Ruprecht Karl University of Heidelberg
Faculty of Mathematics and
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Bachelor Thesis

Reflection length in affine Coxeter groups

(Spiegelungslänge in affinen Coxeter Gruppen)

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Abstract

In any Coxeter group the conjugate of elements in its Coxeter generating set are called reflections. The length of an element with respect to this expanded generating set is its reflection length. This thesis conjectures an explicit formula to compute reflection length in affine Coxeter groups and gives a proof for all groups of rank 1 and 2. Also, it provides a proof for one inequality of the formula in affine Coxeter groups of arbitrary rank.

Zusammenfassung

In einer beliebigen Coxeter Gruppe sind Spiegelungen die zu Coxeter Erzeugern konjugierten Elemente. Die Länge eines Elements unter diesem erweiterten Erzeugendensystem heißt Spiegelungslänge. Die vorliegende Arbeit betrachtet die Spiegelungslänge in affinen Coxeter Gruppen. Es wird eine explizite Formel vermutet und ein Beweis dieser für alle Gruppen von Rang 1 und 2 gegeben. Außerdem wird die eine Ungleichung der Formel für affine Coxeter Gruppen beliebigen Ranges gezeigt.

Contents

Introduction

A Coxeter group W is a group generated by elements of order two, and thus can be seen as 'abstract reflection group'. Define a *reflection* in W as any element conjugate to a generator. The collection R of all reflections is another natural generating set of W as it contains any minimal generating set and is invariant under conjugation. The *reflection length* of an element $w \in W$ is the minimal number of reflection necessary to express w. For proper definitions of these terms see Section 1. Studying the reflection length in Coxeter groups is a direct consequence of introducing the set R of reflections as generating set.

This thesis considers reflection length in spherical and affine Coxeter groups. Those can be constructed using the notion of a root system. A *root system* Φ is a finite collection of vectors α called *roots* fulfilling some technicalities (for details see Section 2). For each root α there is a reflection r_α across the hyperplane H_α orthogonal to α. These reflections r_α generate a *spherical Coxeter group* W_0. Reflection length in these finite Coxeter groups has been studied for example in [Car70].

Affine Coxeter groups are closely related to root systems as well. They are constructed as follows: Take for each root α a family of parallel hyperplanes $H_{\alpha,j},\ j \in \mathbb{Z}$ each orthogonal to α. The reflections $r_{\alpha,j}$ across all those hyperplanes $H_{\alpha,j}$ then generate an *affine Coxeter group* W. For a detailed description of affine Coxeter groups and related terms see Section 3.

The underlying root system Φ captures the structure of some elements in W. Obtain a *coroot* $\alpha^\vee$ by rescaling a root α with $\frac{2}{(\alpha,\alpha)}$ where the denominator is the inner product of α with itself. These coroots form the coroot system $\Phi^\vee$ of Φ. If $t_\lambda \in W$ is a translation by the vector λ, then λ lies in the $\mathbb{Z}$-lattice $L(\Phi^\vee)$ spanned by $\Phi^\vee$.

Each $w \in W$ allows several translation-elliptic factorisations. That is, it can be written as the product $w = t_\lambda u$ of a *translation* $t_\lambda \in W$ and an *elliptic* element $u \in W$. Also, one can associate an *elliptic* and a *differential dimension* to $w \in W$. Then the *dimension* of an element $w \in W$ is defined as the sum of its elliptic and differential dimensions and if w is a translation t_λ one writes $\dim(\lambda)$ for the dimension of t_λ. These different notions of dimensions are used in [LMPS19] to compute the reflection length in an arbitrary affine Coxeter group W.

In this thesis we identify W as a semidirect product $W = T \rtimes W_0$. Here, T is isomorphic to $\mathbb{Z}^n$ where n is the rank of W and W_0 is a spherical Coxeter group over the same root system as W. This identification corresponds to a 'choice of origin' in the hyperplane arrangement of W. It also gives a unique inclusion $\iota : W_0 \hookrightarrow W$ and thereby the elements of W_0 can be regarded as elements of W. Then each $w \in W$ can be written in *normal form* $w = t_\lambda u$ which is a translation-elliptic factorisation with $t_\lambda \in T$ and $u \in W_0$.

Observe that such a normal form is, in most cases, not a minimal length factorisation of w. For example $w = t_{\alpha^\vee} s_\alpha$ is a normal form using three reflections but has reflection length 1. A minimal reflection factorisation is given by $w = s_{\alpha,1}$ which is, in general, not a normal form for any choice of identification $W = T \rtimes W_0$.

This motivates the following.

Conjecture A (Schwer)**.** *Let $W = T \rtimes W_0$ be an affine Coxeter group with spherical Coxeter group W_0 over the same root system Φ. Let $w \in W$ be an arbitrary element of W with normal form $w = t_\lambda u$. Then the reflection length of w can be written as*

$$\ell_R(w) = \frac{1}{2}\ell_R(t_\lambda) + \min_{v \in V_\lambda} \ell_R(vu) = \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

Here, the set V_λ consists of compositions $v_\lambda = r_{\alpha_1} \cdots r_{\alpha_k} \in W_0$ where the coroots $\alpha_1^\vee, \ldots, \alpha_k^\vee$ appear in a minimal integral combination of λ (for details see Definition 4.1).

The formula in Conjecture A would allow to compute the reflection length in an affine Coxeter group by computing the reflection lengths of a translation and elements in a spherical Coxeter group. Both are well understood, see [MP11, Prop. 4.3] and Theorem 2.23 and thus this formula would highly contribute to understanding reflection length in affine Coxeter groups.

Conjecture A is indeed fulfilled in some special cases.

Theorem B. *Conjecture A is true in affine Coxeter groups of rank* 1 *and* 2*.*

This work provides a detailed proof for Theorem B by considering the structure of normal forms in groups of types affine A_1, A_2, B_2 and G_2. This might also help to prove the conjecture in groups of higher rank.

Theorem C (Upper bound)**.** *With the same notation as in Conjecture A holds*

$$\ell_R(w) \leq \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

For the latter Theorem a rather short and technical proof is given. Both proofs can be found in Section 4.

The last section names open problems and gives suggestions for further investigations.

In the end Appendix A addresses the creation of the Ti*k*Z images throughout the thesis.

Acknowledgement

I would like to thank JProf. Dr. Maria Beatrice Pozzetti (University of Heidelberg) and Prof. Dr. Petra Schwer (University of Magdeburg) for supervising me. I am very grateful towards Petra Schwer for introducing me to the concept of reflection length and her conjecture and letting me chose it as topic for my thesis. Thanks to both of you for being there for questions, discussions, tips and feedback regarding content and form of my thesis and its related talks.

1. Coxeter groups and reflections

This section starts with the definition of Coxeter groups definition and some examples that will be revisited throughout the thesis. In the next step Coxeter and Cayley graphs are introduced which both encode some structure of the associated Coxeter group.

Then reflections and the reflection length of an element are defined and basic properties of the reflection length reviewed. The last subsection describes when a Coxeter group is reducible or irreducible and resolves that it suffices to study the reflection length of irreducible Coxeter groups to compute it in arbitrary Coxeter groups.

Definition 1.1 (Coxeter group)**.** A group W with finite generating set $S = \{s_1, \dots, s_n\}$ is called *Coxeter group* if the defining relations are of the form $s_i^2 = \mathbb{1}$ for all i and $(s_i s_j)^{m_{ij}} = \mathbb{1}$ for all $i \neq j$ with $m_{ij} \in \{2, 3, \dots\} \cup \{\infty\}$. That is

$$W = \langle\ s_1, \dots, s_n \mid s_i^2,\ (s_i s_j)^{m_{ij}} \text{ for all } i \neq j\ \rangle$$

Infinite order of $s_i s_j$ is indicated by $m_{ij} = \infty$, meaning there is no relation $(s_i s_j)^k = \mathbb{1}$. The tuple (W, S) consisting of a Coxeter group W and its *Coxeter generating set* S is called *Coxeter system.* Its *Coxeter matrix* M_W is the $n \times n-$matrix with diagonal entries all 1 and m_{ij} in the ith row and jth column for $i \neq j$.

Coxeter groups can be considered as abstract reflection groups as they are generated by elements of order two. The following examples illustrate this.

Example 1.2. (a) The group $\mathbb{Z}/2\mathbb{Z} = \langle\, s \mid s^2 \,\rangle$ is the smallest non-trivial example of a Coxeter group. Its Coxeter matrix is the $1 \times 1-$matrix (1).

(b) The infinite Dihedral group D_∞ is generated by two reflections s, t of the real line $\mathbb{R}$ through two distinct points, say 0 and 1 like in Figure 1. Thus their composition is a translation to the left or right and hence has infinite order. Thus the Coxeter presentation and Coxeter matrix of D_∞ are

$$D_\infty = \langle\, s, t \mid s^2, t^2 \,\rangle \quad \text{and} \quad M_{D_\infty} = \begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix}.$$

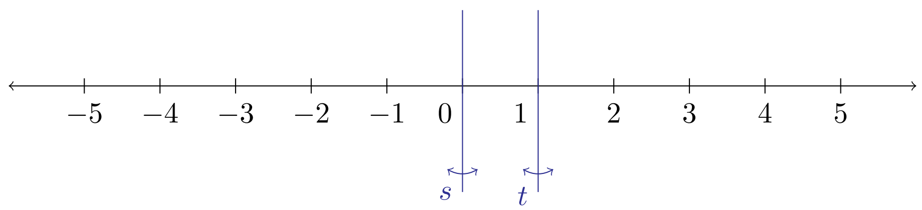


Figure 1: The infinite dihedral group D_∞ acting on the real line.

(c) The symmetric group of an equilateral triangle or equivalently the set $\{1, 2, 3\}$ is

$$S_3 = \{\mathrm{id}, (12), (13), (23), (123), (132)\}.$$

It can be generated by the reflections (12) and (23), whose compositions are the rotations (132) and (321) of order three. Thus the group has Coxeter presentation

$$S_3 = \langle\, s, t \mid s^2,\, t^2,\, (st)^3 \,\rangle.$$

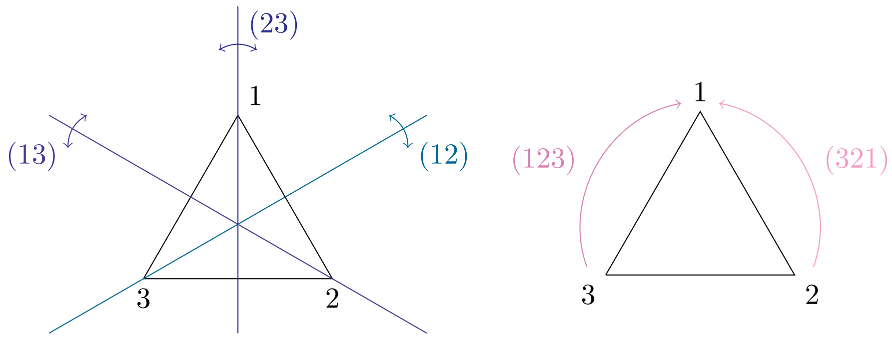


Figure 2: The symmetry group of an equilateral triangle.

(d) Consider the real plane $\mathbb{R}^2$ spanned by the standard basis ε_1, ε_2. Analogous to (a) take s, t the reflections along $\langle \varepsilon_1 \rangle_{\mathbb{R}}$ stabilising the origin and ε_1, respectively. Further, take a third reflection r that reflects about the diagonal $\langle \varepsilon_1 + \varepsilon_2 \rangle_{\mathbb{R}}$ and stabilises the origin, see Figure 3.

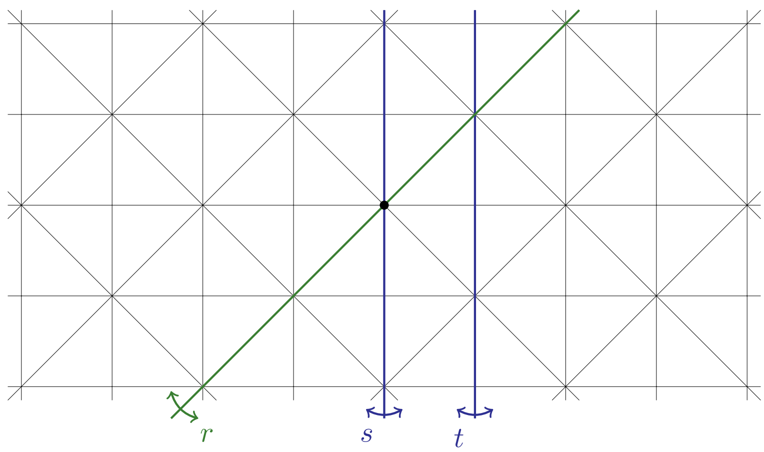


Figure 3: The group $W_B = \langle s, t, r \mid s^2,\ t^2,\ r^2,\ (sr)^4,\ (tr)^4 \rangle$.

The composition st is a translation as in (a) and both sr, tr are rotations of order four. Thus the group generated by s, t, r is a Coxeter group with the following Coxeter presentation and Coxeter matrix.

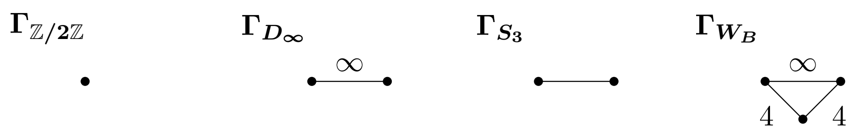
$$W_B = \langle\, s, t, r \mid s^2,\, t^2,\, r^2,\, (sr)^4,\, (tr)^4 \,\rangle \quad \text{and} \quad M_W = \begin{pmatrix} 1 & \infty & 4 \\ \infty & 1 & 4 \\ 4 & 4 & 1 \end{pmatrix}.$$

The group action of W_B on the real plane induces a tessellation with triangles that are in one-to-one correspondence with the elements of W.

Next, assign two graph Γ and $\mathrm{Cay}(W, S)$ to a Coxeter system with both encode some of the groups structure.

Definition 1.3 (Coxeter graph)**.** Given a Coxeter system (W, S) define a graph Γ_W with vertex set $S = \{s_1, \dots, s_n\}$ and s_i, s_j joined by an edge labelled m_{ij} if $m_{ij} \geq 3$. By convention, the label $m_{ij} = 3$ is omitted. Γ is the *Coxeter graph* of (W, S)

Example 1.4. The Coxeter graphs of the groups from the last Example 1.2 are



Definition 1.5 (Cayley graph)**.** Let (W, S) be a Coxeter system. Define a graph $\mathrm{Cay}(W, S)$ by taking W as vertex set and joining $w, v \in W$ by an edge with label $s \in S$ if $w = sv$. This graph is called the *Cayley graph* of W with respect to S.

Note that the Cayley graph of W changes with the choice of a generating set S.

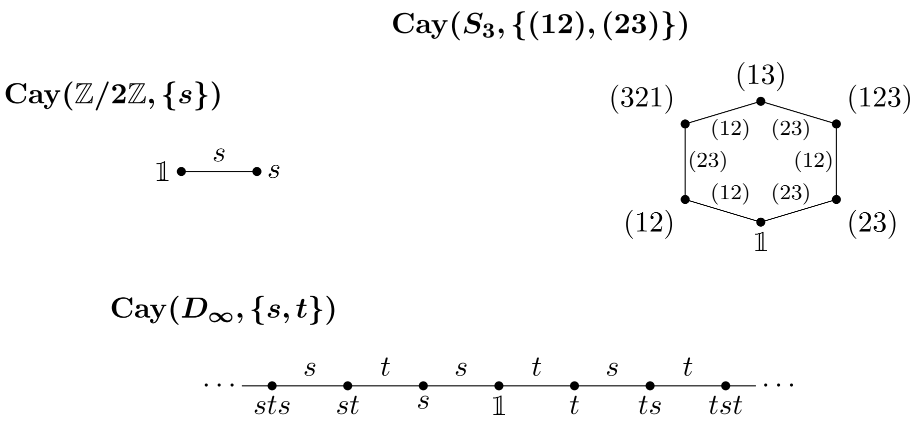


Figure 4: Cayley graphs with respect to S.

Before moving on to the notion of reflection length in the next subsection consider a special kind of element called Coxeter elements.

Definition 1.6 (Coxeter elements)**.** Let W be a Coxeter group with standard minimal generating set $S = \{s_1, \ldots, s_n\}$. A *Coxeter element* in W is any element $w = s_{\pi(1)} \cdots s_{\pi(n)} \in W$ with $\pi \in S_n$ a permutation.

If the set S consist of two generators all Coxeter elements are either rotations or translations.

Example 1.7. (a) The reflections (12), (23) form a standard minimal generating set of the group S_3. Thence both rotations $(123) = (12)(23)$ and $(321) = (23)(12)$ are Coxeter elements in S_3.

(b) Consider the infinite dihedral group D_∞ with standard minimal generating set $\{s, t\}$ as above in Example 1.2(b). Then both translations st and ts are Coxeter elements in D_∞.

1.1. Reflection length

This subsection introduces reflections as another generating set of W. It then defines the length and in particular reflection length of an element and finally reviews some facts about reflection length in arbitrary Coxeter groups.

Definition 1.8 (Reflections)**.** Given a Coxeter system (W, S), a *reflection* is any element in W conjugate to an element in S. Let R denote the set of all reflections in W, that is

$$R = \{\, wsw^{-1} \mid s \in S,\ w \in W \,\}.$$

All generators $s \in S$ of a Coxeter system (W, S) are reflections (by choosing $w = \mathbb{1}$). Thus R is another generating set for its Coxeter group W. Note that W is infinite whenever R is. The converse is also true by definition of R, as W acts by conjugation.

As R generates W every element $w \in W$ can be written with reflections only. Asking for the shortest such representation for a given w is then a natural consequence.

Definition 1.9 (Reflection length)**.** Let G be a group with generating set S. Define the *length* of an element $g \in G$ with respect to S as

$$\ell_S(g) := \min\{k \mid g = s_1 \cdots s_k \text{ with all } s_i \in S\}.$$

Note that by definition $\ell_S(\mathbb{1}) = 0$. In a Coxeter group with reflections R the length ℓ_R is called *reflection length*.

In other words, ℓ_R is the combinatorial distance to $\mathbb{1}$ in the Cayley graph $\mathrm{Cay}(W, R)$ of W with respect to R.

Example 1.10. Consider the group

$$S_3 = \langle (12), (23) \rangle = \{\mathrm{id}, (12), (23), (13), (123), (321)\}$$

from Example 1.2(c). Here, the set of reflections is $R = \{(12), (13), (23)\}$ since

$$(13) = (12)(23)(12) = (23)(12)(23).$$

Thus, the element (13) has *standard length* $\ell_S((13)) = 3$ and reflection length 1. Both rotations $(123) = (12)(23)$ and $(321) = (23)(12)$ have reflection length two, because they can be written with two reflections and are neither the identity nor in the set of reflections.

One can also read these calculations from the Cayley graph shown in Figure 5.

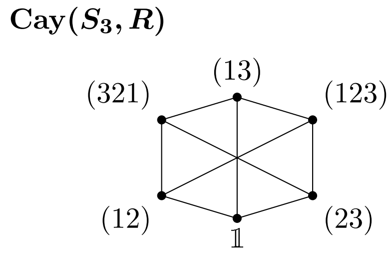


Figure 5: The Cayley graph of S_3 with respect to R (without edge labels).

Proposition 1.11 (Basic properties). *Let W be a Coxeter group. Then for $v, w \in W$:*

(a) $\ell_R(vw) = \ell_R(v) + \ell_R(w) \mod 2$ *(parity restriction)*
and in particular $\ell_R(rw) = \ell_R(w) \pm 1$ *for all* $r \in R$;

(b) $\ell_R(v) - \ell_R(w) \le \ell_R(vw) \le \ell_R(v) + \ell_R(w)$; *(triangle inequality)*

(c) $\ell_R(w) = \ell_R(w^{-1})$;

(d) $\ell_R(vw) = \ell_R(wv)$.

Proof. For (a) consider the homomorphism φ from W onto $\mathbb{Z}/2\mathbb{Z}$ sending each reflection to the non-identity element. As reflection length is invariant under conjugation φ sends w to the identity element if and only if $\ell_R(w) = 0 \mod 2$. Thus the statement (a) follows because φ is a homomorphism.

Both (b) and (c) are based on the fact that ℓ_R is the combinatorial distance in the Cayley graph of W with generating set R.

(d) holds as ℓ_R is constant on conjugacy classes since $R = \bigcup_{w \in W} wSw^{-1}$ and thus $\ell_R(vw) = \ell_R(wvww^{-1}) = \ell_R(wv)$. □

Definition 1.12 (Reflection factorisation). Let W be a Coxeter group and $w \in W$. The representation $w = r_1 \cdots r_n$ is called *reflection factorisation* if all r_i are reflections. It is a *minimal length factorisation* of w if in addition $n = \ell_R(w)$.

Example 1.13 (Continuing Example 1.10). The rotation (123) has reflection factorisations $(123) = (12)(23)$ and $(123) = (23)(12)(23)(12)$. Only the first is a minimal length factorisation as $\ell_R((123)) = 2$.

There are many reflection factorisations for a given $w \in W$. Some of them can be rewritten into others.

Lemma 1.14 (Rewriting reflection factorisations [MP11, Lem. 3.5])**.** *Let $w = r_1 \cdots r_l$ be a reflection factorisation of an element w of a Coxeter group. For any selection $1 \leq i_1 < i_2 < \cdots < i_m \leq l$ of positions, there is a length-l reflection factorisation of w whose first m reflections are $r_{i_1} r_{i_2} \cdots r_{i_m}$ and another length-l reflection factorisation of w where these are the last m reflections.*

Remark 1.15 (Hurwitz moves). The proof on this lemma uses the so called *Hurwitz moves.* Those are given by replacing the reflection product rs by $s(srs)$ or $(rsr)r$ thereby moving s further to the beginning or r further to the end while the reflection length remains invariant.

It is known that the reflection length is unbounded whenever W is neither spherical (for a definition see Section 2) or affine (see Section 3)[Dus12]. This thesis focusses on the affine case which is closely related to the spherical one.

There are also results in arbitrary Coxeter groups, like the following.

Theorem 1.16 ([Dye01, Thm. 1.1])**.** *Let (W, S) be a Coxeter system and $w \in W$. For any expression $w = s_1 \cdots s_n$ with $s_i \in S$ for all i and $\ell_S(w) = n$ one has*

$$\ell_R(w) = \min_p \{1 = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_p} \cdots s_n \text{ with } 1 \leq i_1 \leq \cdots \leq i_p\}$$

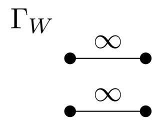
where a hat in this formula indicates that the respective generator is omitted.

1.2. Irreducibility

By taking free or direct product one can obtain new Coxeter groups. Conversely, any Coxeter group can be decomposed as the direct product of so called *irreducible* Coxeter groups. This section addresses these concepts.

Example 1.17. Consider the product of the infinite dihedral group with itself $W = D_\infty \times D_\infty$. It has the following Coxeter matrix and Coxeter graph.

$$M_W = \begin{pmatrix} 1 & \infty & 2 & 2 \\ \infty & 1 & 2 & 2 \\ 2 & 2 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix} \qquad \text{and}$$



W acts on $\mathbb{R}^2$ and tessellates the plane with rectangles that are in bijection with W's elements.

Definition 1.18. A Coxeter group W is called *irreducible* if its Coxeter graph is connected, that is for each two vertices v, u of Γ_W exists a path in Γ_W with starting at v and ending at u. Otherwise W is said to be *reducible.*

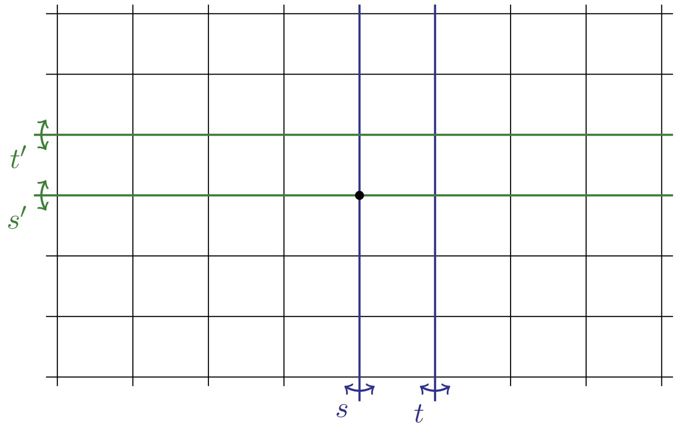


Figure 6: The group $D_\infty \times D_\infty$.

Proposition 1.19 ([Hum90, Section 6.1])**.** *Let W be a Coxeter group. There are irreducible Coxeter groups $W_1, \ldots, W_r$ such that $W = W_1 \times \cdots \times W_r$.*

Example 1.20. The group $D_\infty \times D_\infty$ is reducible and S_3 is irreducible.

Proposition 1.21 ([MP11, Prop. 1.2])**.** *Let W be a reducible Coxeter group. Then the reflection length of $w \in W$ is the sum of the reflection lengths of its factors.*

Thus it suffices to study the reflection length in irreducible Coxeter groups.

2. Root systems and spherical Coxeter groups

In this section root systems are defined which give rise to spherical Coxeter groups. As their name suggests they are a special case of Coxeter groups. Also, root systems and spherical Coxeter groups are closely related to affine Coxeter groups which will be examined in Section 3. Thus in this section the interaction between a root system and its spherical Coxeter group is studied.

The sections starts with the definitions of a root system, the associate coroot system and some basic properties. Then bases are considered as generating sets and positivity and height of roots are defined. In the next subsection reducibility of root systems is considered and special kinds of subsystems and subgroups are studied. The last subsection gives an overview about reflection length in spherical Coxeter groups.

Throughout this section consider examples in dimension one and two. Those will play a prominent role in Section 4 when proving Theorem B.

2.1. Roots and coroots

This subsection starts with the fundamental notions of roots, spherical Coxeter groups and coroots.

Definition 2.1 (Root systems and spherical Coxeter groups)**.** A subset Φ of a finite-dimensional real vector space V is called an *root system* in V if the following properties are satisfied:

(R1) Φ is finite, $0 \notin \Phi$ and $\langle \Phi \rangle_{\mathbb{R}} = V$;

(R2) if $c \in \mathbb{R}$ is such that $\alpha, c\alpha \in \Phi$, then $c = \pm 1$;

(R3) for each $\alpha \in \Phi$ there exists a reflection $s_\alpha \in \mathrm{GL}(V)$ along α stabilising Φ;

(R4) for $\alpha, \beta \in \Phi$, $s_\alpha.\beta - \beta$ is an integral multiple of α (*crystallographic condition*).

Then, the elements of Φ are called *roots* and dimension of V is called the *rank* of Φ. The group $W_0 = W_0(\Phi) := \langle\, s_\alpha \mid \alpha \in \Phi \,\rangle$ is called the *spherical Coxeter group* of Φ. In other contexts, W_0 is also referred to as *Weyl group* of Φ.

Example 2.2. (a) There is only one rank-one root system $\{\alpha, -\alpha\}$ with $\alpha \in \mathbb{R}^1$ (up to rescaling), therefore denote it with Φ_1. Its spherical Coxeter group is the Coxeter group $\mathbb{Z}/2\mathbb{Z}$.

(b) The root system $\Phi_A = \{\pm\alpha, \pm\beta, \pm\gamma\} \subseteq \mathbb{R}^2$ with all roots of equal length as shown in Figure 7 has spherical Coxeter group $D_3 = S_3$.

(c) Roots can also have different length as in the root system $\Phi_B = \{\pm\alpha, \pm\beta, \pm\gamma, \pm\delta\} \subseteq \mathbb{R}^2$ also drawn below. Its spherical Coxeter group is the Coxeter group D_4.

(d) There is another interesting two-dimensional root system denoted with Φ_G consisting of the roots $\{\pm\alpha, \pm\beta, \pm\gamma, \pm\delta, \pm\varepsilon, \pm\zeta\}$. It also has different root lengths. Its spherical Coxeter group is D_6 and it is illustrated below as well.

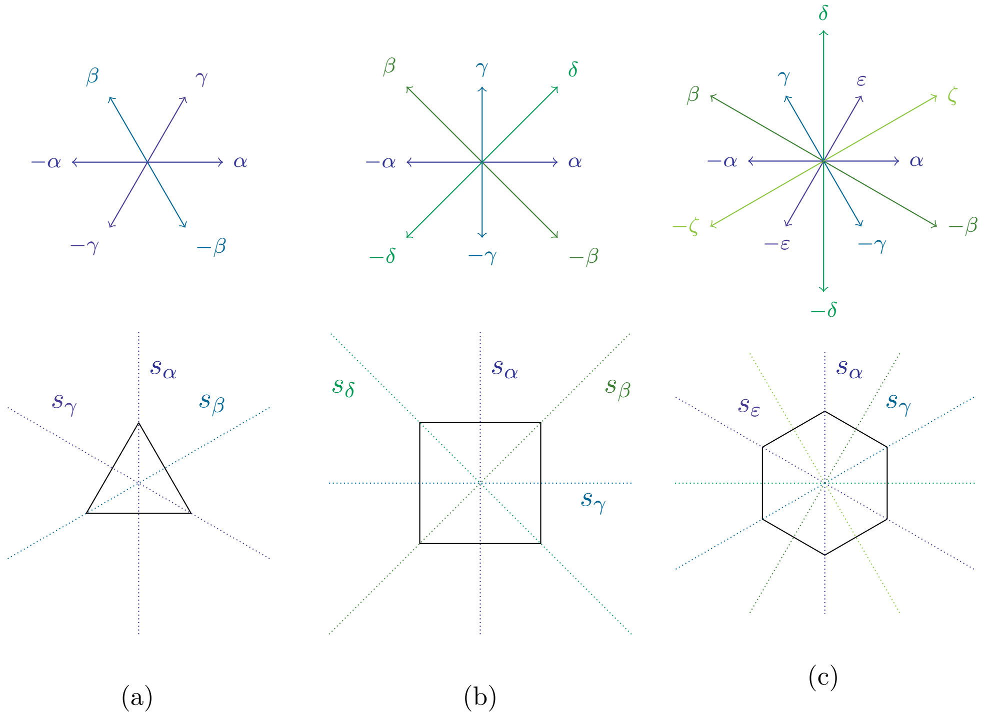


Figure 7: The root systems Φ_A (a), Φ_B (b) and Φ_G (c) are reflection groups of an equilateral triangle, a square and a hexagon, respectively.

Remark 2.3 (Types of root systems). The names Φ_A, Φ_B and Φ_G above are chosen because the respective root system Φ_X is said to be of type X_2. The index 2 indicates the rank of the root system. This is needed because root systems in other dimensions may have the same kind of type. For example Φ_1 is of type A_1 and has rank one. There are only four types of root systems of rank two, namely $A_1 \times A_1, A_2, B_2$ and G_2 [MT11, Prop. A.17].

Spherical Coxeter groups are indeed Coxeter groups (as its generators have order two). The converse is not true in general, for example D_5 is a non-spherical Coxeter group. It is a Coxeter group acting on $\mathbb{R}^2$ but does not appear as the spherical Coxeter group of a root system of type $X_2,\ X = A, B, C$ or $A_1 \times A_1$.

The interaction between a root system and its spherical Coxeter group is very beautiful. The reflections of W_0 in the sense of Definition 1.8 are precisely the elements s_α for $\alpha \in \Phi$. Also, the following is true:

Proposition 2.4 ([MT11, Lem. A.4])**.** *Let Φ be a root system with spherical Coxeter group W_0. Then for all $\alpha \in \Phi$ and all $u \in W_0$ it is*

$$u s_\alpha u^{-1} = s_{u.\alpha}.$$

Let Φ be a root system of V. Assume V is equipped with an W_0-invariant inner product $(\cdot,\cdot)$. This exists since W_0 is finite and stabilises the finite set Φ generating V. With V an inner product space, we can speak of lengths of and angles between vectors.

Proposition 2.5. *Let $s = s_\alpha \in W_0$ be a generating reflection. Then for any $v \in V$*

$$s_\alpha.v = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha.$$

Proof. [MT11, Prop. A.1] Denote the eigenspace of s for the eigenvalue 1 with $H \subset V$, that is H is the fixed space of s. As $(\cdot,\cdot)$ is invariant under s this yields for any $v \in H$ that $(v,\alpha) = (s_\alpha.v, s_\alpha.\alpha) = (v,-\alpha)$. Hence $(v,\alpha) = 0$ holds for any $v \in H$. With $V = H \oplus \langle\alpha\rangle_{\mathbb{R}}$ and $s.\alpha = -\alpha = \alpha - 2\frac{(\alpha,\alpha)}{(\alpha,\alpha)}\alpha$ this gives the claimed statement. $\square$

Therefore $s_\alpha.v$ can be expressed with v and α only, given the inner product on V. This motivates the following. Identify V with its dual space $V^* = \mathrm{Hom}(V,\mathbb{R})$ via $v \mapsto (\cdot, v)$ where $(\cdot,v): V \to \mathbb{R},\ u \mapsto (u,v)$. Further, put $\alpha^\vee := 2\alpha/(\alpha,\alpha)$. Then

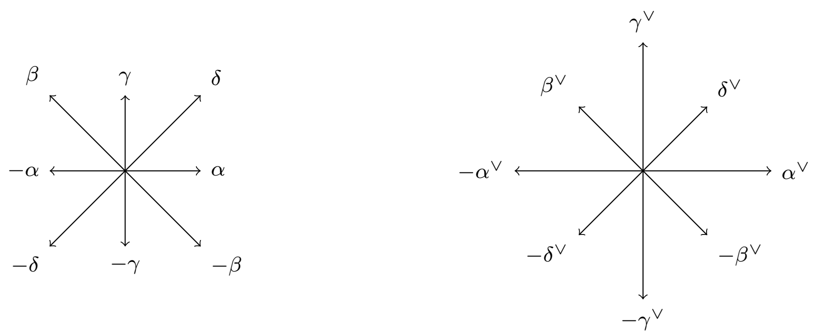
$$s_\alpha.v = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha = v - (v,\alpha^\vee)\alpha = v - \alpha^\vee(v)\alpha.$$

Definition 2.6 (Coroot system)**.** Let Φ be a root system. Define the *coroot system* of Φ as the set $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$.

Note that $\Phi^\vee$ is indeed a root system.

Example 2.7. The coroot system $\Phi_A^\vee$ of Φ_A is the same as Φ_A up to rescaling all roots with the same factor.

In Φ_B and Φ_G there are short and long roots. As a coroot $\alpha^\vee$ is a scaled α by $2/(\alpha,\alpha)$ the long roots give short coroots and conversely the short roots give long coroots. This correlation is illustrated in Figure 8.



(a) (b)

Figure 8: The root system Φ_B (a) with its coroot system $\Phi_B^\vee$ (b).

Definition 2.8 ((Co-)root lattices)**.** Let Φ be a root system. The $\mathbb{Z}$-span $L(\Phi) = \langle \Phi \rangle_{\mathbb{Z}}$ of Φ is called *root lattice*. The *coroot lattice* of $\Phi^\vee$ is defined analogously.

Note that both the root and coroot lattice of Φ are isomorphic to $\mathbb{Z}^n$ where n is the rank of Φ.

Roots, coroots and the coroot lattice play an important roll also in the structure of affine Coxeter groups. For details see Section 3.2.

2.2. Bases, positive systems and height

Here, the term base is introduced as a generating system for a root system. Then, positive systems and the height of a root are defined. Finally, the highest root of a root system is introduced.

Definition 2.9 (Bases and positive systems)**.** A subset $\Delta \subset \Phi$ is called a *base* of Φ if it is vector space basis of V and any $\beta \in \Phi$ is a linear combination $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ with either all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$. For a given base of Φ define the positive system Φ^+ as the collection of the roots with all $c_\alpha \geq 0$. A root in Φ^+ is a *positive* root.

It can be shown that the coefficients c_α are integrals [MT11, Cor. A.12].

Example 2.10. $\{\alpha, \beta\} \subset \Phi_i$ is a base of Φ_i for $i = A, B, G$, compare Figure 9.

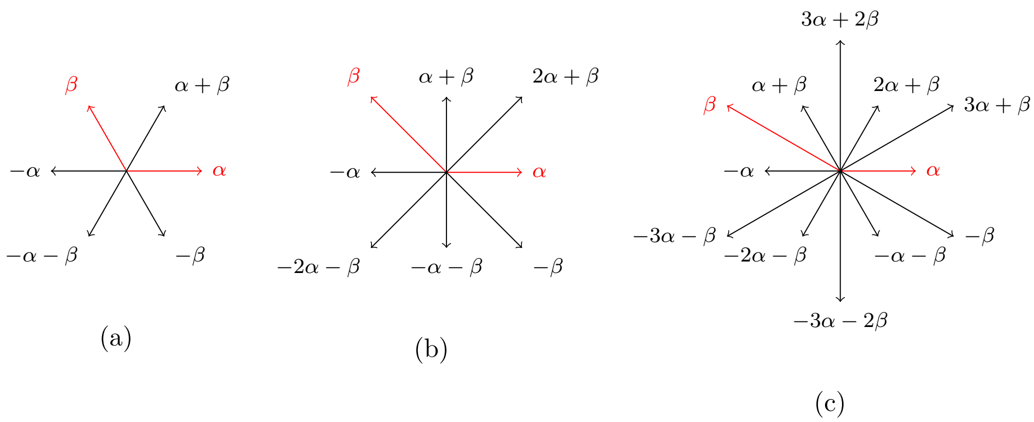


Figure 9: Bases of the root systems Φ_A (a), Φ_B (b) and Φ_G (c).

In the running examples the spherical Coxeter groups $W_0(\Phi_A) = S_3$, $W_0(\Phi_B) = D_4$ and $W_0(\Phi_G) = D_6$ are generated by the *simple reflections* s_α with $\alpha \in \Delta$.

This concept is true in general, that is if Δ is a base, then $W_0 = \langle s_\alpha \mid \alpha \in \Delta \rangle$. Furthermore, for every $\alpha \in \varphi$ there is $u \in W_0$ such that $u.\alpha \in \Delta$. Also, any two bases of a root system Φ are conjugate under W_0 [MT11, Prop. A.9 and Prop. A.11].

Next, we shortly considers the height of a root.

Definition 2.11 (Height and the highest root)**.** Let Δ be a base of the root system Φ. The *height* of a root $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha \in \Phi$ (with respect to Δ) is defined as $\mathrm{ht}(\beta) := \sum_{\alpha\in\Delta} c_\alpha$. There is a unique root $\alpha_0 \in \Phi$ with largest height [MT11, Prop. B.5] which is called *highest root*.

In an indecomposable root system, the coefficients c_α of $\alpha_0 = \sum_{\alpha\in\Delta} c_\alpha \alpha$ can be found in [MT11, Table B.1].

Example 2.12. The highest roots in $\Phi_i,\ i = A, B, G$ with respect to $\{\alpha, \beta\}$ are $\alpha + \beta \in \Phi_A,\ 2\alpha + \beta \in \Phi_B$ and $3\alpha + 2\beta \in \Phi_G$, respectively. They are shown in Figure 10.

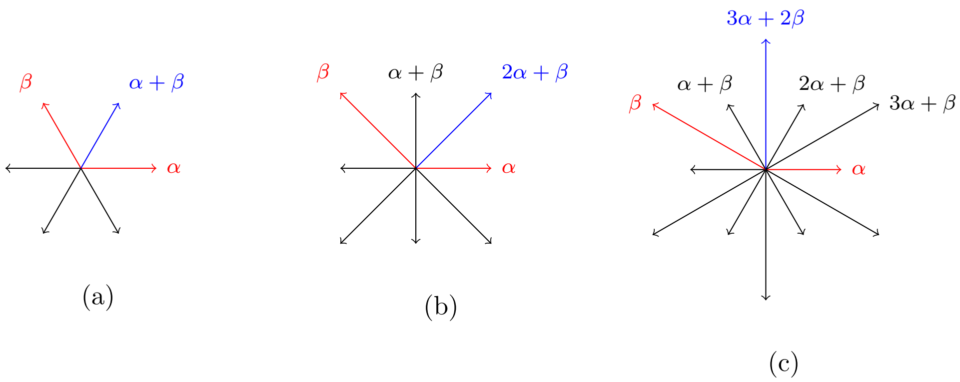


Figure 10: The highest roots of Φ_A (a), Φ_B (b) and Φ_G (c) with respect to their respective base $\{\alpha, \beta\}$.

Note that in general $(\alpha_0)^\vee \neq (\alpha^\vee)_0$. That is, the coroot $(\alpha_0)^\vee$ of highest root α_0 of a root system Φ is not necessarily the highest root $(\alpha^\vee)_0$ in the coroot system $\Phi^\vee$. Though the set of coefficients $c_\alpha \in \mathbb{Z}_{\neq 0}$ of α_0 and $(\alpha^\vee)_0$ are the same since coroot systems always have the same type as their associate root system.

2.3. Decomposition and subsystems

With the notion of a base as generating system for a root system new root systems can be build from the known ones. In other words, a root system may be decomposed into smaller disjoint root systems. This section formalises those ideas.

Definition 2.13. A non-empty root system Φ with base Δ is called *decomposable* if a non-trivial partition $\Delta = \Delta_1 \sqcup \Delta_2$ exists such that $(\alpha_1, \alpha_2) = 0$ for all $\alpha_i \in \Delta_i,\ i = 1, 2$; if no such decomposition exists Φ is said to be *indecomposable*.

Example 2.14. The root system Φ_B is indecomposable. It has two decomposable subsystems with bases $\{\alpha, \alpha + \beta\}$ and $\{\beta, 2\alpha + \beta\}$ which are shown in Figure 11.

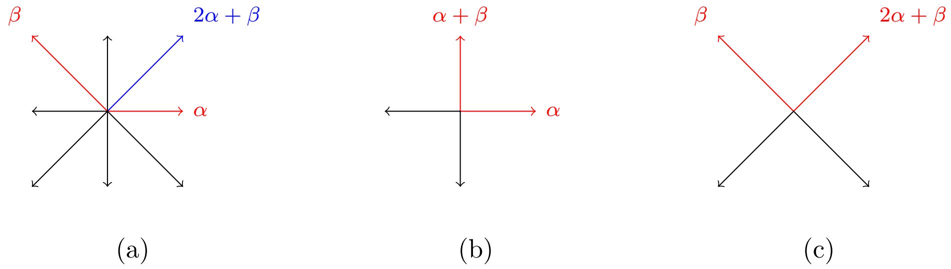


Figure 11: The indecomposable root system Φ_B (a) with two decomposable subsystems in (b) and (c).

Proposition 2.15 ([MT11, Cor. A.16])**.** *Any root system Φ can be decomposed uniquely (up to reordering) into a disjoint orthogonal union $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_r$ of indecomposable root systems Φ_i, and then $W_0(\Phi) \cong W_0(\Phi_1) \times \cdots \times W_0(\Phi_r)$.*

The Φ_i are called *indecomposable components* of Φ. A root system $\Phi \subset V$ is indecomposable if and only if its spherical Coxeter group acts irreducibly on V [MT11, Prop. A.16]. Therefore Φ is indecomposable if and only if W_0 is irreducible. Hence a classification of indecomposable root systems also classifies irreducible spherical Coxeter groups and vice versa. Such a classification can be found in [Bou02, Plates I to IX].

Proposition 2.16 ([MT11, Cor. A.18 and Lem. B.20])**.** *Let Φ be indecomposable. Then:*

(a) There are at most two different root length in Φ. Call the longer ones long roots *and the shorter ones* short roots.

(b) All roots of Φ of the same length are conjugate under W_0.

(c) If Φ contains roots of two different lengths then Φ is generated by its short roots, that is, $\{\alpha \mid \alpha \text{ is a short root}\}\mathbb{Z} \cap \Phi = \Phi$.

A root system can also have other subsystems than indecomposable components. Those are again in one-to-one correspondence to subgroups which are spherical Coxeter groups.

Definition 2.17 (Parabolic subgroups)**.** Let Δ be a base of Φ and $S = \{ s_\alpha \mid \alpha \in \Delta\}$ the simple reflections of W_0. For a subset $J \subseteq S$ call $W_{0,J} := \langle s \in J \rangle$ a *standard parabolic subgroup* of W_0. It is again a spherical Coxeter group with base $\Delta_J := \{ \alpha \in \Delta \mid s_\alpha \in J \}$ and root system $\Phi_J := \Delta_J\mathbb{Z} \cap \Phi$ [MT11, Prop. A.25].

Any conjugate of a standard parabolic subgroup is called *parabolic subgroup* of W_0.

Example 2.18. Consider D_4 with root system Φ_B and base $\{\alpha, \beta\}$. Its simple reflections are $\{s_\alpha, s_\beta\}$. The subgroup $\langle s_\beta \rangle$ is a standard parabolic subgroup with root system $\{\pm\beta\}$ of type A_1. Its conjugate $\langle s_{2\alpha+\beta}\rangle$ is just a parabolic subgroup of type A_1. Both associate subsystems of Φ_B are illustrated in Figure 12.

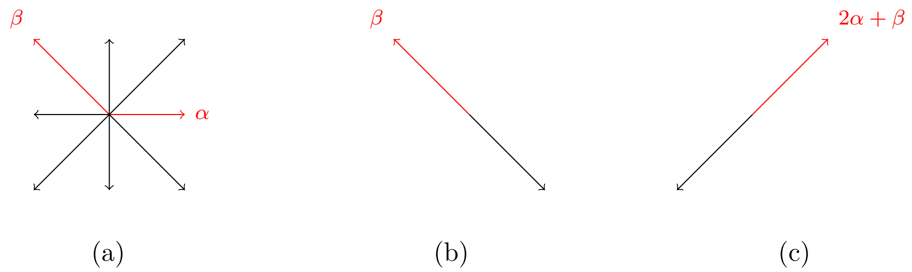


Figure 12: The root system Φ_B (a) with the root systems of a standard (b) and non-standard (c) parabolic subgroup of the spherical Coxeter group D_4.

Definition 2.19 (Closed subsystems)**.** Let Φ be a root system. A subset $\Psi \subseteq \Phi$ is called *closed* if for all $\alpha, \beta \in \Phi$

(C1) $s_\alpha.\beta \in \Psi$ and

(C2) if $\alpha + \beta \in \Phi$ then also $\alpha + \beta \in \Psi$.

Closed subsets are automatically subsystems [MT11, Prop. B.14].

Example 2.20 (Continuing Example 2.14)**.** The decomposable subsystem $\{ \pm\beta, \pm(2\alpha + \beta) \}$ is closed, but $\{ \pm\alpha, \pm(\alpha+\beta) \}$ is not because it does not include $\alpha+(\alpha+\beta) = 2\alpha+\beta$.

There exists a explicit classification of maximal closed subsystems of an indecomposable root system.

Theorem 2.21 (Borel-de Siebenthal [MT11, Thm. B.18])**.** *Let Φ be an indecomposable root system with base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and highest root $\alpha_0 = \sum_{i=1}^{n} c_i\alpha_i$ with respect to Δ. Then the maximal closed subsystems of Φ up to conjugation are those with bases:*

(1) $\Delta \setminus \{\alpha_i\}$ for $1 \leq i \leq n$ with $c_i = 1$ and

(2) $\Delta \setminus \{a_i\} \cup \{\alpha_0\}$ for $1 \leq i \leq n$ with c_i a prime.

Note that the subsystems in Theorem 2.21(1) are of rank $n-1$ where those in (2) have rank n, that is, the same rank as Φ. The coefficients of the highest root in indecomposable root systems of arbitrary rank can be found in [MT11, Table B.1]. Therefore the maximal closed subsystems are easily obtained with this theorem.

Example 2.22 (Continuing Example 2.20)**.** In Φ_B the highest root with respect to the base $\{\alpha, \beta\}$ is $2\alpha + \beta$. Thus, by Theorem 2.21 $\{\pm\beta, \pm(2\alpha+\beta)\}$ is the only closed subsystem of rank two.

For other classifications of non-maximal and/or non-closed subsystems see [MT11, Appendices A and B].

2.4. Reflection length in spherical Coxeter groups

This section considers reflection length in the case of spherical Coxeter groups. A few definitions are given which are later generalised for affine Coxeter groups in Section 3.

Theorem 2.23 ([Car70, Lem. 2])**.** *Let (W_0, S) be a spherical Coxeter system and $u \in W_0$. Then a reflection factorisation $u = r_{\alpha_1}, \dots, r_{\alpha_l}$ is a minimal length factorisation and $\ell_R(u) = l$ if and only if $\alpha_1, \dots, \alpha_l$ are linearly independent.*

In particular, ℓ_R is bounded by the rank of W_0.

Remark 2.24 (Reflection length in rank two). Thus for a given spherical Coxeter group W_0 of rank two the reflection length is bounded by 2. Let $u \in W_0$ with reflection factorisation $u = r_1 \cdots r_m$. If u is the identity it has reflection length 0. Because of the parity restriction Proposition 1.11(a) it is $\ell_R(u) = 1$ if and only if m is odd and $\ell_R(u) = 2$ if and only if m is even and $u \neq \mathrm{id}$.

Theorem 2.23 can be reformulated by introducing the dimension of an element which uses the notion of a move-set.

Definition 2.25 (Fixed space and move-set [LMPS19, Def. 1.7])**.** Let u be an orthogonal transformation of V. Define the *fixed space* $\mathrm{Fix}(u)$ as the set of vectors $\lambda \in V$ with $u(\lambda) = \lambda$. The *move-set* $\mathrm{Mov}(u)$ is the set of vectors $\mu \in V$ for which there is $\lambda \in V$ such that $u(\lambda) = \mu + \lambda$.

In other words, $\mathrm{Fix}(u)$ is the kernel $\mathrm{Ker}(u - \mathbb{1})$ and $\mathrm{Mov}(u)$ is the image $\mathrm{Im}(u - \mathbb{1})$.

Since reflections are orthogonal transformations and the latter are closed under composition, every element of a spherical Coxeter group is a orthogonal transformation. Thus their move-set and fixed space ar orthogonal complements in V.

Example 2.26. Consider the reflection $(23) \in S_3$ acting on the real vector space $\mathbb{R}^2$ by fixing the origin and mapping ε_1 to $-\varepsilon_1$. Its move-set is given by $\mathrm{Mov}((23)) = \langle \varepsilon_1 \rangle_{\mathbb{R}}$ and its fixed space is $\mathrm{Fix}((23)) = \langle \varepsilon_2 \rangle_{\mathbb{R}}$.

The move-set of the rotation $(123) \in S_3$ is all $\mathbb{R}^2$ and its fixed space consist of the origin only. They both move-sets and fixed spaces are illustrated in Figure 13.

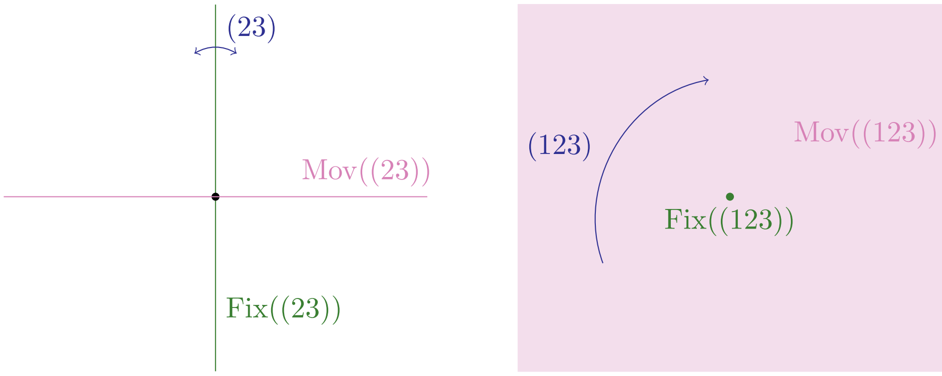


Figure 13: The fixed space and move-set of (23), (123) $\in S_3$.

Definition 2.27 (Dimension of an element)**.** Let W_0 be a spherical Coxeter group. The *dimension* of $u \in W_0$ is defined as $\dim(u) := \dim(\mathrm{Mov}(u))$.

With this notion we can reformulate:

Theorem 2.28 (Alternative version of Theorem 2.23)**.** *Let W_0 be a spherical Coxeter group and $u \in W_0$. Then $\ell_R(u) = \dim(u)$.*

This version is not only shorter but also looks more like Theorem 3.32 we will later see in the affine case.

3. Affine Coxeter groups

Affine Coxeter groups are Coxeter groups acting on Euclidean spaces rather than Euclidean vector spaces. They are closely related to root systems and spherical Coxeter groups. In this section they and related notions are defined and basic properties studied. The last Section 3.4 reviews some results about reflection length in affine Coxeter groups.

This section is completely based on [LMPS19, Sections 1.3 to 1.5].

3.1. Points, vectors and affine hyperplanes

In this subsection affine Coxeter groups are defined and consider their connection with spherical Coxeter groups and root systems. Then, the notion of move-sets and fixed spaces is generalised to the affine case.

When doing linear algebra fixing a specific coordinate system can disguise an underlying geometric structure. In the same way working in an affine Coxeter group with a specific choice of origin can have an disguising effect. To take this into account in the following is distinguished between *points* and *vectors* as in [BM15].

Definition 3.1 (Euclidean space)**.** Let V be an Euclidean vector space. A *Euclidean space* is a set E with a unique transitive V-action, that is, for all $x, y \in E$ there exists $\lambda \in V$ sending x to y. In this case write $\lambda + x = y$.

The core idea of E is that it has no well-defined origin, like V. This is also expressed in the terminology to call the elements of E *points* and denote them with Roman letters, such as x and y, whereas the elements of V are called *vectors* and denoted with Greek letters, such as λ and μ.

Example 3.2 (Euclidean space vs. Euclidean vector space)**.** The set $E := \mathbb{R}$ is a Euclidean space with the V-action from $V := \mathbb{R}$ (as vector space) given by standard addition. An illustration is given in Figure 14.

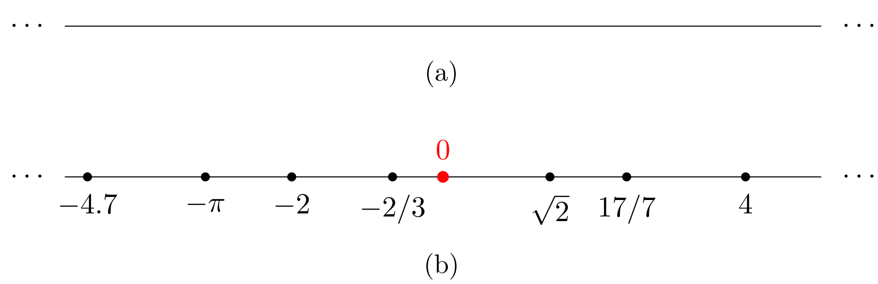


Figure 14: Illustration of $\mathbb{R}$ as Euclidean space (a) and $\mathbb{R}$ as Euclidean vector space (b).

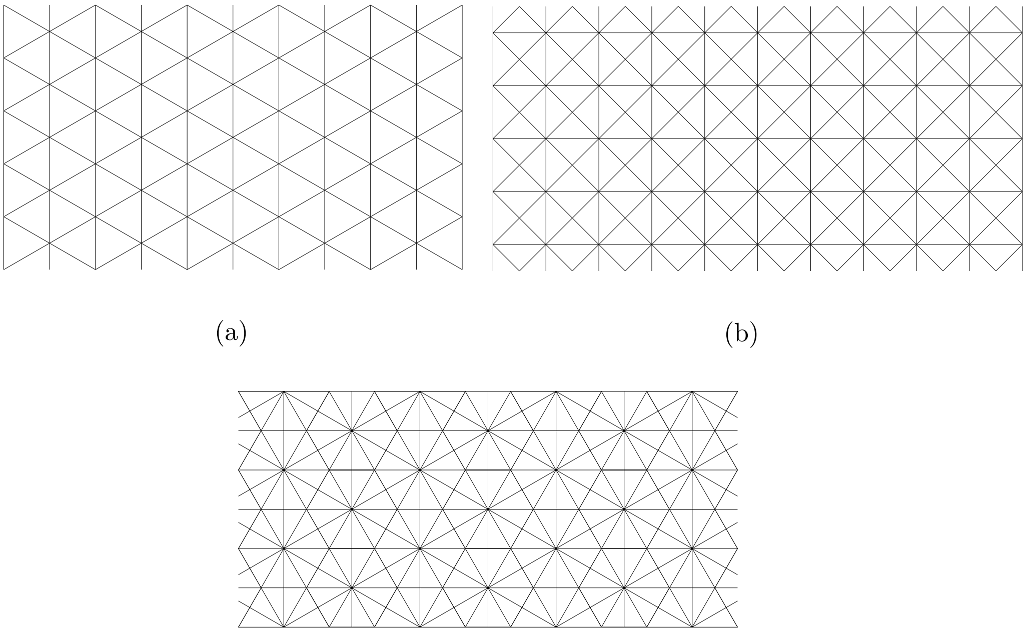
Note that a Euclidean space E can be identified with its Euclidean vector space V by selecting an origin $x \in E$ and sending each vector λ to the point $\lambda + x$. This identification is used to define affine Coxeter groups.

Definition 3.3 (Affine Coxeter group)**.** Let E be a Euclidean space whose associated Euclidean vector space V is equipped with a root system Φ. Construct an affine Coxeter group W from Φ as follows: Fix a point $x \in E$ to temporarily identify E with V. Thereby the inner product $(\cdot,\cdot)$ on V is induced on E, treating x as the origin. For each root $\alpha \in \Phi$ and $j \in \mathbb{Z}$ denote with $H_{\alpha,j}$ the *(affine) hyperplane* in E of points $v \in E$ such that $(v,\alpha) = j$. The unique non-trivial isometry of E that fixes $H_{\alpha,j}$ pointwise is a *reflection* denoted with $r_{\alpha,j}$. The set $R = \{r_{\alpha,j} \mid \alpha \in \Phi,\ j \in \mathbb{Z}\}$ then generates the *affine Coxeter group* W and R are its *reflections* in the sense of Definition 1.8.

Analogous to the spherical case, the *rank* of W is the dimension of V.

A minimal generating set S can be obtained from R by restricting to those reflections reflecting about the facets of a certain polytope in E.

Example 3.4 (Type affine B_2)**.** The group W_B from Example 1.2(d) is an affine Coxeter group. It is of type affine B_2 as its root system is of type B_2. The corresponding hyperplane arrangement is shown below in Figure 15b.



(a) (b)

(c)

Figure 15: Affine hyperplanes for groups of type affine A_2 (a), B_2 (b) and G_2 (c).

The connection of affine Coxeter groups and root systems can be seen directly from Definition 3.3. The connection with spherical Coxeter groups can be expressed as follows.

Definition 3.5 (Translations, quotients and subgroups)**.** Given a root system Φ, the affine Coxeter group W is closely related to the spherical Coxeter group W_0. Sending the reflection $r_{\alpha,j}$ in W to r_α in W_0 extends to a group homomorphism $p:\ W \twoheadrightarrow W_0$. The kernel of p is a normal abelian subgroup T isomorphic to $\mathbb{Z}^n$ whose elements are called *translations*. Furthermore it is $W_0 \cong W/T$.

Let $x \in E$ be the point fixed in the construction of W (the 'choice of origin'). Then the map $\iota : W_0 \hookrightarrow W$ sending r_α to $r_{\alpha,0}$ is a section of the projection p, identifying W_0 with the subgroup of W generated by all reflections fixing x. Thence W may be identified as a semidirect product $W \cong T \rtimes W_0$ and, once such an identification is chosen, an element $u \in W_0$ may be regarded as an element of W via ι.

Of course, such an identification of W_0 with a subgroup of W is non-unique since any conjugate by elements of T yields another such subgroup.

Remark 3.6 (Geometric interpretation of W, W_0 and p). Fix a root system Φ and consider the affine Coxeter group W and the spherical Coxeter group W_0 over Φ.

The spherical group W_0 acts on the Euclidean vector space V and the reflection hyperplanes of W_0 divide V into cones. Every such cone can play the role of a fundamental domain of the action of W_0 on V. By choosing one fundamental cone c the elements of W_0 are in bijection with the cones. Here, the fundamental cone c corresponds to the identity and any $u \in W_0$ corresponds to image cone $w.c$ of c under w. One such bijection is indicated in green in Figure 16 for Φ the root system Φ_B of type B_2.

The *boundary* of V is a sphere S whose points are the parallelism classes of geodesic rays in V. Each reflection hyperplane in V of a reflection in W_0 corresponds to a reflection hyperplane in the sphere S. One can think of such a reflection hyperplane in S as an equator. These hyperplanes in S divide S into *chambers*, that is, a chamber is a maximal connected component in the complement of the reflection hyperplanes in S. Each chamber is also the parallelism class of a cone in V. Thereby the chambers are in bijection with the elements in W_0.

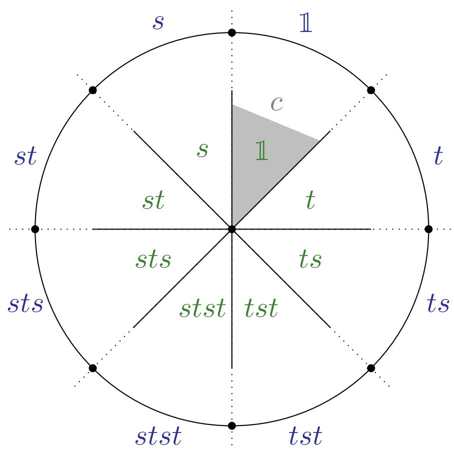


Figure 16: The elements of W_0 of type B_2 are in bijection with the cones which again are in bijection with the chambers in the boundary of V.

Now consider the affine group W. It acts on the Euclidean space E. The affine reflection hyperplanes $H_{\alpha,j}$ divide E: An *alcove* A is (the closure of) a maximal connected component in the complement of all reflection hyperplanes $H_{\alpha,j}$ in the Euclidean space E. These are the small triangles in Figure 15 like the OliveGreen-coloured triangle in Figure 17. Analogously to the spherical case, each such triangle can act as the fundamental domain of the action of W on E. Each choice of a point $x \in E$ as origin and an fundamental alcove A_x having x as vertex determines a bijection between the elements of W and all alcoves in E. Here, the fundamental alcove corresponds to the identity in W and every $w \in W$ corresponds to the alcove $w.A_X$ being the image of A_x under w.

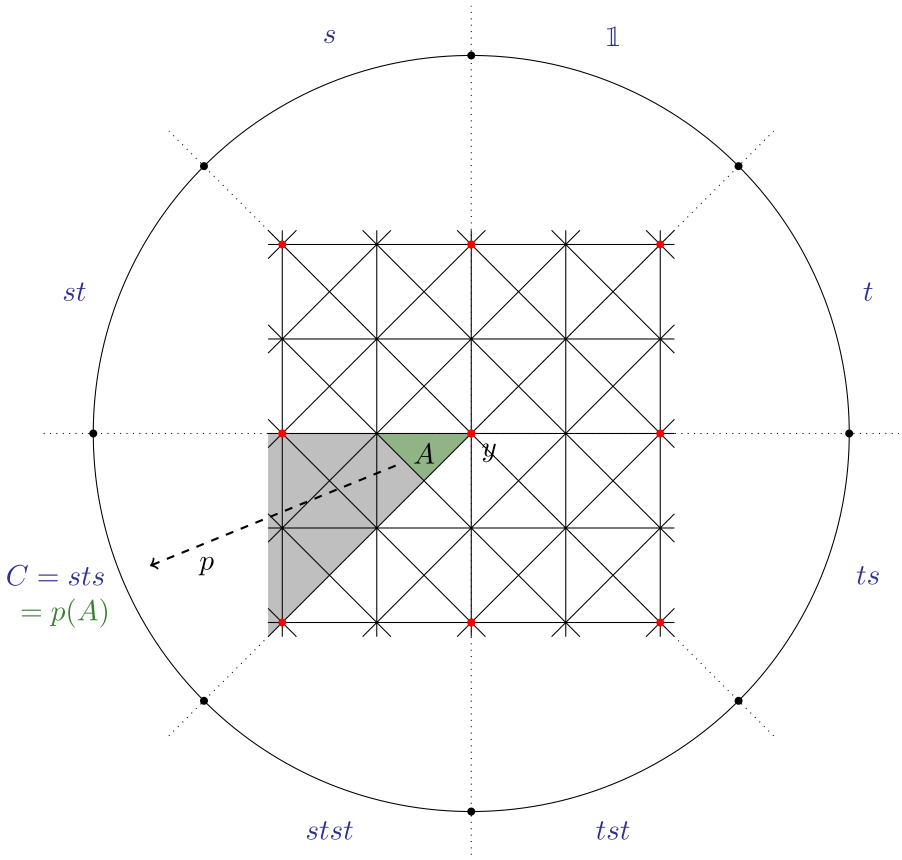


Figure 17: The elements of W of type affine B_2 are in bijection with the alcoves in the tessellated plane E. The projection p maps an alcove A to the chamber C in the boundary of E that points 'in the same direction'.

The *boundary* of E is also a sphere S whose points are again the parallelism classes of geodesic rays. The tessellation of the the Euclidean space E induces a simplicial structure on the sphere S. Here, a parallelism class of hyperplanes in E corresponds to a reflection hyperplane in the sphere S. Again the *chambers* of S are the maximal connected components in the complement of the hyperplanes in S. Here, these are the parallelism classes of simplicial cones in E. Namely, for any of the Red vertices in Figure 17, say y, the complement of all affine hyperplanes going through y divides E into simplicial cones. These are called *Weyl cones* based at y. One of them is indicated in grey. The parallelism classes of these Weyl cones are in bijection with the chambers in S.

A bijection between the elements of W_0 and the chambers of S is obtained from the action of W as follows. Each element $w \in W$ is an affine motion of E mapping parallel rays to parallel rays. As a result it induces an isometry on (the tessellation of) S where translations induce the identity on S. One might want the bijection between W_0 and the chambers of S to be compatible with these induced isometries of W on S. This can be achieved by mapping the identity in W_0 to the parallelism class of the unique Weyl cone based at x containing the alcove A_x.

From this perspective, the projection $p : W \twoheadrightarrow W_0$ can be seen geometrically as the map sending an element $w \in W$ to the induced isometry on S. Using the geometric terms defined above, this is to map an alcove A with vertex y to the chamber C at infinity that is the parallelism class of the cone based at y which contains A. This is indicated by the dotted arrow depicted in Figure 17. One can think of this as 'walking from y to infinity in the direction of A'.

More about the correlation between alcoves and chambers can be found in [AP15].

Next, we generalise the notion of move-sets and fixed spaces we have already seen in the spherical setting (compare Definition 2.25).

Definition 3.7 (Move-set and fixed space)**.** Let w be an Euclidean isometry. The *motion* of a point $x \in E$ under w is a vector $\lambda \in V$ such that $w(x) = \lambda + x$. The collection of the motions of all points $x \in E$ forms the *move-set* $\mathrm{Mov}(w)$ of w. The fixed space of w is the set of all points $x \in E$ with $w(x) = x$, in other words, $\mathrm{Fix}(w)$ consists of all points in E whose motion is the zero-vector. In symbols:

$$\begin{aligned}\mathrm{Mov}(w) &= \{\lambda \in V \mid w(x) = \lambda + x \text{ for some } x \in E\}\\ \mathrm{Fix}(w) &= \{x \in E \mid w(x) = x\}\end{aligned}$$

The move-set is an (affine) subspace of V [BM15, Prop. 3.2]. If $\mathrm{Fix}(w)$ is non-empty it is an affine subspace of E, that is, there exists an affine subspace U of V, such that the U-action is transitive on $\mathrm{Fix}(w)$ and $\mathrm{Fix}(w)$ is closed under it.

Example 3.8 (Move-set and fixed space in type affine B_2)**.** To get a better understanding of the notion of a move-set in the affine setting consider an element in a group of type affine B_2. Assume $w = rst$ with r, s and t the reflections about the lines highlighted in Figure 18 and suppose x, y and z are the labelled points. The points $w(x), w(y)$ and $w(z)$ are also indicated.

First, make a (temporary) identification of E and $\mathbb{R}^2$. The standard basis vectors ε_1 and ε_2 of $\mathbb{R}^2$ are the vectors that send z to x and z to y, respectively. That is $\varepsilon_1 + z = x$ and $\varepsilon_2 + z = y$ with the notation introduced in Definition 3.1 of an Euclidean space. By abuse of notation write $\varepsilon_1 = x - z$ and $\varepsilon_2 = y - z$.

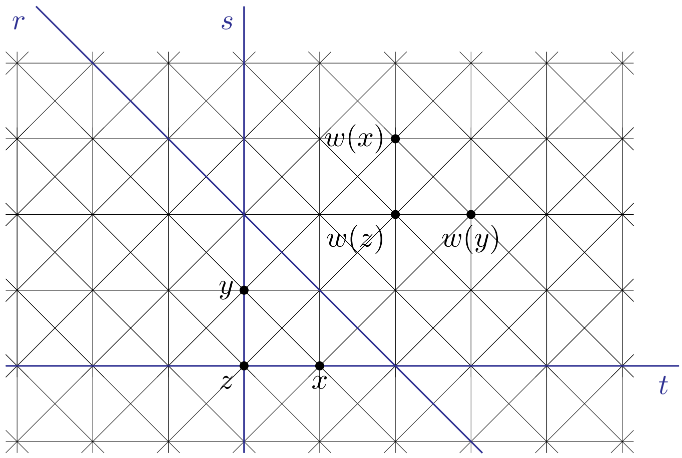


Figure 18: The action of $w = rst$ in the affine group of type B_2 on the points x, y, z in the Euclidean plane E.

Having made this identification every point p in $E \cong \mathbb{R}^2$ can be expressed as

$$p = a(x - z) + b(y - z) + z$$

for some $a, b \in \mathbb{R}$. With this coordinate system rewrite

$$\begin{aligned} w(x) &= 2(x - z) + 3(y - z) + z = 1(x - z) + 3(y - z) + x \\ w(y) &= 3(x - z) + 2(y - z) + z = 3(x - z) + 1(y - z) + y \\ w(z) &= 2(x - z) + 2(y - z) + z \end{aligned}$$

Now it can be see that the motion of x is the vector $(1, 3)$, that of y is $3, 1$ and the motion of z is $(2, 2)$. To compute the whole move-set, we do this again for an arbitrary $p \in E$ using linearity:

$$\begin{aligned} w(p) &= aw(x - z) + bw(y - z) + w(z) \\ &= a(w(x) - w(z)) + b(w(y) - w(z)) + w(z) \\ &= a(y - z) + b(x - z) + 2(x - z) + 2(y - z) + z \\ &= (b + 2)(x - z) + (a + 2)(y - z) + z \\ &= (b - a + 2)(x - z) + (a - b + 2)(y - z) + p \end{aligned}$$

and thus the motion of p is $\lambda = (2, 2) + (a - b)(-1, 1)$ and hence the move-set of w is given by the affine line

$$\mathrm{Mov}(w) = (2, 2) + (-1, 1)\mathbb{R}.$$

The fixed space of w is empty because 0 is not in $\mathrm{Mov}(w)$ and thence no point has trivial motion under w.

3.2. Elliptics and translations

This section considers special kinds of elements in affine Coxeter groups, namely elliptics and translations and records some of their properties.

Let r be a reflection whose fixed space is the hyperplane H. Then the motion of any point under r is in a direction orthogonal to H, and the move-set of r is the line through the origin of V in this direction.

Definition 3.9 (Roots of a reflection)**.** For a reflection r a vector $\alpha \in V$ is called a *root* of r if $\mathrm{Mov}(r) = \langle \alpha \rangle_{\mathbb{R}}$.

For an affine Coxeter group the fixed spaces of the reflections come in a finite number of parallel families. For each such family, those fixed spaces are equally spaced hyperplanes and one can chose a common root α such that its length encodes additional information like the distance between adjacent parallel hyperplanes. Through normalising all roots this way a root system Φ in V is obtained, in the sense of Definition 2.1.

Recall that each root system comes with a coroot system $\Phi^\vee$ and a (co-)root lattice $L(\Phi^{(\vee)})$ which are both isomorphic to $\mathbb{Z}^n$ (for details see Definition 2.6 and Definition 2.8).

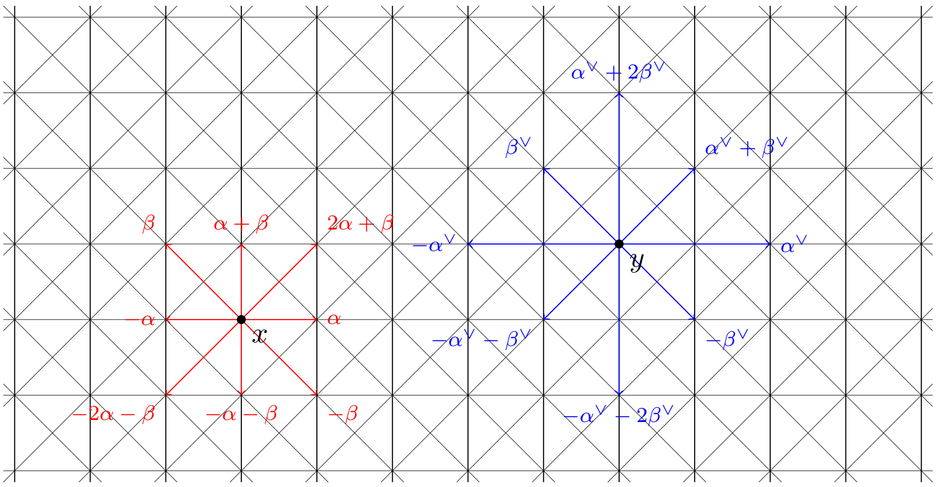


Figure 19: The action of roots and coroots of type B_2 on the points x and y in the Euclidean plane E.

Example 3.10 (Roots and coroots in type affine B_2)**.** Consider the hyperplane arrangement for type affine B_2 illustrated in Figure 19. The affine hyperplanes lie in the Euclidean plane E, that is, the Euclidean space isomorphic to the set $\mathbb{R}^2$.

The root system Φ_B is a subset of the Euclidean vector space $V = \mathbb{R}^2$ and thus acts on E. The action of all roots on the point x is displayed in Red in the figure. Likewise the action of all coroots in $\Phi_B^\vee$ on the point y is highlighted in Blue.

It can be seen that the distance between two adjacent parallel affine hyperplanes is half the length of the perpendicular coroots.

Moving on to elliptics and translations.

Definition 3.11 (Elliptic part and elliptic elements)**.** Let W be an affine Coxeter group and $w \in W$. The *elliptic part* w_e of w is its image $p(w)$ under the projection $p: W \twoheadrightarrow W_0$. Also, w is said to be *elliptic* if its fixed space is non-empty.

Note that the elliptic part of an element in W is an element of W_0 and thus acts naturally on V rather than on E. A characterisation and other nice properties of elliptic elements are given at the end of this subsection, after some more definitions and examples.

Definition 3.12 (Translations)**.** For every vector $\lambda \in V$ there exists an Euclidean isometry t_λ on E called *translation* which sends each point $x \in E$ to $\lambda + x$.

Let W be an affine Coxeter group acting on E with root system $\Phi \subset V$. Then $w \in W$ is a translation in the sense of this definition if and only if it is in the kernel T of the projection $p: W \twoheadrightarrow W_0$. Thus the notions of a translation in Definition 3.12 and Definition 3.5 are equivalent. Furthermore, the set of vectors in V defining the translations in T is identical to the coroot lattice $L(\Phi^\vee)$.

Example 3.13 (Translations and elliptics in type affine B_2)**.** Consider W of type affine B_2. Let $r := s_{\beta,m} \in W$ for some $m \in \mathbb{Z}$ as illustrated in Figure 20. Then r is a reflection and fixes $H_{\beta,m}$ point-wise. Thus its fixed space is non-empty and elliptic. Its elliptic part is $p(s_{\beta,m}) = s_\beta \in W_0$ and thus r is not a translation.

Next, consider $t_{\alpha^\vee} = s_{\alpha,n-1}s_{\alpha,n} \in W$ for some $n \in \mathbb{Z}$. It is also illustrated in Figure 20. $t_{\alpha^\vee}$ lies in the kernel T and thus is a translation, non-elliptic and has trivial elliptic part.

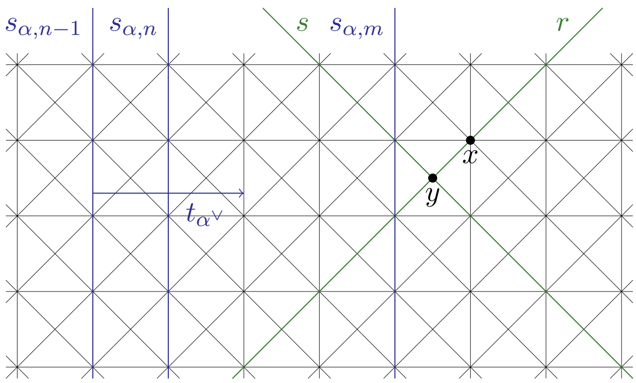


Figure 20: Illustration of Examples 3.13 and 3.16.

Definition 3.14 (Translation-elliptic factorisations)**.** Let W be an affine Coxeter group and $w \in W$. A *translation-elliptic factorisation* of w is any expression $w = t_\lambda u$ as the product of a translation $t_\lambda \in W$ with $\lambda \in V$ and an elliptic element $u \in W$. Given such a factorisation, t_λ is referred to as the *translation part* and u is called the *elliptic part* of w.

For any translation-elliptic factorisation $w = t_\lambda u$ the translation part t_λ is in the kernel T of p (as it is a translation) and thus $w_e = u_e$.

For most elements many possible translation-elliptic factorisations exist. Once an identification $W = T \rtimes W_0$ is chosen, there is a unique inclusion $\iota : W_0 \hookrightarrow W$. Through ι the elements of W_0 can be regarded as elements of W. Some of those many possible translation-elliptic factorisations encode this added structure and are therefore called normal forms.

Definition 3.15 (Normal forms)**.** Let W be an affine Coxeter group and $w \in W$. For a given an identification $W = T \rtimes W_0$ call the unique translation-elliptic factorisation $w = t_\lambda u$ with $u \in W_0$ and $t_\lambda \in T$ a *normal form* of w.

In general, not all translation-elliptic factorisations $t_\lambda u$ are also normal forms, in other words, there exists no identification $W = T \rtimes W_0$ such that $u \in W_0$. As any such identification comes with a choice of origin $x \in E$ which is fixed by $\iota(W_0)$ the element u needs to stabilise such a possible origin x. In general, not all nodes in the hyperplane arrangement can be an origin x, or in other words, can be fixed by $\iota(W_0)$. Consider an example.

Example 3.16 (Translations-elliptic factorizations and normal forms in type affine B_2)**.** Again, put $r = s_{\beta,m}$ as in Example 3.13. Then the product $t_{\alpha^\vee} r$ is a translation-elliptic factorisation whereas $rt_{\alpha^\vee}$ is not. Observe that both $t_{\alpha^\vee} s_{\alpha,m}$ and $s_{\alpha,m} t_{\alpha^\vee}$ are translation-elliptic factorisations as both compositions are reflections and hence elliptic.

All three of these translation-elliptic factorisations can be realised by choosing $x \in E$ as origin where x is the point indicated in Figure 20.

Assume s is the reflection denoted in the figure as well. The point y is stabilised by both r and s. Thus their product rs is elliptic and $t_{\alpha^\vee} rs$ is a translation-elliptic factorisation. But y can not be chosen as origin because there is no reflection $s_{\alpha,j}$ stabilising y. Therefore $t_{\alpha^\vee} rs$ is no normal form.

Until the end of the subsection consider elliptic elements. Start with a characterisation.

Proposition 3.17 (Recognising elliptics [LMPS19, Lem. 1.18 and Prop. 1.24])**.** *Let $w \in W$ be any element of an affine Coxeter group with translation elliptic factorisation $w = t_\lambda u$. Then the following are equivalent.*

(a) w is elliptic

(b) $\mathrm{Fix}(w)$ *is non-empty;*

(c) w is of finite order;

(d) $\mathrm{Mov}(w) \subset V$ *is a linear subspace;*

(e) $\mathrm{Mov}(w)$ *contains the origin;*

(f) $\lambda \in \mathrm{Mov}(u)$*;*

(g) $\mathrm{Mov}(w) = \mathrm{Mov}(u)$ *and*

(h) $\mathrm{Mov}(w) = \mathrm{Mov}(w_e)$*.*

There is also a statement connecting the reflection length of an elliptic element with the roots involved in a reflection factorisation of w. Observe the resemblance with Theorem 2.23 in the spherical setting.

Proposition 3.18 (Minimum elliptic factorisations)**.** *Let W be an affine Coxeter group and $w = r_1 \cdots r_k$ a product of reflections. Let H_i be the affine hyperplane about which r_i reflects and assume α_i is the root orthogonal to H_i.*

If w is elliptic and $\ell_R(w) = k$ then the roots $\alpha_1, \ldots, \alpha_k$ are linearly independent. Conversely, if the roots $\alpha_1, \ldots, \alpha_k$ are linearly independent then w is elliptic, $\ell_R(w) = k$, $\mathrm{Fix}(w) = H_1 \cap \cdots \cap H_k$ *and* $\mathrm{Mov}(w) = \langle \alpha_1, \ldots, \alpha_k \rangle_{\mathbb{R}}$.

The proof of this proposition uses [BM15, Lemmas 3.6 and 6.4] which states these facts in the more general setting of the full isometry group of the Euclidean space E generated by all possible reflections.

Remark 3.19 (Maximal elliptics). Consider an affine Coxeter group W acting cocompactly on an Euclidean space E. Let u be an elliptic element of reflection length $n = \dim(E)$ (such as a Coxeter element of a maximal parabolic subgroup of W). Then the move-set of u is all of V by Proposition 3.18. Thus for any translation $t_\lambda \in T$ the element $t_\lambda u$ is again elliptic by the equivalence of (a) and (f) in Proposition 3.17. Particularly, $\ell_R(t_\lambda u) = \ell_R(u) = n$ for every $t_\lambda \in T$.

3.3. Dimensions of an element

In this section the dimension of an element in an affine Coxeter group is introduced as a generalisation of Definition 2.27 in the spherical case. Thereafter we define the terms of elliptic and differential dimension.

Like in the spherical setting the dimension of an element is defined using move-sets. Here, we need the notion of a root space and root dimension as well.

Definition 3.20 (Root spaces)**.** Let V be a Euclidean vector space equipped with a root system Φ. A subset $U \subseteq V$ is called a *root space* if it is the span of the roots it contains, in symbols $U = \langle U \cap \Phi \rangle_{\mathbb{R}}$. The collection of all root spaces in V is called the *root space arrangement* denoted with $\mathrm{Arr}(\Phi)$.

Note that $U \subseteq V$ is a root space if and only if it is a linear subspace spanned by a set of roots or if U has a basis of roots. Therefore, $\mathrm{Arr}(\Phi)$ is finite since Φ is.

Definition 3.21 (Root dimension)**.** Assume V is a Euclidean vector space equipped with a root system Φ and let $A \subseteq V$ be an arbitrary subset. Define the *root dimension* $\dim_\Phi(A)$ of A as the minimal dimension of a root space containing A, that is,

$$\dim_\Phi(A) = \min\{\dim(U) \mid A \subseteq U \in \mathrm{Arr}(\Phi)\}.$$

For a vector $\lambda \in V$ let $\dim(\lambda) := \dim(\{\lambda\})$.

The root dimension is well-defined for any subset $A \subseteq V$ because V is a root space and thus the minimum is taken over a non-empty finite set.

Example 3.22 (Root spaces in dimension two)**.** Consider the Euclidean vector space $\mathbb{R}^2$ equipped with a root system Φ_A of type A_2 illustrated in Figure 21. Then any line $\langle \alpha \rangle_{\mathbb{R}}$

with $\alpha \in \Phi$ is a root space of root dimension 1. The line $\langle 2\alpha + \beta \rangle_{\mathbb{R}}$ illustrated in Blue is not a root space since $2\alpha + \beta \notin \Phi_A$ and therefore has root dimension two. An affine line A not containing the origin, for example the one indicated in Thistle in Figure 21, is not a linear subspace and thus never a root space. Hence it is of root dimension two as well.

Any vector λ distinct from the origin does not give a root space $\{\lambda\}$ since it is not a linear subspace. Its root dimension is 1 or 2 depending on whether it lies in a root line $\langle \alpha \rangle_{\mathbb{R}}$ with $\alpha \in \Phi$ or not.

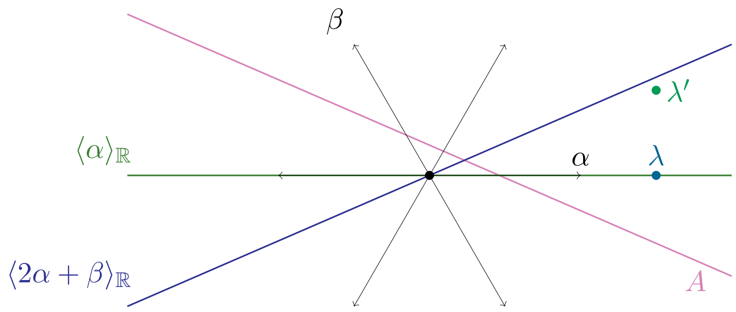


Figure 21: Root spaces and non-root spaces for the root system $\Phi_A \subset \mathbb{R}^2$, illustrating Example 3.22.

Let $w \in W$ be an element of an affine Coxeter group W. Then its move-set $\mathrm{Mov}(w)$ is contained in an Euclidean vector space V which is equipped with the corresponding root system Φ. Thus the root dimension of any move-set is defined.

Definition 3.23 (Dimension of an element)**.** Let W be an affine Coxeter group. The *dimension* of $w \in W$ is defined as the root dimension of its move-set, that is,

$$\dim(w) := \dim_{\Phi}(\mathrm{Mov}(w)).$$

Note that in the spherical setting the root dimension of a move-set coincides with its standard dimension because move-sets are root spaces as a consequence of Proposition 3.18. Thus Definition 2.27 is a special case of Definition 3.23.

Definition 3.24 (Elliptic and differential dimension)**.** Let W be an affine Coxeter group and p the projection onto W_0. For $w \in W$ call the dimension $e(w) := \dim(w_e)$ of its elliptic part $w_e = p(w) \in W_0$ the *elliptic dimension* of w.

Then define the *differential dimension* of $w \in W$ as $d(w) := \dim(w) - \dim(w_e)$.

Note that $\dim(w) = d(w) + e(w)$. With this the dimension of an element can be studied through its elliptic and differential dimensions. These encode some structure of the group. In particular, $w \in W$ is a translation if and only if $d(w) = 0$ and it is elliptic if and only if $e(w) = 0$) [LMPS19, Prop. 1.31]. Therefore it can be said, roughly speaking, that $d(w)$ measures how far w is from being a translation and $e(w)$ measures how far w is from being elliptic.

The reflection length of an element is closely related to its dimensions.

Proposition 3.25 (Inequalities [LMPS19, Prop. 1.33])**.** *Let W be an affine Coxeter group and $w \in W$. Then $\ell_R(w) \geq \dim(w) \geq e(w) = \dim(w_e)$.*

The proof of the latter proposition uses the following Lemma.

Lemma 3.26 (Separation [LMPS19, Lem. 1.32])**.** *Let M and U be linear subspaces of a vector space V and let $\lambda \in V$ be a vector. Then M contains $\lambda + U$ if and only if M contains both U and λ.*

For elliptic elements it can be said even more:

Proposition 3.27 (Elliptic equalities [LMPS19, Prop. 1.34])**.** *Let W be an affine Coxeter group and $w \in W$ elliptic. Then $\ell_R(w) = \dim(w) = \dim(w_e)$.*

Example 3.28 (Euclidean plane)**.** Consider affine Coxeter groups acting on the Euclidean plane. Then there are five different types of move-sets.

At the start consider the elliptic elements: The identity has the origin as move-set, for a reflection with root $\alpha \in \Phi$ the move-set is the root line $\langle \alpha \rangle_{\mathbb{R}}$ and the move-set of any non-trivial rotation is the whole plane $V = \mathbb{R}^2$. For these elements the differential dimension is 0 and the elliptic dimension and reflection length fulfil $\ell_R(w) = \dim(w) = \dim(w_e) = e(w)$ and thus are equal to 0, 1 or 2, respectively.

For a non-trivial translation t_λ the move-set consists of the single non-zero vector λ. Here, the elliptic dimension is 0 and the differential dimension is either 1 (when λ lies in a root line $\langle \alpha \rangle_{\mathbb{R}}$) or 2 (in any other case). In both cases the reflection length is twice the dimension [MP11, Prop. 4.3].

Finally, when w is a glide reflection it has reflection length 3 and its move-set is an affine line not through the origin. Hence it has elliptic dimension 1 and dimension 2. Therefore $d(w) = 2 - 1 = 1$.

An overview of these calculations is given in Table 1.

w	$\mathrm{Mov}(w)$	$d(w)$	$e(w)$	$\dim(w)$	$\ell_R(w)$
identity	the origin	0	0	0	0
reflection	a root line	0	1	1	1
rotation	the plane	0	2	2	2
translation	an affine point	1 or 2	0	1 or 2	2 or 4
glide reflection	an affine line	1	1	2	3

Table 1: Invariants of the five types of elements in affine Coxeter groups acting on the Euclidean plane.

To finish this subsection two remarks are given on how to compute dimensions in general. Let W be an affine Coxeter group and fix an identification $W = T \rtimes W_0$. Assume $w \in W$ is given in normal form $w = t_\lambda u$ with an elliptic element $u \in W_0$ and translation $t_\lambda \in T$ with a vector $\lambda \in L(\Phi^\vee)$ in the coroot lattice.

Computing the elliptic dimension $e(w)$ of w is straight forward.

Remark 3.29 (Computing elliptic dimension). By Definition 3.24 of the elliptic dimension $e(w)$ it suffices to compute the root dimension $\dim_\Phi$ of the move-set of its elliptic part $\mathrm{Mov}(w_e)$. Since $w = t_\lambda u$ is in normal form follows $u = p(w) = p(w) = w_e$ where we identify W_0 with $\iota(W_0)$. Then $e(w) = \dim(w_e) = \dim_\Phi(\mathrm{Mov}(w_e)) = \dim_\Phi(\mathrm{Mov}(u)) = \dim(\mathrm{Mov}(u)) = \ell_R(u)$.

Computing the differential dimension $d(w)$ of w is more complicated. However, it can be reduced to computing the dimension of a point in a simpler arrangement of subspaces in a lower dimensional space. How much lower depends on the value of $e(w)$.

Remark 3.30 (Computing differential dimension). By Definition 3.24 of the differential dimension $d(w)$ to compute it one needs to find the minimal dimension of a root space containing $\mathrm{Mov}(w)$ and then subtract the elliptic dimension of this value.

As $w = t_\lambda u$ it is $\mathrm{Mov}(w) = \lambda + U$ with $U := \mathrm{Mov}(u) = \mathrm{Mov}(w_e)$. By Lemma 3.26 it suffices to consider root spaces containing both λ and U or equivalently, both λ and $\lambda + U$. Thus consider the natural quotient $q : V \twoheadrightarrow V/U$ which is a linear transformation with kernel U. Under this quotient q the coset $\lambda + U$ is sent to a point in V/U denoted with λ/U and the root spaces in $\mathrm{Arr}(\Phi)$ containing U are sent to subspaces in V/U we denote with $\mathrm{Arr}(\Phi/U)$.

Furthermore, let $\dim_{\Phi/U}(\lambda/U)$ denote the minimal dimension of a subspace in $\mathrm{Arr}(\Phi/U)$ containing the point λ/U. By going via q from V to V/U all dimensions have been reduced by $\dim(U) = e(w)$. Thus $\dim_{\Phi/U}(\lambda/U) = d(w)$ is the wanted differential dimension of w.

3.4. Reflection length in affine Coxeter groups

This section reviews some results for reflection length in affine Coxeter groups.

First, observe that the reflection length in these groups is bounded and this bound is sharp.

Theorem 3.31 (Optimal upper bound [MP11, Thm. B]). *Let W be an affine Coxeter group of rank n. Then every element of W has reflection length at most $2n$ and there are elements in W with reflection length equal to $2n$.*

The elements of length $2n$ mentioned above are translations of dimension n [MP11, Prop. 4.3]. For example the translation $t_{3\alpha^\vee + \beta^\vee}$ in a Coxeter group of type affine B_2 with α, $\beta \in \Phi$ has reflection length 4.

There is also a recent result to compute the reflection length of an element with the means of its elliptic and differential dimensions.

Theorem 3.32 (Computing reflection length [LMPS19, Thm. A]). *Let W be an affine Coxeter group and let $p : W \twoheadrightarrow W_0$ be the projection onto its associated spherical Coxeter group. Then the reflection length of $w \in W$ is*

$$\ell_R(w) = 2 \cdot \dim(w) - \dim(p(w)) = 2d(w) + e(w).$$

In the same paper the authors show that for each element $w \in W$ exist a translation-elliptic factorisation of w such that the reflection length of w can be split as the sum of the reflection lengths of its factors:

Theorem 3.33 (Factorisation [LMPS19, Thm. B]). *Let W be an affine Coxeter group and $w \in W$. Then a translation-elliptic factorisation $w = t_\lambda u$ exists such that $\ell_R(t_\lambda) = 2d(w)$ and $\ell_R(u) = e(w)$. In particular, $\ell_R(w) = \ell_R(t_\lambda) + \ell_R(u)$ for this specific factorisation of w.*

Note that such a translation-elliptic factorisation as in Theorem 3.33 does not need to be a normal form. In general, there are elements that do not allow a normal form with these properties.

Example 3.34 (Normal forms are insufficient – Revisiting Example 3.16). Consider the affine Coxeter group W of type affine B_2. Let r and s be the two reflections indicated in Figure 22. Then $w = rs$ is translation-elliptic factorisation as in Theorem 3.33, that is, $\ell_R(t_0) = 0 = 2d(w)$ and $\ell_R(rs) = 2 = e(w)$. But rs is not a normal form for w (compare Example 3.16).

Consider the structure of W in more detail. A standard generating set of W consist of the elements $r_{\alpha,0}$, $r_{\beta,0}$ and $r_{2\alpha+\beta,1}$ which are also indicated in Figure 22. Hence $r_{\beta,0}$ and $r_{2\alpha+\beta,1}$ generate a maximal standard parabolic subgroup of W fixing y which is not isomorphic to W_0. Therefore the subgroup generated by r and s is a parabolic subgroup of W since it is conjugate to this standard parabolic subgroup. It is not isomorphic to W_0 as well. Since it is generated by r and s the rotation rs is a Coxeter element in this parabolic subgroup.

Observe that every Coxeter element in such an alternative maximal parabolic subgroup of W that is not isomorphic to W_0 does not allow a normal form with the properties in Theorem 3.33.

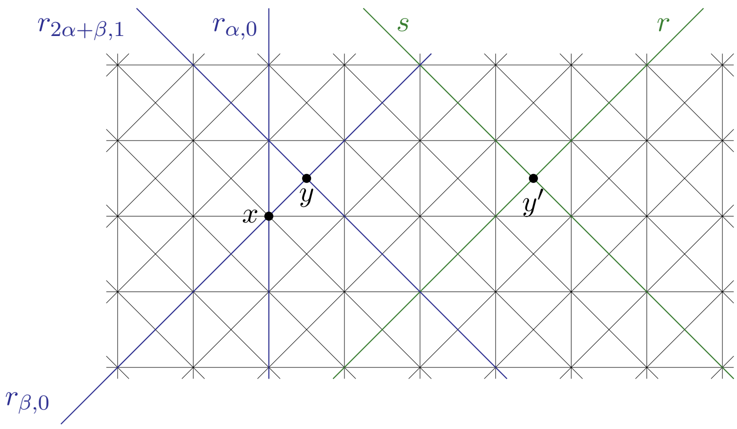


Figure 22: The reflections s, r generate a parabolic subgroup that is conjugate to the standard parabolic subgroup generated by $r_{\beta,0}, r_{2\alpha+\beta,1}$.

The scenario from the last Example 3.34 can appear in any affine Coxeter group allowing a maximal parabolic subgroup that is not isomorphic to W_0. Thus, in general, Theorem 3.33 does not contribute to Conjecture A in a straight forward way.

In type affine A_n there are no maximal parabolic subgroups not isomorphic to W_0.

Corollary 3.35 (Affine symmetric normal form [LMPS19, Cor. 2.5])**.** *Let W be an affine symmetric group. For a given element $w \in W$ there exists an identification of W as a semidirect product $W = T \rtimes W_0$ such that w has normal form $w = t_\lambda u$ and $\ell_R(w) = \ell_R(t_\lambda) + \ell_R(u)$.*

This corollary might contribute to a proof of Conjecture A for type affine A_n.

4. Proof of main statements

This section provides the main results of this work, split up into four subsections.

The first three subsections combined prove Theorem B by considering all types of irreducible Coxeter groups with rank smaller or equal to two. The first subsection gives a proof for affine Coxeter groups of type affine A_1 and A_2, the second for type affine B_2 and the third for type affine G_2. The last subsection provides a proof for Theorem C, that is, for one inequality of the formula in groups of arbitrary rank.

Start with a formal definition of the set V_λ. Recall that the dimension of a vector $\lambda \in V$ is the root dimension of $\{\lambda\}$ (compare Definition 3.21).

Definition 4.1 (The set V_λ)**.** Let W_0 be a spherical Coxeter group over the root system Φ. Let λ be a vector in the coroot lattice $L(\Phi^\vee)$ of dimension k. If λ can be written as an integral combination of the coroots $\alpha_1^\vee, \ldots, \alpha_k^\vee \in \Phi^\vee$, that is,

$$\lambda = \sum_{i=1}^{k} c_i \alpha_i^\vee \quad \text{where all } c_i \in \mathbb{Z}_{\neq 0}.$$

then denote with $v_\lambda = s_{\alpha_1} \cdots s_{\alpha_k}$ the composition of all reflections associated to $\alpha_1, \ldots, \alpha_k$. The collection of all v_λ obtained in this way is denoted with V_λ.

That is, if λ is of dimension 1 the set V_λ consists of one reflection only. For $\dim(\lambda) = 2$ it is a subset of all rotations $rs \in W$ with reflections $r, s \in R$. Note also that V_λ is closed under inverses.

The next observation is a short lemma for all elements whose normal form has trivial translation part.

Lemma 4.2 (Trivial translation parts)**.** *Let $W = T \rtimes W_0$ be an affine Coxeter group and $w \in W$ with normal form $w = t_\lambda u$. If the translation part t_λ is trivial one can write the reflection length of w as*

$$\ell_R(w) = \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

Proof. If $t_\lambda = \mathbb{1}$ then $\dim(\lambda) = 0$ and $V_\lambda = \{\mathbb{1}\}$. Also $w = u$ thus $\ell_R(w) = 0 + \ell_R(\mathbb{1}u)$. $\square$

4.1. Proof for types affine A_1 and affine A_2

Here, a proof for the formula is given in affine Coxeter groups of type affine A_1 and A_2. Beginning with type affine A_1, in other words, W is an infinite dihedral group.

Proposition 4.3 (Type affine A_1)**.** *Let W be an affine Coxeter group of type affine A_1. Assume W is identified as $W = T \rtimes W_0$ with*

$$T \cong \mathbb{R} \quad \textit{and} \quad W_0 \cong \langle s | s^2 \rangle.$$

Then the reflection length of $w \in W$ with normal form $w = t_\lambda u$ can be written as

$$\ell_R(w) = \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

Proof. For $\lambda = 0$ the statement follows directly from Lemma 4.2.

Thus let $\lambda \neq 0$. Then $\lambda = c\alpha^\vee$ for some $c \in \mathbb{Z} \setminus \{0\}$ hence $V_\lambda = \{s_\alpha\}$ and $t_\lambda = s_{\alpha,n}s_{\alpha,m}$ for some $n, m \in \mathbb{Z}$. If $u = s$ then $w = s_{\alpha,n}s_{\alpha,m}s_{\alpha,0}$ is a reflection and therefore $\ell_R(x) = 1 = 1 + 0$. If $u = \mathbb{1}$ then w is a translation and therefore $\ell_R(x) = 2 = 1 + 1$. $\square$

Proposition 4.4 (Type affine A_2)**.** *Let W be an affine Coxeter group of type affine A_2 with $\Phi_A^\vee \subset \mathbb{R}^2$ the coroot system in Figure 23. Denote $s = s_\alpha, t = s_\beta$ and assume W is identified as $W = T \rtimes W_0$ with*

$$T \cong \mathbb{R}^2 \quad \text{and} \quad W_0 \cong \langle s, t \mid s^2, t^2, (st)^3 \rangle.$$

Then the reflection length of $w \in W$ with normal form $w = t_\lambda u$ can be written as

$$\ell_R(w) = \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

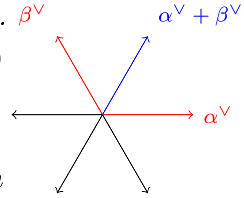


Figure 23: $\Phi_A^\vee$.

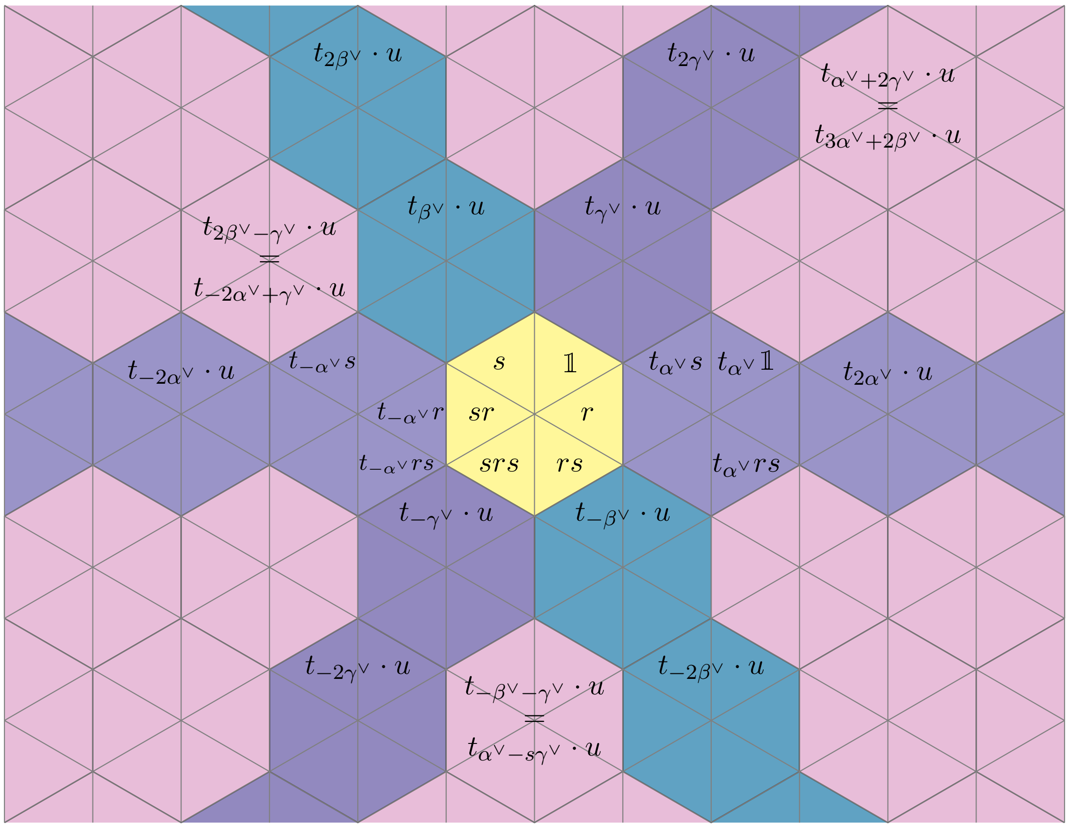


Figure 24: The action of W of type affine A_2 on the Euclidean plane.

Remark 4.5 (Geometric notes on type affine A_2). A group W of type affine A_2 acts on the Euclidean plane and tessellates it with equilateral triangles illustrated in Figure 24. Those alcoves are in one-to-one correspondence with the elements of W.

Via the identification $W = T \rtimes W_0$ the elements of W_0 can be regarded as element of W, they are situated in the Yellow-coloured hexagon.

All other elements $w \in W$ are non-trivial translates $t_\lambda u$ of an element $u \in W_0$. The colour of w's alcove indicates which coroots $\alpha_i^\vee$ can be used to decompose l as $l = \sum_{i=1}^{\dim(\lambda)} \alpha_i^\vee c_i$ where all $c_i \in \mathbb{Z}_{\neq 0}$. Elements with Blue-, MidnightBlue- or BlueViolet-coloured alcoves are translates with λ of dimension 1, namely in directions $\alpha^\vee$, $\beta^\vee$ and $\gamma^\vee := \alpha^\vee + \beta^\vee$, respectively. Those elements with Thistle-coloured alcoves have translation part with $\dim(\lambda) = 2$ and thus λ can be expressed with any two distinct positive roots.

Proof. For $\dim(\lambda) = 0$ the formula holds by Lemma 4.2, thus only the cases $\dim(\lambda) = 1$ and $\dim(\lambda) = 2$ are considered.

First, let $\dim(\lambda) = 1$. Then λ lives in the one dimensional lattice $L(\{\pm\rho^\vee\})$ spanned by a positive co-root $\rho^\vee \in \Phi_A^{\vee,+}$. All co-roots in $\Phi_A^\vee$ have the same length and $\Phi_A^\vee$ is indecomposable. Thence all its co-roots are conjugate by Proposition 2.16. As reflection length is invariant under conjugation assume $\rho = \alpha$ without loss of generality.

Speaking in the geometric terms of Remark 4.5 and Figure 24, consider all elements in Blue-coloured hexagons. They are illustrated in more detail in Figure 25. The horizontally hatched alcoves correspond to translates of the identity in W_0. The vertically hatched regions mark translates of the identity element with respect to the parabolic subgroup of W_0 generated by s_α.

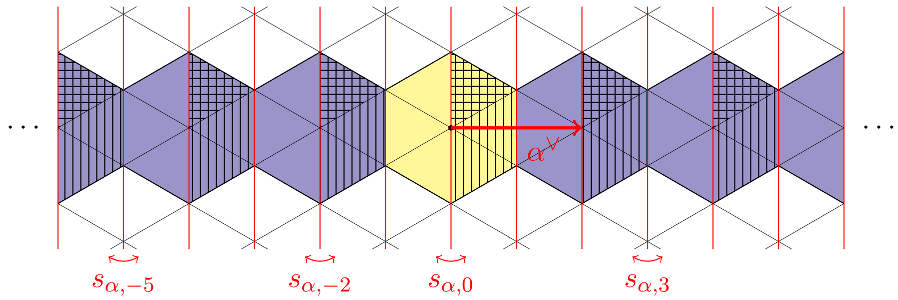
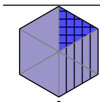
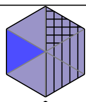
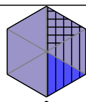
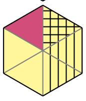
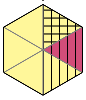
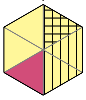
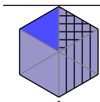
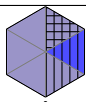
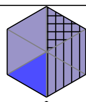
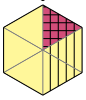
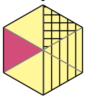
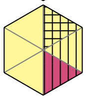


Figure 25: The translates of elements in W_0 in $\alpha^\vee$-direction.

For $\rho^\vee = \alpha^\vee$ we obtain $V_\lambda = \{s_\alpha\} = \{s\}$. Hence the formula simplifies to $\ell_R(w) = 1 + \ell_R(su)$ which is quickly verified for all $u \in W_0$ in the table below.

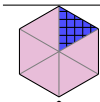
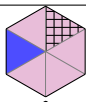
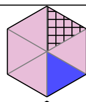
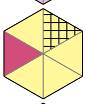
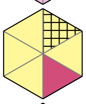
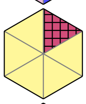
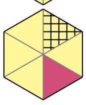
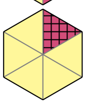
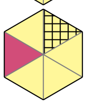
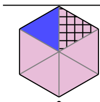
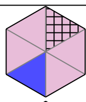
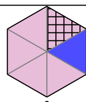
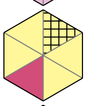
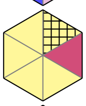
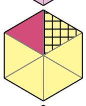
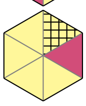
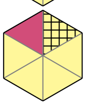
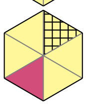
It contains a column for each element $u \in W_0$. The first entry of each column records the type of $w = t_\lambda u$, that is, whether it is a translation, rotation, reflection or glide reflection. Then the reflection length of $w = t_\lambda u$ is stated which is known from Table 1. The second entry considers the element $s \cdot u \in W_0$. The composition is computed and its reflection length is derived from Remark 2.24. In the small images the alcoves of w and $s \cdot u$ are highlighted. The background colour of the displayed hexagon is the same as in Figure 24 therefore the background of $s \cdot u \in W_0$ is Yellow-coloured. The hatched areas denote the same as in Figure 25.

$u = \mathbb{1}$	$u = st$	$u = ts$
 translation $\ell_R(w) = 2$	 rotation $\ell_R(w) = 2$	 rotation $\ell_R(w) = 2$
 $s \cdot u = s$ $\ell_R(s) = 1$	 $s \cdot u = t$ $\ell_R(t) = 1$	 $s \cdot u = sts$ $\ell_R(sts) = 1$
$u = s$	$u = t$	$u = sts$
 reflection $\ell_R(w) = 1$	 glide reflection $\ell_R(w) = 3$	 glide reflection $\ell_R(w) = 3$
 $s \cdot u = \mathbb{1}$ $\ell_R(\mathbb{1}) = 0$	 $s \cdot u = st$ $\ell_R(st) = 2$	 $s \cdot u = ts$ $\ell_R(ts) = 2$

Let $\dim(\lambda) = 2$. Thus λ lies in the span of two positive coroots $\rho^\vee, \sigma^\vee \in \Phi_A^{\vee,+}$. Those translates are the Thistle-coloured ones displayed in Figure 24.

Again, all coroots are conjugate. Thus, for a given λ, the coroots $\rho^\vee$ and $\sigma^\vee$ could be any pair of distinct elements in $\Phi_A^{\vee,+}$. For the pair $\alpha^\vee, \beta^\vee$ follows $s_\alpha s_\beta = st \in V_\lambda$ and the same for its inverse ts. Since those are all elements of W_0 with reflection length two (W_0 contains only two rotations) it is $V_\lambda = \{st, ts\}$. Hence it suffices to show $\ell_R(w) = 2 + \min\{\ell_R(tsu), \ell_R(stu)\}$.

This is true for all $u \in W_0$, as shown in the table below. It is arranged as the table in the one dimensional case, just that each column has three entries: one for $w = t_\lambda u$ and one for each element in V_λ. As the rotations st and ts do not cancel out with both elements of V_λ the cancelling combination is highlighted in Blue.

$u = \mathbb{1}$	$u = st$	$u = ts$
 translation $\ell_R(w) = 4$	 rotation $\ell_R(w) = 2$	 rotation $\ell_R(w) = 2$
 $st \cdot u = st$ $\ell_R(st) = 2$	 $st \cdot u = ts$ $\ell_R(ts) = 2$	 $st \cdot u = \mathbb{1}$ $\ell_R(\mathbb{1}) = 0$
 $ts \cdot u = ts$ $\ell_R(ts) = 2$	 $ts \cdot u = \mathbb{1}$ $\ell_R(\mathbb{1}) = 0$	 $ts \cdot u = st$ $\ell_R(st) = 2$
$u = s$	$u = sts$	$u = t$
 glide reflection $\ell_R(w) = 3$	 glide reflection $\ell_R(w) = 3$	 glide reflection $\ell_R(w) = 3$
 $st \cdot u = sts$ $\ell_R(sts) = 1$	 $st \cdot u = t$ $\ell_R(t) = 1$	 $st \cdot u = s$ $\ell_R(s) = 1$
 $ts \cdot u = t$ $\ell_R(t) = 1$	 $ts \cdot u = s$ $\ell_R(s) = 1$	 $ts \cdot u = sts$ $\ell_R(sts) = 1$

$\square$

4.2. Proof for type affine B_2

This subsection gives a proof of the conjecture in type affine B_2 and considers the action of W of this type on the Euclidean plane.

Proposition 4.6 (Type affine B_2)**.** *Let W be an affine Coxeter group of type affine B_2 with $\Phi_B^\vee \subset \mathbb{R}^2$ the coroot system in Figure 26. Denote $s = s_\alpha, t = s_\beta$ and assume W is identified as $W = T \rtimes W_0$ with*

$$T \cong \mathbb{R}^2 \quad \textit{and} \quad W_0 \cong \langle s, t \mid s^2, t^2, (st)^4 \rangle.$$

Then the reflection length of $w \in W$ with normal form $w = t_\lambda u$ can be written as

$$\ell_R(w) = \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

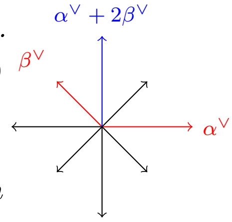


Figure 26: $\Phi_B^\vee$.

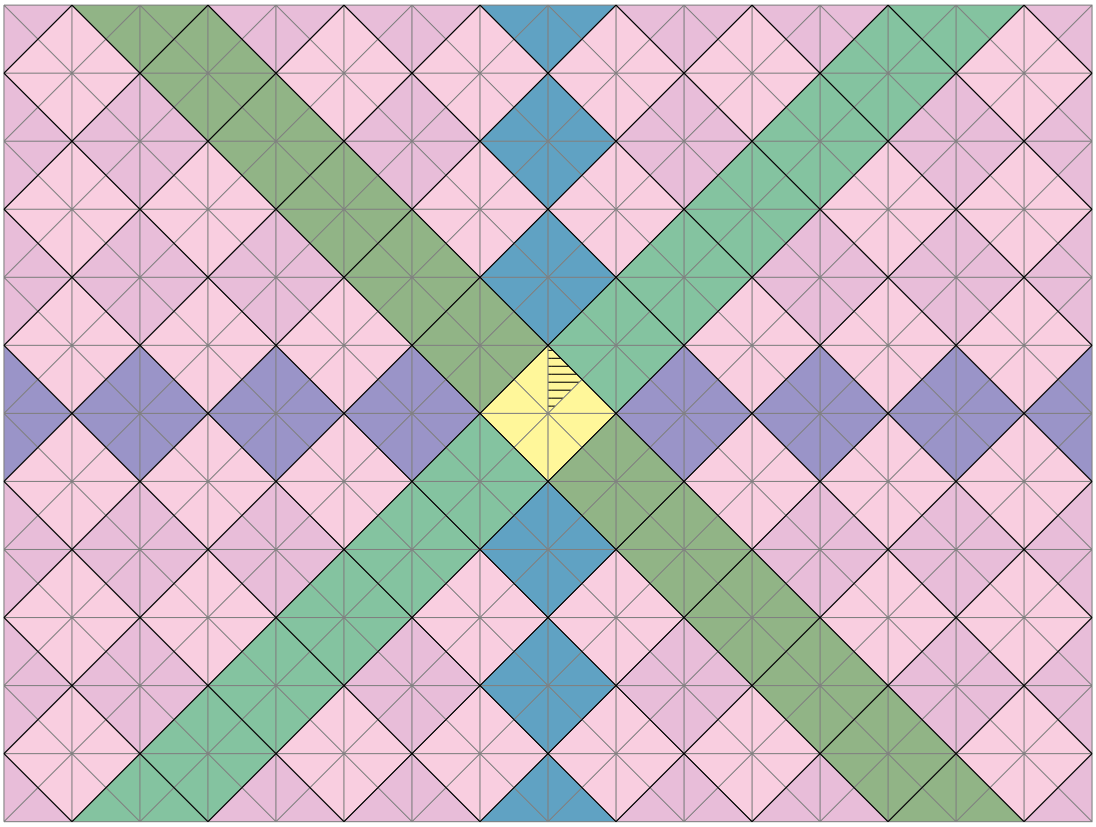


Figure 27: The action of W of type affine B_2 on the Euclidean plane.

Remark 4.7 (Geometric notes on type affine B_2). The action of W of type affine B_2 on the Euclidean plane tessellates it with triangles as shown in Figure 27. Those alcoves are not equilateral triangles like for type affine A_2 but still they are in bijection with the elements of W. Chose the horizontally hatched alcove as fundamental alcove. Thus this triangle corresponds to the identity in W.

Again, the elements of W_0 are considered as elements of W by the identification $W = T \rtimes W_0$ and their alcoves are highlighted in Yellow.

Unlike the root system of type A_2, $\Phi_B^\vee$ has roots of different length. Translates of elements from W_0 in the direction of a long coroot $\alpha^\vee$ or $\alpha^\vee + \beta^\vee$ are coloured in Blue and MidnightBlue, respectively. Conversely, translates by multiples of a short coroots $\beta^\vee$ or $2\alpha^\vee + \beta^\vee$ are the OliveGreen- or ForestGreen-coloured alcoves.

Consider the translates of an element in W_0 by a vector λ of dimension two. If λ can be expressed using any two distinct positive coroots in $\Phi_B^{\vee,+}$ it is coloured in Lavender. Otherwise λ can only be written with specific pairs of two distinct positive coroots in $\Phi_B^\vee$. For example not with the pair consisting of both long coroots $\alpha^\vee$ and $\alpha^\vee + 2\beta^\vee$. Those element's alcoves are Thistle.

Proof. Again, the case $\dim(\lambda) = 0$ is tackled by Lemma 4.2 and only the cases $\dim(\lambda) = 1$ and $\dim(\lambda) = 2$ are considered here.

Start with $\dim(\lambda) = 1$. Then λ lies in the one dimensional lattice $L(\{\pm\rho^\vee\})$ spanned by a positive co-root $\rho^\vee \in \Phi_B^{\vee,+}$. Here, $\rho^\vee$ can be a short or a long co-root. By applying Proposition 2.16 $\rho^\vee$ is conjugate to either the long coroot $\alpha^\vee$ or the short coroot $\beta^\vee$. With the invariance of reflection length under conjugation it thence suffices to consider the cases $\rho^\vee = \alpha^\vee$ and $\rho^\vee = \beta^\vee$, without loss of generality.

These are precisely those elements whose alcoves are highlighted with Blue or Olive-Green in Figure 27. For $\rho^\vee = \alpha^\vee$ they are illustrated in Figure 28 with more details. The horizontally hatched alcoves denote the identity in W_0 or translates of it. The areas vertically hatched are translates of the identity in the parabolic subgroup of W_0 generated by s_α.

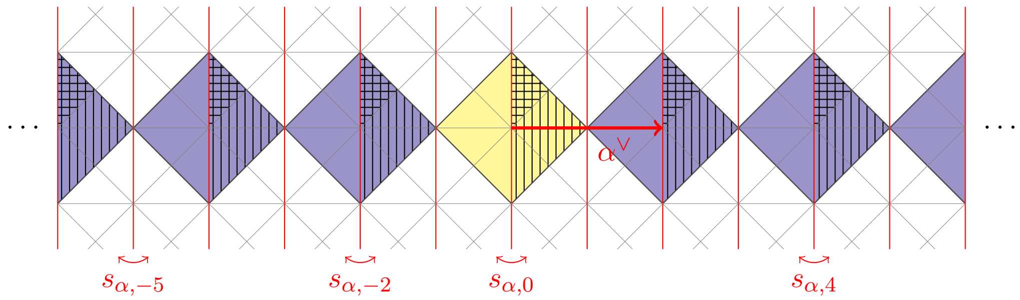
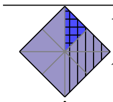
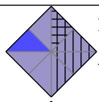
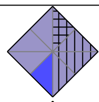
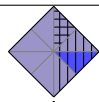
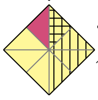
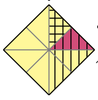
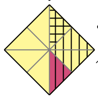
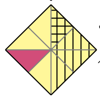
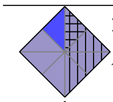
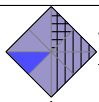
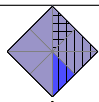
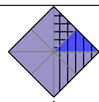
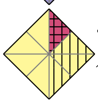
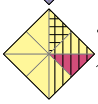
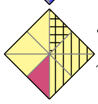
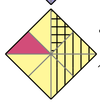


Figure 28: The translates of $u \in W_0$ in $\alpha-$direction.

With $\rho^\vee = \alpha^\vee$ it is $V_\lambda = \{s_\alpha\} = \{s\}$ and for $\rho^\vee = \beta^\vee$ we have $V_\lambda = \{s_\beta\} = \{t\}$. Thus the latter case can be deduced from the first by exchanging s and t below as they generate W_0. That is, assuming $\rho^\vee = \alpha^\vee$ reduces the statement to $\ell_R(w) = 1 + \ell_R(su)$. This holds for all $u \in W_0$ as checked in the table below. Its scheme is the same as for type affine A_2 and the reflection lengths are again determined with Table 1 and Remark 2.24.

$u = \mathbb{1}$	$u = st$	$u = stst$	$u = ts$
 translation $\ell_R(w) = 2$	 rotation $\ell_R(w) = 2$	 rotation $\ell_R(w) = 2$	 rotation $\ell_R(w) = 2$
 $s \cdot u = s$ $\ell_R(s) = 1$	 $s \cdot u = t$ $\ell_R(t) = 1$	 $s \cdot u = tst$ $\ell_R(t) = 1$	 $s \cdot u = sts$ $\ell_R(sts) = 1$

$u = s$	$u = sts$	$u = tst$	$u = t$
 reflection $\ell_R(w) = 1$	 glide refl. $\ell_R(w) = 3$	 glide refl. $\ell_R(w) = 3$	 glide refl. $\ell_R(w) = 3$
 $s \cdot u = \mathbb{1}$ $\ell_R(\mathbb{1}) = 0$	 $s \cdot u = ts$ $\ell_R(ts) = 2$	 $s \cdot u = stst$ $\ell_R(stst) = 2$	 $s \cdot u = st$ $\ell_R(st) = 2$

Let λ be of dimension two.

Then either it can be written with any two distinct positive coroots and is illustrated Lavender-coloured in Figure 29. Or some combinations of two distinct positive coroots are not possible, then it is coloured in Thistle.

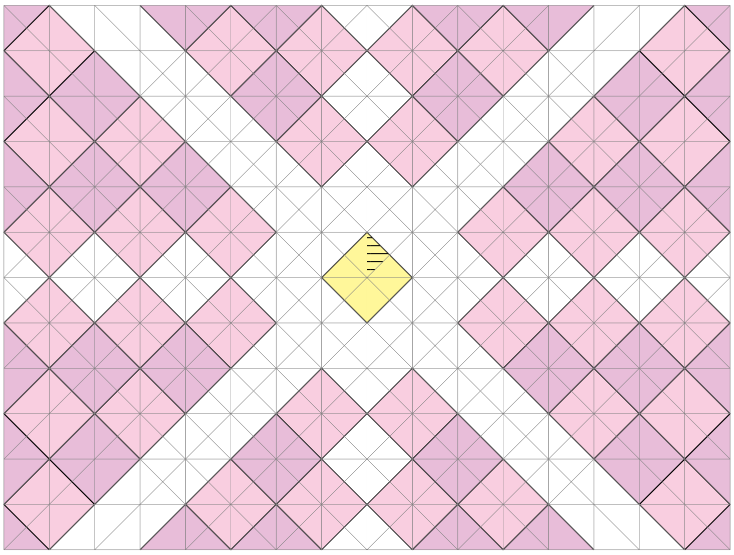
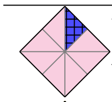
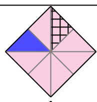
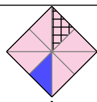
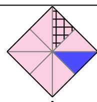
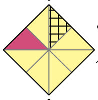
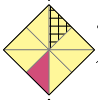
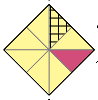
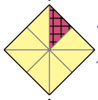
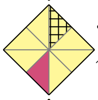
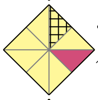
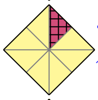
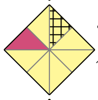
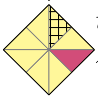
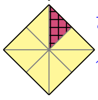
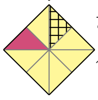
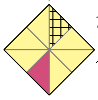
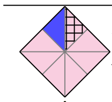
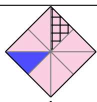
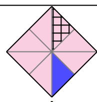
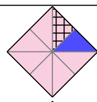
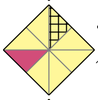
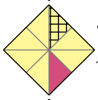
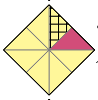
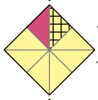
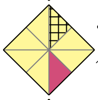
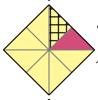
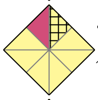
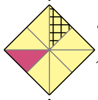
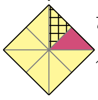
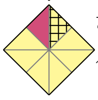
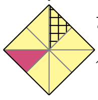
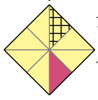


Figure 29: Translates with λ of dimension two.

Since $\alpha^\vee$ and $\beta^\vee$ are a base of $\Phi_B^{\vee,+}$ one can integrally combine λ with these two. That is, $s_\alpha, s_\beta = st \in V_\lambda$ and so does its inverse ts. Further, as the short coroots $\beta^\vee$ and $\alpha^\vee + \beta^\vee$ generate $\Phi_B^{\vee,+}$ by Proposition 2.16 any λ can be expressed as an integral combination of these two coroots. Therefore $s_{\beta^\vee} s_{\alpha^\vee + 2\beta^\vee} = t \cdot sts \in V_\lambda$ which is self-inverse.

There are only three rotations in W_0, in other words, elements of order two. Thus $V_\lambda = \{st, stst, ts\}$ and the equation simplifies to $\ell_R(w) = 2 + \min_{v \in \{sr, rs, srsr\}} \ell_R(v \cdot u)$. This is checked below for all $u \in W_0$.

$u = \mathbb{1}$	$u = st$	$u = stst$	$u = ts$
 translation $\ell_R(w) = 4$	 rotation $\ell_R(w) = 2$	 rotation $\ell_R(w) = 2$	 rotation $\ell_R(w) = 2$
 $stu = st$ $\ell_R(st) = 2$	 $stu = stst$ $\ell_R(stst) = 2$	 $stu = ts$ $\ell_R(ts) = 2$	 $stu = \mathbb{1}$ $\ell_R(\mathbb{1}) = 0$
 $ststu = stst$ $\ell_R(stst) = 2$	 $ststu = ts$ $\ell_R(ts) = 2$	 $ststu = \mathbb{1}$ $\ell_R(\mathbb{1}) = 0$	 $ststu = st$ $\ell_R(st) = 2$
 $tsu = ts$ $\ell_R(ts) = 2$	 $tsu = \mathbb{1}$ $\ell_R(\mathbb{1}) = 0$	 $tsu = st$ $\ell_R(st) = 2$	 $tsu = stst$ $\ell_R(stst) = 2$

$u = s$	$u = sts$	$u = tst$	$u = t$
 glide refl. $\ell_R(w) = 3$	 glide refl. $\ell_R(w) = 3$	 glide refl. $\ell_R(w) = 3$	 glide refl. $\ell_R(w) = 3$
 $stu = sts$ $\ell_R(sts) = 1$	 $stu = tst$ $\ell_R(tst) = 1$	 $stu = t$ $\ell_R(t) = 1$	 $stu = s$ $\ell_R(s) = 1$
 $ststu = tst$ $\ell_R(tst) = 1$	 $ststu = t$ $\ell_R(t) = 1$	 $ststu = s$ $\ell_R(s) = 1$	 $ststu = sts$ $\ell_R(sts) = 1$
 $tsu = t$ $\ell_R(t) = 1$	 $tsu = s$ $\ell_R(s) = 1$	 $tsu = sts$ $\ell_R(sts) = 1$	 $tsu = tst$ $\ell_R(tst) = 1$

The table is structured as before using Table 1 and Remark 2.24 to determine the reflection lengths. Like in the case of type affine A_2 not all elements of V_λ cancel out with all rotations and the cases giving the value of the minimum are highlighted in Blue. $\square$

4.3. Proof for type affine G_2

In this subsection the action of an affine Coxeter group of type affine G_2 on the Euclidean plane is considered. Also, it provides a proof for the formula for this type.

Proposition 4.8 (Type affine G_2)**.** *Let W be an affine Coxeter group of type affine G_2 with $\Phi_G^\vee \subset \mathbb{R}^2$ the coroot system in Figure 30. Denote $s = s_\alpha, t = s_\beta$ and assume W is identified as $W = T \rtimes W_0$ with*

$$T \cong \mathbb{R}^2 \quad \textit{and} \quad W_0 \cong \langle s, t \mid s^2, t^2, (st)^6 \rangle.$$

Then the reflection length of $w \in W$ with normal form $w = t_\lambda u$ can be written as

$$\ell_R(w) = \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

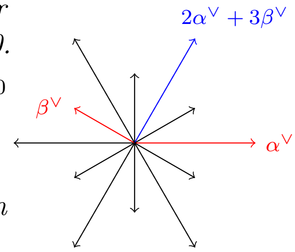


Figure 30: $\Phi_B^\vee$.

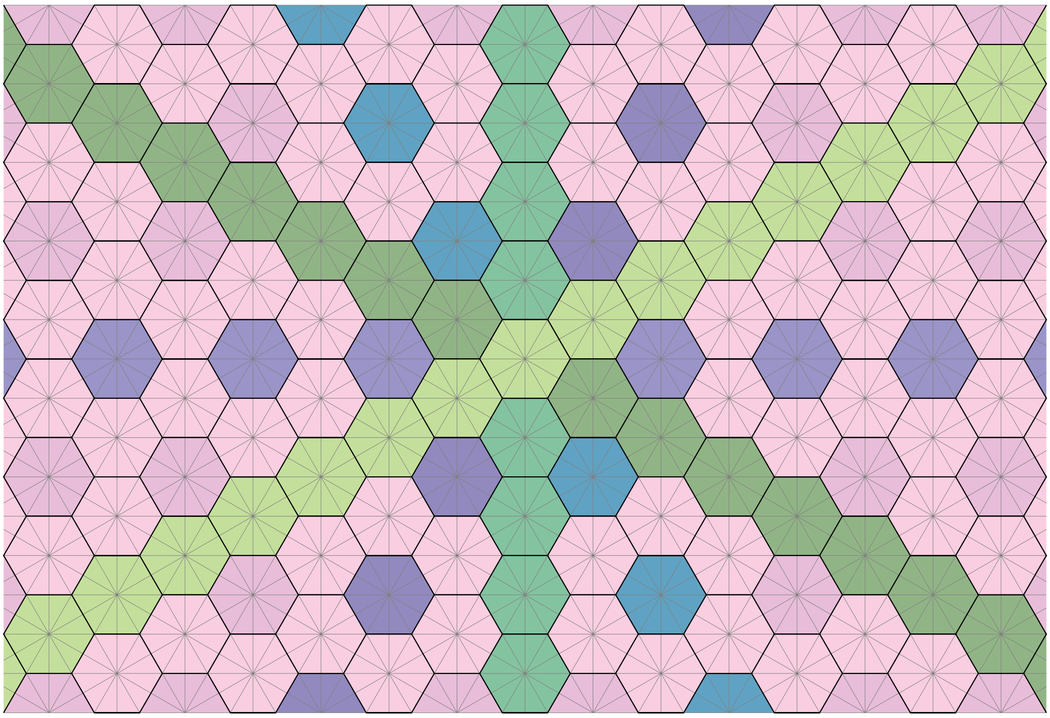


Figure 31: The action of W of type affine A_2 on the Euclidean plane.

Remark 4.9 (Geometric notes on type affine G_2). A group W of type affine G_2 acts on the Euclidean plane and tessellates it with triangles. This tessellation is illustrated in Figure 31. Again the triangles are alcoves being in bijection with the elements of $W = T \rtimes W_0$ and coloured depending on λ in their normal form $t_\lambda u$. Chose the horizontally hatched alcove as fundamental alcove being in correspondence with the identity in W.

If $\lambda = 0$ then w's alcove is among the Yellow-coloured triangles. For λ of dimension one we again distinguish between translations along short and long coroots. Hence those elements with λ in the span of a short coroot are situated in the 'greenish rays' coloured in OliveGreen, ForestGreen and LimeGreen, Respectively, if λ is in the root lattice of a long coroot w's alcove can be found in the 'dotted blueish rays' coloured in Blue, MidnightBlue and BlueViolet, respectively

As in type affine B_2 elements with λ of dimension two are coloured either in Lavender or Thistle.

Proof. Consider only $\dim(\lambda) \neq 0$ since again the case $\dim(\lambda) = 0$ is tackled by Lemma 4.2.

Begin with the case $\dim(\lambda) = 1$. Then λ is an integral multiple of a positive co-root $\rho^\vee \in \Phi^{\vee,+}$. Analogue to this case in type affine B_2 only consider the case $\rho^\vee = \alpha^\vee$, without loss of generality.

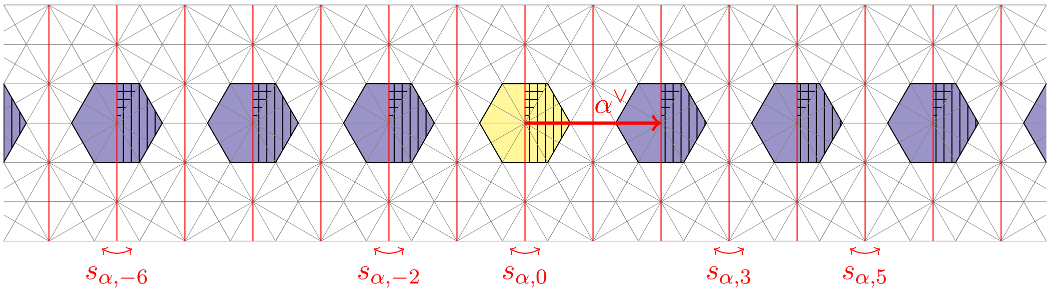
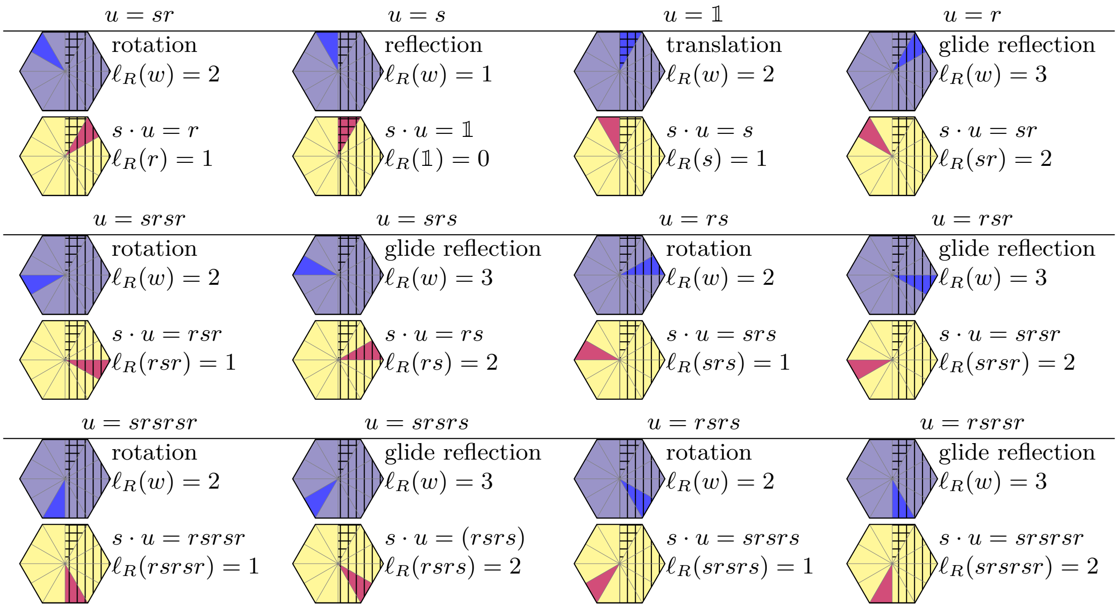


Figure 32: The translates of elements in W_0 in $\alpha^\vee$-direction.

In this case $V_\lambda = \{s_\alpha\} = \{s\}$ and it suffices to show $\ell_R(w) = 1 + \ell_R(su)$. This is fulfilled for all $u \in W_0$ as shown below. The table is organised as in type affine A_2 and the reflection lengths are obtained with Table 1 and Remark 2.24.



$u = sr$	$u = s$	$u = \mathbb{1}$	$u = r$
rotation $\ell_R(w) = 2$	reflection $\ell_R(w) = 1$	translation $\ell_R(w) = 2$	glide reflection $\ell_R(w) = 3$
$s \cdot u = r$ $\ell_R(r) = 1$	$s \cdot u = \mathbb{1}$ $\ell_R(\mathbb{1}) = 0$	$s \cdot u = s$ $\ell_R(s) = 1$	$s \cdot u = sr$ $\ell_R(sr) = 2$
$u = srsr$	$u = srs$	$u = rs$	$u = rsr$
rotation $\ell_R(w) = 2$	glide reflection $\ell_R(w) = 3$	rotation $\ell_R(w) = 2$	glide reflection $\ell_R(w) = 3$
$s \cdot u = rsr$ $\ell_R(rsr) = 1$	$s \cdot u = rs$ $\ell_R(rs) = 2$	$s \cdot u = srs$ $\ell_R(srs) = 1$	$s \cdot u = srsr$ $\ell_R(srsr) = 2$
$u = srsrsr$	$u = srsrs$	$u = rsrs$	$u = rsrsr$
rotation $\ell_R(w) = 2$	glide reflection $\ell_R(w) = 3$	rotation $\ell_R(w) = 2$	glide reflection $\ell_R(w) = 3$
$s \cdot u = rsrsr$ $\ell_R(rsrsr) = 1$	$s \cdot u = (rsrs)$ $\ell_R(rsrs) = 2$	$s \cdot u = srsrs$ $\ell_R(srsrs) = 1$	$s \cdot u = srsrsr$ $\ell_R(srsrsr) = 2$

Now, let λ have dimension two.

Then either λ can be written with any two distinct positive coroots and w's alcove is illustrated in Lavender in Figure 33. Or some combinations of two distinct positive coroots are not possible, then the alcove is indicated Thistle-coloured.

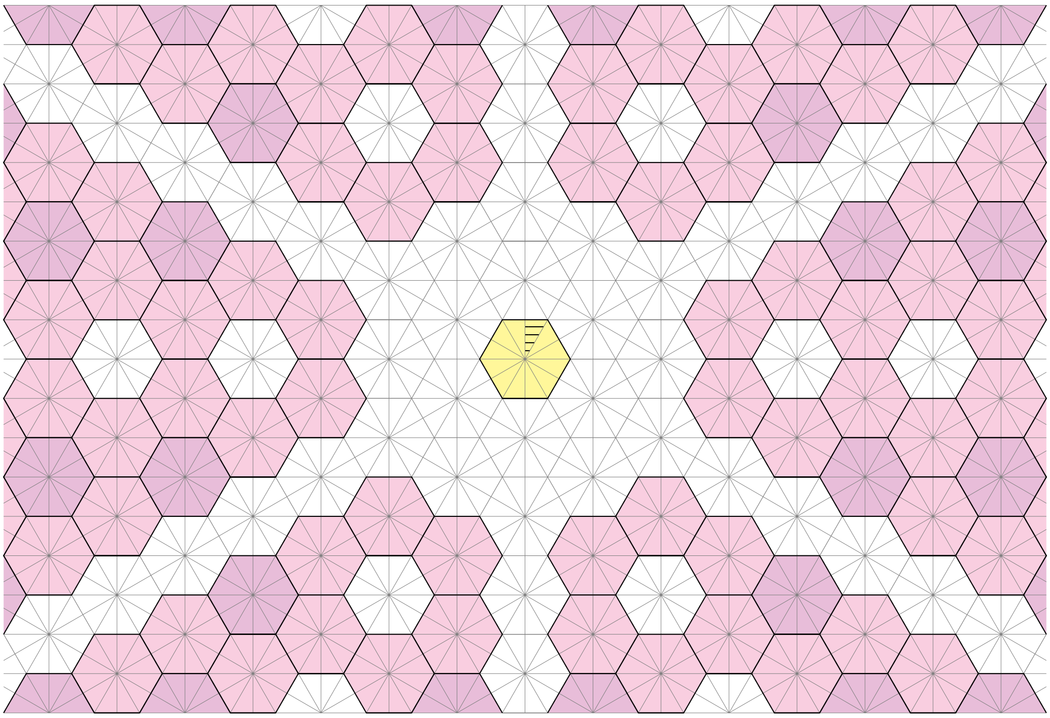


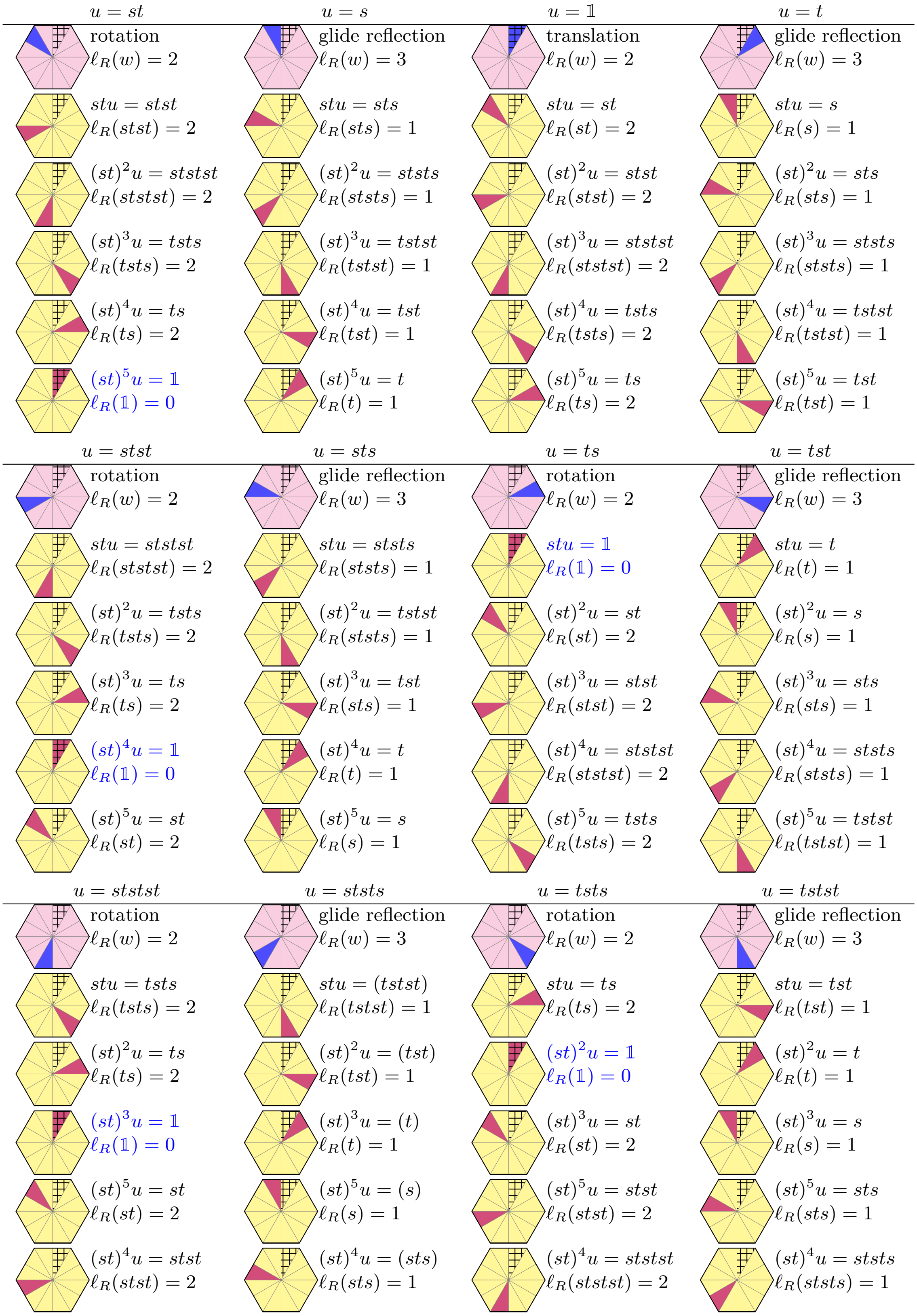
Figure 33: All translates with λ of dimension two.

To determine the set V_λ note that both $\alpha^\vee$ and $\beta^\vee$ generate $\Phi_G^\vee$ and so does any pair of distinct short coroots. The latter follows with the fact that any two short coroots generate a sub-root system of type A_2 and all these short coroots together generate Φ_G by Proposition 2.16. Thence λ is an integral combination of both $\alpha^\vee, \beta^\vee$ and any pair of positive short coroots in $\Phi_B^{\vee,+}$. The first gives $st, ts \in V_\lambda$ (the rotations about $\pm\pi/3$) and the latter $stst, tsts \in V_\lambda$ (the rotations about $\pm 2\pi/3$). Finally, $ststst \in V_\lambda$ since every translation can be written with two orthogonal coroots where one is short and the other long.

Thus $V_\lambda = \{st, stst, ststst, tsts, ts\}$ consists of all length-2 elements and the equation reads

$$\ell_R(w) = 2 + \min_{v \in \{st, stst, ststst, tsts, ts\}} \ell_R(v \cdot u)$$

This is true for all $u \in W_0$ as shown below. The table is structured as before using Table 1 and Remark 2.24 to compute the reflection lengths. The in Blue-highlighted cells indicate elements $v \in V_\lambda$ for which the minimum is taken.


$u = st$
rotation
$\ell_R(w) = 2$
$stu = stst$
$\ell_R(stst) = 2$
$(st)^2u = ststst$
$\ell_R(ststst) = 2$
$(st)^3u = tsts$
$\ell_R(tsts) = 2$
$(st)^4u = ts$
$\ell_R(ts) = 2$
$(st)^5u = \mathbb{1}$
$\ell_R(\mathbb{1}) = 0$
$u = s$
glide reflection
$\ell_R(w) = 3$
$stu = sts$
$\ell_R(sts) = 1$
$(st)^2u = ststs$
$\ell_R(ststs) = 1$
$(st)^3u = tstst$
$\ell_R(tstst) = 1$
$(st)^4u = tst$
$\ell_R(tst) = 1$
$(st)^5u = t$
$\ell_R(t) = 1$
$u = \mathbb{1}$
translation
$\ell_R(w) = 2$
$stu = st$
$\ell_R(st) = 2$
$(st)^2u = stst$
$\ell_R(stst) = 2$
$(st)^3u = ststst$
$\ell_R(ststst) = 2$
$(st)^4u = tsts$
$\ell_R(tsts) = 2$
$(st)^5u = ts$
$\ell_R(ts) = 2$
$u = t$
glide reflection
$\ell_R(w) = 3$
$stu = s$
$\ell_R(s) = 1$
$(st)^2u = sts$
$\ell_R(sts) = 1$
$(st)^3u = ststs$
$\ell_R(ststs) = 1$
$(st)^4u = tstst$
$\ell_R(tstst) = 1$
$(st)^5u = tst$
$\ell_R(tst) = 1$
$u = stst$
rotation
$\ell_R(w) = 2$
$stu = ststst$
$\ell_R(ststst) = 2$
$(st)^2u = tsts$
$\ell_R(tsts) = 2$
$(st)^3u = ts$
$\ell_R(ts) = 2$
$(st)^4u = \mathbb{1}$
$\ell_R(\mathbb{1}) = 0$
$(st)^5u = st$
$\ell_R(st) = 2$
$u = sts$
glide reflection
$\ell_R(w) = 3$
$stu = ststs$
$\ell_R(ststs) = 1$
$(st)^2u = tstst$
$\ell_R(ststs) = 1$
$(st)^3u = tst$
$\ell_R(sts) = 1$
$(st)^4u = t$
$\ell_R(t) = 1$
$(st)^5u = s$
$\ell_R(s) = 1$
$u = ts$
rotation
$\ell_R(w) = 2$
$stu = \mathbb{1}$
$\ell_R(\mathbb{1}) = 0$
$(st)^2u = st$
$\ell_R(st) = 2$
$(st)^3u = stst$
$\ell_R(stst) = 2$
$(st)^4u = ststst$
$\ell_R(ststst) = 2$
$(st)^5u = tsts$
$\ell_R(tsts) = 2$
$u = tst$
glide reflection
$\ell_R(w) = 3$
$stu = t$
$\ell_R(t) = 1$
$(st)^2u = s$
$\ell_R(s) = 1$
$(st)^3u = sts$
$\ell_R(sts) = 1$
$(st)^4u = ststs$
$\ell_R(ststs) = 1$
$(st)^5u = tstst$
$\ell_R(tstst) = 1$
$u = ststst$
rotation
$\ell_R(w) = 2$
$stu = tsts$
$\ell_R(tsts) = 2$
$(st)^2u = ts$
$\ell_R(ts) = 2$
$(st)^3u = \mathbb{1}$
$\ell_R(\mathbb{1}) = 0$
$(st)^5u = st$
$\ell_R(st) = 2$
$(st)^4u = stst$
$\ell_R(stst) = 2$
$u = ststs$
glide reflection
$\ell_R(w) = 3$
$stu = (tstst)$
$\ell_R(tstst) = 1$
$(st)^2u = (tst)$
$\ell_R(tst) = 1$
$(st)^3u = (t)$
$\ell_R(t) = 1$
$(st)^5u = (s)$
$\ell_R(s) = 1$
$(st)^4u = (sts)$
$\ell_R(sts) = 1$
$u = tsts$
rotation
$\ell_R(w) = 2$
$stu = ts$
$\ell_R(ts) = 2$
$(st)^2u = \mathbb{1}$
$\ell_R(\mathbb{1}) = 0$
$(st)^3u = st$
$\ell_R(st) = 2$
$(st)^5u = stst$
$\ell_R(stst) = 2$
$(st)^4u = ststst$
$\ell_R(ststst) = 2$
$u = tstst$
glide reflection
$\ell_R(w) = 3$
$stu = tst$
$\ell_R(tst) = 1$
$(st)^2u = t$
$\ell_R(t) = 1$
$(st)^3u = s$
$\ell_R(s) = 1$
$(st)^5u = sts$
$\ell_R(sts) = 1$
$(st)^4u = ststs$
$\ell_R(ststs) = 1$

4.4. Proof of the upper bound

This subsection gives a proof for one inequality of the formula in groups of arbitrary rank. We give the statement of Theorem C again in detail.

Theorem C (Upper bound)**.** *Let W be an affine Coxeter group of arbitrary rank. Assume W is identified as $W = T \rtimes W_0$ and let $w \in W$ with normal form $w = t_\lambda u$. Then the reflection length of w is bounded from above:*

$$\begin{aligned}\ell_R(w) &\leq \frac{1}{2}\ell_R(t_\lambda) + \min_{v \in V_\lambda} \ell_R(vu) \\ &= \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).\end{aligned}$$

Proof. Denote by k the dimension of λ. Then there exist linearly independent positive roots $\alpha_1, \ldots, \alpha_k$ and $c_1, \ldots, c_k \in \mathbb{Z}_{\neq 0}$ such that $\lambda = \sum_{i=1}^k c_i \alpha_i^\vee$. Write $t_\lambda = t^1 \cdots t^k$ with t^i a translation in α_i-direction.

Those translations t^i can be expressed as the product $t^i = s_i r_i$ of reflections $s_i := s_{\alpha_i} \in W_0$ and $r_i := r_{\alpha_i, c_i/2} \in W$. Thus $t_\lambda = s_1 r_1 s_2 r_2 \cdots s_k r_k$. Applying Lemma 1.14 to this reflection factorization of t_λ we can rewrite it as $t_\lambda = y \cdot s_1 s_2 \cdots s_k$ with $y \in W$ of length $k = \frac{1}{2}\ell_R(t_l) = \dim(\lambda)$. Hence with $v := s_1 \cdots s_k \in V_\lambda$ this gives

$$\begin{aligned}\ell_R(w) &= \ell_R(t_\lambda u) \\ &= \ell_R(s_1 r_1 s_2 r_2 \cdots s_k r_k u) \\ &= \ell_R(y \cdot s_1 s_2 \cdots s_k u) \\ &\leq \ell_R(y) + \ell_R(s_1 s_2 \cdots s_k u) \\ &= \frac{1}{2}\ell_R(t_\lambda) + \ell_R(vu) \\ &= \dim(\lambda) + \ell_R(vu)\end{aligned}$$

This holds for any choice of coroots $\alpha_1, \ldots, \alpha_k$ and any order of $s_1, \ldots, s_k$ as the translations $t^1, \ldots, t^k$ commute. Thus $\ell_R(w) \leq \dim(\lambda) + \ell_R(vu)$ for any $v \in V_\lambda$ and hence the statement. $\square$

This technical proof makes essential use of the formula's shape and thus its approach does not suffice to show equality in general.

Outlook

Can the other inequality of the formula in Conjecture A be proven in general?
An option could be to use induction on the rank of W and use Theorem B as base case. The induction step could then deduce the formula in W from affine Coxeter subgroups of W with smaller rank. Though for this to work it is necessary to verify the compatibility of the reflection length in appearing subgroups with the reflection length in all W.

Does Conjecture A hold in groups of type affine A_n?
It might be easier to prove the conjecture for this type only. One reason is the result from Corollary 3.35, that is, any translation-elliptic factorisation is a normal form (for the right choice of origin). Another reason is that because root systems of type A_n have only one root length every subgroup is closed and thus maximal subgroups are given by Theorem 2.21. Since the coefficients of the highest root are all 1 these subsystems are of rank $n-1$ and type A_{n-1} or $A_l \times A_k$. This might be useful for an approach using induction.

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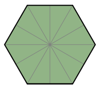
A. The illustrations

All pictures in this thesis were drawn with TikZ and this appendix gives a short insight in how the drawings were created. The code for Figure 31 is considered exemplarily. This image is composed of many small hexagons using the symmetry of the action of W on the Euclidean plane. One such hexagon looks like this:



```
\begin{tikzpicture}
\GHexagonPlain{-3}{3}{\ColrDirLongA}
\end{tikzpicture}
```

Figure 34: One of the many hexagons from Figure 31 with its code.

Of course `\GHexagonPlain{-3}{3}{\ColrDirLongA}` is only a command to plot the entire hexagon. The numbers -3 and 3 specify the position of the hexagon and `\ColrDirLongA` colours the background in OliveGreen!50. The complete code for just one hexagon reads as follows:

```
\newcommand{\GHexagonPlain}[3]{
% #1 :: x-coordinate
% #2 :: y-coordinate
% #3 :: backgroundcolor of hexagon

\fill [color=#3] (-0.5*\hexasize+#1,-\hexaside+#2) --
  ( 0.5*\hexasize+#1,-\hexaside+#2) -- ( \hexasize+#1, 0+#2) --
  ( 0.5*\hexasize+#1, \hexaside+#2) --
  (-0.5*\hexasize+#1, \hexaside+#2) -- (-\hexasize+#1, 0+#2);

% background divisions
\draw [color=\ColrGridLinesBackground, ultra thin]
  (             0+#1,     -\hexaside+#2)--(             0+#1,     \hexaside+#2)
  ( 0.75*\hexasize+#1, -0.5*\hexaside+#2)--(-0.75*\hexasize+#1, 0.5*\hexaside+#2)
  (-0.75*\hexasize+#1, -0.5*\hexaside+#2)--( 0.75*\hexasize+#1, 0.5*\hexaside+#2)
  (     -\hexasize+#1,              0+#2)--(      \hexasize+#1,             0+#2)
  (  0.5*\hexasize+#1,     -\hexaside+#2)--( -0.5*\hexasize+#1,     \hexaside+#2)
  ( -0.5*\hexasize+#1,     -\hexaside+#2)--(  0.5*\hexasize+#1,     \hexaside+#2);

% frame around hexagon
\draw [color=\ColrGridLinesFrame]
   (-0.5*\hexasize+#1,-\hexaside+#2) -- ( 0.5*\hexasize+#1,-\hexaside+#2) --
   ( \hexasize+#1, 0+#2) -- ( 0.5*\hexasize+#1, \hexaside+#2) --
   (-0.5*\hexasize+#1, \hexaside+#2) -- (-\hexasize+#1, 0+#2) --
   (-0.5*\hexasize+#1,-\hexaside+#2) -- ( 0.5*\hexasize+#1,-\hexaside+#2);
}
```

The numbers `\hexasize`$= 0.6$ and `\hexaside`$= \frac{\sqrt{3}}{2}0.6$ are defined globally. Also the colours of the grid lines `\ColrGridLinesBackground` and `\ColrGridLinesFrame` are defined globally and can be redefined locally to yield different behaviour. For example Figure 33 uses the same hexagons, some of them with white background and `\ColrGridLinesFrame` set to grey and `ultra thin`.

Since some of the hexagons are situated at the borders there are versions of `\GHexagon` that only plot a partial hexagon. Also there are non-`Plain` versions plotting more information like the hatched areas or highlighted elements used in the proofs for Theorem B.

The full code for Figure 31 reads:

```
\def\LocalSide{1.5*\hexasize}
\def\LocalUp{\hexaside}

%%% lambda of dimension 0 %%%
\GHexagon{0*\LocalSide}{0*\LocalUp}{\ColrDirNo}

%%% lambda of dimension 1 %%%
% bluish rays ie using one long coroot
% in alpha direction
\foreach \x in {-3,...,3}{
  \GHexagonPlain{-\x*2*\LocalSide}{0*\LocalUp}{\ColrDirShortA}
}
% other long coroots
\foreach \x in {-2,...,2}{
  \GHexagonPlain{ \x*\LocalSide}{-\x*3*\LocalUp}{\ColrDirShortB}
  \GHexagonPlain{ \x*\LocalSide}{ \x*3*\LocalUp}{\ColrDirShortC}
}

% greenish rays ie using one short coroot
% up
\foreach \x in {-4,...,4}
  {\GHexagonPlain{0*\LocalSide}{-\x*2*\LocalUp}{\ColrDirLongB}}
% other short coroots (beta and alpha+beta)
\foreach \x in {-7,...,7}{
  \GHexagonPlain{ \x*\LocalSide}{-\x*\LocalUp}{\ColrDirLongA}
  \GHexagonPlain{-\x*\LocalSide}{-\x*\LocalUp}{\ColrDirLongC}
}

% half or partial hexagons at the boarders:
\GHexagonPlainTop{    3*\LocalSide}{-3*3*\LocalUp}{\ColrDirShortB}
\GHexagonPlainBottom{-3*\LocalSide}{ 3*3*\LocalUp}{\ColrDirShortB}
\GHexagonPlainTop{   -3*\LocalSide}{-3*3*\LocalUp}{\ColrDirShortC}
\GHexagonPlainBottom{ 3*\LocalSide}{ 3*3*\LocalUp}{\ColrDirShortC}

\GHexagonPlainLeft{  8*\LocalSide}{ 0*\LocalUp}{\ColrDirShortA}
\GHexagonPlainRight{-8*\LocalSide}{ 0*\LocalUp}{\ColrDirShortA}
\GHexagonPlainLeft{  8*\LocalSide}{-8*\LocalUp}{\ColrDirLongA}
\GHexagonPlainRight{-8*\LocalSide}{ 8*\LocalUp}{\ColrDirLongA}
\GHexagonPlainLeft{  8*\LocalSide}{ 8*\LocalUp}{\ColrDirLongC}
\GHexagonPlainRight{-8*\LocalSide}{-8*\LocalUp}{\ColrDirLongC}
```

```
%%% lambda of dimension 2 %%%
% complete hexagons sorted by y-coordinate
\foreach \f in {-1,1}
  \foreach \g in {-1,1}{
    \foreach \x in {3,5,7}{
      \GHexagonPlain{ \x*\LocalSide*\f}{ 1*\LocalUp*\g}{\ColrDirLongOther}}
    \foreach \x in {4,6}{
      \GHexagonPlain{ \x*\LocalSide*\f}{ 2*\LocalUp*\g}{\ColrDirLongOther}}
    \foreach \x in {5,7}{ % other colour
      \GHexagonPlain{ \x*\LocalSide*\f}{ 3*\LocalUp*\g}{\ColrDirShortOther}}
    \foreach \x in {2,6}{
      \GHexagonPlain{ \x*\LocalSide*\f}{ 4*\LocalUp*\g}{\ColrDirLongOther}}
    \foreach \x in {1,3,7}{
      \GHexagonPlain{ \x*\LocalSide*\f}{ 5*\LocalUp*\g}{\ColrDirLongOther}}
    \foreach \x in {4}{ % other colour
      \GHexagonPlain{ \x*\LocalSide*\f}{ 6*\LocalUp*\g}{\ColrDirShortOther}}
    \foreach \x in {1,3,5}{
      \GHexagonPlain{ \x*\LocalSide*\f}{ 7*\LocalUp*\g}{\ColrDirLongOther}}
    \foreach \x in {2,4,6}{
      \GHexagonPlain{ \x*\LocalSide*\f}{ 8*\LocalUp*\g}{\ColrDirLongOther}}
}

% half or partial hexagons at the borders:
\foreach \f in {-1,1}
  \foreach \x in {1,5,7}{ % other colour
    \GHexagonPlainTop{    \x*\LocalSide*\f}{-9*\LocalUp}{\ColrDirShortOther}
    \GHexagonPlainBottom{ \x*\LocalSide*\f}{ 9*\LocalUp}{\ColrDirShortOther}
}
\foreach \g in {-1,1}{
  \foreach \y in {2,4}{
    \GHexagonPlainLeft{  8*\LocalSide}{ \y*\LocalUp*\g}{\ColrDirLongOther}
    \GHexagonPlainRight{-8*\LocalSide}{ \y*\LocalUp*\g}{\ColrDirLongOther}}
  \foreach \y in {6}{ % other colour
    \GHexagonPlainLeft{  8*\LocalSide}{ \y*\LocalUp*\g}{\ColrDirShortOther}
    \GHexagonPlainRight{-8*\LocalSide}{ \y*\LocalUp*\g}{\ColrDirShortOther}}
}
```

Eigenständigkeitserklärung

Hiermit bestätige ich, dass ich die vorliegende Bachelorarbeit mit dem Titel „Reflection length in affine Coxeter groups“ selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Darmstadt, 5. Juni 2020

Noam von Rotberg