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Sharkovsky's Theorem
A direct proof and further observations

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Sharkovsky's Theorem

A direct proof and further observations

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Abstract

We present a proof by Keith Burns and Boris Hasselblatt[1] for Sharkovsky's famous theorem regarding possible sets of periods for interval maps. Their proof improves the former proof through the introduction of *Štefan sequences*. Furthermore we present some observations on the family of truncated tent maps, which are used by Burns and Hasselblatt to prove the Sharkovsky realization theorem.

Zusammenfassung

Wir präsentieren einen Beweis von Keith Burns und Boris Hasselblatt1 für Sharkovskys berühmten Satz über mögliche Mengen periodischer Punkte von Interval-Abbildungen. Ihr Beweis verbessert vorherige Beweise durch die Einführung von *Štefan Sequencen*. Weiterhin stellen wir einige Beobachtungen bezüglich der Familie von abgeschnittenen Zelt-Funktionen vor, welche von Burns und Hasselblatt zum Beweis des Realisierungssatzes herangezogen werden.

[1] [2], 2011

Declaration of authorship

Hiermit versichere ich, dass die vorliegende Arbeit mit dem Titel *Sharkovsky's Theorem* selbstständig verfasst worden ist. Es wurden keine anderen Quellen und Hilfsmittel als die angegebenen benutzt und die Stellen der Arbeit, die anderen Werken – auch elektronischen Medien – dem Wortlaut oder Sinn entnommen wurden, wurden unter Angabe der Quelle als Entlehnung kenntlich gemacht.

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1 Introduction

In the late 1970's the american mathematician James A. Yorke was attending a river cruise during a congress in East Berlin. Together with his colleague Tien-Yien Li he had just published his groundbreaking paper "Period three implies chaos" [5], which states that any one-dimensional continuous function f from $\mathbb{R}$ into itself with a 3-period orbit has the following two key properties:
Firstly for any integer T there is one point in $\mathbb{R}$, that is mapped on itself again after T iterations of f. Secondly such a function produces an uncountably infinite set S that is scrambled, meaning any two points in S get arbitrarily close and further apart again and again when applying the function.
With the second part they should contribute largely to the mathematical notion of chaos and introduce their ideas about the topic to a wider audience, making it one of the most popular parts of mathematics, even finding its way into mainstream media.

So now Yorke happened to be on a boat somewhere behind the iron curtain, that divided the eastern and western hemisphere during that time, resulting in little to no exchange between scientists of all disciplines. A Ukrainian mathematician called Oleksandr M. Sharkovsky, who was taking part in the same congress, used the opportunity to approach Yorke. Despite communicational problems due to the lack of a common language, their talk resulted in Yorke being made aware that his results on periodic points of interval mappings had already been discovered by Sharkovsky, in fact more than ten years earlier. Sharkovsky's version of the theorem was even more powerful, introducing the so called Sharkovsky order on the natural numbers, in which 3 happens to be the first number. He could prove that for any one-dimensional continuous function, the presence of an m-period orbit was a sufficient criterion for finding l-period orbits for any l smaller than m in Sharkovsky's order.

The incidental encounter of these two mathematicians should lead to global recognition of Sharkovsky's work and multiple approaches to simplify the proof of what would now be called Sharkovsky's theorem. A first breakthrough consisted in the introduction of so called *Štefan cycles* by P. Štefan.
In this thesis we want to present a further simplification of the proof, first published in 2011 by Keith Burns and Boris Hasselblatt [2]. The genius of this proof lies in the notation of *Štefan sequences* which are inspired by the notation of *Štefan cycles* but represent a slightly more general version.

There is however a second part of Sharkovsky's theorem, the Sharkovsky Realization Theorem which, in a sense, completes the statement described above in a quite astounding way. The proof for this second proposition, presented in [2] (and therefore also in this thesis), makes use of a certain family of functions, the family of truncated tent maps. As part of this thesis we will introduce some observations on this family that make use of Sharkovsky's theorem. Furthermore we use this family as an example to show how the theorem beautifully fits into the theory of chaos that was developed in the 1960's and 70's.

We hope that by now we evoked interest in how to prove this striking theorem whose history, in a way, conveys the value of global exchange and collaboration in science.
We ask the reader kindly to enjoy this beautiful piece of mathematics.

2 Definitions and statement of the theorem

Let f be a continuous function from some interval I into itself. The interval upon which f is defined does not necessarily need to be closed or bounded so whenever we are talking about an interval in this paper what we mean is a connected subset of $\mathbb{R}$, in some cases even $\mathbb{R}$ itself. We can call such a function a discrete dynamical system, as the fact that I is mapped into itself again allows us to look at iterations of f. By f^n we denote

$$\underbrace{f \circ \cdots \circ f}_{n\ times}$$

where f^0 corresponds to the identity. In such a system one iteration of f corresponds to one step in time and a point x in the interval corresponds to one possible initial state of the system.

The Sharkovsky Theorem is a stunning result about a special kind of points in I, namely those points $x \in I$ that are mapped upon themselves again after a certain number of iterations:

$$\exists n \in \mathbb{N} : f^n(x) = x$$

We call such points *periodic points*. Furthermore we refer to the set of iterates

$$\mathcal{O} := \{f^m(x) | m \in \mathbb{N}\}$$

of a periodic point x as the *cycle* or *orbit* of x. Lastly we call the smallest number $m \in \mathbb{N}$ such that x is mapped upon itself again after m iterations the *period* of x and we say the number m is a period for f whenever there is a periodic point x with period m.

Given a dynamical system one may wonder, what periods the underlying function has. As pointed out in the introduction, this question is of relevance to how chaotic the system is. Sharkovky's theorem sheds some light on this question, as it provides us with a result on the structure of periodic points. This structure is given by the following order on the natural numbers, the Sharkovsky order.

2.1 The Sharkovsky Theorem

Definition 2.1. Take $\mathbb{N}$ the set of all positive integers. We can reorder the natural numbers in the following way:

$$3 \lhd 5 \lhd 7 \lhd \cdots \lhd 2{\cdot}3 \lhd 2{\cdot}5 \lhd 2{\cdot}7 \lhd \cdots \lhd 2^2{\cdot}3 \lhd 2^2{\cdot}5 \lhd 2^2{\cdot}7 \lhd \cdots \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1.$$

This total ordering is called the Sharkovsky order. The following doubling property of this order is important to us, as it is needed to conclude the proof that we give:

$$\text{for } l, m \in \mathbb{N} : l \lhd m \iff 2l \lhd 2m$$

We encourage the reader to verify this statement on their own in order for them to gain some insight in how the Sharkovsky order works.

This definition allows us to state Sharkovsky's theorem which consists of the following two parts:

Theorem 2.2. The Sharkovsky Forcing Theorem. If m is a period for f and $l \lhd m$, then l is a period for f aswell.

This is already an astonishing result as it indicates the set of periodic points of some function f to be a *tail* of the Sharkovsky order. By tail we mean a set $\mathcal{T} \subseteq \mathbb{N}$ such that every number that does not lie in $\mathcal{T}$ is greater in the Sharkovsky order than every number that does lie in $\mathcal{T}$.
We can distinguish between three different kinds of tails, namely $\{m\} \cup \{l \in \mathbb{N} : l \lhd m\}$ for some $m \in \mathbb{N}$, $\{2^k : k \in \mathbb{N}\}$ the set of all powers of two and $\emptyset$.
With this notion we can formulate the second part of the theorem which is even more striking:

Theorem 2.3. The Sharkovsky Realization Theorem. Every tail of the Sharkovsky order is the set of periods for some continous map of an interval into itself.

Now the Sharkovsky theorem is the union of both these statements, namely that a subset of $\mathbb{N}$ is the set of periods for f if and only if it is a tail of the Sharkovsky order.

The next five sections are dedicated to proving theorem 2.2, the Sharkovsky forcing theorem. As already mentioned, we are not discussing the original proof as stated by Sharkovsky himself, but a simplification of the standard proof after Keith Burns and Boris Hasselblatt found in [2]. Theorem 2.3, the Sharkovsky realization theorem, is proven in section 8 in a more than elegant way., that is extracted from the same paper as the former one.

3 Foundation: Interval coverings and cycles

Before we start with cycles of continuous maps we first take a look at intervals and interval coverings. The main idea of the proof is to not use the given cycle directly but to derive intervals from the points of the cycle and find covering relations between these intervals that can be used to build new covering relations. In this section we are going to see that a certain kind of covering relation gives rise to periodic points, which enables us to prove the existence of cycles of length smaller in the Sharkovsky order than the one of the cycle given. We begin by introducing our notion of covering relations.

Definition 3.1. Let $J_0, ..., J_{n-1}$ be Intervals. By that we will always mean closed, bounded, connected subsets of $\mathbb{R}$.

(i) We say J_0 covers J_1 under f or write $J_0 \xrightarrow{f} J_1$ if $J_1 \subseteq f(J_0)$.
In case $J_0 = f(J_1)$ we write $J_0 \overset{f}{\rightarrowtail} J_1$.

(ii) A series of covering relations $J_0 \xrightarrow{f} \cdots \xrightarrow{f} J_{n-1} \xrightarrow{f} J_0$ that starts and ends in the same interval is called a loop (or n-loop) of intervals.

(iii) We say that a point $x \in J_0$ follows the loop if
$f^i(x) \in J_i \; for \; i = 0, \ldots, n-1 \; and \; f^n(x) = x.$

(iv) An n-loop of intervals is called elementary if every point that follows it has period n.

3.1 Coverings produce cycles

Now this provides us with the vocabulary needed to formulate the main proposition of this section, which will later form a key part of the proof:

Proposition 3.2. Given an elementary n-loop of intervals $J_0 \xrightarrow{f} \cdots \xrightarrow{f} J_{n-1} \xrightarrow{f} J_0$ there exists a point $x \in J_0$ that follows the loop and has period n.

Proof: To conclude the statement it is sufficient to show that for a given n-loop there is a point that follows it. The loop being elementary implies the proposition as given above. We start off by showing this statement first for the special case $n = 1$, which means nothing more than an interval covering itself has a fixed point. We then proceed with the general case, making use of the first statement.

Case 1: Let $a_1, a_2 \in I$ such that $[a_1, a_2] \xrightarrow{f} [a_1, a_2]$.

We want to show that the function $f(x) - x$ has a zero in $[a_1, a_2]$. Choose $b_1, b_2 \in [a_1, a_2]$ such that $f(b_1) = a_1$ and $f(b_2) = b_2$. As $b_1 \geqslant a_1$ it holds that $f(b_1) - b_1 \geqslant 0$ and equally

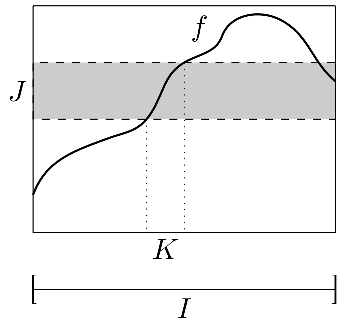


Figure 1

$f(b_2) - b_2 \leqslant 0$. As f and therefore $f - id$ is continuous we can conclude the statement by the intermediate value theorem.

Case 2: Now let $J_0 \xrightarrow{f} \cdots \xrightarrow{f} J_{n-1} \xrightarrow{f} J_0$ be an arbitrary n-loop of intervals.

The idea now is to transform the loop into a series $K_0 \overset{f}{\rightarrowtail} \cdots \overset{f}{\rightarrowtail} K_{n-1} \overset{f}{\rightarrowtail} J_0$. Then for any point x in K_0 it holds true that $f^i(x) \in K_i \subseteq J_i$. Furthermore $K_0 \xrightarrow{f^n} J_0$ and $K_0 \subseteq J_0$ imply $K_0 \xrightarrow{f^n} K_0$. Hence we can conclude the existence of a fixed point $f^n(x) = x$ through *case 1*.

What we need to show now is that for any intervals I and J it is possible to transform a covering relation $I \to J$ into a covering relation $K \rightarrowtail J$ for some $K \subseteq I$:
To find such a K first take $f^{-1}(J)$. This consists of several components. As f is continuous and I covers J we are able to find one component K such that $f(K) = J$ as we can also see in figure 1.

We conclude the proposition by transforming the relation $J_{n-1} \to J_0$ and proceeding inductively to the front to obtain a series of relations as desired.

■

Now this gives us the opportunity to conclude the existence of periodic points of period l given that there exists an elementary l-loop of intervals. To find such loops, we will need some criteria to determine whether a loop of intervals is elementary. This is what the next lemma is about.

Lemma 3.3. An n-loop of Intervals $J_0 \to \cdots \to J_{n-1} \to J_0$ is elementary if the following two conditions are met:

(i) The loop is not followed by either endpoint of J_0.

(ii) The interior of J_0 is disjoint from the other intervals: $Int(J_0) \cap \bigcup_{i=1}^{n-1} J_i = \emptyset$

Proof: This statement can be conlcuded fairly easily. Let $x \in J_0$ be a point that follows the loop.

(i) implies, that x can not be either endpoint of J_0 as those do not follow the loop and therefore has to lie within the Interior of J_0.
By (ii) x itself cannot lie in any of the J_i for $1 \leqslant i \leqslant n-1$, where its iterates under f because of x following the loop. Therefore the first iteration of x under f that coincides with x again has to be $f^n(x)$.

Therefore the period of x is n, the lenght of the loop of intervals given and the loop is proven to be elementary as x was chosen arbitrarily.

■

3.2 Cycles produce coverings

In the next section we see a special case of the theorem, when we provide a proof for the case $m = 3$: "Period three implies all periods."
From now on we are only going to work with *$\mathcal{O}$-intervals*, which refer to intervals with endpoints in a cycle $\mathcal{O}$. Thereby we only need to use information of how f acts on the cycle $\mathcal{O}$.
Furthermore we are exclusively going to discuss relations $I \to J$ of $\mathcal{O}$-intervals that are *$\mathcal{O}$-forced*. By this we mean that the $\mathcal{O}$-interval with endpoints the leftmost and rightmost points of $f(\mathcal{O} \cap I)$ is a superset of J. The continuity of f and the intermediate value theorem then imply the covering relation $I \to J$. Later on this notation is going to simplify the finding of covering relations of intervals as we only need to consider where the points of the cycle are mapped under f.
Lastly we call a whole loop of intervals *$\mathcal{O}$-forced* if every arrow in it is $\mathcal{O}$-forced.

Figure 2: The two versions of 3-cycles

4 Examples

In this section we have a look at two examples, namely the most iconic special case of the Sharkovsky Theorem ("Period 3 implies all periods"), that had also been proven by Yorke and Li, and the example of a 6 cycle. These two cases use different approaches and will shed light on the underlying logic of the two cases that we are going to discuss in sections 5, 6 and 7.

4.1 Period 3 implies all periods

Given a 3-cycle there are just two different ways that the points of the cycle can be ordered. As one can easily see in Figure 2, those two versions are mere mirror images of each other.

The cycle consists of three points x_0, x_1 and x_2 such that

$$f(x_0) = x_1, f(x_1) = x_2 \text{ and } f(x_2) = x_0$$

as the dashed arrows indicate. We now choose J_1 to be the $\mathcal{O}$-interval with endpoints x_0 and x_1 and J_0 the $\mathcal{O}$-interval with endpoints x_0 and x_2. As already indicated in section 3 our first task is to observe what covering relations exist between these two intervals.

First we take a look at the image of J_1. As its endpoints are mapped to the endpoints of the whole interval, we obtain the $\mathcal{O}$-forced covering relations $J_1 \to J_1$ and $J_1 \to J_0$. Secondly we observe that the endpoints of J_0 are mapped to those of J_1 whereby we can conclude the $\mathcal{O}$-forced covering relation $J_0 \to J_1$.
We summarize these relations by writing

$$\circlearrowleft J_1 \rightleftarrows J_0.$$

To conclude the proof we are now going to construct elementary loops of length l using these relations. Proposition 3.2 is then going to provide the existence of a periodic point with the corresponding period.

$l = 1$: This case can easily be concluded by *Case 1* of 3.2 using the covering relation $J_1 \to J_1$, which directly provides us with a fixed point.

$l = 2$: From the above image we can read off the loop

$$J_0 \to J_1 \to J_0$$

It remains to show that this loop is elementary.
First of all the interior of J_0 does not meet J_1 simply by construction and secondly the endpoints of J_0 are chosen to be part of the cycle whereby they are periodic points of period 3 and cannot follow the loop as $f^2(x_i) \neq x_i$ for $i = 0, 1, 2$.
This way we obtain a periodic point of period 2.

$l > 3$: For an l-loop of intervals of length greater than 3 we use the fact that J_1 covers itself, whereby we can just insert $l - 1$ copies of this arrow into the loop above, obtaining the l-loop

$$J_0 \to \underbrace{J_1 \to \cdots \to J_1}_{l-1\ copies\ of\ J_1} \to J_0$$

Again the interior of J_0 does not meet J_1 as argued before. If we take a look at the iterates of the endpoints of J_0 we observe that after two iterions in case of x_0 or one in case of x_1 they are mapped to x_2, which does not lie within J_1. There being at least three copies of J_1 in the middle of the loop makes it impossible for the endpoints of J_0 to follow the loop. This concludes the proof.

■

The idea to first observe some covering relations and then build elementary l-loops for all $l \lhd m$ is going to be generalized in sections 5 and 6 by introducing the notion of so called Štefan sequences. However this kind of sequence cannot be found in any cycle. We are going to see one example now where the method introduced in these later sections will not be applicable. In this special case we are providing an alternative proof of the Sharkovsky forcing theorem making use of the doubling property of the Sharkovsky order introduced in [add ref to def].

4.2 One example of a 6-Cycle

We consider the 6-cycle given in Figure 3. Looking at the cycle we make the observation that the set of points on the left and the set of points on the right are mapped upon each other under f, forming two 3-cycles for the second iterate f^2.

As we have already seen in the example before this accounts to saying that we can construct an elementary l-loop for any given $l \in \mathbb{N}$. We can now define J_0 and J_1 the same way, obtaining the following three cycles:

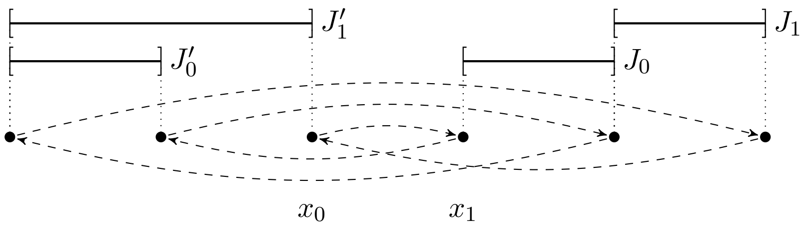


Figure 3: An example of a 6-cycle

$$(1)\ J_1 \xrightarrow{f^2} J_1 \quad (2)\ J_1 \xrightarrow{f^2} J_0 \quad (3)\ J_0 \xrightarrow{f^2} J_1$$

Our idea is to make use of the doubling property of Sharkovsky's Order, telling us that

$$\forall l \in \mathbb{N} : l \lhd 3 \iff 2l \lhd 6$$

With that in mind we only need to transform every elementary l-loop of f^2 into a $2l$-loop of f.

To achieve this we are going to define the two Intervals J_0' and J_1' to be the shortest intervals containing $f(J_0 \cap \mathcal{O})$ or respectively $f(J_1 \cap \mathcal{O})$.

Observing where the endpoints of J_0' and J_1' are mapped we can also conclude the $\mathcal{O}$-forced covering relations

$$(1')\ J_1' \xrightarrow{f} J_1 \quad (2')\ J_1' \xrightarrow{f} J_0 \quad (3')\ J_0' \xrightarrow{f} J_1.$$

By construction we additionally gain the following $\mathcal{O}$-forced covering relations

$$J_0 \xrightarrow{f} J_0' \text{ and } J_1 \xrightarrow{f} J_1'.$$

Now given any elementary l-loop of f^2 we can just replace the covering relations of the form

$$J_i \xrightarrow{f^2} \dots$$

by the relations

$$J_i \xrightarrow{f} J_i' \xrightarrow{f} \dots .$$

This shows, how we can transform an elementary l-loop of f^2 into an $2l$-loop of f. What remains to show is that an $2l$-loop constructed this way gives rise to a $2l$-periodic point.

To ensure this we just need to observe two facts: first of all, the iterates under f^2 have to be pairwise distinct, as the l-loop they follow is elementary. Secondly the iterates under f lie on alternating sites, ensuring, that altogether all $2l$ iterates under f are distinct. In conclusion a point that follows the l-loop under f gives rise to a $2l$-periodic point under f. This concludes the proof as stated above.

■

These examples of Sharkovky's theorem give an idea of the underlying logic of the proof. In the next sections we will discuss the general case and finally conclude the statement given in section 2.
We will proceed now by introducing Štefan sequences and show how this special kind of sequence produces cycles in a way similar to the case of period 3. Again our way to go will be to observe a handful of $\mathcal{O}$-intervals and covering relations between them and to then build loops of the desired length. In fact this already concludes the proof in the cases where we are able to find a Štefan sequence.

5 Štefan sequences produce cycles

5.1 Definition of a Štefan sequence

From now on let $m \geqslant 2$ and $\mathcal{O}$ an m cycle for our continuous map f on the interval I. The case of $m = 1$ is vacuously true, as there is no number smaller than 1 in the Sharkovsky order. Before we proceed as indicated in the end of the last section we first need to introduce some notation that will then allow us to define a Štefan sequence:

Definition 5.1. We first introduce the points p and q as the *middle points* of the cycle $\mathcal{O}$ in the sense that p is the rightmost point such that $f(p) > p$ and q the point to its immediate right. This allows us to talk about the *center* of the cycle by which we mean the point $c := (p + q)/2$.

Furthermore we will want to talk about the points of the cycle between a given point of $\mathcal{O}$ and the center. We will notate this set in the folling way: For $x \in \mathcal{O}$ define $\mathcal{O}_x \subseteq \mathcal{O}$ by:

$$\mathcal{O}_x := \mathcal{O} \cap [x, p] \text{ in case } x \leqslant p$$

$$\mathcal{O}_x := \mathcal{O} \cap [q, x] \text{ in case } x \geqslant q$$

We finally say a point $x \in \mathcal{O}$ switches sides if the center c lies between x and $f(x)$.

The core idea behind a Štefan sequence is to mantain a series of points in $\mathcal{O}$ that are *spiraling outwards* starting next to the center defined as above. This gives rise to a corresponding series of intervals spiraling outwards as well, which is how we are going to ensure some of the desired covering relations later on. But first let us introduce the exact notion of a Štefan sequence with this idea in the back of our head:

Definition 5.2. A sequence $x_0, \dots x_n$ of points in $\mathcal{O}$ is called a Štefan Sequence if the following four conditions are satisfied:

(Š1) $\{x_0, x_1\} = \{p, q\}$

(Š2) $x_0, \dots, x_n$ are on alternating sides of the center c and the sequences (x_{2j}) and (x_{2j+1}) are both strictly monotonous and therefore moving away from c.

(Š3) For $1 \leqslant i \leqslant n - 1$ x_j switches sides and $x_{j+1} \in \mathcal{O}_{f(x_j)}$.

(Š4) x_n does not switch sides.

Now this definition comes with a handful of implications:
First of all what the condition $x_{j+1} \in \mathcal{O}_{f(x_j)}$ in (Š3) basically says is that the next point in the series will be chosen to lie on the same side as the image of the last one under f, but possibly closer to the center.
Furthermore (Š2) implies that the points $x_0, \ldots, x_n$ are pairwise distinct. This ensures, that $n + 1 \leqslant m$, meaning the quantity of the series can not exceed the number of points in the whole cycle. There are cases where a Štefan sequence consists of the entire cycle, but this does not necessarily have to be the case.
Finally we want to note that $n \geqslant 2$, which is implied by (Š1) and (Š4) together and hence the cycle must consist of at least 3 points. We have already discussed the case $m = 1$. If $m = 2$ we just need to find a fixed point of f to conclude the proof, which is provided by *Case 1* of Proposition 3.2.
An example for a Štefan sequence is provided in the next subsection through figure 5.

Next up we see how such a series gives rise to a series of intervals and how these can be used to conclude the theorem.

5.2 Proving the theorem through Štefan sequences

Now as already stated we are going to prove the Sharkovsky theorem provided that the cycle we begin with has a Štefan sequence, so this section will consist of the proof of the following proposition. In figure 5 there is an example of a 9-cycle with a Štefan sequence and the corresponding intervals. We will keep referring to this example in order for the reader to have an image in mind. Later on we will also see the algorithm used to find the Štefan sequence in this example, but for now we content ourselves with its existence.

Proposition 5.3. Suppose that the m-cycle $\mathcal{O}$ has a Štefan sequence. If $l \lhd m$, then f has a periodic point with least period l.

Proof: Given a Štefan sequence $x_0, \ldots, x_n$ in $\mathcal{O}$ we can define the $\mathcal{O}$-intervals $J_0, \ldots, J_{n-1}$ in the following way:

$1 \leqslant i \leqslant n-1$: Define J_i to be the smallest interval, such that

$$x_0, x_1 \in J_i \text{ and } x_i \in J_i.$$

$j = 0$: Define J_0 to be the interval with endpoints x_n and x_{n-2}.

By construction and using (Š2) we can already conclude that $Int(J_0) \cap \bigcup_{i=1}^{n-1} J_i = \emptyset$, as x_{n-1} does not lie on the same side as x_n and x_{n-2} and hence given the monotony of both sides there is no point between x_n and x_{n-2}. To further clarification consult figure 5, where the intervals $J_0, \ldots, J_5$ are derived from the Štefan sequence given in the described manner.

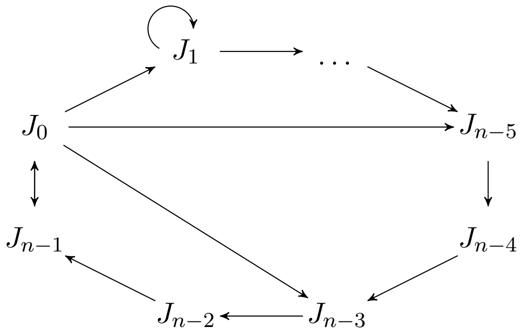


Figure 4: Diagram summarizing the covering relations

This definition leads us to observing the following $\mathcal{O}$-forced covering relations, summarized in Figure 4:

(1) $J_1 \to J_1$ and $J_0 \to J_1$

(2) $J_1 \to J_2 \to \cdots \to J_{n-1} \to J_0$

(3) $J_0 \to J_{n-1}, J_{n-3}, \ldots$

Our first goal is to understand where these covering relations come from:

(1): We will show even more, namely that $J_i \to J_1$ for $i = 0, \ldots, n-1$. As J_1 is the $\mathcal{O}$-interval with endpoints x_0 and x_1 it suffices to show:

$$x_0, x_1 \in f(J_i) \text{ for } i = 0, \ldots, n-1.$$

In the case of $i = 0$ one endpoint of J_0 switches sides, whereas the other one does not. As both points where on the same side before, their images under f will therefore be located on opposite sides of the center.
The same holds true for the cases where $i \geqslant 1$, as here both points lay on opposite sites, but then again both switch sides in the process of applying f. To sum up $f(J_i)$ contains points of $\mathcal{O}$ on both sides of the center c and therefore at least x_0 and x_1, therefore covering the interval J_1.

(2): As argued before it will suffice to show, that

$$x_0, x_1 \text{ and } x_{i+1} \in f(J_i) \text{ for } i = 1, \ldots, n-1.$$

As $x_0, x_1 \in f(J_i)\ for\ i = 1, \ldots, n-1$ has already been discussed, we are left with the task of showing that the image of one of the intervals J_1 to J_{n-1} contains the next point in the sequence, namely that $x_{i+1} \in f(J_i)$ for $i = 1, \ldots, n-1$.
This is the case as $f(x_i) \in f(J_i)$ and as also at least one point on the other side of the center lies in $f(J_i)$ (x_0 or x_1) so does the whole set $\mathcal{O}_{f(x_i)}$.
Lastly (Š3) implies that $x_{i+1} \in \mathcal{O}_{f(x_i)} \subseteq f(J_i)$. In these relations we see, how the spiraling character of the sequence that transfers to the series of intervals comes into play.

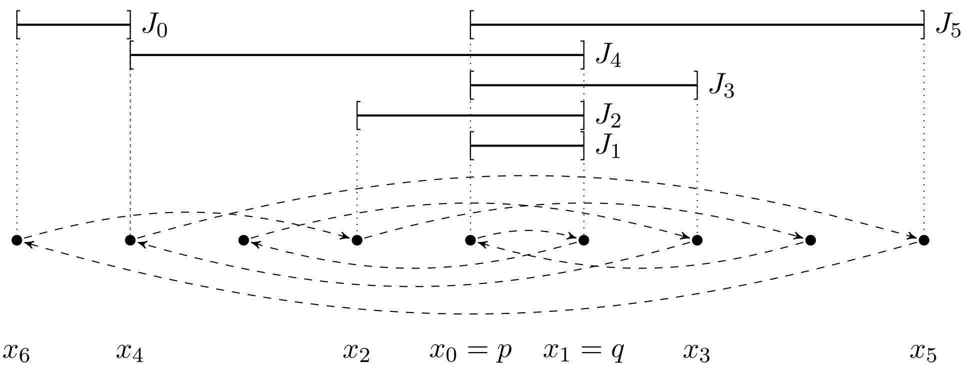


Figure 5: Example of a 9-cycle containing a Štefan sequence

(3): Again as $f(I_0)$ contains x_0 and x_1 it suffices to show that $x_{n-1}, x_{n-3}, \ldots \in f(I_0)$. We know that $x_{n-2} \in J_0$. Hence (Š3) implies $f(x_{n-2})$ lying further away from the center c than x_{n-1}. Furthermore (Š2) implies the same holding true for $x_{n-3}, x_{n-5}, \ldots$.

In conclusion as the other endpoint of J_0, x_n, does not switch sides the points $x_{n-1}, x_{n-3}, \ldots$ lie within $f(J_0)$.

Now that we ensured the covering relations given in figure 4 to be true, we can use them to build cycles in a way similar to the first example in section 4.

Let $l \lhd m$. Then we only need to cover three separate cases:

case 1: $l = 1$ We observe the 1-loop $J_1 \to J_1$. As we have already seen several times such a loop directly provides us with a fixed point.

case 2: $l \leqslant n$ is even For this case we take a look at the l-loop:

$$J_0 \to J_{n-(l-1)} \to J_{n-(l-2)} \to \cdots \to J_{n-1} \to J_0$$

This loop is elementary as it can not be followed by either endpoint of J_0 which where chosen to be part of the m-cycle. The second criterion for a loop to be elementary has already been guaranteed. Therefore this loop provides us with a periodic point of period l through Proposition 3.2.

case 3: $m \neq l \geqslant n$ For this last case we recall the example of period 3 and use the same trick, namely putting in as many copies of the arrow $J_1 \to J_1$ as needed into the loop $J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$, providing us with the loop

$$J_0 \to \underbrace{J_1 \to J_1 \to \cdots \to J_1}_{l-n+1 \text{ copies of } J_1} \to J_2 \to \cdots \to J_{n-1} \to J_0.$$

With $l - n + 1$ copies put in we obtain an l-loop. The endpoints of J_0 will either not land on themselves again in time if $l < m$ or will land on x_n too early if $l > m$. Both cases make it impossible for the endpoints of J_0 to follow the loop. Together with our earlier observation that $Int(J_0) \cap \bigcup_{i=1}^{n-1} J_i = \emptyset$, by construction, we can conclude the loop to be elementary, providing us again with a periodic point of period l through Proposition 3.2.

■

We have now shown that as soon as we have found a Štefan sequence in a given cycle, we can conclude the Sharkovsky Forcing Theorem. The next section will shed light on how to find such a sequence and whether this is always possible.

6 Finding a Štefan sequence

If every cycle had a Štefan sequence, Proposition 5.3 would already imply the theorem altogether. Nevertheless there are cycles that do not contain such a sequence. We have already seen an example of a cycle that cannot have one in section 4: Every point in the 6-cycle we discussed switches sides, which prevents (Š4) from being satisfied, as the last point of a Štefan sequence cannot switch sides. This might be unfortunate, but in fact the case where every point of a cycle switches sides is the only one, that prevents us from finding the desired sequence of points. Providing a proof for this statement will be the goal of this section.
The idea is to start with the two middle points and to then have a look at where the point is mapped by f. Now the next point of the series will be chosen to lie on the same side as this image and at least as close to the center to satisfy (Š3). Not just any point is chosen but the one, that maps furthest away from the center. We proceed in that manner until the algorithm eventually stops in a point that does not switch sides.
We are going to see now, how this approach leads to a Štefan sequence, given that at least one point of the cycle does not switch sides.

6.1 The algorithm to find a Štefan sequence

Proposition 6.1. A cycle $\mathcal{O}$ containing more than one point always contains a Štefan Sequence if at least one point does not switch sides.

Proof: We begin by narrowing down the points of the cycle to a set $\mathcal{S} \subseteq \mathcal{O}$ containing all candidates for the non-final terms of the Štefan Sequence.
For that we need the notion of the set M which we define to be the maximal $\mathcal{O}$-interval containing the middle points p and q of the cycle such that every point in $M \cap \mathcal{O}$ switches sides.

Now as indicated before $\mathcal{S}$ consists of those $x \in M \cap \mathcal{O}$ that are mapped further away from the center c than any other point in $\mathcal{O}_x$. Note that by this $p, q \in \mathcal{S}$.

With $\mathcal{S}$ defined as above we can assign to each of its elements a successor in $\mathcal{O}$ using the map $\sigma : \mathcal{S} \to \mathcal{O}$ as follows:

(i) If $f(x) \notin M$, we assign $\sigma(x)$ to be any point of $\mathcal{O}_{f(x)}$ that does not switch sides. Therefor $\sigma(x) \notin \mathcal{S}$ and there is no more successor.

(ii) As long as $f(x) \in M$, we assign its successor $\sigma(x)$ to be the point in $\mathcal{O}_{f(x)}$ that maps furthest away from the center. Note that the way $\mathcal{S}$ was defined we obtain $\sigma(x) \in \mathcal{S}$. This definition results in the following statement being true:

$$f(\mathcal{O}_{f(x)}) \subseteq \mathcal{O}_{f(\sigma(x))}.$$

As we have seen in the previous section, it is crucial to mantain the outward spiraling of the series for (Š2) to hold true. As $x \in M$ and its successor $\sigma(x) \in \mathcal{O}_f(x)$, x and $\sigma(x)$ are ensured to lie on opposite sides of the center c. Therefore we only need to make sure that $\sigma^2(x)$, which lies on the same side as x again, is further away from the center or put differently $\sigma^2(x) \notin \mathcal{O}_x$, as otherwise we do not obtain the desired monotony. We deal with this issue with help of the next lemma:

Lemma 6.2. If there is an $x \in \mathcal{S}$ such that $\sigma^2(x) \in \mathcal{O}_x$, then all points of the cycle switch sides.

Proof of Lemma: For $\sigma^2(x)$ to be defined in the first place and additionally in $\mathcal{O}_x$ we need $x, y := \sigma(x)$ and $z := \sigma^2(x)$ to be contained in $\mathcal{S}$.

As we have seen above this implies the following two subset relations:

$$f(\mathcal{O}_{f(x)}) \subseteq \mathcal{O}_{f(y)} \text{ and } f(\mathcal{O}_{f(y)}) \subseteq \mathcal{O}_{f(z)} \subseteq \mathcal{O}_{f(x)}$$

where the last one stems from the fact that $z \in \mathcal{O}_x$ and $x \in \mathcal{S}$: x is mapped further away from the center than any of the points on the same as x and closer to c and z happens to be one of these points.

All of this results in the set $\mathcal{O}_f(x) \cup \mathcal{O}_f(y)$ being mapped into itself by f. Now that f acts as a cyclic permutation on the set $\mathcal{O}$ the only f-invariant nonempty subset of $\mathcal{O}$ is $\mathcal{O}$ itself.

We can therefore ensure $\mathcal{O} = \mathcal{O}_{f(x)} \cup \mathcal{O}_{f(y)}$ which together with $x, y \in \mathcal{S}$ leads us to the conclusion, that $\mathcal{O} \subseteq M$ or in other words all points of the cycle switch sides.

■

So suppose there exists at least one point in the cycle that does not switch sides. Then the contrapositive of lemma 6.2 implies that we cannot have both $\sigma(p) = q$ and $\sigma(q) = p$. This allows us to chose the middle points $\{p, q\}$ as the starting points $\{x_0, x_1\}$ of the sequence and $x_2 := \sigma(x_1) \neq x_0$.
We now define $x_2 := \sigma(x_1) \neq x_0$ and as long as $x_i \in \mathcal{S}$ we continue with $x_{i+1} := \sigma(x_i)$. With the series defined we just need to make sure it satisfies the criteria (Š1) to (Š4). As we have already seen that, as soon as (Š2) is ensured, the elements of the series are pairwise distinct and therefore the sequence terminates. We label the last term x_n and now proceed by verifying that the sequence forms in fact a Štefan sequence:

(Š1) $\{x_0, x_1\} = \{p, q\}$ by construction

(Š2) First of all p and q lie on opposite sites of the center. Furthermore the successor of a point in the series is always chosen to lie on the other side of c, which ensures the alternation of successive terms. Next $x_2 \notin \mathcal{O}_{x_0}$ $(= \{x_0\})$ and by lemma 6.2 $x_{i+2} = \sigma^2(x_i) \notin \mathcal{O}_{x_i}$. This ensures the outward spiralling or in other words the series being monotonous on both sides of the center and as already argued the termination of the series, resulting in x_n the last element of the sequence being welldefined.

(Š3) By definition it holds true that

$$\text{for } 0 \leqslant i \leqslant n-1: \; x_i \in \mathcal{S} \subseteq M$$

meaning that all points up to the penultimate switch sides. Again by definition we made sure that $x_{i+1} = \sigma(x_i) \in \mathcal{O}_{x_i}$. Hence the third criterion is met as well.

(Š4) Lastly we made sure that x_n arises necessarily from (i) in the above definition of the sequence and was chosen to be a point that does not switch sides. This concludes the proof.

■

The last two sections combined lead us to the result that the Sharkovsky forcing theorem holds true in case one point of the cycle given in the beginning does not switch sides, as in this case the algorithm explained in the current section provides us with a Štefan sequence. As seen in section 5 such a sequence enforces a series of intervals with a couple of covering relations between them. Using these covering relations we can build loops of all lengths shorter in the Sharkovsky order than the length of the given cycle which then leads us to periodic points of these lengths, as we have seen in section 3.
It remains to show that in the case that every point of a given series switches sides, the theorem holds true as well. This is going to be the content of the next section in which we will finally conclude the theorem altogether.

7 Concluding the Proof

In this section we finally conclude the Sharkovsky forcing theorem. As already explained, the only case missing is the one of a cycle where every point switches sides. Our approach to this issue is already known to the reader: We have seen the core concept in section 4, where we were given a 6-cycle. Similar to this special case we will look at f^2 instead of f using the doubling property of the Sharkovsky order. With this in mind we finally conclude the theorem altogether by the following final proposition:

Proposition 7.1. An m-cycle $\mathcal{O}$ has an $\mathcal{O}$-forced elementary l-loop of $\mathcal{O}$-intervals for each $l \lhd m$.
By Proposition 3.2 this implies the Sharkovsky forcing theorem.

Proof: To proof this proposition we are going to perform an induction over m:

$m = 1$: Proposition 7.1 is vaciously true in this case, as there is no number smaller than 1 in the Sharkovsky Order.

$m > 1$: Suppose now proposition 7.1 is known for all cycles of length less than m. We can assume every point in the cycle $\mathcal{O}$ to switch sides, as proposition 6.1 and 5.3 together conclude the proof otherwise.

First of all we observe, that with $L := min\ \mathcal{O}$ the leftmost point of the cycle and $R := max\ \mathcal{O}$ the rightmost point of the cycle, f switches the two sets $\mathcal{O}_L$ and $\mathcal{O}_R$ as by assumption all points on the left side of the center are mapped to the right side and vice versa. Therefore both $f|_{\mathcal{O}_L}^{\mathcal{O}_R} : \mathcal{O}_L \to \mathcal{O}_R$ and $f|_{\mathcal{O}_R}^{\mathcal{O}_L} : \mathcal{O}_R \to \mathcal{O}_L$ are bijections resulting in both sets to have the same cardinality, meaning that there is an equal amount of points on both sides of the center. With this result in the back of our head we now want to remind the reader of the doubling property of the Sharkovsky Theorem, providing us with the information that $l \lhd m$ if and only if either $l = 1$ or $k \lhd m/2$ for some $k \in NN$ such that $l = 2k$.

Hence it remains to show that f has (i) an elementary 1-loop and (ii) an elementary $2k$-loop for every $k \lhd m/2$

(i) In this case we can just choose the $\mathcal{O}$-interval $[p, q]$: This interval covers itself, as both its endpoints switch sides. We recall that this already provides us with a fixed point.

(ii) For the second case we begin with the observation that $\mathcal{O}_L$ and $\mathcal{O}_R$ ar cycles of length $m/2$ for the second iterate f^2. Using the induction hypothesis we can apply 7.1 to either cycle. Without loss of generality we will focus on $\mathcal{O}_R$.
f^2 has an elementary $\mathcal{O}$-forced k-loop for every $k \lhd m/2$.

Our goal now is to show how these loops give rise to elementary $2k$-loops of f: Consider the k-loop of $\mathcal{O}_R$-intervals:

$$J_0 \xrightarrow{f^2} J_1 \xrightarrow{f^2} \cdots \xrightarrow{f^2} J_{k-1} \xrightarrow{f^2} J_0.$$

Similar to how we proceeded in the case of the 6-cycle in section 4, we will now define the intervals J_i' for $0 \leqslant i \leqslant k-1$ to be the shortest interval that contains $f(J_i \cap \mathcal{O}) \in \mathcal{O}_L$. By construction the endpoints of these intervals are part of the cycle $\mathcal{O}$ and therefore the intervals are $\mathcal{O}$-intervals. Furthermore we directly obtain the $\mathcal{O}$-forced covering relation $J_i \to J_i'$.
The statement of the proposition will follow once we show that this produces an $\mathcal{O}$-forced elementary $2k$-loop for f of the form

$$J_0 \xrightarrow{f} J_0' \xrightarrow{f} J_1 \xrightarrow{f} J_1' \xrightarrow{f} \cdots \xrightarrow{f} J_{k-1} \xrightarrow{f} J_{k-1}' \xrightarrow{f} J_0.$$

There are only two considerations we need to make: (1) that the covering relations $J_i \xrightarrow{f} J_{i+1}$ (for convenience we set $J_k := J_0$) are given and (2) that the loop is in fact elementary:

(1) Since the $\mathcal{O}_R$-forced relation $J_i \xrightarrow{f^2} J_{i+1}$ exists there are some $a_i, b_i \in J_i \cap \mathcal{O}$ such that
$$[f^2(a_i), f^2(b_i)] \supset J_{i+1}.$$
But then there exist $a_i' = f(a_i)$ and $b_i' = f(b_i)$ in $J_i' \cap \mathcal{O}$ which gives us
$$f(J_i') \supset [f(a_i'), f(b_i')] \supset J_{i+1}$$
as required.

(2) Consider a periodic point $x \in J_0$ of f that follows the loop. As x follows also the above elementary k-loop of f^2 it has period k with respect to f^2. This results in all the iterates on the right side to be distinct.
Now as the intervals in the $2k$-loop are alternating, so will be the iterates of x under f. We can therefore conclude the $2k$ iterates to be pairwise distinct, making the point x a periodic point of period $2k$.
This concludes the proof.

■

8 The Sharkovsky realization theorem

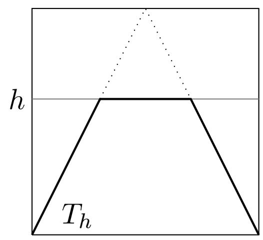


Figure 6: Truncated tent maps

Now as the Sharkovsky forcing theorem is proven we want to show the Sharkovsky realization theorem:

Theorem 8.1. For every possible tail of the Sharkovsky order we can find some continous map f from an interval into itself, such that said tail forms the set of periods of f.

Proof: To prove this statement we take a look at the family of truncated tend maps which actually provides us with one map for every tail of the Sharkovsky order. For $h \in [0,1]$ define:

$$T_h : [0,1] \to [0,1], \quad x \mapsto \min\left(h, 1 - 2\left|x - \tfrac{1}{2}\right|\right)$$

An equivalent and possibly a more accessible definition would be the following:

$$T_h(x) = \begin{cases} \min(h, 2x) & x \leqslant \frac{1}{2} \\ \min(h, 2-2x) & x > \frac{1}{2} \end{cases}$$

First of all we consider the two special cases $h = 0$ and $h = 1$. T_0 only has the point 0 as a fixed point and no further periodic points whereas T_1 has a 3 cycle $\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$ and therefore all natural numbers as periods as we have seen in section 4. Now for all cases in between we observe the following lemma:

Lemma 8.2. Every cycle $\mathcal{O} \subseteq [0,h)$ of T_h is a cycle of T_1 and conversely every cycle $\mathcal{O} \subseteq [0,h]$ of T_1 is a cycle of T_h.

Proof: This is the case, as T_1 and T_h are identical on the interval $[0, h)$. For cycles that include h itself, we need to pay attention however, as these do not translate to T_1[2]. The converse is not a problem though.

∎

Now the idea of the proof is to define the following map, that assignes to every natural number a value between 0 and 1:

$$h : \mathbb{N} \to [0,1], \; h(m) := \min\{\max \mathcal{O} : \mathcal{O} \text{ is an } m\text{-cycle of } T_1\}$$

Note that by inspection of the graph T_1^m we find exactly 2^m fixed points, which is why we can use the notion of "min" instead of "inf". The beauty of the proof lies in the fact that the function $h(\cdot)$ orders the natural numbers in the Sharkosky order resulting in the set of periods of $T_{h(m)}$ to be exactly the tail starting at m. To assure this we observe the following properties:

(1) T_h has an l-cycle $\mathcal{O} \subseteq [0, h)$ if and only if $h(l) < h$. This follows directly from the observed relation of cycles in T_h and T_1 and from the definition of $h(l)$.

(2) The orbit of $h(m)$ is an m-cycle for $T_{h(m)}$. This is also a direct consequence of the definition of $h(m)$.

(3) All other cycles of $T_{h(m)}$ lie within $[0, h(m))$. This is the case as the image of $T_{h(m)}$ is $[0, h(m)]$ and $h(m)$ already has period m.

From these properties we can conclude this next and final lemma:

Lemma 8.3. It holds true that for all $m, l \in \mathbb{N}$:

$$h(l) < h(m) \iff l \lhd m.$$

Proof:

$\underline{\Leftarrow:}$ Property (2) together with the Sharkovsky forcing theorem tells us that $T_{h(m)}$ has an l-cycle for every $l \lhd m$. Through property (3) we know that this cycle needs to lie within $[0, h(m))$. Now $h(l) < h(m)$ follows from property (1).

$\underline{\Rightarrow:}$ This case is equivalent to showing that $l \rhd m \Rightarrow h(l) > h(m)$, making the case symmetrical to the first one.

∎

Now from this key property combined with the property (1) and (2) we conclude the set of periods of the map $T_{h(m)}$ to be the tail of the Sharkovsky order that starts in m.

[2] Chapter 9 provides an example for this problem

There are two special cases remaining: the set of all powers of 2 and the empty set. For the latter we just need to consider the translation $x \mapsto x - 1$ on $\mathbb{R}$. The former can be concluded by defining $h(2^\infty) := \sup_k h(2^k) \in [0, 1]$. Here the above key property applies again such that $h(2^\infty) > h(2^l)$ for all $l \in \mathbb{N}$. By property (1) $T_{h(2^\infty)}$ has 2^l-cycles for all $l \in \mathbb{N}$.
Suppose now that $T_{h(2^\infty)}$ has an m-cycle for some m that is not a power of 2. As m is not a power of 2, the number $2m$ is smaller than m in the Sharkovsky order, assuring that $T_{h(2^\infty)}$ also has a $2m$-cycle.
As these two cycles are necessarily disjoint, at least one of them is contained in $[0, h(2^\infty))$. This is because again the image of $T_{h(2^\infty)}$ is $[0, h(2^\infty)]$ and the value $h(2^\infty)$ can only lie within one of these two cycles. In conclusion one of the two cycles needs to lie in $[0, h(2^l))$ for some $l \in \mathbb{N}$ as the series $h(2^l)$ approaches $h(2^\infty)$.
Now this implies either m or $2m$ is a power of two because of properties (1) to (3), which constitutes a contradiction.

With these last two special cases this concludes the realization theorem altogether.

■

Remark. $h(2^\infty)$ can in fact be computed by computing some values of the series $h(2^l)$. In chapter 9 we will approximate $h(2^\infty)$ in this manner and additionally see, why this value is of interest for chaos theory.

9 Further observations on the Truncated Tent Maps

In the previous chapter, which provided a proof of the Sharkovsky realization theorem, we introduced the family of truncated tent maps and saw only within this special family an example for each possible tail of the Sharkovsky order.
Given a natural number m, the key idea was to find a certain height $h(m)$ such that every member of the family with a height greater than $h(m)$ has at least one m-cycle[3] and proving that $h(\cdot)$ brings the natural numbers into Sharkovsky's order.

One may wonder now what $h(m)$ looks like for different $m \in \mathbb{N}$. When trying to compute $h(m)$ as it is defined in chapter 8, one quickly runs into problems, as all zeros of the function $T_1^m(x) - x$ need to be computed in order to find the periodic points of period m. Unfortunately the mth iteration of T_1 has 2^m zeros, making the algorithm of exponential complexity. This can easily be understood by having a look at figure 7.

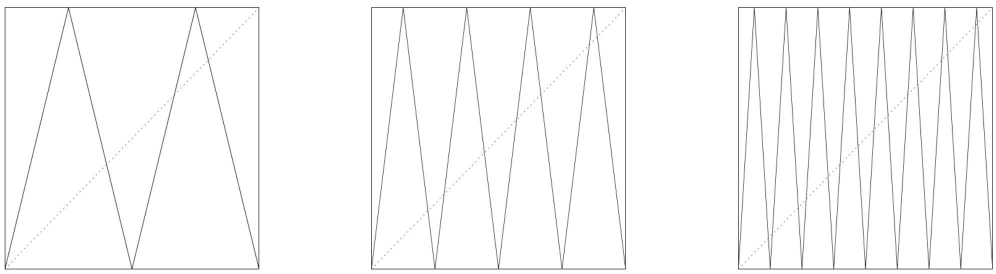


Figure 7: 2,3 and 4 iterations of T_1

As with every iteration we obtain double the amount of tents within the interval, the number of zeros doubles in every step, which explains the complexity being exponential.

In this chapter we provide an algorithm that determines $h(m)$ and display the results at least for small numbers. These results are sufficient to determine an approximate value for $h(2^\infty)$, which is of interest because of how the Sharkovsky order relates to chaos theory. Lastly we will see an example showing that $h > h(m)$ is only a sufficient criterion for finding periodic points and not a necessary one.

This chapter includes some plots of iterated truncated tent maps, so hopefully we can convey the beauty and complexity of these functions.

[3]Unfortunately there can be a periodic point of period m even though h is smaller than $h(m)$. We will later provide an example for such a case.

9.1 Computing the critical height $h(m)$

We remember that given a natural number m the critical height from whereon we will find periodic points of period m was defined in the following way:

$$h(m) := \inf\{\max \mathcal{O} | \mathcal{O} \text{ is an } m\text{-cycle of } T_1\}$$

In order to find $h(m)$ we took the following approach with code written in Python:

1. Calculate a zero of $T_1^m(x) - x$ using Newtons method between 0 and 1 with a stepsize of δ[4]. We call that zero x_0.

   a) As x_0 is a periodic point of period m or smaller, we can compute its cycle and save it in a set. As sets do not save the same entry twice, checking the cardinality of the set will then allow us to check whether x_0 is in fact of period m.

   b) We then copy the set into a list (loc_list), which allows us to sort the elements and therefore have the maximum of the cycle in the last entry.

   c) If the set has the right cardinality and is not listed in a list of all cycles (cycles_list) yet, we append the set to this list and append the last element of loc_list (this was the maximum of our cycle) to a list of all maximum elements called max_list.

   d) We continue by calculating the next zero until we have eventually run through the whole interval $[0, 1]$.

2. Now everything we need to do is to sort the list of all maximums and print out the first entry. This will be the smallest of all maximal elements of cycles of length m or, in other words, $h(m)$.

As already mentioned, this algorithm only works for very small m, as it is of exponential complexity. Luckily enough, the biggest and the smallest number in the Sharkovsky order are monadic, such that we can easily determine the range wherein all of the critical heights can be found. On the following page a table of the numbers up to 16 can be found with their respective critical height as well as the number of periodic points of the respective period. Apart from that, the code realizing the above algorithm can be seen. Unfortunately, for a big number of zeros the algorithm is already running into its limits and does not provide exact solutions, as can be seen in the case of $m = 16$. $h(16)$

[4] The stepsize δ depends on how many zeros the function has for the mth iteration or in other words on the size of 2^m

m	$h(m)$	number of periodic points
3	0.857	8
5	0.839	32
7	0.835	128
9	0.833	512
...	...	...
$2 \cdot 3 = 6$	0.8254	64
$2 \cdot 5 = 10$	0.82534	1024
$2 \cdot 7 = 14$	0.82487	16384
...	...	...
$2 \cdot 2 \cdot 3 = 12$	0.82399	4096
...	...	...
$2^4 = 16$	0.8236	65536
$2^3 = 8$	0.8237	256
$2^2 = 4$	0.8235	16
2	0.8	4
1	0	2

Table 1: Critical height for small periodic points

would have to be bigger than $h(8)$, which is not the case in the results presented here[5]. Nevertheless, for small values of m the algorithm seems to deliver results as expected and the mentioned property of $h(\cdot)$ can be confirmed.

[5] Another plausible explanation for why the algorithm fails could simply be, that the function gets very chaotic. As can be seen in the beginning of this chapter, the nth iteration of T_1 has 2^n tents squeezed into the interval $[0, 1]$. So for the 16th iteration we need to put over 65000 tents into this rather small interval. This means essentially, that for a very small difference in the starting values, there can be enormously different outcomes, even after "only" 16 iterations. Someone familiar with chaos theory could have already surmised this, as the comparably easy tent function has a horseshoe which results in very chaotic behaviour.

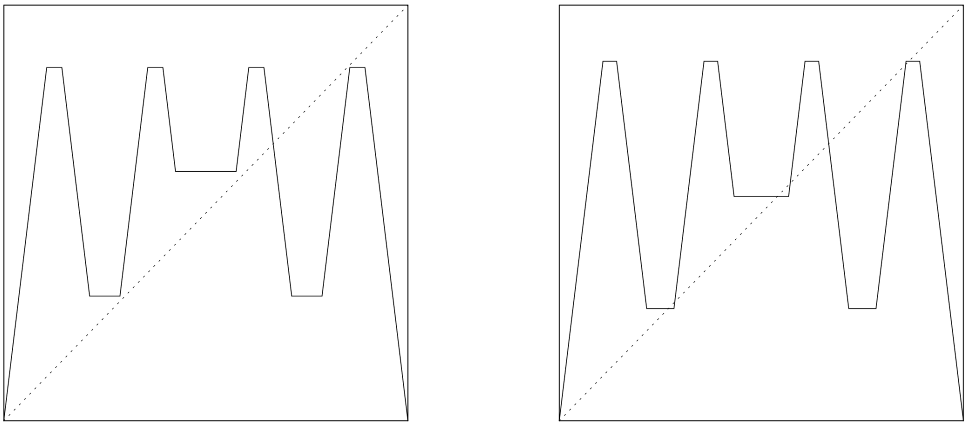
```python
# create empty lists for all cycles and maximum elements
cycle_list = []
max_list = []

for i in tqdm(list(drange(0,1,str(delta)))):
  # compute a zero of T_1^n(x)-x
  x = newtons_method(f,df,i,0.01)
  loc_set = set()
  loc_list = []
  # compute cycle of x and write it into loc_set
  for j in range(0,iterations):
      loc_set.add(round(iterT(j,x),3))
  # copy set into a list and sort the list
  for y in loc_set:
      loc_list.append(y)
  loc_list.sort()
  # if x has period m and the was not mentioned yet, append the cycle to the list of cycles and the maximum to the list of maximums
  if len(loc_set) == iterations and loc_set not in cycle_list:
      cycle_list.append(loc_set)
      max_list.append(loc_list[iterations-1])
# sort the list of maximums and print the first entry (the smallest element)
max_list.sort()
print("h(" + str(iterations) +") = " + str(max_list[0]))
```

Algorithm to determine $h(m)$ written in Python

To get an idea that the values in this table are close to true, we want to give some graphical intuition, namely by plotting some of these iterations, once with a height slightly lower and once with a height slightly bigger than the respective value of $h(m)$ in the table. The results can be seen in figures 8 and 9.

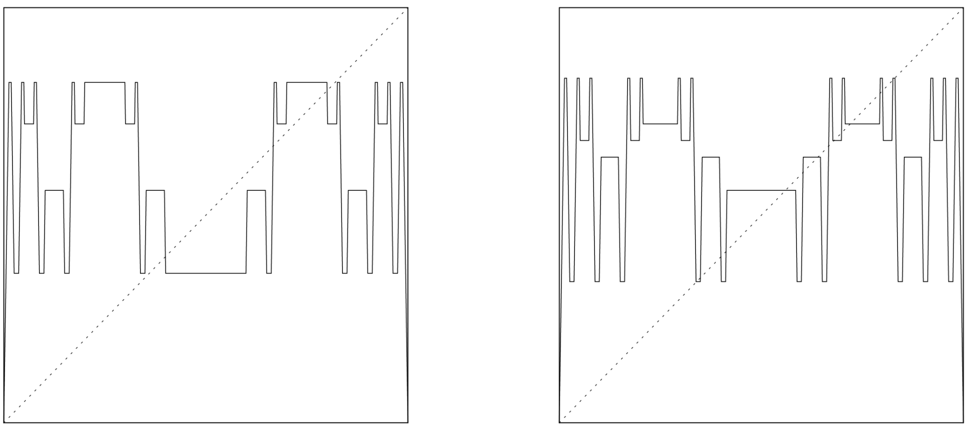
As can be seen in these images, as soon as the critical height $h(m)$ is exceeded, we find new fixed points of T_h^m, suggesting that the values returned by the algorithm are reasonable. In the case of $m = 6$ we even observe the function turning more complex with the increase in h as new spikes appear in the graph, resulting in several new fixed points.

What is probably one of the most interesting observations about the behaviour of $h(\cdot)$ is that $h(\mathbb{N} \setminus \{1\})$ has a diameter of only roughly 0.05. Reading the proof provided in chapter 8 one would suspect the values of $h(\cdot)$ to be found all over the interval $[0, 1]$. Apart from that, the values provided for $m = 2^k$ already give a rough idea of where $h(2^\infty)$ lies, which would be somewhere around 0.8238.
In both cases Sharkovsky's theorem is used to conclude the result, giving an idea of some possible applications. The second result is especially interesting when one knows more about chaos theory, namely about topological entropy. [7] can be consulted for exact information. The part interesting for us is the relation between topological entropy and Sharkovsky's order, which we will explore to some extent in the next section.



(a) Plot of $T^3_{0.85}$ (b) Plot of $T^3_{0.865}$

Figure 8: Comparison of 3 iterations with h slightly below and sligthly above $h(3) = 0.857$



(a) Plot of $T^6_{0.82}$ (b) Plot of $T^6_{0.83}$

Figure 9: Comparison of 6 iterations with h slightly below and sligthly above $h(6) = 0.8254$

9.2 Topological entropy and Sharkovsky's order

First of all we want to introduce the notion of topological entropy. The concept was first introduced by Adler, Konheim and McAndrew [6] and is expressed as a nonnegative extended real number that measures the complexity of a discrete dynamical system[6]. For the initial definition the underlying space X of the discrete dynamical system only needs to be a compact topological Hausdorff space.
However we are going to focus on a later definition that requires the additional structure of a metric on X. This definition by Bowen [1] might be less general but it has the benefit of being more comprehensible and actually clarifying the meaning of topological entropy.

Definition 9.1. Let $(X, d)X$ be a metric space and $f : X \to X$ a continuous function.

(i) For $n \in \mathbb{N}$ and $\varepsilon > 0$ we call a set $E \subseteq X$ (n, ε)-*separated* for f if for all distinct points $x, y \in E$ we find $k \in \{0, \ldots, n-1\}$ such that

$$d\left(f^k(x), f^k(y)\right) > \varepsilon$$

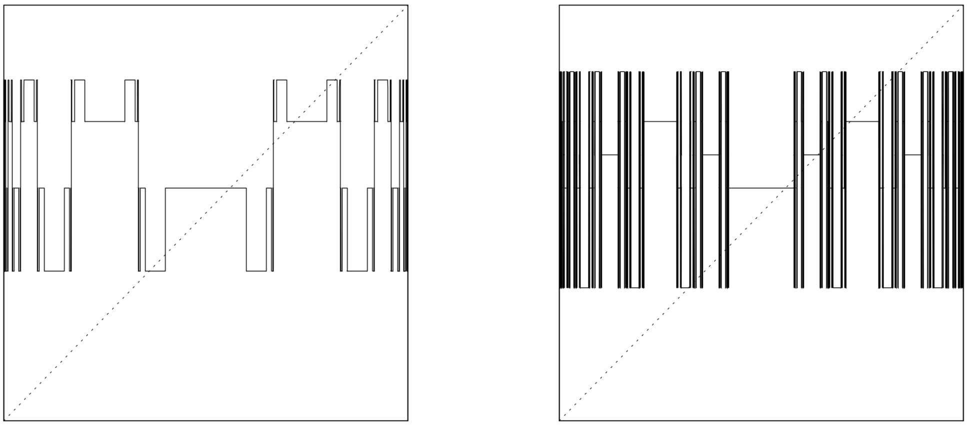
We refer to the maximal cardinality of an (n, ε)-separated set in X for f with $s_n(f, \varepsilon)$.

(ii) Finally the *topological entropy* of a discrete dynamical system (f, X) is defined by

$$h_{top}(f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \tfrac{1}{n} \log s_n(f, \varepsilon) \in [0, \infty]$$

Now one would have to show that this definition is welldefined by checking wether it actually, but as this section is just meant to give a rough idea of topological entropy and how it describes chaotic behavior we just included the definition for the sake of completeness and hence do not want to go too much into detail. In case the reader is interested in more information, [7] can be consulted.
Basically h_{top} describes the rate in which the number of distinguishable orbits grows. In that sense it is more than suited to give a sense of how complex the system gets. It is fair to say that, as long as the topological entropy is equal to 0, the function is quite manageable. But as soon as we arrive at a value great than 0, chaos arises and the function turns far more complicated.
Now one may wonder how this definition relates to Sharkovsky's order. In [7] the answer to that question is described as "one of the most striking results in interval dynamics":

[6] In this context by discrete dynamical system we mean a self-map f on some topological space X.



Twenty iterations with $h = 0.82$ Twenty iterations with $h = 0.84$

Figure 10: Comparing higher iterations of T_h around $h(2^\infty) = 0.8238$

Theorem 9.2. [7] For I a compact interval and $f : I \to I$ a continuous interval map the follwing two assertions are equivalent:

(i) f has a periodic point whose period is not a power of 2,

(ii) f has positive topological entropy.

This means that as long as the Sharkovsky type of a function is smaller than 2^∞ its topological entropy is equal to 0. This result is in fact so powerful that Coppel uses it as his definition for chaos, saying that a dynamical system is *chaotic* if it has a periodic point of period other than a power of 2 [4].
The proof for this statement requires a deeper dive into chaos theory as it uses (among others) the notion of horseshoes. It exceeds the purpose of this section. Again, if one is interested in the complete proof, it is found in [7].
Now hopefully the reader got an idea what makes the value of $h(2^\infty)$, which is approached in the previous section, so interesting. It represents the border between chaos and non chaos. Coming back to our example of the truncated tent maps we can see quite beautifully how this theorem plays out in praxis[8]. Figure 10 shows the plots of a higher number of iterations of T_h with values of h slightly above and below the critical value $h(2^\infty)$. It illustrates clearly, how the tiny step over the "frontier of chaos" can make a huge difference regarding complexity on the long run.

This result was simply too interesting to us to not be included in this thesis and we hope that the reader agrees with us.

[7] This theorem cannot be attributed to a single person as it was proven in several steps by different mathematicians. Among them Bowen and Sharkovsky.

[8] if you want to call it praxis

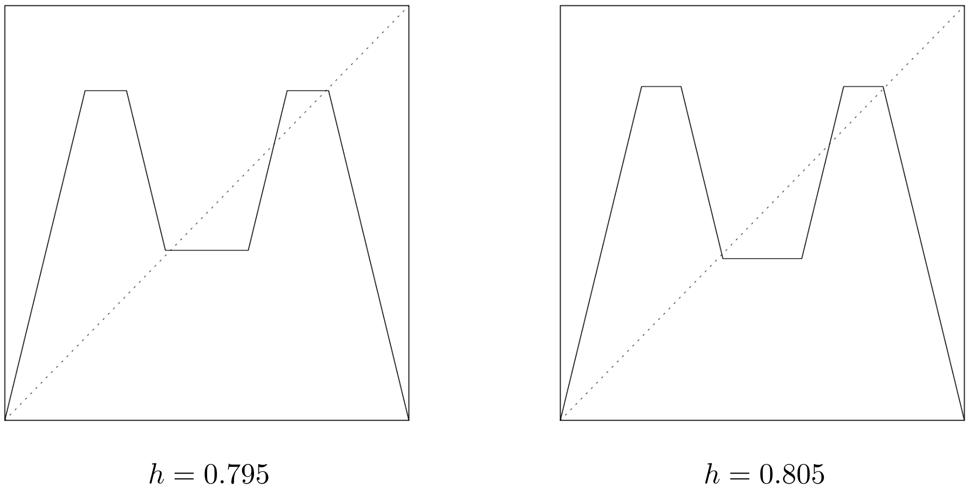


Figure 11: 2 iterations with h slightly below and above $h(2)$ =0.8

9.3 Finding points of period 2

Now that we caught a glimpse of chaos theory and how this simple family of truncated tent maps can illustrate one of its most striking theorems, we want to come back to the initial question of when the function starts to have periodic points with a period greater than 1 and present a small theorem and its proof which arose from my own contemplations of these truncated tents.

Unfortunately, the height of a truncated tent map being above $h(m)$ for some $m \in \mathbb{N}$ is only a sufficient criterion and not a necessary one when it comes to finding periodic points of period m. This means that, in some cases, even though the height in which we cut off the tent is below $h(m)$, we are still able to find an m-cycle. This is already true for the case $m = 2$, as can be seen in figure 11.
We know however, thanks to Lemma 8.2, that any m-cycle found beneath a height of $h(m)$ has to contain the height h as a point. If it would not contain h, than it would be a cycle for T_1 as well and its maximal element would be necessarily smaller than $h(m)$, which would lead to a contradiction due to the definition of $h(m)$. This can also be seen in figure 11, where, in the first picture with $h = 0.795$, one of the fixed points lies on the plateau, which means that this point has to be h. The plateaus are in fact the only areas, in which T_1 and T_h are different from each other.
With this in mind we can formulate and prove the following theorem.

Theorem 9.3. Let $h < h(2)$. Then the following statement holds true:

$$T_h \text{ has a periodic point of period 2 if and only if } T_h(h) < h.$$

Proof:

⇒: Let $\mathcal{O}$ be a 2-cycle of T_h. If it was $\mathcal{O} \subseteq [0, h)$, then according to 8.2, $\mathcal{O}$ would transfer to a cycle of T_1. But as all its elements (and hence particularly $\max \mathcal{O}$) are smaller or equal to h and $h < h(2)$ this would form a contradiction. As $\mathcal{O} \subseteq [0, h]$ and $\mathcal{O} \nsubseteq [0, h)$, the height h has to lie in $\mathcal{O}$.
This means that h is a 2-periodic point and is hence not a fixed point of T_h. As $T_h([0,1])$=$[0, h]$ we can directly conclude $T_h(h) < h$.

⇐: For this part we prove the contraposition. So let T_h not have any 2-periodic points. We need to prove that $T_h(h) = h$. Without loss of generality we can assume $h > \frac{1}{2}$, as otherwise

$$T_h(h) = \min(h, 2h) = h.$$

And we are already finished. Therefore $T_h(h) = \min(h, 2 - 2h)$. We can now easily calculate the values with $T_h(h) < h$:

$$T_h(h) < h \iff 2 - 2h < h \iff h > \frac{2}{3}$$

So again w.l.o.g. we can assume $T_h(h) = 2 - 2h$. We go on by performing a second iteration of T_h, observing that

$$T_h^2(h) \in \{h, 4 - 4h, 4h - 2\}.$$

As $4 - 4h < h$ is equivalent to $h > 0.8 = h(2)$[9] we can eliminate this case due to our initial assumption. Furthermore $4h - 2 < h$ is equivalent to $h < \frac{2}{3}$, already resulting in $T_h(h) = h$ as we have seen above. We therefore remain with the case $T_h^2(h) = h$. But by our premise, T_h does not have points of period 2, so the period of h has to be a proper divisor of 2. Hence h is a fixed point of T_h. ■

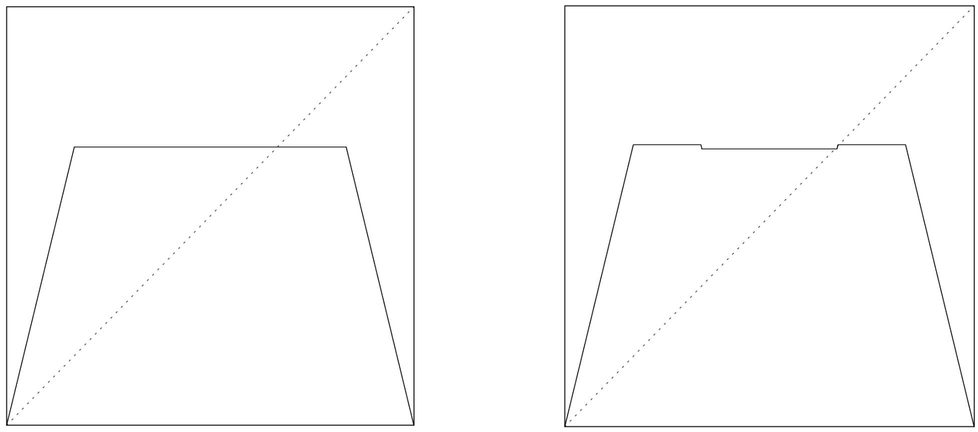
Now from this proof we can extract the following corollary:

Corollary 9.4. T_h has a 2-periodic point if and only if $h > \frac{2}{3}$.

Proof: In the case that $h \geqslant h(2)$ this is clear and follows directly from our considerations in chapter 8. Else the statement is a direct consequence of theorem 9.3. ■

In figure 12 it can be seen that as soon as the height of the truncation is above $\frac{2}{3}$ the function gets addiotional bumps, which cause the 2-periodic points.

[9] If the reader mistrusts the algorithm stated in section 9.1, it is a fairly easy task to compute the value of $h(2)$ by hand.



$h = 0.665$ $h = 0.67$

Figure 12: 2 iterations with h slightly below and above $\frac{2}{3}$

If we apply further iterations on the left side, we can observe the funtion approaching the constant function $f(x) = h$. This is a nice example for another early theorem of chaos theory, which was proven by Coppel [3] in the mid 1950's. It states that every point converges to a fixed point under iteration of a continuous map of a closed interval if the map does not have a 2-periodic point. This is in fact the least chaotic way a function can behave, further constituting to the association between chaos and the advance in Sharkovsky's order, which we caught a first glimpse of in section 9.2. So it is fair to say that the truncated tent maps with a height less or equal to $\frac{2}{3}$ are easily predictable, as all its points approach h, whereas higher truncations get more and more complicated with increasing height.

This concludes our little collection on further information on the truncated tent maps, Sharhovsky's theorem and chaos theory. We hope that this little excursion shed some light on how Sharkovsky's order is intertwined with chaos theory and how it can be useful to predict the longtime behaviour of a function.

But above all we hope that this last chapter (and of course the whole thesis) was delightful, as Georg Cantor put it so eloquently:

"The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful."

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