^2\sigma_1 \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \begin{pmatrix}-\phi\\-1\end{pmatrix}, \ \rho^2\sigma_2 \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \begin{pmatrix}-\phi\\-\phi\end{pmatrix}, \ \rho^2\sigma_3 \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \begin{pmatrix}-1\\-\phi\end{pmatrix}.$$

This works similarly for the ***second quadrant****, with the same matrix ρ^2 and Theorem 4.5. This rotates all vectors from the fourth quadrant point-symmetrically around the origin by the angle π, i.e.,*

$$\rho^2\sigma_0^{-1} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \begin{pmatrix}-1\\0\end{pmatrix}, \ \rho^2\sigma_1^{-1} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \begin{pmatrix}-\phi\\1\end{pmatrix}, \ \rho^2\sigma_2^{-1} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \begin{pmatrix}-\phi\\\phi\end{pmatrix}, \ \rho^2\sigma_3^{-1} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \begin{pmatrix}-1\\\phi\end{pmatrix}.$$

5 The graph of slopes

In this section, all previous findings are put together to construct a part of the graph of slopes.
The vertices of the graph are equivalence classes of parallel saddle connections. The algorithm from section 4 delivers one of the two long saddle connections in each direction that admits a saddle connection. Its slope presents one vertex in the graph of slopes.

Proposition 5.1. *The tree of saddle connections and the expanded tree with the inverse matrices determine all directions of saddle connections.*

Proof. By Theorem 4.5 and 4.3, we find all long saddle connections in the first and fourth quadrant, i.e., all directions of saddle connections. In Remark 4.7 is explained how the saddle connections in the second and third quadrant can be generated. However, this doesn't deliver any new saddle connections, they have the same length and direction. □

There is an edge between two different equivalence classes θ and $\bar{\theta}$ if there is a saddle connection with slope θ that doesn't intersect a saddle connection with slope $\bar{\theta}$.
Recall that the matrices $\sigma_i, i \in \{0, 1, 2, 3\}$ not only generate the tree of long saddle

connections, but also they generate the Veech group. Its properties help us to reduce the problem in the following way:

Proposition 5.2. *If two saddle connections in the tree don't intersect, they still don't intersect after applying the same matrix $\sigma_i, i \in \{1,2,3,4\}$, on both. In other words: If $m(\sigma)\binom{1}{0}$ and $m(\sigma')\binom{1}{0}$ for $\sigma, \sigma' \in A^*$ or $\bar{A}^*$ don't intersect, $m(\sigma,\sigma_i)\binom{1}{0}$ and $m(\sigma',\sigma_i)\binom{1}{0}$, respectively $m(\sigma,\sigma_i^{-1})\binom{1}{0}$ and $m(\sigma',\sigma_i^{-1})\binom{1}{0}$, don't intersect for $\sigma_i \in \{\sigma_0,\sigma_1,\sigma_2,\sigma_3\}$ and $\sigma_i^{-1} \in \{\sigma_0^{-1},\sigma_1^{-1},\sigma_2^{-1},\sigma_3^{-1}\}$.*

Proof. The Veech group Γ is defined to be the stabilizer of the group action of $GL^+(2,\mathbb{R})$ on the golden L. Furthermore, by Lemma 2.26 saddle connections are sent to saddle connections by $GL^+(2,\mathbb{R})$ and because $SL(2,\mathbb{R})$ is a subgroup of $GL^+(2,\mathbb{R})$, by Proposition 2.27 this also applies to Γ.
This means that for $A \in \Gamma$ and two non intersecting saddle connections u and $v \in \mathcal{SC}(X)$, $A(u)$ and $A(v)$ are again saddle connections. The translation surface $A(X)$ is isomorphic to X and can be transformed into X by cutting and gluing. Then $A(u)$ and $A(v)$ don't intersect in X.

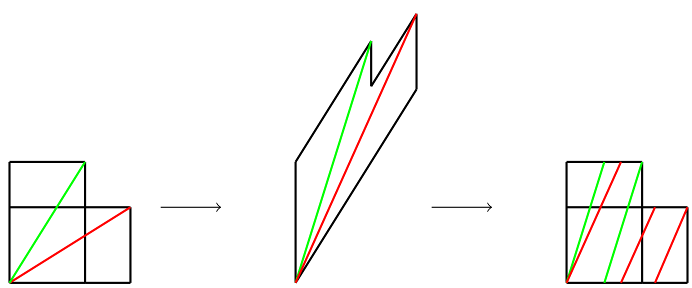


Figure 7: On the left: $\sigma_3\begin{pmatrix}1\\0\end{pmatrix}$ (green) and $\sigma_1\begin{pmatrix}1\\0\end{pmatrix}$ (red). Application of σ_3 to both saddle connection vectors and the golden L illustrates that the vectors still connect corners. On the right: $\sigma_3\sigma_3\begin{pmatrix}1\\0\end{pmatrix}$ (green) and $\sigma_3\sigma_1\begin{pmatrix}1\\0\end{pmatrix}$ (red) back don't intersect in golden L.

$\square$

The σ_i divide the first quadrant into four parts, $\Sigma_0, \Sigma_1, \Sigma_2$ and Σ_3 (see previous section). This means that a vector resulting from a tree word $\sigma = \sigma_{k_1}, ..., \sigma_{k_n} \in A^*$ lies in section Σ_{k_n}. This leads to the following results and we can further reduce the problem.

Proposition 5.3. *Saddle connections corresponding to a tree word ending with σ_3 intersect with those that correspond to a word ending with σ_0 (except for the saddle connections $\sigma = \sigma_0$ and $\sigma' = \sigma_3$, which don't intersect).*

Proof. A saddle connection u of the form $\sigma = \hat{\sigma}, \sigma_3$ with an arbitrary $\hat{\sigma} \in A^*$ and word length > 1 has a slope $> \phi$.
A saddle connection v of the form $\sigma' = \hat{\sigma}, \sigma_0$ with an arbitrary $\hat{\sigma} \in A^*$ has a slope $< \frac{1}{\phi}$.
For both, there are three possible "starting corners": the left bottom corner of the 1×1-square, the left bottom corner of the upper small rectangle and the left bottom corner of the right small rectangle.
If u starts from the left bottom corner of the 1×1-square or the left bottom corner of the the upper small rectangle, it will meet the upper edge of the upper small rectangle and continue from the bottom edge of the 1×1-square. If u starts from the left bottom corner of the 1×1-square or the left bottom corner of the right small rectangle, it will meet the right edge of the right small rectangle and continue from the left edge of the 1×1-square. Then v and u intersect.
Notice the symmetry with respect to the axis $x = y$ of σ_0 and σ_3 in $\mathbb{R}^2$. Then the above argumentation holds for all other cases, since the comparison of saddle connections u with slope $\frac{1}{\phi}$ "starting" from the left bottom corner of the upper small rectangle and v with slope ϕ "starting" from the left bottom corner of the right small rectangle is symmetric to the comparison of the same saddle connections with switched "starting" corners.
This applies for long saddle connections as well as short saddle connections.
In the case that the saddle connections lie entirely in one small rectangle each and therefore don't intersect, these can only be short saddle connections. This is because saddle connections are contained in the boundary of a cylinder, and only the short cylinder can fit entirely into the small rectangles. Then the long saddle connection in one direction intersects the short saddle connection in the other direction. $\square$

For the first three levels of the tree, the pairwise comparisons, whether two saddle connections intersect, can be calculated manually with the help of **Geogebra** and **Python**, which delivers the following picture:

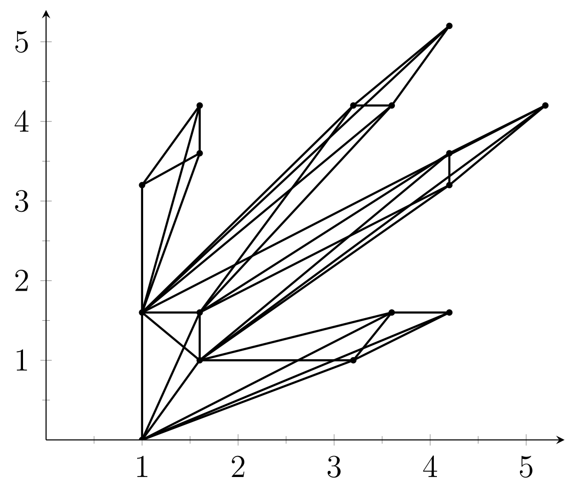


Figure 8: The first three levels of the tree of saddle connections in the graph of slopes

Figure 10 shows only the edges that result from the non intersecting long saddle connections from Figure 5. It is possible that edges are missing because in each direction there is one other long saddle connection and one short saddle connection that might not intersect a saddle connection from another equivalence class.
You can find a version of the graph with labeled vertices in the Appendix B.
Analogously to the Farey Graph we can project the holonomy vectors onto the unit circle and picture the edges of the graph as circle segments. The result is Figure 9.

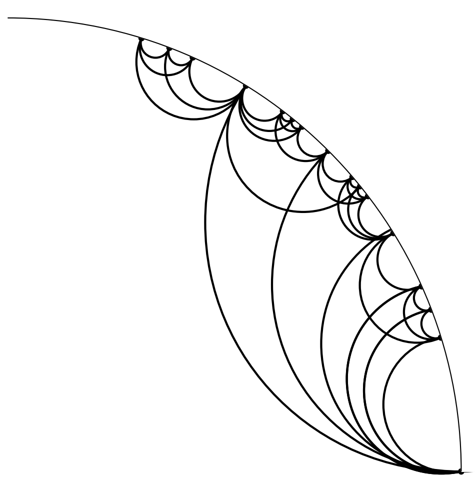


Figure 9: The graph of slopes visualized in the unit circle with circle segments as edges.

Remark 5.4. *We want to expand the graph by the saddle connections with negative slope, that we get with Theorem 4.5. These have the same slope although with negative sign, but we can't simply mirror the above graph along the horizontal axis. This is because the saddle connections with positive slope are not symmetric to the ones with negative slope as Figure 6 shows. The saddle connection* $\sigma_1 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ *doesn't intersect* $\sigma_3 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$*, but* $\sigma_1^{-1} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ *intersects* $\sigma_3^{-1} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ *for example.*
Furthermore it remains to determine the edges between the positive and the negative slopes to complete the picture.

Remark 5.5. *The presentation of the graph is different to the Farey Graph in figure 1. The latter shows the saddle connection* $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ *with slope* ∞ *on the left in the circle and the saddle connection* $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ *with slope* 1 *on top of the circle. Then the saddle connections with positive slope are in the upper half circle and the ones with negative slope are in lower half. With our approach, we only get one half circle with all slopes of saddle connections.*

6 Outlook

Since the graph we constructed is relatively small, it might be interesting to see what happens if we add more vertices and edges to the picture. It requires 120 pairwise comparisons to construct the graph of slopes as pictured above and already 2016 pairwise comparisons for the next level in the tree. Calculating this manually is inefficient. In order to extend the graph with more vertices, we need an algorithm that first generates the saddle connections until a certain level $N \in \mathbb{N}$ in Figure 5 and Figure 6. Then the algorithm has to understand the translation surface, such that it recognizes identified edges. According to these edge identifications, it knows how a saddle connection and its associated cylinder look like on the golden L and is able to search for disjoint directions. Here we can rule out the comparisons, whose slope is too different, see Proposition 5.3, because they intersect. By Proposition 5.2, we can exclude whole paths of non-intersecting directions in the tree.

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A Auxiliary calculations from Proposition 3.2

Let ϕ be the golden ratio, i.e., the positive solution to the equation $x^2 - x - 1 = 0$. Recall the matrices from section 3 with its inverses:

$$\sigma_0 = \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} \phi & \phi \\ 1 & \phi \end{pmatrix}, \sigma_2 = \begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix},$$

$$\sigma_0^{-1} = \begin{pmatrix} 1 & -\phi \\ 0 & 1 \end{pmatrix}, \sigma_1^{-1} = \begin{pmatrix} \phi & -\phi \\ -1 & \phi \end{pmatrix}, \sigma_2^{-1} = \begin{pmatrix} \phi & -1 \\ -\phi & \phi \end{pmatrix}, \sigma_3^{-1} = \begin{pmatrix} 1 & 0 \\ -\phi & 1 \end{pmatrix}$$

Recall also the matrix P from Lemma 3.1:

$$P = \begin{pmatrix} 1 & \cos(\frac{\pi}{5}) \\ 0 & \sin(\frac{\pi}{5}) \end{pmatrix}$$

The matrices τ and σ_0 are the generators of the Veech group of the golden L. This is because conjugating σ_0, respectively τ with P, delivers the generators R and T of the Veech group of the double pentagon:

$$\begin{aligned}
\tau =& \sigma_0\sigma_3^{-1}\sigma_0\sigma_3^{-1}\sigma_0\sigma_0 \\
=& \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\phi & 1 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\phi & 1 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} \\
=& \begin{pmatrix} -\phi & \phi \\ -\phi & 1 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\phi & 1 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} \\
=& \begin{pmatrix} -\phi & -1 \\ -\phi & -\phi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\phi & 1 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} \\
=& \begin{pmatrix} 0 & -1 \\ 1 & -\phi \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & \phi \end{pmatrix}
\end{aligned}$$

$$P\tau P^{-1} = \begin{pmatrix} 1 & \cos(\frac{\pi}{5}) \\ 0 & \sin(\frac{\pi}{5}) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \phi \end{pmatrix} \begin{pmatrix} 1 & -\frac{\cos(\frac{\pi}{5})}{\sin(\frac{\pi}{5})} \\ 0 & \frac{1}{\sin(\frac{\pi}{5})} \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\frac{\pi}{5}) & -1 + 2\cos^2(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) & 2\cos(\frac{\pi}{5})\sin(\frac{\pi}{5}) \end{pmatrix} \begin{pmatrix} 1 & -\frac{\cos(\frac{\pi}{5})}{\sin(\frac{\pi}{5})} \\ 0 & \frac{1}{\sin(\frac{\pi}{5})} \end{pmatrix} = \begin{pmatrix} \cos(\frac{\pi}{5}) & -\sin(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) \end{pmatrix} = R$$

$$P\sigma_0 P^{-1} = \begin{pmatrix} 1 & \cos(\frac{\pi}{5}) \\ 0 & \sin(\frac{\pi}{5}) \end{pmatrix} \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\cos(\frac{\pi}{5})}{\sin(\frac{\pi}{5})} \\ 0 & \frac{1}{\sin(\frac{\pi}{5})} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3\cos(\frac{\pi}{5}) \\ 0 & \sin(\frac{\pi}{5}) \end{pmatrix} \begin{pmatrix} 1 & -\frac{\cos(\frac{\pi}{5})}{\sin(\frac{\pi}{5})} \\ 0 & \frac{1}{\sin(\frac{\pi}{5})} \end{pmatrix} = \begin{pmatrix} 1 & 2\cot(\frac{\pi}{5}) \\ 0 & 1 \end{pmatrix} = T$$

We can reduce τ by one σ_0 and call this matrix ρ:

$$\rho = \tau\sigma_0^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & \phi \end{pmatrix} \begin{pmatrix} 1 & -\phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We find products of σ_0 and ρ to show that σ_1, σ_2 and σ_3 are in the Veech group of the golden L:

$$\sigma_0\rho\sigma_0 = \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \phi & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \phi & \phi \\ 1 & \phi \end{pmatrix} = \sigma_1$$

$$\sigma_0\rho\sigma_0\rho\sigma_0 = \sigma_1\rho\sigma_0$$

$$= \begin{pmatrix} \phi & \phi \\ 1 & \phi \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \phi & -\phi \\ \phi & -1 \end{pmatrix} \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix} = \sigma_2$$

$$\sigma_0\rho\sigma_0\rho\sigma_0\rho\sigma_0 = \sigma_2\rho\sigma_0$$

$$= \begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\phi \\ \phi & -\phi \end{pmatrix} \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix} = \sigma_3$$

B The graph of slopes with labeled saddle connections

In section 5, we have constructed a part of the graph of slopes of the golden L. In this figure, the saddle connections are labeled with the words that they result from. For example, the word 13 refers to the saddle connection $\sigma_3\sigma_1 \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \begin{pmatrix} \phi \\ \phi^2+1 \end{pmatrix}$.

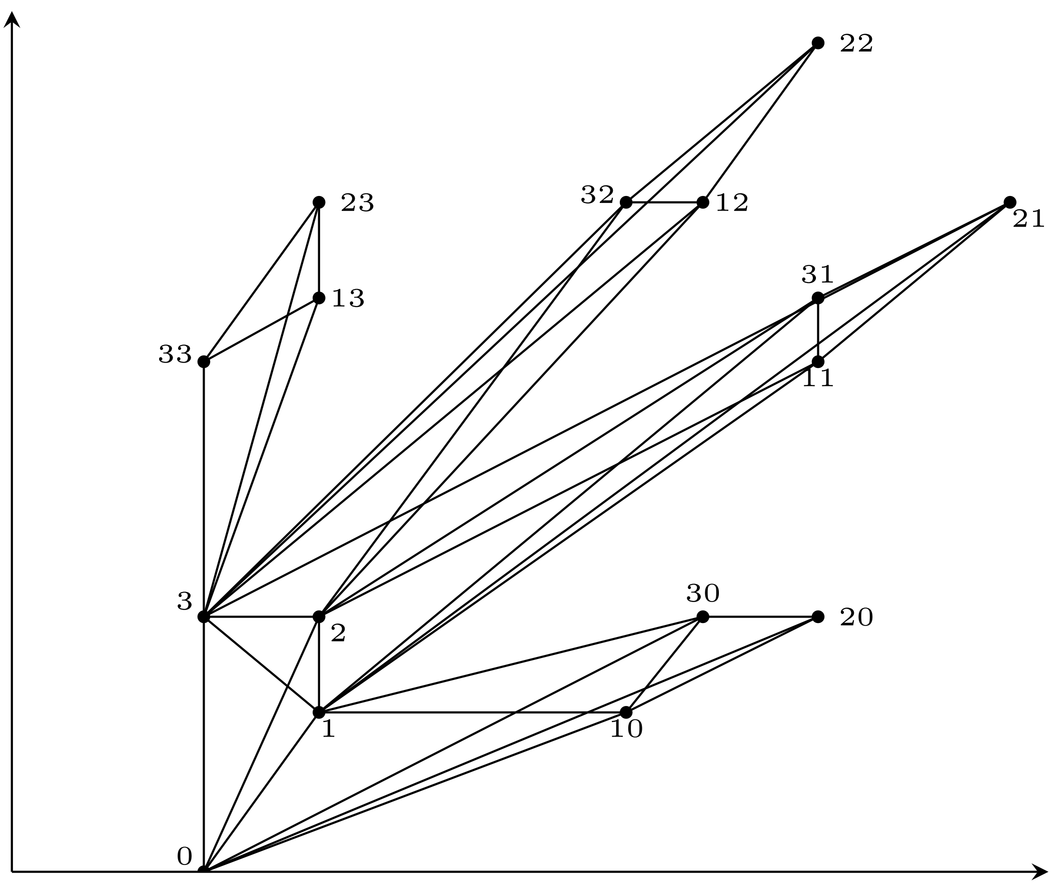

Figure 10: The first three levels of the tree of saddle connections in the graph of slopes with labels