Universal Constants in 3-Dimensional Hyperbolic Manifolds

Bachelor's Thesis

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Declaration of Authorship

I hereby declare, that this thesis is my own, unaided work. All used sources are acknowledged as references.

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1 Introduction

The topic of this Bachelor's thesis are 3-dimensional hyperbolic manifolds. These are all 3-dimensional manifolds, with an hyperbolic metric, meaning a metric such that the space is *negatively curved* at every point and in very direction. These spaces are in some sense the strangest of the standard three geometries: Euclidean, spherical and hyperbolic.
Usually it is hard to explain to a person, what the defining factor for a space being hyperbolic is. Euclidean geometry is easy to describe, because we are confronted with it regularly, as the space we live in, is at least locally Euclidean. (The global geometry of the universe is not yet known.) Also there are other well known Euclidean shapes like a flat torus, which everyone familiar with a bit of differential geometry can explain.
With spherical geometry it already becomes a bit harder, but at least in dimensions one and two there are vivid examples, the circles and spheres. Here intuition can still guide us pretty far, since we are already somewhat used to the geometry of spheres because of the spherical shape of earth's surface.
In contrast to those, there seems to be no space lending itself to hyperbolic geometry quiet in the same way as for the other two. A problem with visualizing hyperbolic spaces is their immense growth. Conversely to a circle on a sphere, where the the area of the circle grows less with the radius than in Euclidean space, the area a hyperbolic circle contains, grows exponentially with its radius. So there is not enough space to show a hyperbolic plane in Euclidean 3-space without it "crinkling up". Interestingly though, such crinkled planes can be seen in nature more than a few times for example with some jellyfishes, sponges and lettuces. So, also hyperbolic shapes appear in nature in some ways.
Mathematically seen, they even offer the richest variety of geometric three-dimensional shapes. It is a basic result in Riemannian geometry, that every smooth manifold can be equipped with a Riemannian metric. Today we know as well, that "most" three-dimensional, smooth and compact manifolds carry actually a hyperbolic metric. This alone tells us, that there must be much to discover when studying these spaces.

Interest in and research on hyperbolic geometry goes back already a long time. One inspiration for many early geometers was Euclid's *Elements* from ca. 300 B.C.. For many centuries geometers tried to either prove Euclid's famous parallel postulate, which in modern words states, that *"For every straight line l and every point P not on l there is at most one straight line l_2 in the plane determined by l and P, that contains p and does not intersect l."* from the other axioms given in *Elements* or tried to show its independence from the other axioms. This lead to the discovery of first results on other forms of geometry as early as the end of the 18th century. Then, since the middle of the 19th century hyperbolic geometry, which can be based on negating Euclid's parallel postulate, became a recognized field of study chiefly influenced by the work of mathematicians Nikolai Ivanovich Lobachevsky, Carl Friedrich Gauss, Eugenio Beltrami and Felix Klein, and has been very active ever since.

My motivation to study specifically hyperbolic geometry comes from its richness and its great importance in many areas of mathematics. In number theory there are modular forms, which are functions invariant under subgroups of the group of isometries of $\mathbb{H}^2$. In the realm of complex analysis, we have Riemann surfaces, which mostly can only wear an hyperbolic metric. Even in applied mathematics hyperbolic geometry appears in special relativity theory, when studying relativistic velocity. On the whole. certain aspects of hyperbolic geometry appear in so many distinct mathematical areas, which makes it arguably the most interesting and most important of the non-standard geometries. Also there is a vast amount of extremely interesting results on it, yet research on hyperbolic geometry is not nearly closed, as so many questions remain open.
Prerequisites for reading this thesis are only what an introductory course on differential geometry teaches you about differential and Riemannian geometry and some general topology. Also basic knowledge of groups and group actions is expected, because we will be using a lot of terms from these fields. Especially, there is no need to have previous knowledge of hyperbolic geometry, as we will discuss almost everything we are going to need from the start. If something is left out, there will be given a source, where one can read the details of what was left unproven. The proofs in this thesis will mostly use knowledge about the isometries of the standard simply connected hyperbolic space in three dimensions $\mathbb{H}^3$, which will be introduced in detail. With some group theory and some complex analysis this can be used to proof quiet a few interesting results. Obviously we will also make use of some geometric and topological arguments as well as basic real analysis.

This thesis is mainly based on chapters one to three of the book *Outer Circles, An Introduction to Hyperbolic 3-manifolds* by Albert Marden from 2007, [Mar07], but also uses other sources. The first chapter of this book gives also a short introduction to general 2- and 3-dimensional hyperbolic geometry from the perspective of differential geometry. For a discussion of hyperbolic space, which is not restricted to 3-dimensions the reader may take a look at the book *Foundations of Hyperbolic Manifolds* by John G. Ratcliffe, [Rat19]. The main theorems of this thesis in chapter 5 were first proven in the 1960's and 1970's. The pages 114-115 of [Mar07] contain some information on the original proofs.
As already mentioned, the goal of this thesis is to look at complete and connected manifolds with an hyperbolic metric, meaning the curvature of these manifolds at every point is the same as that of $\mathbb{H}^3$, the standard, simply connected, hyperbolic space. This makes them locally isometric to $\mathbb{H}^3$, but not globally. For their actual geometry and also topology in 3 dimensions there are rich possibilities, which are subject to ongoing research. In contrast to this Euclidean or spherical 3-manifolds are in some sense all known and classified. Actually, in 3 dimension there are also other more exotic types of geometry on manifolds besides hyperbolic, spherical and flat, but we will not mention these. Of course, we are not going to be able to reach current research levels in hyperbolic geometry in this thesis because of how deep this field is and how long people have already done research on it.
Let us now talk about, what we will do in this thesis regarding hyperbolic manifolds.

Each of them is isomorphic to $\mathbb{H}^3/G$, where G is a discrete subgroup of the group of isometries of hyperbolic 3-space $\mathbb{H}^3$, Isom($\mathbb{H}^3$). This group is isomorphic to the quotient of the matrix group $SL(2,\mathbb{C})$ by $\pm$Id, called $PSL(2,\mathbb{C})$, and acts as Möbius transformations on the Riemann sphere $\mathbb{C}\cup\infty$, which can be seen as a type of boundary of $\mathbb{H}^3$. This way we are given a relatively easy way to understand the qualitative behaviour of these isometries. The final goal is, using our understanding of Möbius transformations and subgroups of $PSL(2,\mathbb{C})$, to proof the existence of constants, which limit the maximal complexity of the geometric and topological structure of every possible hyperbolic 3-manifold in certain "small" regions of these manifolds. For example we will proof that a neighbourhood of a small enough closed geodesic, is locally equivalent to a solid tube, whose ends have been glued together. Moreover, we will show that for regions, where the manifold is "thin" in some way, we can exactly determine the topological and geometric structure. This is known as the *Thick-Thin-composition.* As the most important result on the way to discuss this, we will also proof a version of the Margulis lemma for 3-dimensional hyperbolic space.
Working towars that goal, we will in section 2 set up the definitions and knowledge needed about hyperbolic 3-space and especially its isometries, the group of Möbius transformations of the Riemann sphere $\mathbb{C}\cup\infty$. We will show how they act geometrically on $\mathbb{C}\cup\infty$ and via Poincaré extension on our model of $\mathbb{H}^3$. Doing this, we will discover the existence of three different types of transformations. Two of them are analogous to the Euclidean isometries translation and rotations around an axis, while one type is new and unique to hyperbolic geometry. We will discuss in section 4 some of its geometric properties.
In the third section of this thesis we will concern ourselves with sequences of Möbius transformations and their convergence behaviour on $\mathbb{C}\cup\infty$ and $\mathbb{H}^3$, discussing some cases of subgroups we are particularly interested in, namely discrete and discrete elementary groups.
These groups will be important in the next section, section 4. Here we take a look at the quotient of $\mathbb{H}^3$ by discrete subgroups of Isom($\mathbb{H}^3$). If the group action is additionally free on $\mathbb{H}^3$, then these are the hyperbolic manifolds, that we are interested in. The fundamental group of the hyperbolic manifold can then be identified with the discrete group used to construct it. If the group action is not free, the quotient is an *orbifold*, a generalization of a manifold. We will not always restrict us to the case of manifolds and also prove some results in the more general regime of orbifolds, because many simple examples of discrete groups contain elements that fix points, resulting in the quotient being an orbifold. Also, we discuss some distinctive topological features of these manifolds/orbifolds, where we will come back to the third kind of isometries from section 2.
Section 5 is an application of the results so far to so called *Schottky groups.* We construct these kind of discrete groups concretely, and see what we can say about them and their associated quotient manifold with the knowledge we gained.
The penultimate section, section 6, is the heart of the thesis: We will prove 5 theorems, giving us constants mostly independent of the actual hyperbolic manifold in question, which determine geometrically "simple" regions of the manifold. So we can identify these regions across all kinds of hyperbolic manifolds, and limit their complexity. The first, Theorem 6.3 shows us the shape of two types of regions that stretch out infinitely

far away from the rest of the manifold. The second one, Theorem 6.8, assures us that around any short geodesic in any hyperbolic manifold there is a region isomorphic to a solid closed tube of with constant radius. The third one, Theorem 6.7, is only for orbifolds and tells us that we can control the number of axis of rotational isometries, whose axes can intersect a small enough region. The second-to-last one, Theorem 6.11, is the Margulis lemma for $\mathbb{H}^3$ and shows, colloquially speaking, that every neighbourhood of radius less than the Margulis constant in any hyperbolic manifold or orbifold is of relatively easy structure. Lastly, the final theorem, Theorem 6.12, uses this to state the existence of an embedded ball of fixed radius in any hyperbolic manifold, which for example gives us a minimal volume of each hyperbolic manifold. Finitely, the last section is about an application of the Margulis lemma to hyperbolic manifolds. It shows neatly the geometric significance of the Margulis lemma and actually gives us a more holistic understanding of the ”thin” parts of the manifold than the separate theorems from section 5, because we look at the whole manifold at once, rather than only small regions.

2 Preliminaries

2.1 Hyperbolic Space

Hyperbolic n-space $\mathbb{H}^n$ is the n-dimensional, connected, simply connected and complete Riemannian manifold of constant sectional curvature -1 and the base space for hyperbolic geometry. As such it is the analogue to the n-sphere $\mathbb{S}^n$ for spherical geometry, which has constant sectional curvature $+1$ and the Euclidean space $\mathbb{R}^n$ for Euclidean geometry, in which the sectional curvature is 0 everywhere.
There are many concrete models for hyperbolic n-space. Here we use only the so called **upper half-space** for three dimensions and the **upper half-plane** for two dimensions.

Definition 2.1. The **upper half-space** model for hyperbolic 3-space $\mathbb{H}^3$ is the set $\mathbb{C} \times (0, \infty)$ with the Riemannian metric $g \in T^{(0,2)}\mathbb{H}^3$

$$g_{(z,t)}(X, Y) = \frac{\langle X, Y\rangle}{t^2}, \quad (z, t) \in \mathbb{C} \times (0, \infty); \quad X, Y \in T_{(z,t)}\mathbb{H}^3 = \mathbb{R}^3 \tag{1}$$

where $\langle\cdot, \cdot\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^3$.
The **upper half-plane** model for hyperbolic 2-space $\mathbb{H}^2$ is the set $\{z \in \mathbb{C} : \Im z > 0\}$ with the Riemannian metric $h \in T^{(0,2)}\mathbb{H}^2$

$$g_z(U, W) = \frac{\langle U, W\rangle}{\Im(z)^2}, \quad z \in \{z \in \mathbb{C} : \Im z > 0\}; \quad U, W \in T_z\mathbb{H}^2 = \mathbb{R}^2 \tag{2}$$

where $\langle\cdot, \cdot\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^2$.

Basic theory on hyperbolic space shows that it is uniquely geodesic in any dimension. This means that for any two points $a, b \in \mathbb{H}^n$ there exists exactly one geodesic up to reparametrization, on which both points lie. The length of the geodesic segment between the two points gives us their distance from each other. We will later prove that the geodesics in the half-plane and half-space model are the vertical straight lines and the half circles orthogonal to the $\mathbb{R}$- respectively $\mathbb{C}$-part of the boundary. (Proposition 2.10). With this we can easily state a general formula of the hyperbolic distance between two points in the half-plane and half-space models:

Proposition 2.2. *Let $(z,t),(w,s) \in \mathbb{H}^3$ in the upper half-space model. Then their hyperbolic distance is given by:*

$$d((z,t),(w,s)) = \operatorname{arcosh}\left(1 + \frac{|z-w|^2 + (t-s)^2}{2ts}\right) \tag{3}$$

For $z = x_1 + iy_1, w = x_2 + iy_2 \in \mathbb{H}^2$ in the upper half-plane model, the hyperbolic distance is given by:

$$d(z,w) = \operatorname{arcosh}\left(1 + \frac{|w-z|^2}{2y_1y_2}\right) \tag{4}$$

Proof. Use the hyperbolic line element $d(z,t)^2 = dx^2 + dy^2 + dt^2/t^2$ repsectively $dz^2 = (dx^2 + dy^2)/y^2$ and a parametrization of the unique geodesic between the two points to compute its length. This is the distance of the two points. □

2.2 The Boundary at Infinity

We introduce the so called **boundary at infinity** $\partial\mathbb{H}^3$ and $\partial\mathbb{H}^2$ on $\mathbb{H}^3$ and $\mathbb{H}^2$, which will help us to visualize the action of isometries on the spaces by their much simpler action on the respective boundary.

Definition 2.3. Consider the Riemannian manifold $\mathbb{H}^n$. Choose a point $O \in \mathbb{H}^n$. A geodesic ray starting at O is a path $\gamma : [0,\infty) \to \mathbb{H}^n$, such that $d(O,\gamma(t)) = t$ for all $t > 0$ and $\gamma(0) = O$. On the set of geodesic rays, we define the equivalence relation $\gamma \sim \sigma$ for rays, that stay at most a fixed distance apart, meaning $d(\gamma(t),\sigma(t)) \leq K$ for some $K \in \mathbb{R}$, and all $t > 0$. The boundary at infinity $\partial\mathbb{H}^n$ is now the set of equivalence classes

$$\partial\mathbb{H}^3 = \{[\gamma] \mid \gamma \text{ is a geodesic ray in } \mathbb{H}^n\}. \tag{5}$$

Remark 2.4.

1. That the relation defined above is indeed an equivalence relation, follows directly from the properties of the metric, $d(x,x) = 0$, $d(x,y) = d(y,x)$ and the triangle inequality.

2. In our cases of interest $\mathbb{H}^2$ and $\mathbb{H}^3$ we can easily identify the boundary at infinity with $\mathbb{R} \cup \infty$ respectively $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. Just consider the upper half-plane and upper half-space model. Then geodesic rays stay at most a bounded distance apart, if and only if their endpoint is the same point $p \in \mathbb{R}$ respectively $p \in \mathbb{C}$ or they go to infinity on the imaginary axis respectively the $\mathbb{R}_+$- component.
   This is because near $\mathbb{R}$ and $\mathbb{C}$ the Riemannian metric increases the distance of

any two non-converging lines through the factor $1/t^2$ resp. $1/y^2$ to infinity. On the other hand, if they go to infinity in the boundary the distance between them decreases monotonically and is bounded by the distance of the starting points.

Now we want to define a topology on the boundary at infinity that extends the topology on $\mathbb{H}^n$. We will choose the so called **cone-topology**:

Definition 2.5. We define a base of the topology given by the sets

$$C(p, v, \epsilon, r) = \{ \; [\exp(tw)] \mid w \in B_v(\epsilon) \subseteq T_p\mathbb{H}^n \} \tag{6}$$
$$\cup \{\exp(tw) \mid w \in B_v(\epsilon) \subseteq T_p\mathbb{H}^n, \; t > r\} \tag{7}$$

for $p \in \mathbb{H}^n$, $v \in T_p\mathbb{H}^n$, $\epsilon, r \in \mathbb{R}^+$ and the open sets in $\mathbb{H}^n$

Figure 1 shows a sketch of a basis $C(p, v, \epsilon, r)$ set of this topology, that contains points of $\mathbb{H}^3$ and of the boundary. For more on the boundary at infinity and for proofs of the statements regarding it we refer to Maubon's lecture notes [Mau] pp. 17-18. Visually one can think of the basis sets that include boundary points as truncated geodesic cones, with apex $p \in \mathbb{H}^n$, where the geodesic rays start at p and go in a direction $w \in B_v(\epsilon) \subseteq T_pM$. They are truncated at distance r from p.

Remark 2.6. With this topology $\mathbb{H}^n$ is homeomorphic to a closed ball $\overline{B^n}$. If one restricts the topology to the boundary, the boundary becomes homeomorphic to the standard sphere $\mathbb{S}^n$. [Mau] p.18.

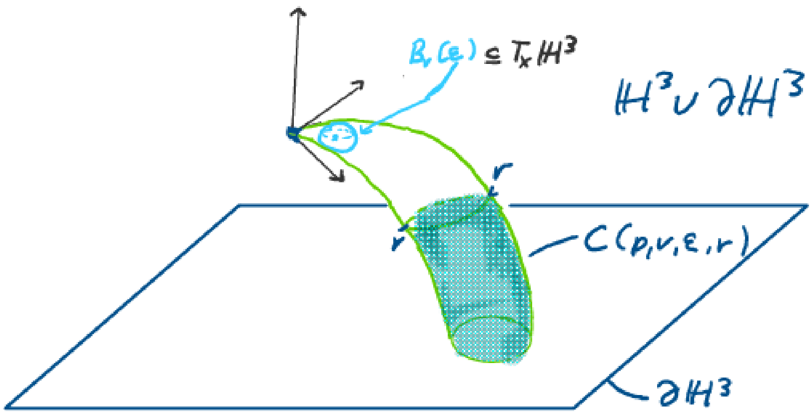


Figure 1: A basis set of the cone-topology containing points of $\mathbb{H}^3$ and $\partial\mathbb{H}^3$. The cone begins in the point $p \in \mathbb{H}^3$

2.3 Möbius Transformations

We now turn to a class of very important maps called *Möbius transformations*:

Definition 2.7. A **Möbius transformation** is a rational function that maps the Riemann sphere bijectively onto itself. It is given by

$$\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \ z \mapsto \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{C}, \ ad - cb = 1 \tag{8}$$

The values a, b, c, d can be thought of as complex entries of a 2×2-matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Because the result of the composition of two Möbius transformation is the same as matrix multiplying their matrices and then using the product matrix as Möbius transformation, we can identify the group of Möbius transformations with the group $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/ \pm \mathrm{Id}$. We do not need the full group $SL(2, \mathbb{C})$, because multiplication of a matrix with $-\mathrm{Id}$ will result in the same Möbius transformation. Usually, we will not distinguish between a Möbius transformation as element of $PSL(2, \mathbb{C})$ and its representation as a matrix in $SL(2, \mathbb{C})$. Hence, we write down a matrix and mean the corresponding Möbius transformation from the equivalence class of this matrix in $PSL(2, \mathbb{C})$.

Right away, we state one important fact about Möbius transformations of $\hat{\mathbb{C}}$:

Proposition 2.8. *Möbius transformations act transitively on triplets of points in $\hat{\mathbb{C}}$. Additionally, Möbius transformations preserve generalized circles in $\hat{\mathbb{C}}$. Combined, this means that Möbius transformations act transitively on the set of generalized circles in $\hat{\mathbb{C}}$*

Proof. The first part of the proof is straightforward. We will simply give a formula for a transformation, that sends any three points $p_2, p_3, p_4 \in \hat{\mathbb{C}}$ to $1, 0, \infty \in \hat{\mathbb{C}}$. This is the so called *cross ratio* of four points:

$$z \mapsto \frac{(z - p_3)(p_2 - p_4)}{(z - p_4)(p_2 - p_3)} = (z, p_2, p_3, p_4) \tag{9}$$

A matrix representing this map, is $T = \left(\begin{smallmatrix} (p_2 - p_4) & -(p_2 - p_4)p_3 \\ (p_2 - p_3) & -(p_2 - p_3)p_4 \end{smallmatrix}\right)$. If we choose $\zeta \in \mathbb{C}$ with $\zeta^2 = \det(T)$, then the matrix $\zeta^{-1}T$ has determinant 1 and consequently is in $SL(2, \mathbb{C})$. However, this yields the same Möbius transformation, because the factor appears in numerator and denominator. Thus, we have proven the claim.
To prove the second claim one first needs to know that generalized circles in $\hat{\mathbb{C}}$ are Euclidean circles in $\mathbb{C}$ and Euclidean lines plus the point ∞. The points z on a generalized circle are given by the formula

$$\alpha z\bar{z} + u\bar{z} + \bar{u}z + \beta = 0, \quad \text{for } \alpha, \ \beta \in \mathbb{R}, \ u \in \mathbb{C}, \ u\bar{u} - \alpha\beta > 0.$$

If $\alpha = 0$ this in an Euclidean line (without ∞), otherwise it is a circle in $\mathbb{C}$. We can rewrite this as all z satisfying

$$(\bar{z}, 1) \cdot H \cdot (z, 1)^T = 0$$

for $H = \left(\begin{smallmatrix} \alpha & u \\ \overline{u} & \beta \end{smallmatrix}\right)$ an indefinite, hermitian matrix with negative real determinant. Moreover, if we identify $(\infty, 1)$ with the vector $(1, 0)$ this formula also shows, whether the point at infinity is on the circle. Now for ϕ, a Möbius transformation with matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{C})$, we get

$$|cz + d|^2 \left((\overline{\phi(z)}, 1) \cdot H \cdot (\phi(z), 1)^T \right) = (\overline{z}, 1) \cdot (\overline{M}^T \cdot H \cdot M) \cdot (z, 1)^T$$

where $\overline{M}^T \cdot H \cdot M$ is again hermitian with negative real determinant. If necessary, we replace the vector $(\infty, 1)$ again with $(1, 0)$. This way we have shown that, if z lies on the circle given by the matrix H, $\phi(z)$ is on the circle given by $\overline{M}^T HM$. Consequently, we have also proven the second claim. From [Wei] pp. 8-9.

□

Furthermore, the action of a Möbius transformation on $\hat{\mathbb{C}} = \partial\mathbb{H}^3$ can be extended to an action on all of $\mathbb{H}^3$ by the formula

$$(z, t) \mapsto \left(-\frac{\overline{z + d/c}}{c^2(|z + d/c|^2 + t^2)} + \frac{a}{c}, \frac{t}{|c|^2(|z + d/c|^2 + t^2)} \right) \quad \text{when } c \neq 0, \tag{10}$$

$$(z, t) \mapsto \left(\frac{a}{d}(z + b/a), \left|\frac{a}{d}\right| t \right) \quad \text{when } c = 0 \tag{11}$$

for a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{C})$. As one can easily see the formula for the action on the boundary is just the boundary case of the formula above for $t \to 0$. This way, the extension is continuous on $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. Also, if we restrict us to Möbius transformations ϕ in $PSL(2, \mathbb{R})$ and their domain to $\mathbb{R} \times (0, \infty) \subseteq \mathbb{H}^3$, the action is again the same as in $\mathbb{R} \times (0, \infty) \subseteq \hat{\mathbb{C}}$.

What makes these maps so special for hyperbolic space, is the statement of our next theorem:

Theorem 2.9. *1. The group $PSL(2, \mathbb{C})$ is the full group of orientation preserving isometries of $\mathbb{H}^3$, by acting as Möbius transformation on the upper half-space model.*

2. The subgroup $PSL(2, \mathbb{R}) \subseteq PSL(2, \mathbb{C})$ is the full group of orientation preserving isometries of $\mathbb{H}^2$, by acting as Möbius transformation on the upper half-plane model of $\mathbb{H}^2$.

Proof. The proof, that the Möbius transformations are isometries of $\mathbb{H}^3$ via the extension formula (10) is quite a lot of work, so we will not give it here. For a complete proof see [MT98], pp. 15-21. That Möbius transformations from $PSL(2, \mathbb{R})$ are also the isometries of $\mathbb{H}^2$, then follows if we restrict their domain to the vertical plane arising from $\mathbb{R} \subseteq \mathbb{C}$ in $\mathbb{H}^3$ and realising, that the extension formula restricted to $\mathbb{R} \times (0, \infty)$ is again equivalent to the formula for Möbius transformations in $\mathbb{H}^2 \subset \hat{\mathbb{C}}$.

Now we show, that there are no other orientation preserving isometries: Let $d_1, d_2, d_3 > 0$, such that they satisfy triangle inequality. Choose $z \in \mathbb{H}^2$ and a geodesic l through z. Let Δ be the triangle, that has one side of length d_1 on l, a side of length d_2 and a side

of length d_3, so that the sides are ordered in positive direction. This way Δ is uniquely determined. Let ψ be an orientation preserving isometry. Then $\Delta' = \psi(\Delta)$ is not a line since ψ is an isometry. So there exists a Möbius transformation $A \in PSL(2,\mathbb{R})$, such that $A \circ \psi$ fixes the three corners of Δ. Since A is also an isometry it fixes all of Δ pointwise. This means $\psi(z) = A^{-1}(z)$ on Δ. Now, for a any point $w \in \mathbb{H}^2$ there is a triangle Δ_w that shares the edge on l with Δ and has w as a third corner. Consequently, there is a Möbius transformation A_w that satisfies $A_w^{-1}(z) = \psi(z)$ on Δ_w. Necessarily, $A_w = A$ since they are the same on the edge on l. It follows that $\psi = A^{-1}$ everywhere. (Proof taken from [Mar07], p.10.)
For $\mathbb{H}^3$ we can do the same thing in a hyperbolic plane (see Proposition 2.10), because every plane is preserved by the subgroup of $PSL(2,\mathbb{C})$ conjugated to $PSL(2,\mathbb{R})$, that preserves its bounding circle on $\hat{\mathbb{C}}$ and every hyperbolic plane is isometric to $\mathbb{H}^2 \cong \mathbb{R} \times (0,\infty) \subseteq \mathbb{H}^3$. □

With this we can proof another statement we will regularly use regarding the shape of geodesics and hyperbolic planes. These are the 2-dimensional totally geodesic submanifolds of $\mathbb{H}^3$. This means geodesics starting in a hyperbolic plane with direction tangential to it stay in it for all time.

Proposition 2.10. *The geodesics in $\mathbb{H}^3$ are vertical Euclidean half-lines and semicircles starting and ending at $\mathbb{C} \subseteq \partial\mathbb{H}^3$. Hyperbolic planes, the 2-dimensional totally geodesic subspaces, are hemispheres orthogonal to $\mathbb{C}$ and vertical Euclidean half-planes.*

Proof. Initially, we will show that the vertical Euclidean half-line l starting at $z = 0$ is itself a geodesic. Given $x = (z,t) \in \mathbb{H}^3$ we define the map $r : \mathbb{H}^3 \to l,\ x \mapsto (0,t)$. This map is a retraction, since it reduces distances. $d(r(x), r(y)) \leq d(x,y)$, with equality only if x and y lie on the same vertical line. This can be seen directly from the inequality

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2} \geq \frac{dt^2}{t^2} \tag{12}$$

where ds is the hyperbolic line element. Now suppose γ is a differentiable path, both of whose endpoints lie on l. Its length strictly exceeds the length of $r(\gamma)$, unless the path is the segment of l between the two endpoints. This means l is a geodesic: the unique shortest path between two points on l is the segment of l between them. Since Möbius transformations are the isometries of $\mathbb{H}^3$, the images of l under some $\phi \in PSL(2,\mathbb{C})$ are geodesics in $\mathbb{H}^3$. They are consequently given by

$$\phi(l(t)) = \left(-\frac{\overline{d/c}}{c^2|d/c|^2 + c^2t^2} + \frac{a}{c}, \frac{t}{|d|^2 + |c|^2t^2} \right) \quad \text{if } c \neq 0$$
$$\phi(l(t)) = \left(\frac{b}{d}, \left|\frac{a}{d}\right| t \right), \quad \text{if } c = 0$$

for the Möbius transformation ϕ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{C})$. Those are exactly the halfcircles with center $\frac{ad+bc}{2cd}$ and radius $|2cd|^{-1}$ if $d \neq 0$ and $c \neq 0$, the vertical lines from infinity to a/c if $d = 0,\ c \neq 0$ and the vertical straight lines with base point b/d if $c = 0$.

In both cases the geodesic is contained in the vertical Euclidean plane determined by it any pair of points on the geodesic. The geodesics one gets this way are all vertical lines and all half-circles orthogonal to $\mathbb{C}$ as can be seen from the discussion above. So it follows, that these are all geodesics of $\mathbb{H}^3$, since a geodesic is uniquely determined by one point on it and a tangent vector at this point, which is the derivative of the parametrization of the geodesic in that point. In the same way a half-cirlce or vertical line is uniquely determined by a point on it and a direction vector at the point tangential to it, and one exists for every combination of point and direction vector. This is a basic result of Euclidean geometry.
The statement about hyperbolic planes follows with little extra work: Consider the vertical Euclidean plane bounded by $\mathbb{R} \cup \infty$ in $\hat{\mathbb{C}}$. This is a totally geodesic subspace by the characterization of geodesics from above, because any geodesic going through a point of it, with derivative at the point tangential to it, is either a vertical straight line with starting point in $\mathbb{R}$ or a half-circle starting and ending in $\mathbb{R}$. In conclusion all its images under Möbius transformations are also totally geodesic subspaces of dimension 2. These are because Möbius transformations act transitively on the generalized circles in $\hat{\mathbb{C}}$, and because of the shape of geodesics shown above, exactly all the half-spheres orthogonal to $\mathbb{C}$ and the vertical Euclidean planes. Proof after [Mar07] pp.10-11. □

For more on totally geodesic subspaces, which hyperbolic planes are, the reader may take a look at [Hel72], section 1.14.

2.4 Characterization of Möbius Transformations

Our next goal is to understand how Möbius transformation act on $\mathbb{H}^3$ geometrically. This is most easily done by looking at how they act on the boundary at infinity $\hat{\mathbb{C}}$. By solving the fixed point equation $z = \phi(z) = \frac{az+b}{cz+d}$ for the Möbius transformation $\phi \neq \mathrm{Id}$ induced by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{C})$ one gets that ϕ has one or two fixed points in $\hat{\mathbb{C}}$. In the case that there are two fixed points, since ϕ is an isometry, ϕ must preserve the geodesic γ which ends at the two fixed points on $\hat{\mathbb{C}}$. Also, because ϕ preserves distances, ϕ must act by translation along the geodesic, by rotation around it or both combined.

Definition 2.11. Let $\phi \neq \mathrm{Id}$ be a Möbius transformation. We call it

- **elliptic**, if it preserves a geodesic $\gamma \subseteq \mathbb{H}^3$ pointwise, and thus acts purely by rotation around γ,
- **loxodromic**, if it performs a translation along γ, possibly with rotation, meaning there are no fixed points on γ.
- **parabolic**, if it has only one fixed point on $\hat{\mathbb{C}}$.

Proposition 2.12. *Möbius transformations, that are not the identity, are characterized up to conjugation, by being parabolic, elliptic or loxodromic.*

Proof. That the characterization is complete follows, if we look at the fixed points of a given transformation ϕ. As mentioned before, it has to have either one or two of those on

$\hat{\mathbb{C}}$. If it has only one, it is parabolic. If it has two on $\hat{\mathbb{C}}$ and preserves γ, the geodesic in $\mathbb{H}^3$ between them, pointwise, it is elliptic. Otherwise it is loxodromic, since it has to preserve γ, but there is a point p on γ, that is mapped to a different point q. Now, there can not be any fixed point on γ. The reason for this is, that $d(x,p) = d(\phi(x),\phi(p)) = d(\phi(x),q)$ and if $\phi(x)$ were equal to x, then p and q are on different sides of x on γ, such that because of continuity and ϕ being an isometry, ϕ would have to be a reflection about x. But this would exchange the fixed endpoints on $\hat{\mathbb{C}}$, which is a contradiction. So, ϕ is loxodromic.
To show invariance under conjugation, we choose two transfomations ϕ, ψ with $\phi \neq \psi$. Let ϕ be parabolic with the single fixed point $w \in \hat{\mathbb{C}}$. Then $\psi(w)$ is the fixed point of the conjugate transformation $\psi \circ \phi \circ \psi^{-1}$:

$$\psi \circ \phi \circ \psi^{-1}(\psi(w)) = \psi \circ \phi(w) = \psi(w) \tag{13}$$

Let $u \in \hat{\mathbb{C}}, u \neq \psi(w)$. Then

$$\phi \circ \psi^{-1}(u) \neq \psi^{-1}(u) \quad \text{and} \quad \psi \circ \phi \circ \psi^{-1}(u) \neq u \tag{14}$$

If ϕ on the other hand is elliptic or loxodromic with fixed points $w_1, w_2 \in \hat{\mathbb{C}}$, we get that, as in the first equation, $\psi(w_1), \psi(w_2)$ are the fixed points of the conjugate on $\hat{\mathbb{C}}$. The same argument holds for the points on the rotation axis in $\mathbb{H}^3$ of an ellipic transformation. Also, similarly to the second equation, if $x \in \mathbb{H}^3$ is no fixed point of ϕ, $\psi(x)$ is no fixed point of the conjugate with ψ. This means conjugates of elliptics have to stay elliptics and conjugates of loxodromics stay loxodromics as well. □

Next we want to list some useful equivalent conditions by which we can charaterize a Möbius transfomations as parabolic, loxodromic or elliptic. This way we don't always have to look at the fixed points. Moreover, we will see really simple examples of transformations in each conjugacy class:

Proposition 2.13. *Let $\phi \neq Id$ be a Möbius transformation given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{C})$.*

1. *It is parabolic, if and only if either of the following conditions hold:*
   - *ϕ has exactly one fixed point in $\hat{\mathbb{C}}$*
   - *ϕ is conjugate to $z \mapsto z+1$*
   - $\mathrm{tr}^2(\phi) = 4$
2. *It is elliptic if and only if*
   - *ϕ has exactly two fixed points $z, w \in \hat{\mathbb{C}}$ and $|\phi'(z)| = |\phi'(w)| = 1$*
   - *ϕ is conjugate to $z \mapsto e^{i\theta}z$, with $\theta \in (0, 2\pi)$*
   - $\mathrm{tr}^2(\phi) \in [0,4]$
3. *Finally, ϕ is loxodromic if and only if*
   - *ϕ has one attracting fixed point z and one repelling fixed point w in $\hat{\mathbb{C}}$, meaning $|\phi'(w)| > 1 > |\phi'(z)|$*

- *ϕ is conjugate to $z \mapsto \lambda^2 z$ with $|\lambda| > 1$*
- $\operatorname{tr}^2(\phi) \in \mathbb{C} \setminus [0, 4)$

The map tr^2 *is well defined for Möbius transformation unlike* tr*, since it only depends on the equivalence class of the associated matrix in $PSL(2, \mathbb{C})$.*

Regarding the statements dealing with the derivative of a Möbius transformation at the fixed points, we need to define what the derivative at ∞ is: This will be defined as the derivative of $z \mapsto \phi(1/z)$ at $z = 0$, if ∞ is not a fixed point of ϕ, and as the derivative of $(z \mapsto 1/\phi(1/z))$ at $z = 0$, if it is a fixed point of ϕ, These formulas are derived from the standard chart of $\hat{\mathbb{C}}$ around ∞.

Proof.

1. The first property is the definition of a parabolic Möbius transformation. Next we show that the trace property is equivalent to the first property. Let $\phi = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \neq \pm \mathrm{Id}$, $ad - cb = 1$.
   Case 1: $c = 0$. Looking at the fixed point equation we have
   $$(a/d)z + b/d = z. \tag{15}$$
   For $a \neq d$ this has exactly two solutions $z = \frac{b}{d-a}$ and $z = \infty$. If $a = d$ we have $a^2 = 1$ from the determinant condition. This means $b \neq 0$ since we excluded $\pm\mathrm{Id}$. It follows that ϕ has exactly the one fixed point ∞, if and only if $\operatorname{tr}(\phi) = a + d = 2a \in \{-2, 2\}$
   Case 2: $c \neq 0$. In this case we get the quadratic fixed point equation
   $$cz^2 + (d - a)z - b = 0, \tag{16}$$
   which has one solution, if and only if the discriminant $(a - d)^2 + 4cb = 0$. Again using $ad - cb = 1$, we get $4 = a^2 + 2ad + d^2 = (a + d)^2 = \operatorname{tr}^2(\phi)$.
   Let now ϕ be conjugated to $z \mapsto z+1$. One matrix associated to this transformation is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and that has trace 2. Since the trace is invariant under conjugation, we get that ϕ is parabolic. Conversely, if ϕ is parabolic we can assume without loss of generality that ϕ fixes ∞. This means $c = 0$ and $\phi = \pm\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some $b \neq 0$ because of determinant and trace condition. Conjugating this matrix with $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ for $\lambda \in \mathbb{C},\ \lambda^2 = b$ we get the desired map $z \mapsto z + 1$. This concludes the parabolic case.

2. In the elliptic case, we can conjugate the Möbius transformation, so that the two fixed points of the conjugate ϕ are 0 and ∞. This works without loss of generality, because Möbius transformations act transitively on point triplets in $\hat{\mathbb{C}}$. Consequently, the vertical half line $\mathbb{R}^+ \subseteq \mathbb{H}^3$ is pointwise preserved by ϕ. From the fixed point formulas (15) and (16) we get $c = b = 0$ and from equation (10) and the determinant condition that $|a| = |d| = 1$. This means $\phi = (z \mapsto e^{i\theta}z)$ for some $\theta \in (0, 2\pi)$.
   Otherwise, if ϕ is conjugated to $z \mapsto e^{i\theta}z$, we have again $c = b = 0$ and a satisfies $a^2 = e^{i\theta}$ by the determinant being 1, meaning $|a| = |d| = 1$. So ϕ preserves the

vertical half-line $z = 0$ pointwise and is therefore elliptic. We have shown the equivalence of the second property and the definition, and continue now with the first property.
The derivative of the Möbius transformation $\phi = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{C})$ is given by

$$\phi'(z) = \frac{1}{(cz + d)^2}, \quad \text{for } z \in \mathbb{C}, \tag{17}$$

$$\phi'(z) = \frac{d}{a}, \quad \text{for } z = \infty \text{ a fixed point}, \tag{18}$$

$$\phi'(z) = \frac{-1}{c^2}, \quad \text{for } z = \infty \text{ else}. \tag{19}$$

We assume ϕ is conjugated to $\phi_0 = (z \mapsto e^{i\theta} z)$. It is clear that for ϕ_0 the derivative at every point in $\mathbb{C}$ is $e^{i\theta}$. The derivative at ∞ is actually $e^{-i\theta}$ by considering the matrix form of ϕ_0, $\left(\begin{smallmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{smallmatrix}\right)$. Let us then examine $\phi = \psi \circ \phi_0 \circ \psi^{-1}$, where ψ is any Möbius transformation. Let $w \in \mathbb{C}$ be a fixed point of ϕ. Then

$$|\phi'(w)| = |\psi'(\phi_0 \circ \psi^{-1}(w)) \cdot \phi_0'(\psi^{-1}(w)) \cdot \frac{1}{\psi'(\psi^{-1}(w))}| \tag{20}$$

$$= |\psi'(\psi^{-1}(w)) \cdot \frac{1}{\psi'(\psi^{-1}(w))} \cdot \phi_0'(\psi^{-1}(w))| \tag{21}$$

$$= |\phi_0'(\psi^{-1}(w))| \tag{22}$$

which shows that the absolute value of the derivative at fixed points is invariant under conjugation and in this case equal to 1. We will skip the case that $w = \infty$, it is going to have the same result.
Again conversely, if the first property holds, we can find a conjugated transformation fixing 0 and ∞, $\phi_0 = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \in SL(2, \mathbb{C})$. So $\phi_0'(0) = \lambda^2$ and from the assumption it follows that $|\lambda| = 1$. Therefore, ϕ satisfies the already proven second property and we have also proven the equivalence of the first and second one.
Now, only the trace property is missing. However, this one immediately follows from the second property since $\mathrm{tr}^2((z \mapsto e^{2i\theta} z)) = (e^{i\theta} + e^{-i\theta})^2$. The imaginary parts cancel out and the real parts are smaller than one each, such that the result lies in the interval $(0, 4)$.
Regarding the inverse statement, we can replace the given Möbius transformation by one with fixed points $0, \infty$ without loss of generality. So $\phi = \pm\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ with $\lambda \in \mathbb{C}$. The trace stays invariant, so we have $\lambda + \lambda^{-1} \in (-2, 2)$, which is equivalent to $\Re(\lambda) + \Re(\lambda) \cdot |\lambda|^{-2} \in (-2, 2)$ and $\Im(\lambda) - \Im(\lambda) \cdot |\lambda|^{-2} = 0$. From the second condition we get that either $|\lambda| = 1$, in which case we have proven that ϕ is elliptic, or $\Im(\lambda) = 0$. In this case $|\lambda| = |\Re(\lambda)|$ and the first condition becomes $|\lambda| + |\lambda^{-1}| < 2$, which is impossible.

3. Finally, there is the loxodromic case. By elimination, if ϕ is loxodromic, $\mathrm{tr}^2(\phi) \in \mathbb{C} \setminus [0, 4]$. Continuing with the assumption that $\mathrm{tr}^2(\phi) \in \mathbb{C} \setminus [0, 4]$, we replace ϕ by a conjugate ϕ_0 with fixed points $0, \infty$. Then again we get a transformation of

the form $\phi_0 = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$. By elimination, since ϕ is not elliptic, $|\lambda|$ can not be 1. If $|\lambda| < 1$ we can conjugate ϕ_0 with $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to receive the inverse of ϕ_0. Now, we have the map $z \mapsto \lambda^{-2}z$ with $|\lambda^{-1}| > 0$.
Starting from what we have proven last, let ϕ be conjugated to $\phi_0 = (z \mapsto \lambda^2 z)$, $|\lambda| > 1$. By using (20) and the formula for the derivative (17) we can see that the derivatives at the fixed points are the same as in the base transformation ϕ_0, meaning they are equal to λ^2 at 0 and λ^{-2} at ∞.
Last but not least, we need to get the definition of being loxodromic from the fixed point property. But since we know from $|\phi'(z)| \neq 1$ at a fixed point z that ϕ is not elliptic, and we know that there are two fixed points, ϕ has to be loxodromic.

□

This characterization yields a useful corollary regarding when two elements share a fixed point:

Corollary 2.14. *Let ϕ and ψ be Möbius transformations different from Id. Then A and B share a fixed point if and only if*

$$\operatorname{tr}(\phi\psi\phi^{-1}\psi^{-1}) = 2. \tag{23}$$

Proof. This can be checked using the standard forms up to conjugation from above. We will exemplarily do the case for ϕ, ψ elliptic, because it is one we will need, and the others are quite similar.
Without loss of generality we assume ϕ to fix ∞ and 0. So $\psi = e^{2i\theta}z$ for $\theta \in (0, \pi)$, ϕ given by $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{C})$. Now we go over to the respective matrices, denoting them again by ϕ and ψ. Then

$$\phi\psi = \pm\begin{pmatrix} ae^{i\theta} & be^{i\theta} \\ ce^{-i\theta} & de^{-i\theta} \end{pmatrix} \in SL(2, \mathbb{C})$$
$$\phi^{-1}\psi^{-1} = \pm\begin{pmatrix} de^{-i\theta} & -be^{-i\theta} \\ -ce^{i\theta} & ae^{i\theta} \end{pmatrix} \in SL(2, \mathbb{C})$$

and the sign of these two matrices is the same. So

$$\operatorname{tr}(\phi\psi\phi^{-1}\psi^{-1}) = \operatorname{tr}\begin{pmatrix} ad - bce^{2i\theta} & -ab + abe^{2i\theta} \\ cde^{-2i\theta} - cd & -cbe^{-2i\theta} + ad \end{pmatrix} = 2ad - bc(e^{-2i\theta} + e^{2i\theta}),$$

which is equal to 2 if and only if $b = 0$, $c = 0$ or $e^{-2i\theta} + e^{2i\theta} = 2\cos(2\theta) = 2$. In the first case ϕ fixes 0, in the second one it fixes ∞, and in the third case $\theta = \pi k$ for some k in $\mathbb{Z}$, which is a contradiction to $\theta \in (0, \pi)$. □

With everything we have shown above on elliptic, loxodromic and parabolic transformations, it makes sense to think of elliptic ones as the analogue of Euclidean rotations around an axis and of loxodromic ones as the analogue of a translation along a Euclidean line, while also potentially rotating around the same line. What exactly parabolic transformations do in $\mathbb{H}^3$ is probably not yet clear. We will discuss these further in section 3.

3 Convergence and Discrete Subgroups

3.1 Convergence of Möbius Transformations

In this section we will examine infinite sequences of distinct Möbius transformations and their action on $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. Initially, we will discuss which form the "limit" of a sequence of Möbius transformations can take on.

Lemma 3.1. *Suppose $\{T_n\}$ is an infinite sequence of distinct Möbius transformations, such that the corresponding fixed points $p_n, q_n \in \hat{\mathbb{C}}$ (here either $p_n = q_n$, T_n elliptic, or p_n the repelling and q_n the attracting fixed point) converge to $p, q \in \hat{\mathbb{C}}$. Then there is a subsequence $\{T_k\}$ with one of the following properties:*

1. *There exists a Möbius transformation T such that $\lim T_k(z) = T(z)$ uniformly on $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ in the Euclidean metric. Equivalently suitable matrices associated to the transformations converge to a new matrix in $SL(2, \mathbb{C})$ in any matrix norm.*
2. *For all $z \neq p$, $\lim T_k(z) = q$ uniformly on compact subsets of $\mathbb{H}^3 \cup \hat{\mathbb{C}} \setminus \{p\}$. Also $\lim T_k^{-1}(z) = p$ for all $z \neq q$, uniformly on compact subsets like above. Possibly $p = q$.*

Remark 3.2. Actually, we do not need to require that the fixed points p_n and q_n converge to some $p, q \in \hat{\mathbb{C}}$, as for any sequence of distinct Möbius transformations, there is a subsequence, such that the sequences $\{p_n\}, \{q_n\}$ have this property, because $\hat{\mathbb{C}}$ is compact. We only do so here to have p and q fixed and we are able to talk about them.

Proof. Assume $\{T_n\}$ is a sequence as described and there is no subsequence converging to a Möbius transformation.
Case 1: $p \neq q$. Choose $\zeta \in \mathbb{C}$ distinct from p, q, p_n, q_n for all n. Set $R_n(z) = (z, \zeta, p_n, q_n)$ so that $\lim R_n(z) = R(z) = (z, \zeta, p, q)$ uniformly on $\hat{\mathbb{C}}$. The transformation $S_n(z) = R_n T_n R_n^{-1}(z)$ fixes $0, \infty$ and has the same convergence properties as $\{T_n\}$. We have for large indices $S_n(z) = a_n z$ with $|a_n| \geq 1$. If $|a_n|$ is bounded for infinitely many indices, then a subsequence converges to a Möbius transformation, a contradiction. Otherwise there exists a subsequence $\{S_m\}$ for which $\lim a_m = \infty$. In this case, $\{S_m\}$ converges uniformly to ∞ outside any given neighborhood U of $(z, t) = 0$ in the Euclidean metric of $\mathbb{C} \times [0, \infty)$. This means for every $C > 0$ there exists N such that for any $m > N$ and any $x \notin U$, $\|S_m(x)\| > C$, which is clear since $|a_m|$ becomes arbitrarily large.
Case 2: $p = q$. Choose $\zeta_1, \zeta_2 \neq q_n, q$ and $\zeta_1 \neq \zeta_2$. Set $R_n(z) = (z, \zeta_1, \zeta_2, q_n)$. Again $\lim R_n(z) = R(z) = (z, \zeta_1, \zeta_2, q)$. Set $S_n(z) = R_n T_n R_n^{-1}(z)$. This fixes ∞ and has the same convergence properties as $\{T_n\}$. So $S_n(z) = a_n z + b_n$. The other fixed point of S_n is $-b_n/(a_n - 1)$. If for a subsequence $\lim b_m = b \neq \infty$, then $\lim a_m = 1$, because the limits of the sequences of fixed points has to be ∞. In this case $\lim S_m(z) = z + b$ is a Möbius transformation. If instead $\lim b_n = \infty$, rewrite S_n as

$$S_n(z) = b_n \left(\frac{(a_n - 1)z}{b_n} + 1 \right) + z.$$

Since $\lim(a_n - 1)/b_n = 0$, we have $\lim S_n(z) = \infty$ for all $z \in \mathbb{C}$, uniformly on compact subsets not containing 0.
As for the inverse,

$$S_n^{-1}(z) = \frac{b_n}{a_n}\left(\frac{z}{b_n} - 1\right).$$

Because

$$\lim \frac{a_n - 1}{b_n} = \lim\left(\frac{a_n}{b_n} - \frac{1}{b_n}\right) = 0$$

and $\lim b_n = \infty$, we find $\lim a_n/b_n = 0$. Therefore $\lim S_n^{-1}(z) = \infty$ as well, for all $z \in \mathbb{C}$, uniformly on compact sets not containing ∞. Proof taken from [Mar07], pp. 50-51. □

3.2 Discrete Groups

Now that we know the convergence behaviour of sequences of Möbius transformations, we can discuss what makes a subgroup of them discrete:

Proposition 3.3. *Let $G \subseteq PSL(2,\mathbb{C})$ be a group of Möbius transformations. If one of the following equivalent conditions hold, then G is discrete:*

1. *No infinite sequence of distinct elements of G converges to the identity.*
2. *No infinite sequence of distinct elements of G converges to a Möbius transformation.*
3. *G acts* properly discontinuously *in $\mathbb{H}^3$: Given any closed ball $B \subseteq \mathbb{H}^3$, the set $\{g \in G \mid g(B) \cap B \neq \emptyset\}$ is finite.*
4. *G has no limit points in $\mathbb{H}^3$: Given $x \in \mathbb{H}^3$, there is no point $y \in \mathbb{H}^3$ with an infinite sequence of distinct elements $\{g_n\}$ in G such that $\lim g_n(y) = x$.*

*A discrete $G \subseteq PSL(2,\mathbb{C})$ also called a **Kleinian group**.*

Proof. 1. ⇔ 2.: If there is a sequence $\{g_n\}$ of distinct elements of G, which converges to a Möbius transformation g, then $\{g_{n+1}^{-1} \circ g_n\}$ converges to Id. Because $\{g_n\}$ is a sequence of distinct elements, $g_{n+1}^{-1} \circ g_n \neq \text{Id}$ for all n. This means there are infinitely many pairwise distinct elements in the second sequence, otherwise it could not converge.
The inverse statement is clear.
2. ⇒ 3.: By contradiction, let $\{g_n\}$ be an infinite sequence of distinct element of G such that $g_n(B) \cap B \neq \emptyset$. Then by Lemma 3.1 and Remark 3.2 there is a subsequence converging to either a Möbius transformation or a point $p \in \hat{\mathbb{C}}$. We assumed that there is no sequence in G converging to a Möbius transformation, so $\{g_n\}$ converges to some $p \in \hat{\mathbb{C}}$, uniformly in B. But this is a contradiction since $p \notin B$ and B closed, so there has to be an N such that $g_n(B) \subseteq \mathbb{H}^3 \cup \hat{\mathbb{C}} \setminus B$ for all $n > N$.
3. ⇒ 4.: Again by contradiction, assume such a limit point $x \in \mathbb{H}^3$, $x = \lim g_n(y)$ exists for $y \in \mathbb{H}^3$, $\{g_n\} \subseteq G$ pairwise distinct. Choose B to be the closure of an open ball

containing x, and $h \in \{g_n\}$ such that $h(y) \in B^{int}$. This way $g_n \circ h^{-1}(B) \cap B$ can not be empty for all n large enough, because $g_n(y)$ has to be in B for all large n. However, this gives us a contradiction to 3.).
<u>4. ⇒ 2.:</u> We show that from the negation of 2.) follows the negation of 4.). But this is clear. If $g_n \to g$, where g is a Möbius transformation in G, then of course $g_n(x) \to g(y)$, where x and $g(y)$ are in $\mathbb{H}^3$.

□

3.3 Elementary Groups

A special class of Kleinian groups, which often have to be treated separately, are the so called *elementary* groups:

Definition 3.4. A discrete group $G \subseteq PSL(2,\mathbb{C})$ is called **elementary**, if and only if it either

- fixes a point on $\hat{\mathbb{C}}$,
- preserves a pair of points on $\hat{\mathbb{C}}$ (not pointwise),
- or it fixes a point in $\mathbb{H}^3$.

We will mostly use this definition to exclude elementary groups from statements we make, or to talk about elementary subgroups of discrete, non-elementary groups. A full characterization of elementary groups was originally done in [For29]. For more a more modern source, which also characterizes discrete elementary subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ see [Rat19], Chapter 5.5.

We will only state and prove some select properties of them:

Lemma 3.5. *Let $\phi, \psi \in PSL(2,\mathbb{C})$ be Möbius transformations having exactly one common fixed point. If ϕ is loxodromic, then then group $\langle \phi, \psi \rangle$ is not discrete.*

This Lemma will be helpful in many cases, especially to restrict the possible structures of the elementary *stabilizer* groups later on.

Proof. By conjugation we can assume that the common fixed point is $\infty \in \hat{\mathbb{C}}$. Further conjugating these elements, we also assume that the second fixed point of ϕ is 0. That this is possible without loss of generality is the content of Proposition 2.13. Now ϕ has the form $(z \mapsto az)$, while ψ is given by $(z \mapsto bz + d)$ for some $a, b, d \in \mathbb{C} \setminus \{0\}$, $|a| \neq 1$. If $|a| > 1$, we pass to ϕ^{-1} without changing notation, such that $|a| < 1$. Then we have

$$\phi^m \psi \phi^{-m} = a^m b a^{-m} z + a^m d = bz + a^m d \text{ for all } m \in \mathbb{N}$$

Because $|a| < 1$ the sequence $\{\phi^m \psi \phi^{-m}\}$ converges for $m \to \infty$ to $(z \mapsto bz)$, which is clearly a Möbius transformation too. So the claim follows. This proof is from [Rat19], Theorem 5.5.4. □

Another really useful and really simple result is the following:

Lemma 3.6. *A group consisting only of elliptics and fixing a single axis pointwise is finite.*

Proof. By conjugation, choose the axis to be the vertical axis l from $0 \in \hat{\mathbb{C}}$ to $\infty \in \hat{\mathbb{C}}$. Assume there is an infinite, purely elliptic group fixing l pointwise. Let $\{\phi_n\}$ be an infinite sequence of distinct elliptics from it. Then $\phi_n = (z \mapsto e^{i\theta_n} z)$ for appropriate $\theta_n \in [0, 2\pi]$ by Proposition 2.13. Since $[0, 2\pi]$ is compact there is a convergent subsequence of $\{\theta_n\}$ converging to some $\theta \in [0, 2\pi]$. So the corresponding elliptics converge to the Möbius transformation $(z \mapsto e^{i\theta} z)$. However, this means the group containing all of the ϕ_n can not be discrete. □

3.4 Stabilizer of a Parabolic Fixed Point

To close our discussion of elementary groups we want to take a more detailed look at one special case of elementary groups, the so-called *stabilizer group* of a parabolic fixed point:

$$\mathrm{Stab}_\zeta(G) = \{g \in G \mid g(\zeta) = \zeta\} \tag{24}$$

By definition, this is an elementary group. We will show that the possible structures of this group are very limited. Initially, we note that the parabolic elements in $\mathrm{Stab}_\zeta(G)$ form a subgroup: Take $\zeta = \infty \in \hat{\mathbb{C}}$, then all parabolics have the form $(z \mapsto z + c)$ for $c \in \mathbb{C} \setminus \{0\}$. That these elements form a subgroup is clear.

Theorem 3.7. *The subgroup H of $\mathrm{Stab}_\zeta(G)$ consisting of all parabolics fixing ζ is either cyclic and conjugate to $\langle z \mapsto z+1 \rangle$, or it is the free abelian group of rank two and conjugate to $\langle z \mapsto z+1, z \mapsto z+\tau \rangle$ for some $\tau \in \mathbb{C}$ with $\Im\tau > 0$.*

Remark 3.8. More precisely, we could also show that in the second case τ can be chosen to be in a fundamental domain of the modular group $PSL(2, \mathbb{Z})$, which is much smaller. But since we will not need this here, so we will not bother to do so.

Proof of Theorem 3.7. We have already shown in Lemma 3.5, that there cannot be a loxodromic in $\mathrm{Stab}_\zeta(G)$. Due to this H is purely parabolic. Naturally, one option is, that $H = \langle g \rangle$, where $g \in G$ is parabolic, for example if $G = G_\infty := \langle z \mapsto z+1 \rangle$. In the general case, if H is cyclic, it is conjugated to this G_∞, because the generator of H is conjugated to $\langle z \mapsto z+1 \rangle$ by Proposition 2.13.
This concludes the cyclic case and we move on to the two generator case. Without loss of generality let $H = \langle z \mapsto z+1, z \mapsto z+c \rangle$.
<u>Case 1: $c \in \mathbb{Q}$.</u> By conjugation with an element $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$ where α is equal to a square root of the denominator of c we get an isomorphic group with the translation constants $a, b \in \mathbb{Z}$. Then the group is again cyclic, where the generator is the greatest common divisor of a, b. The reason is, that in $\mathbb{Z}$ every ideal is a principal ideal.
<u>Case 2: $c \in \mathbb{R} \setminus \mathbb{Q}$.</u> Then $Nc - \lfloor Nc \rfloor$ are always distinct for distinct $N \in \mathbb{Z}$. This means

there are infinitely many distinct elements, that perform a translation with constant $d = Nc - \lfloor Nc \rfloor \in [0,1]$. But this way $\langle z \mapsto z+1, z \mapsto z+c \rangle$ is not discrete, a contradiction.
Case 3: $c \in \mathbb{C} \setminus \mathbb{R}$. In this case we can choose c such that $\Im c > 0$. Of course this group is free abelian as well, satisfying our statement. For example $\langle z \mapsto z+1, z \mapsto z+i \rangle$ is a valid stabilizer group of ∞. This concludes the two generator case as well.
Finally, we look at n-generator groups for $n > 2$. Choose three generators with translation constants $1, b, c$ such that $b, c \in \mathbb{H}$, the upper half plane, without loss of generality. Set $M := \lfloor \Im(Na/b) \rfloor \in \mathbb{Z}$. Then the set $\{Na - Mb - \lfloor \Re(Na - Mb) \rfloor\}_{N \in \mathbb{N}}$ is generated by $1, b, c$ and has all its values in $[0,1]^2 \subseteq \mathbb{C}$. If there are not infinitely many elements in this set, $1, b, c$ are $\mathbb{Z}$-linearly dependent and the parabolic group generated by translation about these constants is also generated by $n-1$ generators. Contrarily, if the set is infinite, the group generated by these translations is not discrete, yielding a contradiction. That is caused by the fact, that the H-orbit of $(0,1) \in \mathbb{H}^3$ has infinitely many points in the compact set $[0,1] \times [0,1] \times \{1\}$. This procedure can be repeated until only two generators are left over, or we get a non-discrete subgroup. □

Definition 3.9. Following the results from the theorem above we call a parabolic fixed point ζ of a Kleinian group G

- a **rank-one** parabolic fixed point, if the subgroup of parabolics in $\mathrm{Stab}_\zeta(G)$ is cyclic
- a **rank-two** parabolic fixed point, if the subgroup of parabolics in $\mathrm{Stab}_\zeta(G)$ is isomorphic to the free abelian group on two generators.

Finally we will now take another look at one special case of elliptics in $\mathrm{Stab}_\zeta(G)$, which we are going to need later.

Corollary 3.10. *If a parabolic fixed point $\zeta \in \mathbb{C}$ is of rank one and there are elliptics in* $\mathrm{Stab}_\zeta(G)$*, then* $\mathrm{Stab}_\zeta(G)$ *is the extension of the cyclic parabolic subgroup by an elliptic of order* 2.

Proof. We take without loss of generality $\zeta = \infty \in \hat{\mathbb{C}}$ and an elliptic $g = (z \mapsto e^{i\theta}z + d) \in \mathrm{Stab}_\zeta(G)$ with $d \in \mathbb{C}$. Let $T = (z \mapsto z + c)$ be a generator of the cyclic parabolic subgroup of $\mathrm{Stab}_\zeta(G)$, $c \in \mathbb{C} \setminus \{0\}$. Then

$$gTg^{-1}(z) = e^{i\theta}(e^{-i\theta}z - e^{-i\theta}d + c) + d = z + e^{i\theta}c$$

This means $e^{i\theta} \in \mathbb{Z}$ and since $\theta \in (0, 2\pi)$, we get that $\theta = \pi$ and $e^{i\theta} = -1$. This means g is of order 2. If there is another $h = (z \mapsto -z+a) \in \mathrm{Stab}_\zeta(G)$ of order 2 with $a \in \mathbb{C} \setminus \{c\}$, then consider the parabolic element $gTh = (z \mapsto z - c + (d-a))$. In order to still be cyclic, the subgroup of parabolics must include this element, so that $d - a = kc \in \mathbb{Z}c$. Thus, $a = d - kc$. However, if we now consider the element

$$T^{-k}g = (z \mapsto -z + d - kc) = (z \mapsto -z + a) = h,$$

we see that $h \in \langle g, T \rangle$. Hence, we have shown that $\mathrm{Stab}_\zeta(G) = \langle g, T \rangle$ is the extension of the subgroup of parabolics by an elliptic of order 2. □

Remark 3.11. With similar methods one can show, that in the in the case of a rank-two parabolic fixed point, the full stabilizer group can be the extension of the parabolic subgroup by three or four elliptics of fixed finite order in four ways. For some of these to arise τ must have special values. See [For29].

3.5 The Limit Set and the Ordinary Set

In this subsection we will only focus on nonelementary discrete groups. Since we have shown that they act on $\mathbb{H}^3$ properly discontinuously, we want to examine their action on $\hat{\mathbb{C}}$, where they cannot act this way, since $\hat{\mathbb{C}}$ is compact and nonelementary groups cannot be finite. This is because only purely elliptic groups can be finite. That all purely elliptic finite groups do actually preserve a point in $\mathbb{H}^3$, which makes them elementary, is a fact, we will not proof here. For a proof see [Mar07], Corollary 4.1.5.

Definition 3.12. Let $G \subseteq PSL(2, \mathbb{C})$ be discrete. A point $\zeta \in \hat{\mathbb{C}}$ is a **limit point**, of G if there exists $\xi \in \hat{\mathbb{C}}$ and a sequence $\{T_n\} \subseteq G$, such that $\lim T_n(\xi) = \zeta$.
The set

$$\Lambda(G) = \{\zeta \in \hat{\mathbb{C}} | \ \zeta \text{ is a limit point}\}$$

is called the *limit set*. It necessarily contains all fixed point of elements in G, such that for nonelementary G it consists of at least 3 points.

To better understand the shape of the limit set we will prove the following statements:

Proposition 3.13. *Let G be discrete and nonelementary.*

1. *$\Lambda(G)$ is invariant under G*
2. *The G-orbit of any $\zeta \in \Lambda(G)$ is dense in $\Lambda(G)$.*
3. *$\Lambda(G)$ is the closure of the set of loxodromic fixed pointsn when there are loxodromics in G. And if there are parabolics, it is also the closure of the set of parabolic fixed points.*
4. *$\Lambda(G)$ is a closed set*
5. *$\Lambda(G)$ has no isolated points.*
6. *Either $\Lambda(G) = \hat{\mathbb{C}}$ or its interior is empty.*

Proof.
1. This one is trivial. If ζ is a limit point of $\{g_n(z)\}$ for some $z \in \hat{\mathbb{C}}$, $g_n \in G$ then $g(\zeta) = \lim g \circ g_n(z)$ for any $g \in G$.
2. We show that for every $x \in \Lambda(G)$ and every $y \in \Lambda(G)$ there is a sequence $\{g_m\} \subseteq G$, such that $g_m(x) \to y$. This proves the statement. Let $x, y \in \Lambda(G)$. Then, since y is a limit point, there is some $z \in \hat{\mathbb{C}}$ with the property, that there is a sequence $\{g_n\} \subseteq G$ of pairwise distinct Möbius transformations with $g_n(z) \to y$. By Lemma 3.1 there exists a subsequence $\{g_m\}$, that converges to a constant on $\hat{\mathbb{C}} \setminus \{q\}$ for

some $q \in \hat{\mathbb{C}}$. If $q \neq z$, then that constant is y of course. Otherwise we replace the $\{g_m\}$ with g_m^{-1}, because again by Lemma 3.1 $g_m^{-1}(z) = y$ then. So either $x = q$ or $\lim g_m(x) = y$. In the second case, we are finished. In the first case using that G is non-elementary, we can pick a $g \in G$ such that $g(x) \neq x$. The sequence $\{g_m \circ g\}$ then satisfies $g_m \circ g(x) \to y$ and we have concluded the proof.

3. With the preceding point we have shown that the closure of the set of loxodromic, respectively parabolic fixed points contains $\Lambda(G)$. Next, we will prove the other inclusion. Let z be an element in the closure of the set of loxodromic or parabolic fixed points. Then there is an sequence $\{z_n\}$ of loxodromic fixed point converging to z. We can choose z_n to be the attracting/only fixed point of an loxodromic/parabolic element $g_n \in G$. If we now look at the set $\{g_n^m \, | n, m \in \mathbb{N}\} \subseteq G$, there is a sequence $\{h_l\}$ in it with $h_l(w) \to z$ for $l \to \infty$. The reason for this is, that for any open neighbourhood U of z there is an $N \in \mathbb{N}$ with $z_N \in U$. Likewise U is then an open neighbourhood of z_N and as such, $M \in \mathbb{N}$ exists, satisfying $g_N^M(z) \in U$. Therefore, $z \in \Lambda(G)$. [Mar07] Lemma 2.4.1.

4. Already proven with 3.

5. Suppose $z \in \hat{\mathbb{C}}$ is an isolated point of $\Lambda(G)$. By 3. the point z is an element of the set of loxodromic or parabolic fixed points L. Hence, there is some $h \in G$, loxodromic or parabolic, that fixes z. Without loss of generality, if h loxodromic, let z be the attracting fixed point. Also since $\#L \geq 3$, there is $w \in L$, that is not fixed by h. But the sequence $\{h^k(w)\}$ converges to z, which is a contradiction. Proof from [Rat19] Theorem 12.2.5.

6. If $\Lambda(G) \neq \hat{\mathbb{C}}$, its complement in $\hat{\mathbb{C}}$, Ω, is open. Now, every loxodromic fixed point is a limit point of the G-orbit of Ω, and so every limit point is. However, this way for every limit point, we have found a sequence of points in Ω converging to it by 1. We conclude that $\Lambda(G)$ has no interior. [Mar07] Lemma 2.4.1

□

Figure 2 shows two limit sets of Kleinian groups with fractal-like structure. These groups are two-generator groups. The formulas for generators, that produce such limit sets are from [MSW02], p.229, Box 21 for the left picture and p.118, Project 4.2 for the right picture. Playing with the free parameters in these formulas gives you a wide variety of more or less interestingly looking limit sets. The algorithm generating the limit sets from these generators builds Möbius transformations from all combinations of the generators of less than a maximal length and computes the fixed points of these Möbius transformations. The pixels corresponding to fixed points in a subset of the complex plane are then colored in red, generating the pictures in Figure 2.

We now turn our attention to the complement of the limit set. As we will see, this also has some interesting properties:

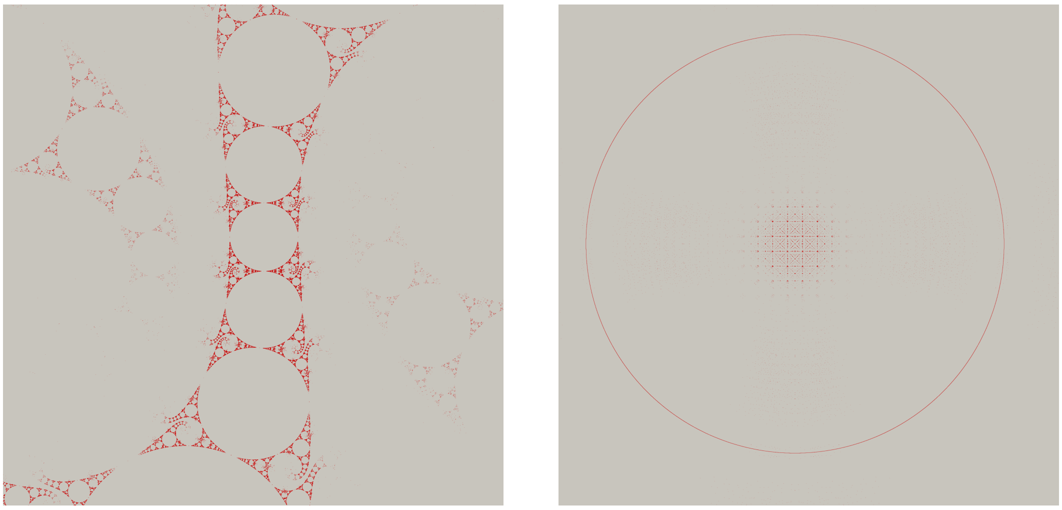


Figure 2: Approximations of the limit sets of two Kleinian groups. Formulas from [MSW02] p.118 and p.229

Definition 3.14. The complement of the limit set

$$\Omega(G) = \hat{\mathbb{C}} \setminus \Lambda(G)$$

is called the **ordinary set** or **set of discontinuity**.

The for us most important and most interesting properties of this set, that are not immediately obvious, are given by the following proposition.

Proposition 3.15. *Let $G \subseteq PSL(2, \mathbb{C})$ be discrete.*

1. *$\Omega(G)$ is the largest open subset of $\hat{\mathbb{C}}$ on which G acts properly discontinuously.*
2. *If G is additionally finitely generated and nonelementary, then $\Omega(G)$ has either one, two or infinitely many components.*

Proof. 1. On a set containing a loxodromic or parabolic fixed point ζ of $g \in G$, the group G cannot act properly discontinuously, as $g^n(\zeta) = \zeta$ for all n. So the largest open subset, on which G acts properly discontinuously, has to be in the complement of the closure of the set of loxodromic fixed points.
Let $K \subseteq \Omega(G)$ be compact. Assume $g_n(K) \cap K \neq \emptyset$ for infinitely many pairwise distinct $g_n \in G$. Then there exists a sequence $\{k_n\} \subseteq K$, where $k_n \in K \cap g_n(K)$. Let $c_n = g_n^{-1}(k_n) \in K$. Because K is compact, we can pass to subsequences indexed by m, such that $c_m \to c$, $k_m \to k$ for $c, k \in K$ and g_m converges to a constant, uniformly outside of a neighbourhood of some point $p \in \hat{\mathbb{C}}$ (Lemma 3.1). If $c, k \neq p$ then k has to be the limit of attracting (or parabolic) fixed points of g_m, because $g_m(c_m) \to k$ and there is a neighbourhood U of p, such that $c_m \notin U$ for m large enough. This yields a contradiction to $k \notin \Lambda(G)$. Otherwise, if $k = p$

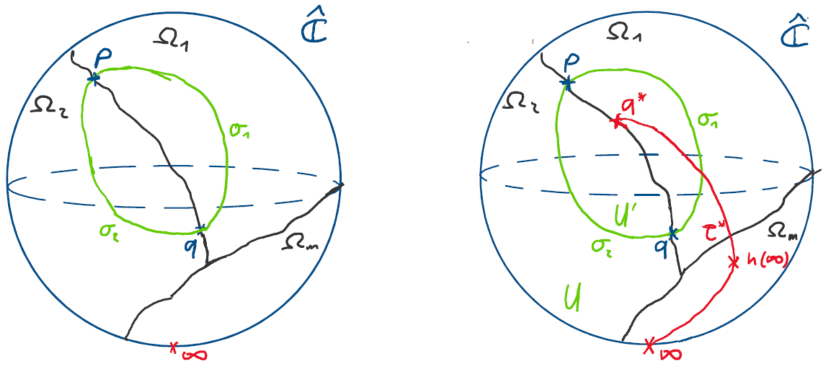


Figure 3: The construction from the proof of Proposition 3.15 2.). The limit set $\Lambda(G)$ is in black.

or $c = p$, we get the same contradiction, since p is the limit of repelling loxodromic (or parabolic) fixed points of g_n and as such in $\Lambda(G)$.

2. Suppose there is a finite number of components $\Omega_1, ..., \Omega_m$, $m > 2$, such that $\infty \in \Omega_m$. There is a subgroup G_0 of finite index and with the same limit set that preserves each of them. The reason for the index of G_0 being finite is, that there are only finitely many possibilities regarding which component is mapped to which. On the other hand, because of this for any loxodromic $g \in G$ there is $k \in \mathbb{N}$ with $g^k \in G_0$. Therefore G_0 has the same set of loxodromic fixed points as G, causing the limit sets to be the same by Lemma 3.3, 3.).
Choose a loxodromic transformation $g \in G_0$. Since g in particular preserves Ω_1 and Ω_2, we can find simple arcs $\sigma_i \in \Omega_i$, $i = 1, 2$, such that $\sigma_i^* = \bigcup_{k=-\infty}^{\infty} g^k(\sigma_i)$ forms a simple arc in Ω_i between the two fixed points of g. This is most easily done by using the quotient surface $\Omega_i / \operatorname{Stab}(\Omega_i)$ and picking any closed, not 0-homotopic curve. Then $\sigma^* = \sigma_1^* \cup \sigma_2^* \cup \{p, q\}$ forms a simple closed curve meeting $\Omega(G)$ only in Ω_1 and Ω_2 like in the left part of Figure 3. Consider the two components of $\hat{\mathbb{C}} \setminus \sigma^*$. One of them, say U, contains Ω_m and ∞. The other, U', contains points of $\Lambda(G)$, for otherwise σ_1^* and σ_2^* could be connected by an arc that does not meet $\Lambda(G)$. Therefore we can find an loxodromic element $h \in G_0$ with attracting fixed point q^* in U'. Connect ∞ to $h(\infty) \in \Omega_m$ by an arc $\tau \subseteq \Omega_m$ and set $\tau^* = \bigcup_{k=0}^{\infty} h^k(\tau)$. Now τ^* is an arch in Ω_m connecting $\infty \in U$ to the attracting fixed point q^* of h in U', so τ^* must cross σ^*, giving a contradiction. This can be seen in the right part of Figure 3. Proof from [Mar07], Lemma 2.4.2.

□

Having written down this lemma we now know enough to advance to the heart of this thesis: the geometry and topology of hyperbolic manifolds.

4 Hyperbolic Manifolds and Orbifolds

During this section we will start to examine hyperbolic manifolds and orbifolds.

Definition 4.1. Let G be a Kleinian group, that is torsionfree, meaning it has no elliptic elements. Then the quotient

$$\mathcal{M}(G) := {}^{\mathbb{H}^3 \cup \Omega(G)}\!\diagup_{G}, \quad \partial\mathcal{M}(G) = {}^{\Omega(G)}\!\diagup_{G} \tag{25}$$

is a smooth, even conformal manifold with boundary $\partial\mathcal{M}(G)$. We will call it a **Kleinian manifold**. The interior $\mathbb{H}^3/G$ has the structure of a **hyperbolic manifold** and the boundary $\partial\mathcal{M}(G)$ is a Riemannian surface.

$\mathcal{M}(G)$ is a manifold since G acts properly discontinuously and freely on $\mathbb{H}^3 \cup \Omega(G)$. Moreover, the projection $\pi : \mathbb{H}^3 \to \mathbb{H}^3/G$ is a local homeomorphism. $\mathcal{M}(G)^{int}$ has a hyperbolic structure induced by projection of the hyperbolic metric on $\mathbb{H}^3$ via π. The fundamental group $\pi_1(\mathcal{M}(G))$ of the quotient is isomorphic to G and $\mathbb{H}^3$ is the universal cover of $\mathcal{M}(G)^{int}$.

4.1 Orbifolds

On the other hand, we can consider the case that G has elliptic elements. Since then the action of G on $\mathbb{H}^3$ is not free, what we get can not be a manifold at the projection of the elliptic axes. That is why we will introduce the concept of an orbifold. Informally, an orbifold is like a manifold, but instead of being locally similar to $\mathbb{R}^n$ it is locally similar to the quotient of $\mathbb{R}^n$ over a finite group, allowing reflection points or rotation axis to exists in our space

Definition 4.2. An orbifold is a tuple (X, A), where X is called the underlying space and A is called orbifold atlas, that satisfy the following properties:
X is a topological Hausdorff space with a covering by a collection of open sets U_i, which is closed under finite intersections. For each U_i there is

- an open subset V_i of $\mathbb{R}^n$, invariant under a faithful action of a finite group Γ_i
- a continuous map $\varphi_i : V_i \to U_i$ invariant under Γ_i called **orbifold chart**

The collection of maps φ_i is an orbifold atlas A, if the following properties are satisfied:

- for each inclusion $U_i \subseteq U_j$ there is an injective group homomorphism $f_{ij} : \Gamma_i \to \Gamma_j$ and a f_{ij}-equivariant homeomorphism ψ_{ij} of V_i onto an open subset of V_j called **gluing map**
- the gluing maps are compatible with the charts: $\varphi_j \circ \psi_{ij} = \varphi_i$
- the gluing maps are unique up to translation by Γ_j meaning for gluing maps ψ_{ij}, ψ'_{ij} there exists a $g \in \Gamma_j$ with $\psi'_{ij} = g\psi_{ij}$.

Figure 4 shows the constructions necessary to define an orbifold in the case of a Kleinian group with two disjoint elliptic rotation axes. The injective group homomorphisms f_{31}, f_{32} are here simply the inclusion of the identity in the cyclical groups C_5 resp. C_3. To make it easier to understand we do not show the quotients but rather their universal coverings $\mathbb{R}^3$ and $\mathbb{H}^3$.

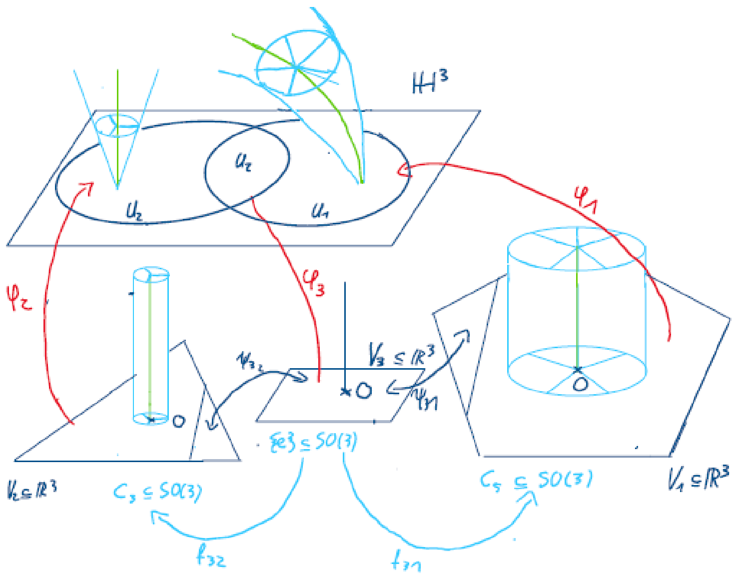


Figure 4: The constructions defining an orbifold with notation from Definition 4.2. The sets U_1, U_2, U_3 and V_1, V_2, V_3 should be thought of as 3-dimensional and arising from the drawn boundary in the plane.

Example 4.3. The easiest example of an orbifold is simply the quotient of $\mathbb{R}$ by the identification $x \sim -x$. In a neighbourhood of zero this is not a manifold, since the reflection does not act freely on it, instead it has a fixed point at zero.

Quotients of hyperbolic manifolds by a group of rotations about a single axis are orbifolds. Around a rotation axis we can define an orbifold chart with a finite subgroup of $O(n)$, otherwise we just use regular charts with the 0-group. Then the given properties are satisfied. The general case of Kleinan groups G acting on $\mathbb{H}^3$ is a bit harder since the elliptics in G do usually not form a subgroup and the rotation axis need not be disjoint. Nontheless the quotient is an orbifold. In this case $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ is a *branched cover* of $\mathcal{M}(G)$, which we will not discuss in more detail. The projection of a rotation axis l is called a *cone axis*, because it is reminiscent of the construction of a (3d-) cone by wrapping up a wedge of angle $< 2\pi$. Locally in this area the projection has the form $(z, t) \to (z^r, t)$, where z is the coordinate in a plane orthogonal to the axis and t goes along the axis.

One basic fact we will be using all the time is the following lemma:

Lemma 4.4. *If the Kleinian groups G, H are conjugated, their Kleinian manifolds $\mathcal{M}(G)$, $\mathcal{M}(H)$ are diffeomorphic. More exactly their interiors are isomorphic as Riemannian manifolds, and their boundaries are conformally equivalent.*

Proof. Let $H = hGh^{-1}$ for a Möbius transformation h. Then the map

$$\phi : \mathcal{M}(G) \to \mathcal{M}(H),\ G \cdot x \mapsto H \cdot h(x)$$

is well defined: Let $y \in G \cdot x$. So there exists $g \in G$, with $g(x) = y$ and

$$\begin{aligned}\phi([y]_G) &= [h(y)]_H = [hg(x)]_H = [hg^{-1}h^{-1}hg(x)]_H \\ &= [h(x)]_H = \phi([x]_G).\end{aligned}$$

That ϕ is additionally an isometry respectively a conformal map, follows since h is an isometry respectively a conformal map on $\mathbb{H}^3$ respectively $\Omega(G)$. □

4.2 Closed Geodesics

Right away, we want to dive deeper into geometric and topological aspects of general Kleinian manifolds and orbifolds $\mathcal{M}(G) = \mathbb{H}^3 \cup \Omega(G)/G$, where G is a discrete subgroup of $PSL(2, \mathbb{C})$. At first we want to familiarize the reader with a couple important topological features in hyperbolic manifolds.

Remark 4.5 (Closed geodesics)**.** Let γ^* be the axis of a primitive loxodromic $g \in G$. This means, there is no loxodromic $h \in G$ with the same axis and $h^n = g$ for $n > 1$. At first, suppose there is no elliptic in G with the same rotation axis, and there is no elliptic of order two interchanging the endpoints of g. Then the projection of γ^* in the quotient, called γ, is a closed geodesic. Its length is given by $d(x, g(x))$ for any $x \in \gamma^*$. Conversely, every closed geodesic is the projection of the rotation axis of a primitive loxodromic element of G. Additionally, the geodesic γ is *simple*, meaning it does not intersect itself, if the orbit $G \cdot \gamma^*$ consists of pairwise disjoint geodesics in $\mathbb{H}^3$.
On the other hand, if γ^* is also the axis of an elliptic element, its projection γ is a closed geodesic and also a cone axis, a feature we talked about in the orbifold section. And if there is an elliptic element of order two interchanging the endpoints of γ^*, then γ projects to a finite geodesic segment of length $d(x, g(x))/2$ with endpoints on cone axes, the degenerate case of a closed curve, as we go back and forth along the segment. From [Mar07] pp. 106-107.

4.3 Horospheres and Horoballs

A second feature, we would like to discuss, are neighbourhoods of a parabolic fixed point on $\hat{\mathbb{C}}$.

Remark 4.6. Here we only consider torsionfree Kleinian groups G. Neighbourhoods of parabolic fixed points in $\mathbb{H}^3 \cup \Omega(G)$ are the domains, where the projection $\mathcal{M}(G)$ fails to be compact. This means they cause "holes" in the boundary at infinity of the quotient. If there are no parabolics like in our discussion of the purely loxodromic Schottky groups in the following section, the boundary is a closed surface without punctures or other special features and the Kleinian manifold is compact.

To examine more thoroughly, what the structure near parabolic fixed points is, we need to define the concept of a *horosphere*.

Proposition 4.7. *Let ϕ be a parabolic Möbius transformation with fixed point $\zeta \in \hat{\mathbb{C}}$. Then there is a family of 2-dimensional surfaces $\{H_{\zeta,r}\} \subseteq \mathbb{H}^3$, for $r \in \mathbb{R}_+$, where each surface $H_{\zeta,r}$ is left invariant by ϕ and tangent to $\partial\mathbb{H}^3$ in ζ. The parameter r can be seen as a measure for the distance of the surface to the boundary point ζ relative to other points in $\mathbb{H}^3$.*

Proof. Because of Proposition 2.13 we know, that each parabolic Möbius transformation ϕ is conjugated to $\phi_0 : z \mapsto z+1$, given by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Since Möbius transformations are isometries of $\mathbb{H}^3$ and homeomorphisms of $\mathbb{H}^3 \cup \hat{\mathbb{C}}$, we can without loss of generality restrict us to the case $\zeta = \infty$ and $\phi = \phi_0$. Using formula (8) for extension to $\mathbb{H}^3$ on ϕ_0 we get that ϕ_0 acts by $(z,t) \mapsto (z+1,t)$ on $\mathbb{H}^3$. The invariant surfaces are therefore

$$H_{\infty,r} = \{(z,t) \mid z \in \mathbb{C}, t = r^{-1}\} \subseteq \mathbb{H}^3 \tag{26}$$

and they are all tangent to $\infty \in \hat{\mathbb{C}}$.
The invariant surfaces at a point $w = \psi(\infty)$ for a Möbius transformation $\psi = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are given by $H_{w,r} = \psi(H_{\infty,r})$. If we calculate this with formula (7), we can see that in this case the surfaces have the form of Euclidean spheres with center $(a/c, |c|^{-2}0.5r) \in \mathbb{H}^3$ tangent to $w = a/c$. ☐

Definition 4.8. The surfaces $H_{\zeta,r}$ for $\zeta \in \mathbb{C}$, ζ fixed by a parabolic $g \in G$ and $r \in \mathbb{R}$ are called **horospheres**. On them we can define metrics $d_{H_{\zeta,r}}$ where $d_{H_{\zeta,r}}(x,y)$ is given as the hyperbolic length of the shortest path on $H_{\zeta,r}$ between x and y. The union $\mathcal{H}_{\zeta,r} := \bigcup_{0<r<R} H_{\zeta,r}$ for some $R \in \mathbb{R}_+$ is called a **horoball**.

Figure 5 shows two horospheres, one tangent to ∞ and one tangent to a point $\zeta \in \hat{\mathbb{C}}$.

Proposition 4.9. *For any horosphere and any Kleinian group G, there is a* least translation length parabolic $T^{\pm 1} \in \mathrm{Stab}_\zeta(G)$, *that satisfies $d_{H_{\zeta,r}}(x, T(x)) \leq d_{H_{\zeta,r}}(x, h(x))$ for any other parabolic $h \in \mathrm{Stab}_\zeta(G)$ and all $x \in H_{\zeta,r}$. This T is the same for all $r \in \mathbb{R}$ and only depends on $\zeta \in \hat{\mathbb{C}}$.*

Proof. If the base point on the boundary is $\zeta = \infty$, this is clear, because then $H_{\zeta,r} = \{(z,t) \mid t = r^{-1}\}$ and the metric on $H_{\zeta,r}$ is Euclidean. Using Theorem 3.7 for the structure of the parabolic stabilizer, we get that a least length parabolic for all $r \in \mathbb{R}$ must exist.
If $\zeta \neq \infty$, we can replace G by a conjugate with some $\phi \in PSL(2,\mathbb{C})$, such that the fixed point of $\phi \circ g \circ \phi^{-1}$ is ∞. Using that ϕ is an isometry of $\mathbb{H}^3$, we are finished. ☐

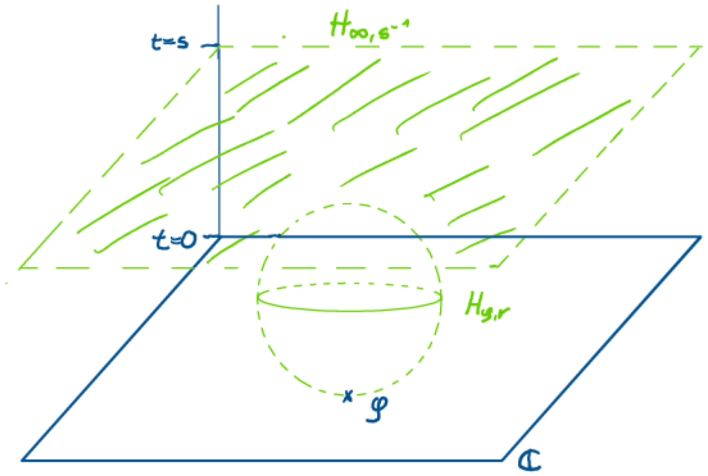


Figure 5: Two horospheres. One tangent to $\zeta \in \mathbb{C}$, one tangent to ∞. The interior of the sphere is the horoball $\mathcal{H}_{\zeta,r}$, what lays above the horosphere at $H_{\infty,s^{-1}}$ is the horoball $\mathcal{H}_{\infty,s^{-1}}$.

We want to use horospheres to separate parts of our quotient manifold, where the structure only depends on the subgroup $\operatorname{Stab}_\zeta(G) \subseteq G$. The quotient $\mathbb{H}^3/\operatorname{Stab}_\zeta(G)$ is a Riemannian covering of $\mathbb{H}^3/G$, because $\operatorname{Stab}_\zeta(G)$ is a subgroup of G.
Let

$$\pi_{\zeta,1} : \mathbb{H}^3 \to \mathbb{H}^3 / \operatorname{Stab}_\zeta(G)$$

and

$$\pi_{\zeta,2} : \mathbb{H}^3 / \operatorname{Stab}_\zeta(G) \to \mathbb{H}^3 / G$$

be the associated covering maps, which are local isometries. Then $\pi_{\zeta,2} \circ \pi_{\zeta,1} = \pi$. If we can find a region $U \subseteq \mathbb{H}^3/\operatorname{Stab}_\zeta(G)$, such that the projection of the horoball $\pi_{\zeta,1}(\mathcal{H}_{\zeta,r}) \subseteq U$ and $\pi_{\zeta,2}|_U$ is an isometry, then we have found such a part. In this case we also say $\mathcal{H}_{\zeta,r}/\operatorname{Stab}_\zeta(G)$ is *embedded*. Naturally every projection of a smaller horoball at ζ is also embedded.

Proposition 4.10. *For any parabolic fixed point ζ of any $g \in G$, there is a horoball $\mathcal{H}_{\zeta,r}$ such that*

$$(G \setminus \operatorname{Stab}_\zeta(G)) \cdot \mathcal{H}_{\zeta,r} \cap \mathcal{H}_{\zeta,r} = \emptyset \tag{27}$$

Thus, there always exists an embedded projection of a horoball for every parabolic fixed point in $\mathcal{M}(G)$.

This statement follows from Theorem 6.3 in the next section of this thesis.

4.3.1 Cusp Tubes and Cusp Tori

Finally, we determine the exact structure of these neighbourhoods of parabolic fixed points discussed above. Let the projection $\pi_{\zeta,1}(\mathcal{H}_{\zeta,r})$ be embedded. Without loss of generality we can choose $\zeta = \infty$ and $z \mapsto z+1$ to be a generator of $\operatorname{Stab}_\zeta(G)$ whenever we want.

Remark 4.11.

1. Let ∞ be a rank-one parabolic fixed point. Then $\operatorname{Stab}_\infty(G) = \langle z \mapsto z+1 \rangle$. With the formula for extension to $\mathbb{H}^3$ (10) from section 1 we get that the strip
$$\{(z,t) \mid 0 \leq \Re z < 1\}$$
is a fundamental region for the action of $\operatorname{Stab}_\infty(G)$ on $\mathbb{H}^3$. Via the map
$$(z,t) \mapsto (e^{2\pi i z}, \log(t))$$
the quotient $\mathbb{H}^3 / \operatorname{Stab}_\infty(G)$ is homeomorphic to the doubly infinite tube with a hole $\mathbb{C} \setminus \{0\} \times (-\infty, \infty)$. Hence, when $\pi_{\zeta,1}(\mathcal{H}_{\zeta,r})$ is embedded,
$$\pi(\mathcal{H}_{\zeta,r}) \cong \mathbb{C} \setminus \{0\} \times (c, \infty) \tag{28}$$
topologically in $\mathbb{H}^3/G$ for any rank one parabolic fixed point ζ. We refer to this as a *solid cusp tube*, although the equivalence is only topological and not a metric one. If we were to compactify $\mathbb{H}^3/G$ around ζ and cut this domain of, we would get a boundary component isomorphic to $\pi(H_{\zeta,r}) \cong \mathbb{C} \setminus \{0\}$.

2. Now let ∞ be a rank-two parabolic fixed point. In this case by Theorem 3.7 $\operatorname{Stab}_\infty(G) = \langle z \mapsto z+1, z \mapsto z+\tau \rangle$ for some $\tau \in \mathbb{C}, \Im\tau > 0$. Thus a fundamental region of $\operatorname{Stab}_\infty(G)$ in $\mathbb{H}^3$ is the inside of the parallelogram with vertices $\{0, 1, 1+\tau, \tau\}$ plus two of its non-parallel borders times $(0, \infty)$. The quotient then is homoemorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$ where the last factor comes from the infinite vertical axis. Thus, if $\pi_{\zeta,1}(\mathcal{H}_{\zeta,r})$ is embedded,
$$\pi(\mathcal{H}_{\zeta,r}) \cong \mathbb{S}^1 \times D^*, \quad \text{where } D^* = \{x \in \mathbb{R}^2 \mid 0 < |x| < \infty\} \cong \mathbb{S}^1 \times (r^{-1}, \infty) \tag{29}$$
in $\mathbb{H}^3/G$ for any rank two parabolic fixed point ζ. This, we call a *solid cusp torus*, since it is topologically a solid torus where one circle in the interior is missing. Again one has to remember that this is only a topological equivalence Again if we wanted to compactify $\mathcal{M}(G)$ around ζ, by cutting the solid cusp torus off, we would get a boundary component isomorphic to $\pi(H_{\zeta,r}) \cong \mathbb{T}^2$.

In Figure 6 a Kleinian manifold, whose conformal boundary is a once-punctured torus is shown. The puncture is associated to a rank-one parabolic fixed point and the region in its interior to the right of the green disc, which represents the projection of a horosphere, is a solid cups tube.

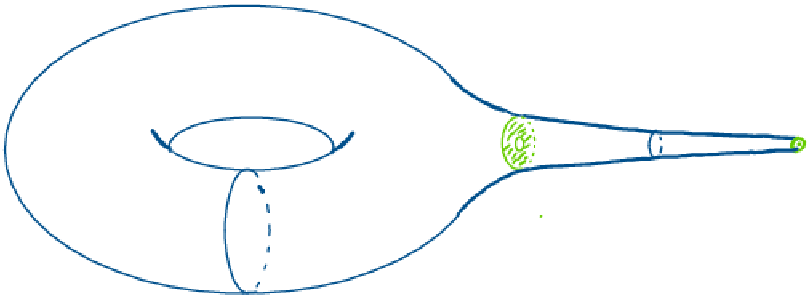


Figure 6: A Kleinian manifold with a rank-one parabolic fixed point. The region right from the green disk is a solid cusp tube.

5 Schottky groups

5.1 Algebraic Properties

We will now focus our attention for a bit on a special kind of Kleinian manifolds as an example for what we have done so far.

Definition 5.1. Let $p \in \hat{\mathbb{C}}$ be a point. If a generalized circle does not contain p, it divides $\hat{\mathbb{C}}$ in two regions; the one not containing p is for us the interior. Let $C_1, ..., C_k$, $C_1', ..., C_k'$ be $2k$ pairwise disjoint circles with disjoint interior. Then, there exist Möbius transformations $M_1, ..., M_k$, such that M_1 maps the inside of C_1 bijectively to the outside of C_1'. The group $G = \langle \phi_1, ..., \phi_k \rangle$ is then called a **classical Schottky group**.

At first, we want the reader to remember that such Möbius transformations always exist, as the group of Möbius transformations acts transitively on the set of all generalized circles in $\hat{\mathbb{C}}$ and inversions in circles are also Möbius transformations. We will give one concrete example of a Schottky group:

Definition 5.2. A θ-Schottky group is the group generated by the transformations

$$\frac{1}{\sin(\theta)}\begin{pmatrix} 1 & \cos(\theta) \\ \cos(\theta) & 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sin(\theta)}\begin{pmatrix} 1 & \cos(\theta)i \\ -\cos(\theta)i & 1 \end{pmatrix} \quad \text{for } \theta \in (0, \pi/4). \tag{30}$$

The Schottky circles of this group are orthogonal to the unit circle with centers $-1/\cos(\theta)$ and $1/\cos(\theta)$ for the paired circles C_1 and C_1', while the other pair of circles C_2, C_2' has their centers at $-i/\cos(\theta)$ and $i/\cos(\theta)$. All circles have a radius of $\tan(\theta)$. This yields a Schottky group for $\theta \in (0, \pi/4)$

In Figure 8 you can see the circles for $\theta = 0.75$. In the same way almost all other figures in this section also show different aspects of the θ-Schottky group for $\theta = 0.75$. The formulas for this groups are from [MSW02], p.118, while the algorithms generating the pictures are my own work.

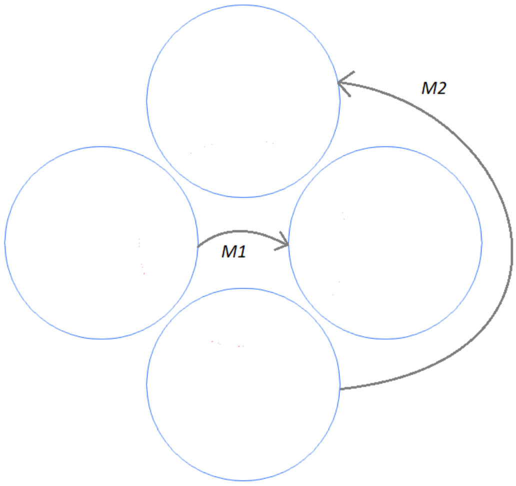


Figure 7: The original 4 Schottky circles of the so called θ-Schottky group with $\theta = 0.75$. The circles are orthogonal to the unit circle in $\mathbb{C}$.

Immediately, we are able to proof a theorem giving us the group theoretic structure of a Schottky group.

Theorem 5.3 (Ping-Pong lemma for Schottky groups). *The Schottky group $G = \langle M_1, ..., M_k \rangle$ is the free group on k generators.*

This statement can also be proven under more general conditions and not only for Schottky groups, but we are not interested in these general cases.

Proof. Proof after [dlH00]. The only need thing we need to proof is, that any reduced composition of the generators and their inverses is different from the identity. Reduced means here, that we have to remove pairs of transformations, where a transformation is immediately followed by its inverse, until there are no more such pairs left. Of course, this will not alter the element of the group we describe, only the way we write it down. Let now

$$g = \prod_{j=1}^{m} g_j^{n_j}, \quad \text{where } g_j \in \{M_1, ..., M_k\},\ n_j \in \mathbb{Z},\ \text{for all } j,\ m \in \mathbb{N} \tag{31}$$

such that $g_i \neq g_{i+1}$ for all i. Denote by $\tilde{C}_i, \tilde{C}'_i$ the respective interiors of the circles C_i, C'_i. We consider the action of g on one of the $\tilde{C}_i \cup \tilde{C}'_i$'s. Choose here $i = 2$. For this we assume initially that $g_1 = g_m = M_1$ and that $m \geq 3$. Otherwise we have simply a power of M_1, which is not the identity. For $i \neq 1$, $\tilde{C}_i \cup \tilde{C}'_i$ is mapped into $\tilde{C}_1 \cup \tilde{C}'_1$ by $g_1^{n_1}$, since n_1 can be positive or negative. Consequently, we get the following chain of containments:

$$g(\tilde{C}_2 \cup \tilde{C}_2') \subseteq \left(\prod_{j=1}^{m-1} g_j^{n_j}\right)(\tilde{C}_1 \cup \tilde{C}_1') \subseteq \left(\prod_{j=1}^{m-2} g_j^{n_j}\right)(\tilde{C}_{\alpha_{m-1}} \cup \tilde{C}'_{\alpha_{m-1}}) \tag{32}$$

$$\subseteq ... \subseteq g_1^{n_1}(\tilde{C}_{\alpha_2} \cup \tilde{C}'_{\alpha_2}) \subseteq \tilde{C}_1 \cup \tilde{C}_1' \tag{33}$$

Here α_j is the index, so $g_j = M_{\alpha_j}$

By using that the circles are disjoint, we get that g cannot act trivially on $\tilde{C}_2 \cup \tilde{C}_2'$, so $g \neq \mathrm{Id}$. Now to finish the argument we need to get rid of the assumptions on g_1, g_m. There are three cases to consider:

- $g_1 = M_1 \neq g_m$: define $h = g_1^{n_1}$;
- $g_1 \neq M_1 = g_m$: then let $h = g_m^{-n_m}$;
- $g_1 \neq M_1 \neq g_m$: then let $h = M_1$;

In each case $h \circ g \circ h^{-1}$ is a reduced composition satisfying the conditions from above. And since $h \circ g \circ h^{-1}$ is not trivial, neither is g. □

By this theorem, each element in G can be written as a unique composition

$$w = \prod_{n=1}^{N} M_{i_n}^{s_n} \quad \text{where } s_n \in \{-1, 1\},\ i_n \in \{1, ..., k\} \text{ for all } n,\ N \in \mathbb{N} \tag{34}$$

and we forbid that $M_{i_{n+1}}^{s_{n+1}} = M_{i_n}^{-s_n}$.
As now the structure of a Schottky group is known, let us see what we can say about the orbits of the circles $C_1, ..., C_k, C_1', ..., C_k'$ under the group action. To do this properly we change notation. A generator M_a now maps the inside of the circle C_A to the outside of the circle C_a. Its inverse is M_A, which pairs C_a with C_A the same way. We denote the set of indices by

$$\mathcal{C} = \{A, a, B, b, ...\}$$

up to the k-th letter in the alphabet. For an index $x \in \mathcal{C}$ we write X for the "inverse index", such that $M_X = M_x^{-1}$ is satisfied.

Remark 5.4. Applying the generator M_a to any of the $2k-1$ circles C_x for $x \in \mathcal{C} \setminus \{A\}$, creates a new circle $C_{ax} := M_a(C_x)$ located inside C_a. Moreover, M_a maps the inside of A to the inside of C_{ax}, because C_x lies in the exterior of C_A. The same works for any of the other generators, if the respective "starting" circle of the generator is excluded. If we now apply a generator M_z with $M_z \neq M_y^{-1}$, $y, z \in \mathcal{C}$ to such a circle C_{yx} ($M_y \neq M_x^{-1}$) we receive again a new circle $C_{zyx} \subseteq C_z$. More exactly, we have the chain of containments

$$C_{zyx} \subseteq C_{zy} \subseteq C_z, \tag{35}$$

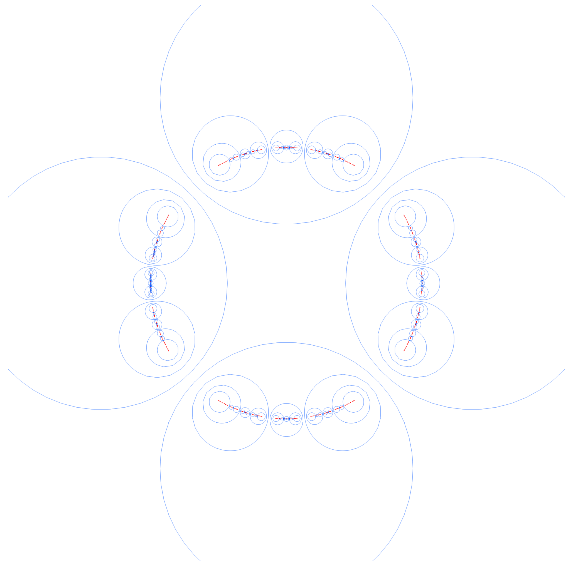


Figure 8: The nested Schottky circles of the θ-Schottky group with $\theta = 0.75$. In red we have its limit set.

because $C_{yx} = M_y(C_x)$ is a circle inside C_y and $M_z(C_y)$ is a circle inside C_z, as C_y is in the exterior of C_Z and C_x is in the exterior of C_Y. Also

$$M_{zy}(C_x) := M_z \circ M_y(C_x) = M_z(C_{yx}) = C_{zyx}. \tag{36}$$

For longer finite compositions of generators and their inverses, we get similar, but longer chains of nested circles, as long as the rules from above are followed. It is important to note that are all these disjoint and located inside the $2k$ original circles $C_A, C_a, ..., C_\Omega, C_\omega$. They form a G-invariant set which we call the set of *Schottky circles*.
Theorem 5.3 showed us, that we can write each element g of G as a unique reduced product of the generators. This discussion now extends this property to the Schottky circles, meaning we can write each Schottky circle C as $C_{x_1 \dots x_{r-1} x_r} = M_{x_1 \dots x_{r-1}}(C_{x_r})$ for $x_1, ..., x_r \in \mathcal{C}$. Here $M_{x_1 \dots x_{r-1}}$ has to be a reduced composition of generators and C_{x_r} is one of the original $2k$ circles, satisfying the additional rule $x_r \neq X_{r-1}$. (Ideas from [MSW02], pp.103-107).

Figure 8 shows the first 4 levels of Schottky circles from the $\theta = 0.75$ Schottky group, generated as the image of the original 4 circles under the elements of G, which compositions of up to 3 generators. The relation of the circles with the limit set, which is also shown, is explored in Remark 5.6. For the time being we want to see, which type of Möbius transformation Schottky groups are made of.

Lemma 5.5. *Let G be a Schottky group generated by the Möbius transformations $M_A, ..., M_\Omega$ and their inverses $M_a, ..., M_\omega$, with circles $C_a, ..., C_\omega$ and $C_A, ..., C_\Omega$. Then G only consists of loxodromic elements and the identity.*

Proof. Let $g \in G \setminus \{\mathrm{Id}\}$, $g = M_{x_1 \ldots x_r} := \prod_{i=1}^{r} M_{x_i}$ for $x_1, ..., x_r \in \mathcal{C}$, such that g is written down in reduced form ($M_{x_i} \neq M_{x_{i+1}}^{-1}$). At first assume $M_{x_1} \neq M_{X_r} := M_{x_r}^{-1}$ as well. We denote by D_w the closed disk on $\hat{\mathbb{C}}$ bounded by the generalized circle C_w for any w, a reduced combination of indices from $\mathcal{C}$. Then

$$g(D_{x_1}) = M_{x_1 \ldots x_r}(D_{x_1}) = D_{x_1 \ldots x_r x_1} \subseteq D_{x_1}$$

and

$$g^{-1}(D_{X_r}) = M_{x_1 \ldots x_r}^{-1}(D_{X_r}) = M_{X_r \ldots X_1}(D_{X_r}) = D_{X_r \ldots X_1 X_r} \subseteq D_{X_r}$$

by the previous remark. So we can apply Brouwer's fixed point theorem to g and the disk D_{x_1}, and g^{-1} with the disk D_{X_r} to get the existence of two fixed points of g one inside of $D_{x_1 \ldots x_r x_1} \subseteq D_{x_1}$ and one inside $D_{X_r \ldots X_1 X_r} \subseteq D_{X_r}$. Since both are the attracting fixed points of the respective transformations g, g^{-1}, a fact that is made clear by looking at g^n for some $n > 1$, g must be loxodromic.
In order to complete the proof we have to get rid of the assumption that $M_{x_1} \neq M_{X_r}$. But if $x_1 = X_r$, it follows that $g = M_{x_1} h M_{x_1}^{-1}$, and as a result g has the same type as $h := \prod_{i=2}^{r-1} M_{x_i} \in G$. This can be repeated until first and last generator are not inverses of each other, or we arrive at the identity, which would be a contradiction to g being in reduced form.

□

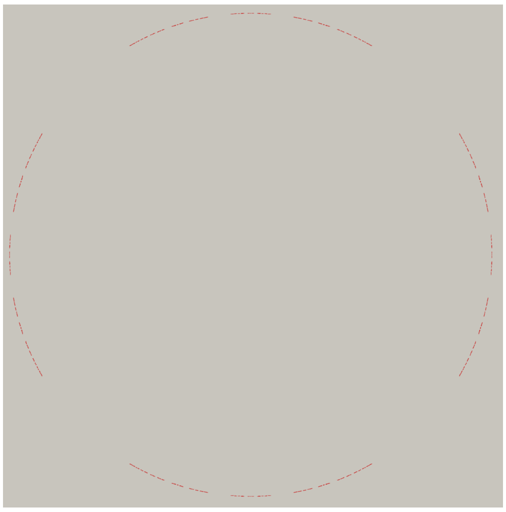
To close out the more algebraic part of the discussion of Schottky groups, we want to discuss some further properties of the Schottky circles, that are interesting and/or will come in handy later.

Remark 5.6.

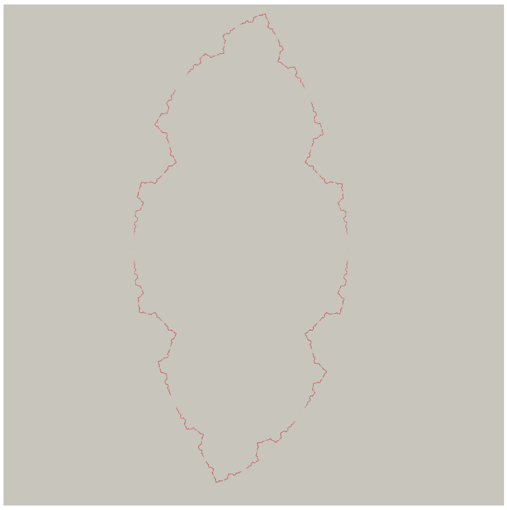
1. Extending the approach from the proof of Lemma 5.5, we can determine the shape of the limit set from the location of the Schottky circles. If $g = M_{x_1 \ldots x_r}$ where $M_{x_1}^{-1} \neq M_{x_r}$, we have proven that g has its attracting fixed point in the closed disk bounded by $C_{x_1 \ldots x_r x_1}$. Because g^n has the same fixed points for any n we can determine the position of its attracting fixed point more exactly by computing the much smaller circles $g^n(C_1) = Cx_1 ... x_r ... x_1 ... x_r x_1$ for $n > 1$. In conclusion, each loxodromic fixed point is at the bottom of an infinite chain of nested Schottky circles. By Proposition 3.13 iii.), so is the whole limit set. Figure 8 shows an approximation of this situation.

2. Furthermore, we also know now that for any circle $S = S_r = C_{x_1 \ldots x_r}$, there is a unique finite sequence of Schottky circles $\{S_n\}_{1 \leq n \leq r-1}$, $S_n = C_{x_1 \ldots x_n}$ that we call predecessors of S. This sequence satisfies, that S_n is contained in the interior of S_{n-1} for each $n = 2, .., r$, that there is no Schottky circle S', which contains S_n in its interior and is contained in the interior of S_{n-1} for any n, and that S_1 is equal to C_{x_1}, one of the original $2k$ circles.

3. Finally, just like there is a unique sequence of predecessors for every Schottky circle S, there are also $2k - 1$ immediate distinct successors, which have S as their



(a) The Cantor set-like limit set of our θ-Schottky group, where $\theta = 0.75$.



(b) Limit set of a group with touching original circles. This group contains parabolics.

Figure 9: Limit sets.

first predecessor. Let $S = C_{x_1 \dots x_r}$. Then the $2k-1$ Schottky circles $C_{x_1 \dots x_r x_s}$ for $x_s \in \mathcal{C} \setminus \{X_r\}$ have S at their first predecessor.

Figure 9 is a visualization of 2 different limit sets. One of them is the limit set of the $\theta = 0.75$-Schottky group, but this time without the Schottky circles, and the other one is the limit set of a group constructed similarly to a Schottky group but with touching circles. This group then contains parabolic elements. The formula for the second one was taken from [MSW02], p.229, Box 21.

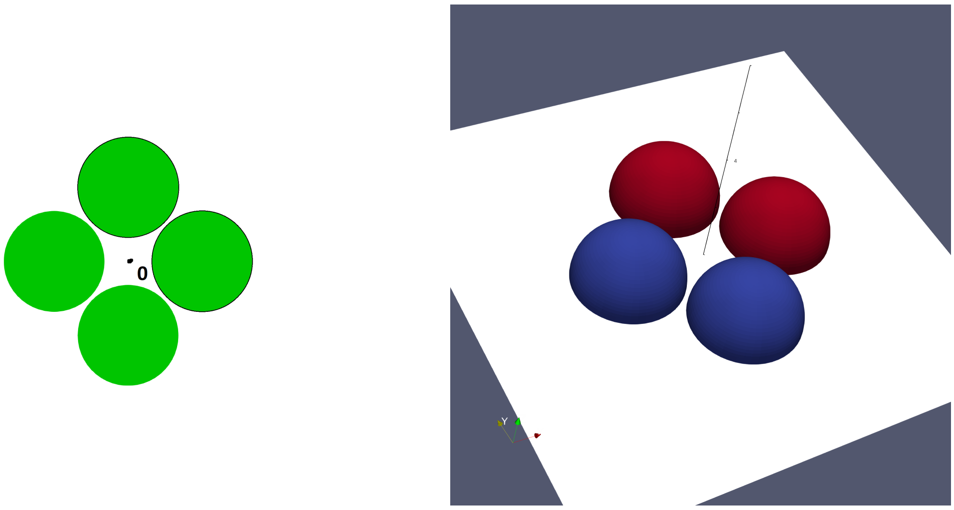
5.2 Geometry of Schottky Groups

Continuing with the more geometric properties, we want to get an idea how the quotient of $\mathbb{H}^3$ by a Schottky group looks like, and verify that this is indeed a manifold. We will start by determining a fundamental domain for the action of a Schottky group G on $\mathbb{H}^3$ and $\Omega(G)$. Again we will use the notation, that $G = \langle M_A, M_B, \dots M_\Omega \rangle \subseteq PSL(2, \mathbb{C})$, where $M_a = M_A^{-1}, \dots, M_\omega = M_\Omega^{-1}$ and the index set $\mathcal{C} = \{A, a, B, b, \dots, \Omega, \omega\}$ contains the first k letters of the alphabet twice. A transformation M_X with $X \in \mathcal{C}$ maps the interior of a circle C_X to the exterior of the circle C_x.

Definition 5.7. For our purpose a fundamental domain of the action of a group H on topological space X is a region $F \subseteq X$ such that for every $x \in X$, the orbit $H \cdot x$ intersects F in exactly one point.

Let U be the common exterior of the circles $C_A, C_a, C_B, C_b, \dots, C_\Omega, C_\omega$. We want to show that,

$$\mathcal{F} := U \cup C_A \cup C_B \cup \dots \cup C_\Omega \tag{37}$$



(a) The complement of the open green disks and the black circles bounding two of the disks in $\hat{\mathbb{C}}$ is a fundamental region in the ordinary set for our $\theta = 0.75$-Schottky group.

(b) The outside of the shown, disjoint half-spheres including the two blue half-spheres themselves is a fundamental region of the $\theta = 0.75$-Schottky group action in $\mathbb{H}^3$.

Figure 10: The respective fundamental regions.

is a fundamental domain for the action of G on $\Omega(G) \subseteq \hat{\mathbb{C}}$.
Let $P_x \subseteq \mathbb{H}^3$ be the hyperbolic plane, that is bounded by the circle $C_x \subseteq \hat{\mathbb{C}}$ for every $x \in \mathcal{C}$. In $\mathbb{H}^3$ we call the region $Q \subseteq \mathbb{H}^3$, which is bounded by $P_A, P_a, P_B, P_b, ..., P_\Omega, P_\omega$ and touches all of them, the common exterior of those planes. There we want to prove that,

$$\hat{\mathcal{F}} := Q \cup P_A \cup P_B \cup ... \cup P_\Omega \tag{38}$$

is a fundamental domain for the action of G on $\mathbb{H}^3$. Pictures of the supposed fundamental regions in $\hat{\mathbb{C}}$ and $\mathbb{H}^3$ of the $\theta = 0.75$ Schottky group can be seen in Figure 10.

Lemma 5.8. *The regions $\mathcal{F}$ and $\hat{\mathcal{F}}$ contain at most one point of any G-orbit $G \cdot z$ for $z \in \hat{\mathbb{C}}$, respectively $G \cdot p$ for $p \in \mathbb{H}^3$.*

Proof. At first we consider the action of G on $\hat{\mathbb{C}}$. Let g be an element in $G \setminus \{\text{Id}\}$, then $g = M_{x_r \dots x_1}$ with $x_r, ..., x_1 \in \mathcal{C}$, is written as a reduced composition of generators. Let $z \in \mathcal{F}$. Then $M_{x_1}(z) \subseteq D_{x_1}$. However, $M_{x_r \dots x_2}(D_{x_1}) = D_{x_r \dots x_2 x_1} \subseteq D_{x_r}$ and this last inclusion is also an inclusion in the interior of D_{x_r} if $r > 1$. So if $r > 1$, then $g(z) \notin \mathcal{F}$. On the other hand if $r = 1$, we have $g(z) = M_{x_1}(z) \in D_{x_1}$, and $z \in \partial D_{x_1} = C_{x_1}$ if and only if $z \in C_{X_1}$. But in this case, as $z \in \mathcal{F}$, it follows that $C_{X_1} \subseteq \mathcal{F}$ which means $g(z) \in C_{x_1} \subseteq \mathcal{F}^C$.
Because of the continuity of the action on $\mathbb{H}^3 \cup \hat{\mathbb{C}}$, the same statement holds true for $p \in \hat{\mathcal{F}}$ with the adjusted regions in $\mathbb{H}^3$: the hyperbolic planes P_x bounded by circles C_x for all $x \in \mathcal{C}$, and their "interiors". These are the pairwise disjoint regions in $\mathbb{H}^3$, which

are bounded only by one plane P_x each. So for $p \in \hat{\mathcal{F}}$, we have that $g(p)$ is either in the "interior" of one of these planes, or potentially on a plane P_x for $x \in a, b, ..., \omega$, meaning $g(p) \notin \mathcal{F}$.

□

Before we can finish proving, that $\mathcal{F}$ and $\hat{\mathcal{F}}$ are fundamental regions, we need to prove a different statement, which we also need independently.

Lemma 5.9. *Let G be a Schottky group. Then G is discrete.*

Proof. Proof taken from [Rat19], Theorem 12.2.16. Let z be in $U \subseteq \mathcal{F}$, the common exterior of the Schottky circles. From the previous lemma we know $\{z\}$ is open in $G \cdot z$ with the relative topology, since U is open in $\hat{\mathbb{C}}$ and contains no other $w \in G \cdot z$. Let $\epsilon : G \to G \cdot z$ be the evaluation map at z. Then ϵ is continuous. This is a general result for example for the isometry group of symmetric spaces ([Hel72], Chapter IV, Theorem 2.5). Therefore $\epsilon^{-1}(\{z\}) = \{\mathrm{Id}\}$ is open in G and so G is discrete by Proposition 3.3. □

As a result, we now know that the quotient $\mathbb{H}^3/G$ for any Schottky group G is a hyperbolic manifold. To complete our picture of it, we return to the fundamental domains:

Theorem 5.10. *In $\Omega(G)$ respectively $\mathbb{H}^3$ the regions $\mathcal{F}$ respectively $\hat{\mathcal{F}}$ are fundamental domains for the G-action.*

Proof. After Lemma 5.8 what is left to show, is that every G-orbit of a point $z \in \Omega(G)$ or $p \in \mathbb{H}^3$ contains a point in $\mathcal{F}$ respectively $\hat{\mathcal{F}}$. We first consider the case of $\Omega(G)$. Initially, we want to prove, that there are only finitely many Schottky circles containing any $z \in \Omega(G)$. Assume the opposite. Then either the radii of the circles have to shrink down to 0, such that z would have to be in $\Lambda(G)$ by Remark 5.6, i), or there are infinitely many nested Schottky circles $\{S_n : n \in \mathcal{N}\}$, $S_n = w_n(C_{x_n})$ for $x_n \in \mathcal{C}$, whose radius is larger than some $r \in \mathbb{R}$. (It makes sense here to speak of the radii of Schottky circles, since at most one of them can be an Euclidean line plus infinity.) Important is that the $w_n \in G$ are pairwise distinct. But this is clear after transitioning to a suitable subsequence, because there are only $2k-1$ possible Schottky circles created by one w_n, as there are $2k-1$ allowed original circles C_{x_n} to which w_n can be applied.
But then we can find a subsequence $\{w_m\}$ of $\{w_n\}$, such that w_m converges to a constant u on $\hat{\mathbb{C}} \setminus \{v\}$ for some $u, v \in \hat{\mathbb{C}}$ by Lemma 3.1, because G is discrete. This obviously is a contradiction to the set of circles $\{S_m\}$ having radius greater than r.
With this we know there is a Schottky circle S, such that z is located inside or on S, but there is no successor of S containing z. Let w_S be the associated reduced composition from Remark 5.4, satisfying $S = w_S(C_x)$ with $x \in \mathcal{C}$. Then w_S maps the inside of C_x to the inside of S and $w_S \circ M_X$ maps the outside of C_X to the inside of S. This means $z' = M_x \circ w_C^{-1}(z) \in \mathcal{F}$. The reason for that is, if z' was element of some Schottky circle E or its interior, then $w_S \circ M_x(E)$ would be a successor of C containing z, a contradiction.
For $p = (w, t) \in \mathbb{H}^3$ the situation is the same. We are just dealing with planes instead of circles, which we will call Schottky planes. Again, p can only be in the "interior" of finitely many such planes. This is because on the one hand, there can only be finitely

many Schottky circles with radius $r > 1/n$ for any n because their radius is bounded by the maximal radius of the original $2k$ circles. And if p is in the interior of a plane P, than $1/n < t \leq r_P$ for some n, where r_P is the radius of the boundary circle of the plane in $\hat{\mathbb{C}}$. On the other hand, there can also only be one Schottky plane arising from a Euclidean line plus ∞ in $\hat{\mathbb{C}}$ and containing p in its "interior", since its boundary then includes ∞ and the Schottky planes are disjoint. The rest follows just like before. $\square$

Now, we can look at the hyperbolic manifold $\mathbb{H}^3/G$ and its conformal boundary $\Omega(G)/G$ and make a few statements about their shape. Before however, we need one more definition.

Definition 5.11. A **handlebody** M of genus $k \geq 1$ is a 3-dimensional orientable manifold with boundary, defined by the following two properties:

- ∂M is a closed surface of genus k
- There exist k mutually disjoint curves on ∂M, each of which bounds a disk within M, such that if is M is cut along these disks, what results is connected and homeomorphic to a ball.

Theorem 5.12. *The Kleinian manifold $\mathcal{M}(G) = \mathbb{H}^3 \cup \Omega(G)/G$ associated to a classical Schottky group G, is a handlebody of genus k with hyperbolic metric on its interior. It satisfies the additional property, that $\mathcal{M}(G)$ contains k mutually disjoint hyperbolic planes, that are bounded by simple, pairwise non-homotopic and not zero-homotopic loops on the boundary.*

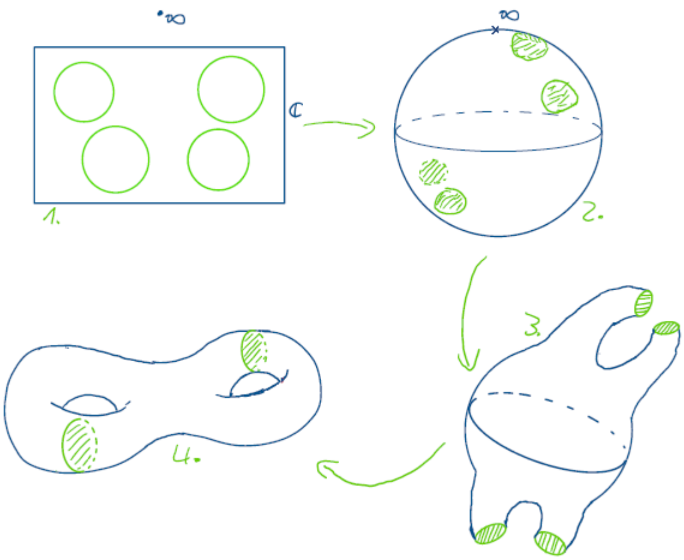


Figure 11: A way of gluing a fundamental region from the ordinary set of a Schottky group into the boundary of its associated handlebody and detecting the additional embedded hyperbolic planes. In the proof of Theorem 5.12 we used $1 \to 2 \to 3 \to 4$ to identify the boundary of $\mathcal{M}(G)$ as genus $2k$ surface.

Proof. We start by considering the boundary $\Omega(G)/G$ of $\mathcal{M}(G)$. This boundary can be modelled as the fundamental region $\mathcal{F} \cup \partial\mathcal{F}$ where $\partial\mathcal{F}$ contains the additionally k circles $C_a, C_b, ...C_\omega$, which do not belong to $\mathcal{F}$, and we identify C_A with C_a, C_B with C_b and so on via the G-action. Before this identification we have a $2k$ times punctured sphere with $2k$ boundary components. Through the identification of the respective boundary components we get a closed surface of genus k.
On the interior of $\mathcal{M}(G)$ we work the other way around. We cut $\mathbb{H}^3/G$ along the embedded planes $\pi(P_x)$, where P_x is the plane bounded by the circle C_x for x from the index set $\{A, B, ..., \Omega\}$. These are embedded planes because P_x is contained in the fundamental region $\hat{\mathcal{F}}$. The result is homeomorphic to the domain $\hat{\mathcal{F}} \cup P_a \cup P_b \cup ... \cup P_\omega \subseteq \mathbb{R}^2 \times \mathbb{R}_+$. Forgetting the boundary we receive an open set in $\mathbb{R}^3$, clearly homeomorphic to $\mathbb{R}^2 \times \mathbb{R}_+$, and because $(0, \infty) \cong (-\infty, \infty)$ topologically, also homeomorphic to $\mathbb{R}^3$. Finally, $\mathbb{R}^3$ is homeomorphic to $B_1(0) \subset \mathbb{R}^3$ via the map

$$f : \mathbb{R}^3 \to B_1(0), \quad x \mapsto \frac{x}{\|x\| + 1},$$

because clearly f is continuous and its inverse

$$f^{-1} : B_1(0) \to \mathbb{R}^3, \quad y \mapsto \frac{y}{1 - \|y\|}$$

is continuous as well.
The planes from the additional property can be chosen to be the projection $\pi(P_x)$ for x in $\{A, B, ..., \Omega\}$ as well. It is immediately obvious that their boundaries satisfy all stated properties.

□

Hopefully, this chapter was a good illustration to the reader how Kleinian groups relate to the topology and geometry of their quotient manifolds, and how we can gain knowledge about the geometry and topology of these manifolds by studying the groups and conversely.

6 Universal Constants

Now we have arrived at the section about the main goal of this thesis. We will be able to proof the existence of constants, (mostly) independent of the hyperbolic manifold or orbifold we are dealing with, that give us certain geometrically and topologically "easy" regions. For these regions we can limit their possible structures and sometimes even describe them exactly. So we get some insights on the "black-box" that a general hyperbolic manifold/orbifold has been for us up until now.
This procedure is structured into five theorems. As a last prerequisite we will set up some notation and define a number quantifying to size of embedded neighbourhoods of points x inside $\mathbb{H}^3/G$.
In this section G will always be a nonelementary Kleinian group, which might also contain elliptic elements. If we want G to be torsionfree somewhere, it will be stated explicitly.

Definition 6.1. Given a discrete group G and $x \in \mathbb{H}^3$ set

$$\delta_x(r) = \{g \neq \mathrm{Id} \in G \mid d(x, g(x)) \leq 2r\} \tag{39}$$

for $r > 0$. We then define the **injectivity radius** at x as

$$r_x = \mathrm{Inj}(x) = \mathrm{Inj}(G; x) = \inf\{r > 0 \mid \delta_x(r) \neq \emptyset\} \tag{40}$$

Remark 6.2. The reason for the distance $2r$ instead of r is, that it allows us to have mutually disjoint balls $B_{g(x)}(r_x) \subseteq \mathbb{H}^3$ for all $g \in G$. Moreover, $\pi(B_x(r_x))$ is exactly the maximal embedded ball centered at $\pi(x)$. As long as x does not lie on an elliptic rotation axis of G, $\mathrm{Inj}(x) > 0$.

6.1 The Universal Horoball

At last, we have arrived at the first theorem about universal constants, the *universal horoball* property. These universal horoballs are neighbourhoods of parabolic fixed points $\zeta \in \hat{\mathbb{C}}$ in $\mathbb{H}^3$ of a size only dependent on the elementary stabilizer subgroup $\mathrm{Stab}_\zeta(G)$, whose projection to $\mathbb{H}^3/\operatorname{Stab}_\zeta(G)$ is embedded in $\mathbb{H}^3/G$.

Theorem 6.3. *Suppose $\zeta \in \hat{\mathbb{C}}$ is a parabolic fixed point and $g \in G$ a least length parabolic (cf. Proposition 4.9). Then the universal horoball $\mathcal{H}_\zeta$ is the horoball bounded by the horosphere $H_{\zeta,r}$, $r > 0$, such that in the intrinsic metric $d_{H_{\zeta,r}}(x, g(x)) = 1$. It satisfies $g(\mathcal{H}_\zeta) \cap \mathcal{H}_\zeta = \emptyset$ for all $g \in G \setminus \mathrm{Stab}_\zeta(G)$.*
If $\mathcal{H}_{\zeta'}$ is the universal horoball at a parabolic fixed point $\zeta' \neq \zeta$ of G, then $\mathcal{H}_{\zeta'} \cap \mathcal{H}_\zeta = \emptyset$.

In order to prove this statement we have to use the *Jørgensen Inequality*:

Lemma 6.4 (Jørgensen's Inequality)**.** *If $G = \langle A, B\rangle$ is discrete and nonelementary, then*

$$|\operatorname{tr}^2(A) - 4| + |\operatorname{tr}(ABA^{-1}B^{-1}) - 2| \geq 1 \tag{41}$$

We will not prove it here, since the proof is very technical and relies on further characterization of elementary groups. For a proof see [Mar07] pp. 52-55.

Proof of Theorem 6.3. As usual, we will without loss of generality work with $\zeta = \infty$ and $T = (z \mapsto z + 1)$ as least length parabolic. From the definition of $\mathcal{H}_\zeta$, we obtain that the universal horoball for this constellation is

$$\mathcal{H}_\infty = \{(z, t) \in \mathbb{H}^3 \mid t > 1\}. \tag{42}$$

Of course, the transformation T can be given by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let A be an element of G that does not fix ∞, given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{C})$. This means $c \neq 0$. Since, naturally, the group generated by T and A is discrete and nonelementary, we can apply Jørgensen's inequality (41). Using $\operatorname{tr}^2(T) - 4 = 0$ and

$$\operatorname{tr}(TAT^{-1}A^{-1}) = \operatorname{tr}\left(\left(\begin{smallmatrix} a+c & b+d \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} d+c & -b-a \\ -c & a \end{smallmatrix}\right)\right) = c^2 + 2$$

we get that $|c|^2 > 1$. As a result, with the formula for extension to $\mathbb{H}^3$, equation (10) from section 2, and $A : (z,t) \mapsto (z', t')$, we obtain

$$t' = \frac{t}{|cz+d|^2 + |c|^2 t^2} \leq \frac{t}{|c|^2 t^2} \leq \frac{1}{t}.$$

So if $t > 1$, it follows that $t' < 1$, which means $\mathcal{H} \cap A(\mathcal{H}) = \emptyset$.
Choose a least length parabolic S for a parabolic fixed point $\xi \neq \infty$. We can replace G by a conjugate with the translation $T' : z \mapsto z - \xi$. As T' fixes ∞, $T'TT'^{-1}$ is still a least translation length parabolic for ∞, while $S' = T'ST^{-1}$ fixes 0. Consequently S' has the form $S' : z \mapsto: z/(cz+1)$. We already know $|c| > 1$. To see that the universal horoball at 0 is disjoint from the one at ∞ note that $(z \mapsto -z^{-1})$ conjugates S' to $(z \mapsto z - c)$ and its universal horoball there is $\{(z,t) \ |t > |c| \geq 1\}$. Returning to S' we see that the boundary of the universal horoball at 0 meets the vertical axis at $1/|c| < 1$, confirming that $\mathcal{H}_\xi \cap \mathcal{H}_\infty = \emptyset$.
The only thing left to consider is what happens if there are elliptics fixing $\zeta = \infty$, since there cannot be loxodromics. But such elliptics have the form $E(z) = e^{2i\theta} + a$ on $\hat{\mathbb{C}}$ by Proposition 2.13 and preserve the horospheres at ∞ as well.
(Taken from [Mar07], Theorem 3.3.4.) □

Remark 6.5. As we have already seen for a bit in the proof, the universal horoball for a general least length parabolic $(z \mapsto z + a)$ at ∞ is given by $\mathcal{H}_\infty = \{(z,t) \in \mathbb{H}^3 \mid t > |a|\}$.

6.2 Universal Tubular Neighbourhoods about Short Geodesics

The next universal property concerns "short" closed geodesics and their *tubular neighbourhoods.*

Definition 6.6. The **tubular neighbourhood of radius r** about a geodesic $\gamma \subseteq \mathbb{H}^3$ is the set

$$N_r(\gamma) = \{x \in \mathbb{H}^3 \mid d(x, \gamma) < r\}.$$

If $N_r(\gamma)$ is embedded in $\mathcal{M}(G)$, then we also call the projection a tubular neighbourhood of radius r about the geodesic $\pi(\gamma) \subseteq \mathbb{H}^3/G$.

Example 6.7. For the geodesic $\gamma : (0, \infty) \mapsto \mathbb{H}^3$, $\gamma(t) = (0, t)$ the r-tubular neighbourhood is the vertical cone arising from $0 \in \hat{\mathbb{C}}$ with cone angle 2θ, where θ is given by $\sinh(r) = \tan(\theta)$. Such a cone can be seen in Figure 12 next to another tubular neighbourhood.

Theorem 6.8. *There exist universal constants $r > 0$ and $L_0 > 0$ such that in any $\mathcal{M}(G)$:*

1. *The tubular neighbourhood of radius r about any closed geodesic of length $\leq L_0$ is embedded, if the geodesic is not also a cone axis, and any geodesic of length smaller than L_0 is simple.*

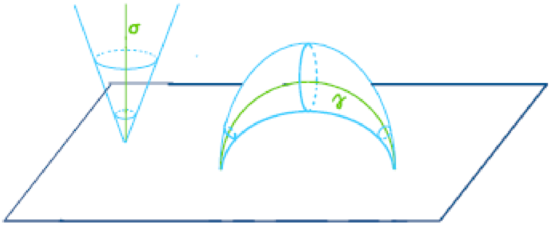


Figure 12: Tubular neighbourhoods about the geodesics γ and σ in $\mathbb{H}^3$.

2. *The r-tubular neighbourhoods about different geodesics of length smaller than L_0 are mutually disjoint.*

3. *The r-tubular neighbourhoods about geodesics of length smaller than L_0 do not intersect the universal horoballs.*

Like before, we have to draw knowledge from a lemma, which we will not prove here, regarding limits of nonelementary two-generator groups:

Lemma 6.9. *Suppose $\{\langle A_n, B_n\rangle\}$, $A_n, B_n \in PSL(2,\mathbb{C})$ is a sequence of nonelementary, discrete groups, such that $\lim A_n = A$, $\lim B_n = B$, $A, B \in PSL(2,\mathbb{C})$. Then $\langle A, B\rangle$ is also a nonelementary, discrete group. The corresponding conclusion also holds for sequences of three-generator nonelementary groups.*

For a proof of the Theorem by Jørgensen, from which this is a special case, see [Mar07], pp. 188-193, Theorem 4.1.1 and its proof.

Proof of Theorem 6.8. We start proving (i) by contradiction. Assume there are no such $r, L_0 > 0$. Naturally, this mean there exist sequences $\{r_n\}, \{L_n\} \subseteq \mathbb{R}$, $r_n \to 0, L_n \to 0$, a sequence of corresponding Kleinian groups G_n and geodesics $\gamma_n \subseteq \mathcal{M}(G_n)$, such that the tube of radius r_n about the closed geodesic γ_n of length $\leq L_n$ is not embedded in $\mathcal{M}(G_n)$. By replacing each G_n by a conjugate, we choose each γ_n to be a projection of the vertical straight line $l = \{(0,t) : t > 0\} \subseteq \mathbb{H}^3$. This way there is for each $n \in \mathbb{N}$ a primitive loxodromic transformation $A_n \in G_n$, that preserves l and translates along it a distance smaller or equal to L_n, meaning $A_n(z) = a_n z$, where $|a_n| > 1$ and

$$L_n \geq d((0,t),(0,|a_n|t)) = \operatorname{arcosh}\left(1 + \frac{(|a_n| - 1)^2}{2|a_n|}\right) = \ln(|a_n|) \to 0$$

See equations (3), (10) from section 2.
Because we only handle closed geodesics here, we do not need to worry about the possibility that there could be an elliptic of order two exchanging the endpoints of our geodesics of interest. However, our proof works regardless whether there are elliptics with the same axis as the A_n or not, but in the former case there will be a cone axis in the tubular neighbourhood around γ_n and and the tubular neighbourhood will not be

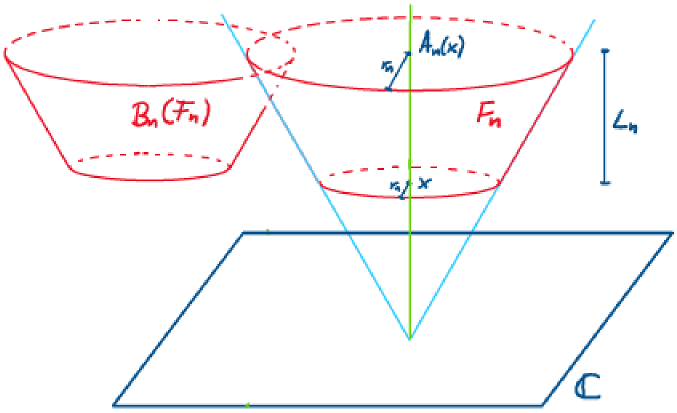


Figure 13: Constructions from the proof of Theorem 6.8.1.

embedded, only its quotient by the finite elliptic subgroup with the same axis will be. Let C_n be the tubular neighbourhood about l of radius r_n. It is a Euclidean cone like in Example 6.7. We then call the compact $F_n = \{x \in C_n \mid 1 \leq |x| \leq |a_n|\}$ a fundamental chunk of C_n with respect to the A_n-action. By assumption, since $\pi_n(C_n)$ is not embedded there is an element $B_n^* \in G_n$, which does not preserve l with $B_n^*(C_n) \cap C_n \neq \emptyset$. We translate along γ_n with A_n, such that $B_n = A_n^q B_n^* A_n^p \in G$ satisfies $B_n(F_n) \cap F_n \neq \emptyset$ for some $p, q \in \mathbb{Z}$. As a result there are $x_n \in F_n$ with $B_n(x_n) \in F_n$ and $\langle A_n, B_n \rangle$ is not elementary. This arrangement is shown in Figure 13

After passing to a subsequence if necessary, we find A, B, x so that $A_n \to A$, $B_n \to B$ and $x_n \to x$ (Lemma 3.1), where A, B are Möbius transformations, since neither of them can converge to a point on the boundary on all of $\mathbb{H}^3$. But A, B fix $x = \lim B_n(x_n) \in \bigcap_{n\in\mathbb{N}} F_n$ because $\bigcap_{n\in\mathbb{N}} F_n = \{(0,1)\}$ only contains this one point. So $\langle A, B \rangle$ is elementary, yielding a contradiction to Lemma 6.9.

We continue with (ii). Again assume by contradiction there are sequences $\{r_n\}, \{L_n\} \subseteq \mathbb{R}$ and groups G_n, whose associated Kleinian manifolds $\mathcal{M}(G_n)$ contain the corresponding short geodesics $\gamma_n \neq \sigma_n$, of length smaller or equal to L_n, such that the tubular neighbourhoods $N_{\sigma_n}(r_n), N_{\gamma_n}(r_n)$ are not disjoint. The exact same argument as above gives us a contradiction here as well. Just consider the $2r_n$-neighbourhood of γ_n and choose as B_n^* the primitive generator of the respective conjugation of the short closed geodesic σ_n. Obviously, this B_n^* does not preserve l, because $\sigma_n \neq \gamma_n$, and by assumption the $2r_n$-neighbourhood of γ_n intersects σ_n for all $n \in \mathbb{N}$.

Finally, we dedicate ourselves to statement (iii). Without loss of generality, suppose that $T := (z \to z+1)$ is an element of G and the least length parabolic associated to the fixed point $\infty \in \hat{\mathbb{C}}$. So the associated universal horoball is $\mathcal{H}_{\infty,1} = \{(z,t) \in \mathbb{H}^3 \mid t > 1\}$. Let $A \in G$ be the generator of the closed geodesic $\gamma \in \mathcal{M}(G)$ of length L. We conjugate the group by $(z \to z + c)$ for $c \in \mathbb{C}$ such that the fixed points of A on $\mathbb{C}$ are symmetric with respect to $0 \in \mathbb{C}$. For example if $p, q \in \mathbb{C}$ are the fixed points of A we can conjugate by $(z \mapsto z - (p+q)/2)$ and the new fixed points are by equation (13) from section 2 the points $(p-q)/2$ and $(q-p)/2$. Continuing, we get that the conjugate of A, which we denote with abuse of notation again by A is represented by a matrix

$$\begin{pmatrix} a & bd \\ d/b & a \end{pmatrix} \in SL(2,\mathbb{C}), \text{ for } a,b,c,d \in \mathbb{C},\ a^2 - d^2 = 1,\ \operatorname{tr} A = 2a \notin [0,4].$$

The fixed points of A are then b and $-b$. Remember that the geodesics in $\mathbb{H}^3$ that start and end in $\mathbb{C}$ are semicircles. So its highest point is $(0,|b|)$ and the highest point of its r-tubular neighbourhood is given by $s|b|$, such that

$$r = d((0,|b|),(0,s|b|)) = \operatorname{arcosh}\left(1 + \frac{|b|^2(s-1)^2}{2|b|^2 s}\right) = \log(s) \text{ for } s > 1. \tag{43}$$

This means $s|b| = |b|e^r$, which is smaller than or equal to 1 if and only if $|b| \leq e^{-r}$. So the geodesic intersects the universal horoball, if and only if $|b| > e^{-r}$ Suppose there is no $L_0 \geq 0$ such that for geodesics of length $L < L_0$ with endpoints $\pm b$, it follows that $|b| < e^{-r}$. Then there is a sequence of groups G_n containing T and a loxodromic A_n with fixed points $\pm b_n$ symmetric about $z = 0 \in \mathbb{C}$ and translation length L_n, such that $\lim L_n = 0$ while $\lim b_n = b^* \geq e^{-r}$. If $b^* = \infty$, then for n big enough a segment of the axis of A_n with length bigger than L_n intersects $\mathcal{H}_\infty$, such that $A_n(\mathcal{H}_\infty) \cap \mathcal{H}_\infty \neq \emptyset$ in contradiction to the universal horoball property. Conversely, since $b^* \neq 0$, $\lim A_n = A$ exists with A either elliptic or the identity. This stands in contradiction to Lemma 6.9. We conclude that there must be $L_0 \in \mathbb{R}$, for which the r-tubular neighbourhood about any geodesic in any $\mathcal{M}(G)$ of length smaller or equal to L_0 does not intersect the projection of the universal horoball. Proof from [Mar07] p.110-111. □

By this theorem we have shown that for every $\mathcal{M}(G)$ and every closed geodesic γ, which is short enough, there is a neighbourhood of fixed radius $r > 0$ that is geometrically and topologically only dependant on the length of γ.

6.3 Isolated Cone Axis

This next statement deals with cone axes in an $\mathcal{M}(G)$ with elliptic elements in G. As we already partly discussed in the orbifold section, the elliptic elements in G generate so called *cone axis* in the quotient, but a priori not much can be said about their number or location. It is easy to see that there are Kleinian groups with intersecting cone axis. For example the symmetry groups of the five platonic solids are also purely ellipitc, finite subgroups of $PSL(2,\mathbb{C})$, with intersecting rotation axes in the center. Another possibility is, that nonidentical cone axes end in the same point of $\hat{\mathbb{C}}$. By Corollary 2.14 their commutator is parabolic, so the endpoint is a parabolic fixed point. There are exactly four possibilities up to group isomorphism, in which way the stabilizer of a parabolic fixed point can contain elliptic elements. In each case multiple elliptic rotation axes end at the fixed point. For more on this the we refer to [For29], where elementary discrete subgroups of $PSL(2,\mathbb{C})$ were characterized originally, which is what these subgroups, that preserve points in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ are.

We here want to discuss the other case: How close can non-intersecting cone axis, that end in disjoint pairs of points, come to each other? If they could accumulate at a point in $\mathbb{H}^3$, there would be a problem, because then $\mathcal{M}(G)$ could not be an orbifold. Luckily we have the following theorem:

Theorem 6.10. *There exists $\delta > 0$ such that the distance between any nonintersecting rotation axes in $\mathcal{M}(G)$ is at least δ, except if they have a common endpoint at a rank two cusp, of perhaps if they are axes of order two.*

The proof is quite similar to the proof of the tubular neighbourhood property. We will again use Lemma 6.9.

Proof. Let $\{G_n\}$ be a sequence of Kleinian groups. Suppose there is a sequence $\{d_n\} \in \mathbb{R}$, $d_n > 0$, $d_n \to 0$, such that there are elliptics E_n, F_n in G_n with rotation axes l of E_n and l_n of F_n. The l_n do not intersect l, the vertical Euclidean half line starting at 0, but come within distance d_n of it. That all E_n have rotation axis l can be chosen without loss of generality by replacing the G_n with appropriate conjugates. Additionally, we are able to further conjugate the G_n with loxodromics preserving l, such that there are points p_n on l_n with the property $\lim p_n = p \in l$.
When n is large enough, there are two cases for $\langle E_n, F_n \rangle$. In the first one $\langle E_n, F_n \rangle$ is nonelementary. Then after passing to a suitable subsequence, such that $E_n \to E$, $F_n \to F$, $E, F \in PSL(2, \mathbb{C})$, we know by Lemma 6.9, that $\langle E, F \rangle$ is non-elementary as well. We have shown we can find these subsequences in Lemma 3.1 and it also tells us that E, F are Möbius tranformations, because these sequences of elliptics cannot converge to a point on the boundary, as they fix the points p_n converging to $p \in \mathbb{H}^3$. But since the point p is fixed by both E and F, this is a contradiction.
In the second case, on the other hand $\langle E_n, F_n \rangle$ is elementary for all n big enough. Then there are the three cases of elementary groups:
It could be, that E_n and F_n fix a point on $\zeta \in \hat{\mathbb{C}}$. But then by Corollary 2.14 their commutator $K = E_n F_n E_n^{-1} F_n^{-1}$ is parabolic, meaning ζ is a parabolic fixed point of G_n. Consequently, by Corollary 3.10 ζ is a rank two cusp.
It could also be that E_n, F_n preserve a pair of points. But then of course, since they cannot be preserved pointwise by $\langle E_n, F_n \rangle$, at least one on the two is of order 2. Actually, both of them are of order two. Suppose without loss of generality E_n is of order larger than 2. Then E_n has to fix the two points $0, \infty$ pointwise. Because F_n exchanges them, F_n acts by reflection on the geodesic l. But this means there is one point on l, which is fixed by F_n as well. Consequently, we have a common fixed point inside $\mathbb{H}^3$, which is discussed next.
In the final case they fix a point inside $\mathbb{H}^3$. However, because the only fixed points of an elliptic in $\mathbb{H}^3$ lie on its rotation axis, the rotation axis of E_n, F_n need to intersect, which stand in contradiction to the assumption. Proof idea from [Mar07] p. 111. □

As a consequence we know, that the only non-intersecting rotation axis, that do not end in the same point $\zeta \in \hat{\mathbb{C}}$ and can be a distance smaller than δ apart, are pairs of order-2 elliptics. With this we close our discussion of cone axis definitely.

6.4 Universal Elementary Neighbourhood

In this section we will prove our penultimate theorem about universal constants. It is possibly the most important of the five theorems and also in a more general form known as the Margulis Lemma. We are now dealing with general neighbourhoods of points in $\mathbb{H}^3$ and the structure of groups, that are generated by elements, which move x only inside these neighbourhoods. In the quotient this gives us the structure of small neighbourhoods around any point $p \in \mathbb{H}^3/G$.

Theorem 6.11. *There exists $\delta > 0$ such that for every $x \in \mathbb{H}^3$ and every Kleinian group G the subgroup generated by*

$$\{A \in G : d(x, Ax) < 2\delta\} \tag{44}$$

*is elementary. We call this δ the **universal elementary constant**.*
If a generator A is loxodromic it represents a simple, closed geodesic.

Proof. We denote by $G_x(r)$ the group generated by the set $\delta_x(r)$ from Definition 6.1 that contains any $g \in G$, for which $d(x, g(x)) \leq 2r$. We claim that there exists $r_0 > 0$, such that for any $x \in \mathbb{H}^3$ and any nonelementary Kleinian group G, the group $G_x(r_0)$ is elementary. This is equivalent to the statement that the group generated by the elements g_i, that satisfy $g_i(\overline{B_r(x)}) \cap \overline{B_r(x)} \neq \emptyset$ for the ball $B_r(x)$ of radius r around x, is elementary for any x.
If we choose a fixed $x \in \mathbb{H}^3$ and a fixed group G , there must be an r with the properties from above. Because for a sequence $\{r_n\} \subseteq \mathbb{R}_+,\ r_n \to 0$ any infinite sequence of distinct elements $g_n \in \delta_x(r_n),\ g_n(x) \neq x$ converges either to an elliptic transformation fixing x or to the identity. However, since G is discrete, no such sequence can exist. So for all r small enough the set $\delta_r(x)$ is independent of r and either empty or it contains only elliptic transformations fixing x.
We will proof this theorem by contradiction. Assume that for every $r > 0$ there is $x \in \mathbb{H}^3$ and a Kleinian group G, such that the subgroup $G_x(r)$ is nonelementary. This means there is a sequences $r_n \subseteq \mathbb{R}^+$ with $r_n \to 0$ and sequences $x_n \subseteq \mathbb{H}^3$, G_n of Kleinian group, such that $G_{n,x_n}(r_n)$ is nonelementary. By conjugating the G_n we can assume that $x_n = x := (0, 1) \in \mathbb{H}^3$ for every n. Moreover, for fixed G_n there exists $\rho \in \mathbb{R}$ with $0 < \rho < r_n$, such that $G_{n,x}(\rho)$ is elementary. As we increase ρ closer to r_n, the elementary groups $G_{n,x}(\rho)$ are nested. There has to be a first number τ_n satisfying $G_{n,x}(\tau_n)$ is elementary and $G_{n,x}(\tau_n) = G_{n,x}(\rho)$ for all ρ with $\tau_n \leq \rho < r'_n \leq r_n$, but $G_{n,x}(r'_n)$ is not. Differently said τ_n is the smallest number such that $G_{n,x}(\tau_n)$ is the largest possible elementary group generated this way and r'_n is the smallest number, for which the group $G_{n,x}(r'_n)$ is nonelementary. We chose r_n as r'_n. Next, we look at the groups $G_{n,x}(\tau_n)$ and $G_{n,x}(r_n)$ and try to find nonelementary two or three generator subgroups inside $G_{n,x}(r_n)$.
Firstly, if $G_{n,x}(\tau_n)$ is finite but not cyclic, it is purely elliptic and there are elements with distinct yet intersecting rotation axes. Reason is that two elliptic elements with the same axis either are generated by one elliptic element if their rotation angles are integer multiples of some $\theta \in \mathbb{R}$ or they generate a non-discrete group, which can not

happen here. The set $\delta_x(r_n)$ must contain an element X_n, which does not fix the common fixed point of A_n, B_n. As a consequence $\langle A_n, B_n, X_n\rangle$ is not elementary.
Secondly, if $G_{n,x}(\tau_n)$ is finite and cyclic let $A_n \in \delta_x(\tau_n)$ be one of the elliptic generators. We then find again an $X_n \in \delta_x(r_n)$, which does not fix the axis of A_n. If now this X_n is elliptic and its axis intersects that of A_n, there must be a second $Y_n \in \delta_x(r_n)$, which does not fix the common fixed point of A_n and X_n in $\mathbb{H}^3$. So either $\langle A_n, X_n\rangle$ is nonelementary or we find Y_n such that $\langle A_n, X_n, Y_n\rangle$ is.
A third possibility is, that $G_{n,x}(\tau_n)$ is an infinite group that preserves a geodesic line $l \subseteq \mathbb{H}^3$. In this case $\delta_x(\tau_n)$ either contains a loxodromic A_n, or it contains two elliptics A_n, B_n of order 2 interchanging the endpoints of l and with A_nB_n loxodromic, since there cannot be infinitely many elliptics in the discrete G_n sharing one rotation axis (Lemma 3.6). Like before, in both cases there must be an element $X_n \in \delta_x(r_n)$ not preserving the line l, so that $\langle A_n, X_n\rangle$ respectively $\langle A_n, B_n, X_n\rangle$ is not elementary.
Finally, suppose that $G_{n,x}(\tau_n)$ falls into neither of these categories. Then by definition of elementary groups it fixes a point $\zeta \in \hat{\mathbb{C}}$. Using Lemma 3.5 we can rule out that there are loxodromics in $G_{n,x}(\tau_n)$. Clearly $\delta_x(\tau_n)$ can contain a parabolic transformation A_n or two elliptics E_n, F_n with exactly one common fixed point. Then by Corollary 2.14 and the property that only parabolic transformations and the identity can have a trace of 2, $E_nF_nE_n^{-1}F_n^{-1}$ is parabolic, because it is clear that two elliptics with exactly one common fixed point cannot commute. Either way the set $\delta_x(r_n)$ includes an element X_n that does not fix ζ. Hence $\langle A_n, X_n\rangle$ or $\langle E_n, F_n, X_n\rangle$ is not elementary.
In all cases we have found two or three generator subgroups generated by elements of $\delta_x(r_n)$. As $n \to \infty$, convergent subsequences of theses generator sequences converge to Möbius transformations that fix x and are therefore elliptic or the identity. Once again we have reached a contradiction to Lemma 6.9.
Now the only thing left to show is the second part of the statement, that if a generator g in $G_x(\delta)$ is loxodromic, it represents a simple geodesic in the quotient. Suppose $\pi(\gamma)$ is closed but not simple. Then by our discussion in Remark 4.5 the G-orbit of γ is not disjoint, meaning there is an $h \in G$ not fixing γ and not interchanging its endpoints with $h(\gamma) \cap \gamma \neq \emptyset$. Let g be the primitive loxodromic fixing γ. We can find $k, l \in \mathbb{Z}$ such that $g^khg^l(\gamma)$ maps a point of the segment $[x, g(x)]$ of γ to a different point on $[x, g(x)]$. But the group $\langle g, h\rangle$ can not be elementary, since we ruled all possible elementary groups with one loxodromic element out. So by the part of the theorem proven directly, the length of $\pi(\gamma)$ must be greater than 2δ. Proof from [Mar07], pp.111-112. □

6.5 The Universal Ball

Finally we have come to the last theorem of this section. In essence, it tells us that in any torsionfree $\mathcal{M}(G)$ there is a domain, which is isometric to a hyperbolic ball of a fixed minimal radius. This also proves, that there exists a lower bound on the volume of any hyperbolic manifold.

Theorem 6.12. *There exists $\delta > 0$ such that $\mathcal{M}(G)$ for any torsionfree and nonelementary Kleinian group G contains an embedded hyperbolic ball of radius δ.*

The proof of this statement is understood more easily if we structure it as sequence of lemmas. Here we only prove this theorem in case of manifolds, meaning only for groups G without elliptic elements. Nevertheless, it can also be proven for general $\mathcal{M}(G)$, see [Mar07], pp.109-115, Theorem 3.3.4. Anyway, because we are only dealing with torsionfree groups G it is necessary to take a look at which kind of elementary subgroups can still occur.

Lemma 1. *For a torsionfree Kleinian group G, every elementary subgroup $H \subseteq G$, which is not the trivial group, fixes exactly one or two points on the boundary $\hat{\mathbb{C}}$ and no points inside $\mathbb{H}^3$.*

Proof. This is an immediate consequence of the definition of elementary groups in 3.4. Loxodromic and parabolic Möbius tranformations have no fixed points inside $\mathbb{H}^3$ by Proposition 2.13 and the only way a pair of points on $\hat{\mathbb{C}}$ is preserved, but not fixed pointwise by a group H, is if there is some element $h \in H$ that interchanges the points. But then h^2 has at least 3 fixed points on $\hat{\mathbb{C}}$, so it can only be the identity and h is elliptic of order 2. □

Continuing, we need a couple of definitions:

Definition 6.13. Let G be torsionfree. We define the set

$$V(G,r) := \{x \in \mathbb{H}^3 \mid \exists g \in G : \ d(x, g(x)) < 2r\} \tag{45}$$

as the set of all points of $\mathbb{H}^3$, whose G-orbit intersects the $2r$-ball around them in more than only the point itself.
In the same vein, set

$$V(G_\zeta, r) := \{x \in \mathbb{H}^3 \mid \exists g \in \mathrm{Stab}_\zeta(G) : \ d(x, g(x)) < 2r\}. \tag{46}$$

A basic fact about any $V(G_\zeta, r)$ is:

Lemma 2. *Let $\zeta \in \hat{\mathbb{C}}$ and $r > 0$. Then then set $V(G_\zeta, r)$ is open.*

Proof. Let $x \in V(G_\zeta, r)$ as stated. By definition there exists $g \in \mathrm{Stab}_\zeta(G)$ and $\epsilon > 0$ with $d(x, g(x)) = 2r - 2\epsilon < 2r$. Choose $y \in B_x(\epsilon)$. Then

$$d(y, g(y)) \leq d(y, x) + d(x, g(x)) + d(g(x), g(y)) < 2\epsilon + 2r - 2\epsilon = 2r$$

□

Lemma 3. *If $r > 0$ and G as before we have*

$$V(G, r) = \bigcup_{\zeta \in \hat{\mathbb{C}}} V(G_\zeta, r) \tag{47}$$

Naturally $V(G_\zeta, r) = \emptyset$ for all $\zeta \in \hat{\mathbb{C}}$ not fixed by some $g \in G$.

Proof. The first inclusion $V(G,r) \supseteq \bigcup_{\zeta\in\hat{\mathbb{C}}} V(G_\zeta, r)$ is clear. For every $\zeta \in \hat{\mathbb{C}}$, the set $V(G_\zeta, r)$ is a subset of $V(G,r)$ by definition. To prove the other inclusion, choose any $x \in V(G,r)$. Then there must be some $g \in G$ that fulfills $d(x, g(x)) < 2r$. But g has at least one fixed point w on $\hat{\mathbb{C}}$, such that $g \in \mathrm{Stab}_w(G)$. □

Now our next goal is to create the preconditions we need, to show that if we take r to be smaller or equal to the universal elementary constant δ from Theorem 6.11, the set $V(G,r)$ cannot be all of $\mathbb{H}^3$. To do this we will at first prove the following statement:

Lemma 4. *Let δ be the universal elementary constant from Theorem 6.11, and G a nonelementary, torsionfree Kleinian group. For $u, v \in \mathbb{C}$ and $0 < r < \delta$ the sets $V(G_u, r)$ and $V(G_v, r)$ are either disjoint or equal. They are equal if and only if the corresponding stabilizer groups $\mathrm{Stab}_u(G)$ and $\mathrm{Stab}_v(G)$ are equal or both sets are empty.*

Proof. By the previous lemma we can write $V(G,r) = \bigcup_{\zeta\in\hat{\mathbb{C}}} V(G_\zeta, r)$. Let $u, v \in \hat{\mathbb{C}}$ be fixed points of elements from $G \setminus \{\mathrm{Id}\}$. Suppose that there is $x \in V(G_u, r) \cap V(G_v, r)$. Then by definition there is $g \in \mathrm{Stab}_u(G)$ and $h \in \mathrm{Stab}_v(G)$ such that $d(x, g(x)) < 2r \leq 2\delta$ and $d(x, h(x)) < 2r \leq 2\delta$. Hence g, h are in the elementary universal neighbourhood of x, and $\langle g, h\rangle$ is elementary. But by the first lemma every elementary group fixes a point on $\hat{\mathbb{C}}$, so there exists $\zeta \in \hat{\mathbb{C}}$, such that $g, h \in \mathrm{Stab}_\zeta(G)$. Now, we look at multiple cases:
Case 1: $u = v$. Then of course also the stabilizer groups are equal.
Case 2: $u \neq v$, but $u = \zeta$ or $v = \zeta$. Without loss of generality let $u = \zeta$. In this case h has the distinct fixed points v, $\zeta = u$ and is loxodromic, while g fixes the point u. Consequently g is either loxodromic with another fixed point $w \in \hat{\mathbb{C}} \setminus \{u\}$ or parabolic. In the former case we can apply Lemma 3.5 with the common fixed point u of g and h to ensure that $w = v$, meaning g and h have the same axis, and use the lemma one more time to prove $\mathrm{Stab}_u(G)$ and $\mathrm{Stab}_v(G)$ only consist of loxodromics fixing the common axis of g and h. As a result $\mathrm{Stab}_u(G) = \mathrm{Stab}_v(G)$. In the latter case the same Lemma 3.5 tells us that the group generated by g, h can not be discrete, so we have a contradiction.
Case 3: u, v, ζ are pairwise distinct. So g is loxodromic with fixed points u, ζ and h is loxodromic with fixed points v, ζ. But just like before, the group generated by g, h can not be elementary by Lemma 3.5, yielding again a contradiction.
Overall we have shown that, if $x \in V(G_u, r) \cap V(G_v, r)$, it must follow that $\mathrm{Stab}_v(G) = \mathrm{Stab}_u(G)$ and as a result $V(G_u, r) = V(G_v, r)$. Of course this also proves that $V(G_u, r)$ being equal to $V(G_v, r)$ implies either $\mathrm{Stab}_v(G) = \mathrm{Stab}_u(G)$ or both sets are empty.
Conversely, if $\mathrm{Stab}_v(G) = \mathrm{Stab}_u(G)$ for some $u, v \in \hat{\mathbb{C}}$, by definition $V(G_u, r) = V(G_v, r)$. As a consequence, $V(G_u, r), V(G_v, r)$ are either equal or disjoint and equal if and only if they are both empty or the corresponding stabilizer groups are equal too. □

To finalize the argument, we must rule out that for one $\zeta \in \hat{\mathbb{C}}$, $V(G\zeta, r)$ is all of $\mathbb{H}^3$

Lemma 5. *For any $\zeta \in \hat{\mathbb{C}}$ and any $r > 0$ it holds that $V(G_\zeta, r) \neq \mathbb{H}^3$.*

Proof. Let $\zeta \in \hat{\mathbb{C}}$ be the fixed point of some $g \in G$. If ζ is a parabolic fixed point we conjugate G to have $\zeta = \infty$. Then there is a least translation length parabolic of the

form $T = (z \mapsto z + c)$, $c \in \mathbb{C} \setminus \{0\}$. On $\mathbb{H}^3$ it takes on the form $T = ((z,t) \mapsto (z + c, t))$. If we choose t small enough the distance $d((z,t),(z+c,t))$ is larger than 2δ. The least translation length parabolic already minimizes the distance, that a point in $\mathbb{H}^3$ is mapped, over all parabolics in $\mathrm{Stab}_\zeta(G)$. Additionally, $\mathrm{Stab}_\zeta(G)$ has to be purely parabolic (Lemma 3.5), so that $V(G_\zeta, r) \neq \mathbb{H}^3$.
If ζ is a loxodromic fixed point, $\mathrm{Stab}_\zeta(G) = \langle g \rangle$ is a cyclic loxodromic group by Lemma 3.5. We can conjugate G to have the vertical line $l = \{(0,t) : t > 0\}$ be the axis of g and $g = ((z,t) \mapsto (az, |a|t)$ with $|a| > 1$. Define points $x_n = (n, 1) \in \mathbb{H}^3$ for all $n \in \mathbb{N}$. They satisfy

$$\begin{aligned} d(x_n, g(x_n)) &= \operatorname{arcosh}\left(1 + \frac{|an - n|^2 + (|a| - 1)^2}{2|a|}\right) \\ &= \operatorname{arcosh}\left(\frac{|a - 1|^2}{2|a|} n^2 + \frac{|a|^2 + 1}{2|a|}\right) \\ &\leq \operatorname{arcosh}(cn^2 + d) \end{aligned}$$

for some constants $c \in (0, \infty)$, $d \in (1, \infty)$. The function arcosh has the asymptotic behaviour $\lim_{n\to\infty} \operatorname{arcosh}(cn^2 + d) = \infty$, such that the distances, $d(x_n, g(x_n))$ become arbitrarily large. As a result, $x_n \notin V(G_\zeta, r)$ for n large enough.

□

Finitely, we proceed to proving the actual theorem at hand.

Proof Theorem 6.12. Let us now put together what we know about $V(G, r)$ for $r \leq \delta$. Lemma 2 and Lemma 3 give us an open covering of the open $V(G, r)$ by the sets $V(G_\zeta, r)$. By Lemma 4 we know that $V(G, r)$ is either a disjoint union of these sets or equal to $V(G_\zeta, r)$ for some $\zeta \in \hat{\mathbb{C}}$. In the first case, there must be a point x in $\mathbb{H}^3 \setminus V(G, r)$ because $\mathbb{H}^3$ is connected. In the second case we apply Lemma 5 to see that $V(G, r) = V(G_\zeta, r) \subsetneq \mathbb{H}^3$. So we find $x \in \mathbb{H}^3 \setminus V(G, r)$ as well.
Consequently $d(x, g(x)) > 2\delta$ for all $g \neq \mathrm{Id} \in G$, and the injectivity radius r_x at x is larger than δ. This means the projection $\pi(B_x(\delta)) \subseteq \mathcal{M}(G)$ is embedded. □

Remark 6.14. As you can see in the proof of the final theorem, the constant for the universal ball property is can be chosen to be at least as large as the universal elementary constant from Theorem 6.11

7 Thick-Thin-Decomposition

In this final section we are going to split a hyperbolic manifold into two parts: The ϵ-thick part and the ϵ-thin part.

Definition 7.1. Let $\epsilon > 0$. The **ϵ-thin part** $\mathcal{M}_\epsilon^{thin}(G)$ of $\mathcal{M}(G)$ is defined as

$$\{x \in \mathbb{H}^3/G \mid \operatorname{Inj}(x) < \epsilon\}. \tag{48}$$

Conversely, the **ϵ-thick part** $\mathcal{M}_\epsilon^{thick}(G)$ is the complement of the thin part in the interior of $\mathcal{M}(G)$:

$$\{x \in \mathbb{H}^3/G \mid \operatorname{Inj}(x) \geq \epsilon\}. \tag{49}$$

With all the work we have done beforehand, there is now quite a bit we can say about these parts:

Theorem 7.2. *Let G be a torsionfree, nonelementary Kleinian group and δ the universal elementary constant from Theorem 6.11.*
For every $\epsilon < \delta$ the ϵ-thin part $\mathcal{M}^{thin}(G)$ is an open set and the union of mutually disjoint components consisting of

1. *The tubular neighbourhood about a geodesic of length $< 2\epsilon$, which is a solid torus,*
2. *The* solid cups tube *coming from the projection of the ϵ-horoball at a rank one parabolic fixed point,*
3. *The* solid cusp torus *coming from the projection of the ϵ-horoball corresponding to a rank two parabolic fixed point.*

The δ-thick part is closed.

Proof. For any $g \in G \setminus \{\mathrm{Id}\}$ we set

$$P_g = \{x \in \mathbb{H}^3 \mid d(x, g(x)) < 2\epsilon\}.$$

These sets are open for any $g \in G$ by triangle inequality.
If g is parabolic, $g = (z \mapsto z + 1)$, the set P_g is a horoball defined by the property that its boundary horosphere $H_{\infty,t^{-1}}$ satisfies, $d((z,t),(z+1,t)) = 2\epsilon$ for all $z \in \mathbb{C}$ in the hyperbolic distance.
If g is loxodromic with axis $\gamma \subseteq \mathbb{H}^3$, P_g is empty, if the length of $\pi(\gamma)$ given by $d(x, g(x))$ is larger or equal to 2ϵ for any $x \in \gamma$. To see this take $\gamma = l$ the vertical straight line starting at $0 \in \mathbb{C}$. Then $g = ((z,t) \mapsto (az, |a|t))$ with some $a \in \mathbb{C} \setminus \{0\}$. Choose now any $y = (w, s) \in \mathbb{H}^3 \setminus l$. Then $|w| > 0$ and

$$\begin{aligned} d(y, g(y)) &= \operatorname{arcosh}\left(1 + \frac{|aw - w|^2 + (|a|s - s)^2}{2|a|s^2}\right) \\ &> \operatorname{arcosh}\left(1 + \frac{(|a|t - t)^2}{2|a|t^2}\right) = d(x, g(x)) \geq 2\epsilon \end{aligned}$$

where $x = (0, t) \in \gamma$.
On the other hand, if $d(x, g(x)) < 2\epsilon$, P_g is a tubular neighbourhood of γ. To prove this let without loss of generality $\gamma = \{(0,t) : t > 0\}$ and $g = (z \mapsto az)$ for $a \in \mathbb{C}\setminus\{0\}$. Choose $x = (w,t) \in \mathbb{H}^3$ and $y = (v, s) \in \mathbb{H}^3$, such that $d(x, \gamma) = d(y, \gamma) = r > 0$. The goal is showing $d(x, g(x)) = d(y, g(y))$, because then we know P_g is a tubular neighbourhood.

Consider the Möbius transformation $T : (z \mapsto \frac{s}{t} z)$. With this

$$d(x, g(x)) = d(T(x), T \circ g(x)) = d(T(x), g \circ T(x))$$

since g and T commute. Moreover,

$$d((s/t\ w, s), \gamma) = d(T(x), T(\gamma)) = d((w, t), \gamma) = d((v, s), \gamma)$$

as T preserves γ. If we write this out and apply *cosh* on both sides, we get

$$\begin{aligned}\min_{l>0} \left(1 + \frac{|(s/t)w|^2 + (t-l)^2}{2tl}\right) &= \cosh d((s/t\ w, s), \gamma) = \cosh d((v, s), \gamma) \\ &= \min_{l>0} \left(1 + \frac{|v|^2 + (t-l)^2}{2tl}\right).\end{aligned}$$

This means obviously $|(s/w)w| = |v|$. Now we can apply an elliptic transformation $T' = (z \mapsto e^{i(\arg(v) - arg(w))} z)$, that fixes γ and satisfies $T' \circ T(x) = y$. Again this is an isometry and commutes with g, so

$$\begin{aligned}d(x, g(x)) &= d(T' \circ T(x), T' \circ T \circ g(x)) = d(T' \circ T(x), g \circ T' \circ T(x)) \\ &= d(y, g(y)).\end{aligned}$$

The radius of this tubular neighbourhood depends on both, the translation length of g along γ and the rotation angle of g around γ.

Now to the rest of the theorem. By definition, we see

$$\pi^{-1}(\mathcal{M}_\epsilon^{thin}(G)) = \bigcup_{g \in G \setminus \{\mathrm{Id}\}} P_g$$

and so $\mathcal{M}_\epsilon^{thin}(G)$ is open.

Suppose there is $x \in P_g \cap P_h$. By Theorem 6.11 this means $\langle h, g \rangle$ is elementary. So, from the discussion of elementary groups, especially Lemma 3.5, $\langle h, g \rangle$ is either a cyclic loxodromic or a parabolic subgroup. As a consequence either $P_g \subseteq P_h$ or $P_h \subseteq P_g$. So we know x belongs to a tubular neighbourhood about a geodesic, with length smaller than 2ϵ or an ϵ-horoball and these features are disjoint.

Let F be one of the connected components of $\pi^{-1}(\mathcal{M}_\epsilon^{thin}(G))$. Consequently, $F = P_\varphi$ for some $\varphi \in G \setminus \{\mathrm{Id}\}$. We look at the group G_F, that is generated by all $g \in G$, for which $g(F) = F$. This group is elementary, since every 2 generator subgroup of it is elementary as was shown in the last paragraph. So the quotient F/G_F is isometric to one of the three components from the theorem. Lastly each of these F/G_F is embedded in $\mathbb{H}^3/G$. Because, if there was $g \in G \setminus G_F$ with $g(F) \cap F \neq \emptyset$, then $g(F) = P_{g\varphi g^{-1}}$ would be not equal, but isometric to $F = P_\varphi$. So $P_{g\varphi g^{-1}}$ would neither be contained in P_φ (or conversely) nor disjoint to it yielding a contradiction to what was proven in the last paragraph.

□

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