HOLONOMIC APPROXIMATION

Plan :

- INTRODUCTION (LUCAS)
- JETS BUNDLE AND HOLONOMIC SECTION
- APPROXIMATION THEOREM: THE "WIGGLING" METHOD

" Thank you Lucas for this nice Introduction " :)

Remark 1 : " NO CAKE " :(

Jets Bundle and Holonomic sections

Let $f: \mathbb{R}^m \longrightarrow \mathbb{R}^q$ be a smooth function, the string of its derivatives up to order r at the point x is:

$$J_f^r(x) = \left(f(x), f^{(1)}(x), f^{(2)}(x), \dots, f^{(r)}(x)\right)$$

where $f^{(s)}$ consists of all partial derivatives $\partial_{x_{i_{\alpha_1}}^{\alpha_1} \cdots x_{i_{\alpha_m}}^{\alpha_m}} f$ with $i_{\alpha_i} \in \{1 \dots m\}$ and $\sum_{i=1}^{m} \alpha_i = s$.

We want to construct a space which "a priori" collects all the possible strings build up throught smooth functions from $\mathbb{R}^m$ to $\mathbb{R}^q$.

Let $d_s = d(s,m)$ the number of all derivatives at the order s (we are not counting repetition du to $\partial x_i \partial x_j = \partial x_j \partial x_i$) and $N_r = \sum_{i=0}^{r} d(i,m)$.
Note that $d(i,m) = \frac{(m+i-1)!}{(m-1)!\,i!}$ and $N_r = \frac{(m+r)!}{m!\,r!}$

Let $x \in \mathbb{R}^m$,

$$x \times \mathbb{R}^{qN_r} = x \times \mathbb{R}^q \times \mathbb{R}^{qd_1} \times \dots \times \mathbb{R}^{qd_r}$$

is the space of all possible derivatives (up to order r) for a function $f: \mathbb{R}^m \longrightarrow \mathbb{R}^q$

The r-jet space for the trivial bundle $\mathbb{R}^m \times \mathbb{R}^q \xrightarrow{P} \mathbb{R}^m$ is

$$J^r(\mathbb{R}^m, \mathbb{R}^q) = \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^{qd_1} \times \dots \times \mathbb{R}^{qd_r}$$

$P_0^n : J^n(\mathbb{R}^m, \mathbb{R}^q) \longrightarrow \mathbb{R}^m \times \mathbb{R}^q$ the projection

The composition $p^n = p \circ P_0^n$ gives the natural structure of the n-Jet Bundle:

$$p^n : J^n(\mathbb{R}^m, \mathbb{R}^q) \longrightarrow \mathbb{R}^m \quad ; \quad \begin{tikzcd} \mathbb{R}^m \times \mathbb{R}^q \arrow[d, "p"] & J^n(\mathbb{R}^m, \mathbb{R}^q) \arrow[l, "P_0^n"'] \arrow[dl, "p^n"] \\ \mathbb{R}^m & \end{tikzcd}$$

exemple

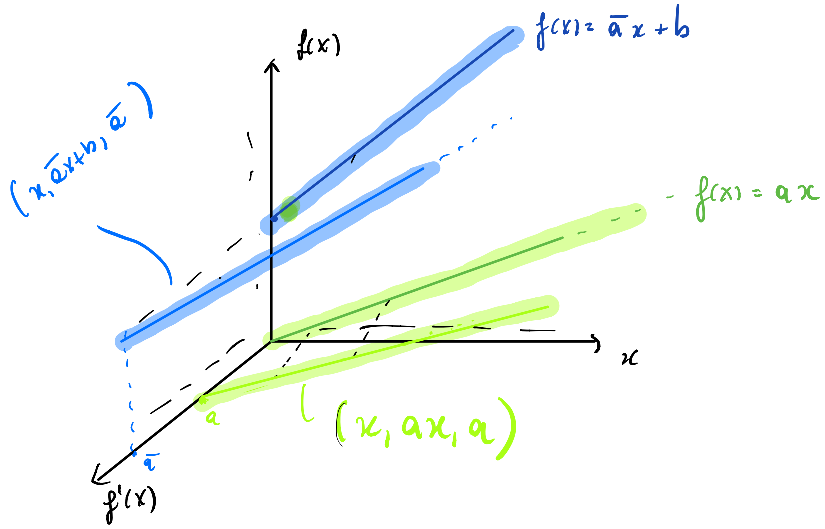
$$J^1(\mathbb{R}^m, \mathbb{R}^q) = \mathbb{R}^m \times \mathbb{R}^q \times M_{m,q}$$
$$(\text{where } M_{m,q} = \mathbb{R}^{mq})$$

Given a function (section) $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m \times \mathbb{R}^q$ the n-Jet section or the n-Jet extension of f is:

$$J_f^n : \mathbb{R}^m \longrightarrow J^n(\mathbb{R}^m, \mathbb{R}^q)$$
$$x \longrightarrow J_f^n(x) = (x, f(x), f^{(1)}(x), \dots, f^{(n)}(x))$$

exemple $f : \mathbb{R} \longrightarrow \mathbb{R}$
$x \longrightarrow ax + b$

$J^1(\mathbb{R}, \mathbb{R})$
$\shortparallel$
$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$



Definition A section $F : \mathbb{R}^n \longrightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$ is said to be holonomic (or holonomic section) if $\exists\, f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^q$ such that

$$J^r_f = F$$

Remark $\forall z \in J^r(\mathbb{R}^n, \mathbb{R}^q)$ $\exists!$ $p^q_r : \mathbb{R}^n \longrightarrow \mathbb{R}^q$ q-polynomial function of degree $\deg(P^q_r) \leq r$ such that

$$J^r_{p^q_r}(p^r(z)) = z$$

In other words every section $F : \mathbb{R}^n \longrightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$ is "punctually" holonomic.

In the following every thing is Smooth. Given a subset $A \subset V$, (V manifold) we denote by $\mathcal{O}_p A$ an arbitrarily small open neighborhood of A.

Jets Bundle for General Fibration (Fibration = Fib. Bundle)

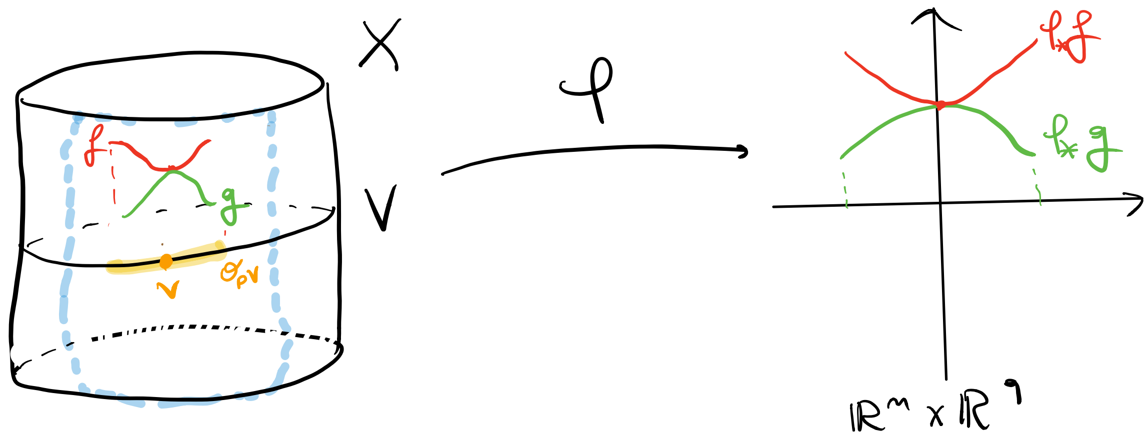
Let $p : X \longrightarrow V$ a fibration of rank q over $V = V^m$, $v \in V$

Two local section $f, g : \mathcal{O}_p v \longrightarrow X$ are called r-tangent at the point v if

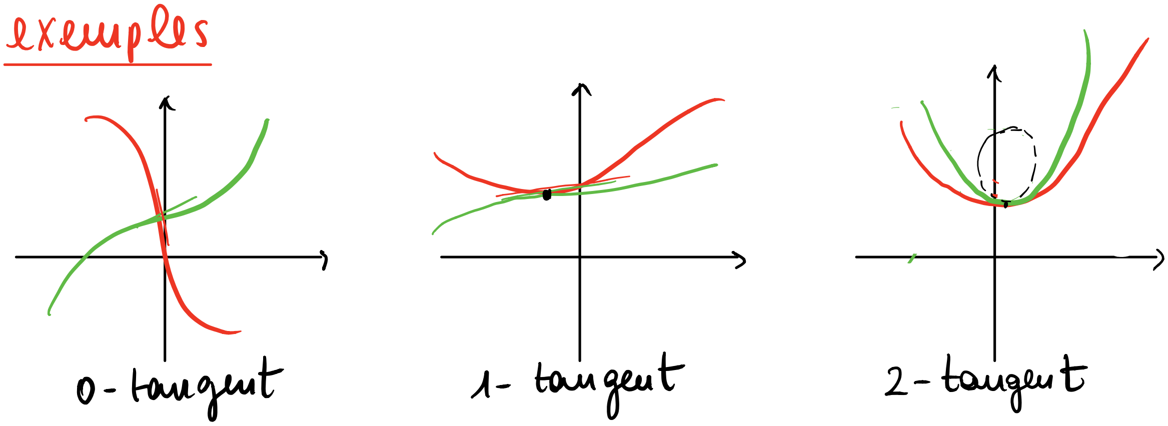
(1) $f(v) = g(v)$

(2) $J^r_{\varphi_* f}(\varphi(f(v))) = J^r_{\varphi_* g}(\varphi(g(v)))$

where $\varphi : \pi^{-1}(\mathcal{O}_p v) \longrightarrow \mathbb{R}^m \times \mathbb{R}^q$ a local trivialization



exemples



As consequence of the Chain Rule, being r-tangent is an equivalence over the local section of $X \longrightarrow V$.

The r-tangency class of a local section $f : \mathcal{O}_p v \longrightarrow X$ at the point v is called the r-jet of f at v.

We define $X^{(r)}$ as the set of all r-jets of section $V \longrightarrow X$.

The smooth structure of this set is given by the charts on the form

$$\varphi^r : (p_0^r)^{-1}(U) \longrightarrow J^r(\mathbb{R}^m, \mathbb{R}^q)$$

which are constructed throught the local trivializations of $X \longrightarrow V$. $(\varphi : U \longrightarrow \mathbb{R}^n \times \mathbb{R}^q)$

In a similar way as the euclidian case:

$$\begin{tikzcd} X \arrow[d, "p"] & X^{(r)} \arrow[l, "p_0^r"'] \arrow[dl, dashed, "p^r = p \circ p_0^r"] \\ V & \end{tikzcd}$$

(with this smooth structure the projections are smooth)

Again: given a section $f : V \longrightarrow X$ the r-Jet extention of f is

$$J_f^r : V \longrightarrow X^{(r)}$$
$$v \longrightarrow J_f^r(v)$$

Def A section $F : V \longrightarrow X^{(r)}$ is said to be holonomic if $F = J_f^r$ for some section $f : V \to X$.

Remind: Holonomic section of $J^r(\mathbb{R}^n, \mathbb{R}^q)$ are all the form $x \longrightarrow (x, f(x), f^{(1)}(x), \dots, f^{(r)}(x))$ for some smooth function $f : \mathbb{R}^n \to \mathbb{R}^q$

We have the correspondence:

$\mathrm{Sec}\, X \qquad\qquad \mathrm{Sec}\, X^{(r)}$

$$J^n : \left\{ \begin{array}{c} \text{sections} \\ f : V \to X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{sections} \\ F : V \to X \end{array} \right\}$$

$$f \longrightarrow J^n_f$$

and is 1-1 with the image

$$J^n(\operatorname{Sec} X) \simeq \operatorname{Hol}(X^{(n)})$$

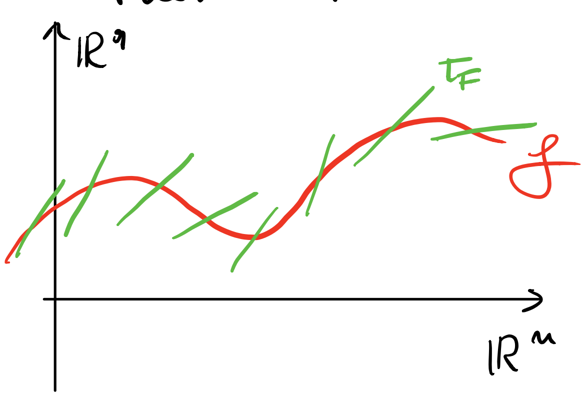
Holonomic sections of $X^{(n)}$

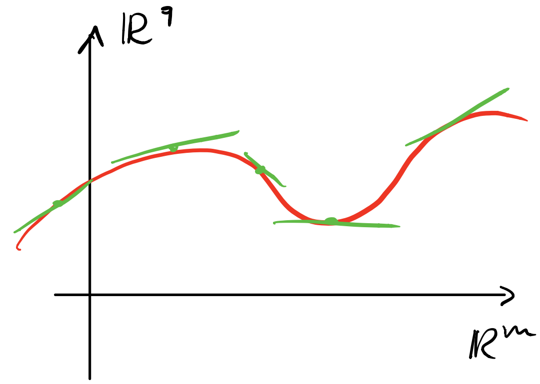
In general it's very important to understand the homotopical properties of $\operatorname{Hol}(X^{(n)})$ in $\operatorname{Sec}(X^{(n)})$.

$p : X \to V$

GEOMETRIC INTERPRETATION OF $F : V \longrightarrow X^{(n)}$:

We can see a section $F : V \longrightarrow X^{(1)}$ as a pair (f, τ_F) where $f = p^n_0 \circ F$ is a section of X and τ_F is a field of non-vertical n-planes over the graph of f. The holonomic sections are those for which these n-planes are tangent to the graph.





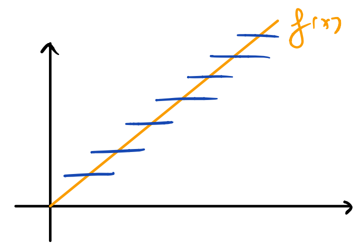
$F_{n-1} = p^n_{n-1}(F_n)$

In the general case $F : V \longrightarrow X^{(n)}$ can be view as (F_{n-1}, τ_{F_n}) where $F_{n-1} : V \longrightarrow X^{(n-1)}$ is obtained by $p^n_{n-1} \circ F$ $\left(p^n_{n-1} : X^{(n)} \longrightarrow X^{(n-1)} \right)$ and τ_{F_n} is a n-planes field in $TX^{(n-1)}$ over the graph of F_{n-1}.

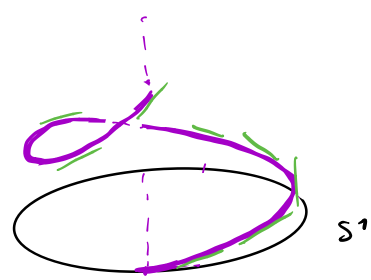
HOLONOMIC APPROXIMATION (THE WIGGLING METHOD)

QUESTION 1 Is every section $F : V \longrightarrow X^{(n)}$ holonomic? (i.e is $\mathrm{Hol}(X^{(n)}) = \mathrm{Sec}(X^{(n)})$). The answer is obviously NO

$$F : \mathbb{R} \longrightarrow J^1(\mathbb{R}, \mathbb{R})$$
$$x \longrightarrow (x, x, 0)$$



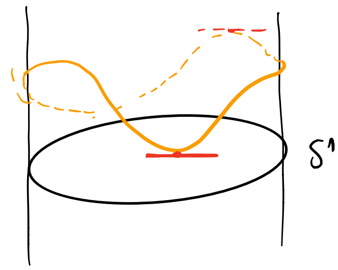
$$F : S^1 \longrightarrow J^1(S^1, \mathbb{R})$$
$$\theta \longrightarrow (\theta, f(\theta), d\theta)$$



"$d\theta$ is not exact over S^1"

QUESTION 2 Given a section $F : V \longrightarrow X^{(n)}$, is there any holonomic section C^0-close to F?
$\left(\text{i.e } \exists f : V \longrightarrow X \mid \| J^n_f - F \|_{C^0} < \varepsilon \right)$

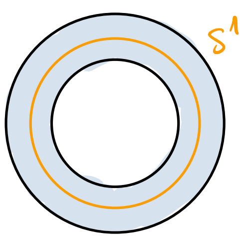
The answer is still NO. Again think about S^1, Any periodic function over $\mathbb{R}$ has at least one critical point so there is not hope to be close to do. (9)

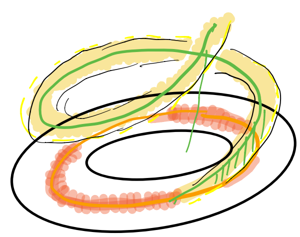


QUESTION 3 Let $A \subset V$ a positive codimensional sub Manifold, $F: V \longrightarrow X^{(n)}$ a section. Is there any holonomic section $J^n_f: \mathcal{O}_p A \longrightarrow V$ such that

$$\| F|_A - J^n_f|_A \|_{C^0} < \varepsilon$$

Thinking about the example of S^1, regarding it as 1-dimen. sub Manifold of the annulus the answer is still **NO**



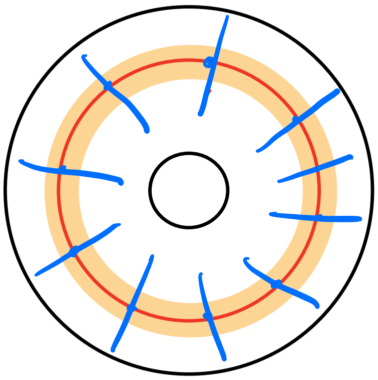


QUESTION 4 Given a section $F: \mathcal{O}_p A \longrightarrow X^{(n)}$, are there holonomic sections close to F and close to A.
The answer is finally: YES!, MANY!, but we need to deform (wiggle) A and we need at least one free direction where make it possible

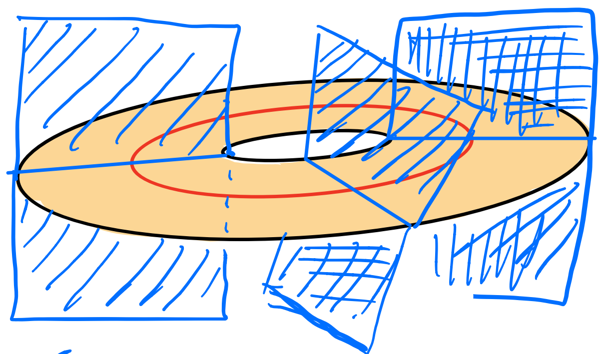
Before the main statement we look again the example of the annulus. ($\mathbb{A}$).

Let $S^1 \subset \mathbb{A}$ and we consider the section $F : \mathcal{O}_p S^1 \to J^1(\mathbb{A}, \mathbb{R})$

$$F(\theta, p) = ((\theta, p), 0, d\theta)$$

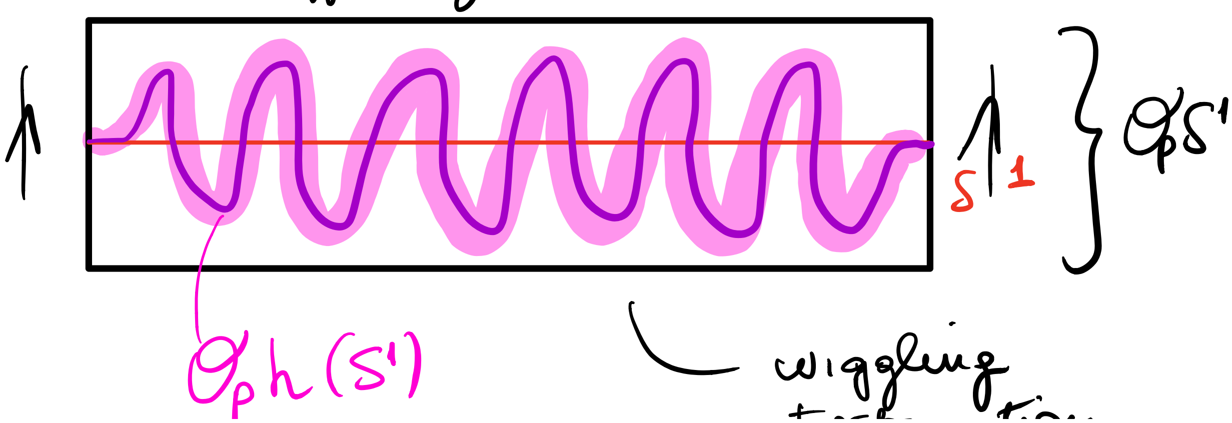


$\mathbb{A}$

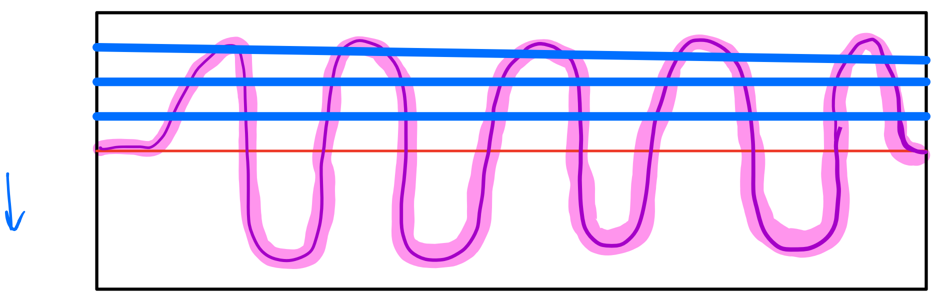


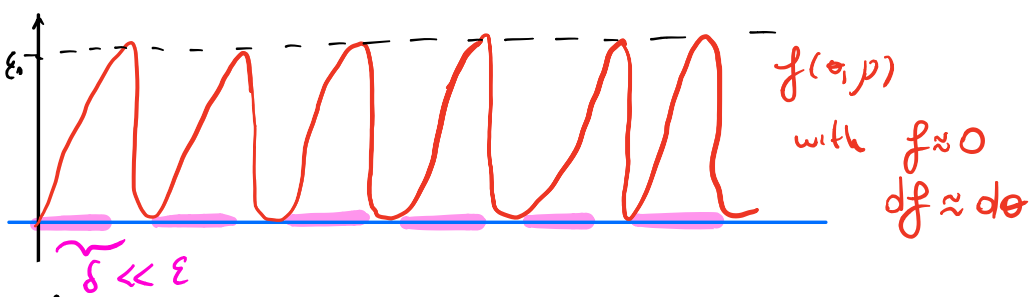
the graph of $(\theta, p) \longrightarrow 0$
where we glue the 2-planes field.
(non transversal).

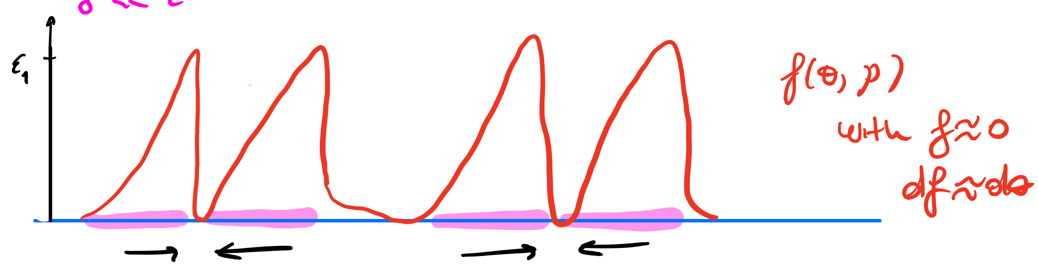
Look $\mathbb{A}$ with the following identification and we define S^1 in this manner throught a diffeotopy $h_\tau : \mathbb{A} \longrightarrow \mathbb{A}$, $\tau \in [0,1]$



We want to build a function over $\mathcal{O}_p h(S^1)$ such that $\| J^1_f - F|_{\mathcal{O}_p h(S^1)} \| < \varepsilon$ i.e we want $|f(\theta, p)|$, and $\| df - d\theta \|$ small.







the problem is when we are close to the turning points of the wiggling.
Here the main importance of the the free direction because we can deform f trought the variable p such that

$$f(\theta,p) \approx 0 \quad \text{and} \quad df \overset{\sim}{\approx} d\theta \pm \underset{\substack{\downarrow \\ \text{small} \\ \text{correction.}}}{\delta}\, dp$$

Thanks to the free direction we can glue together continuously all the little "waves" and then we get f such that

$$\|f\| \leq \varepsilon_1 \quad \text{and} \quad \|df - d\theta\| < \varepsilon_2$$

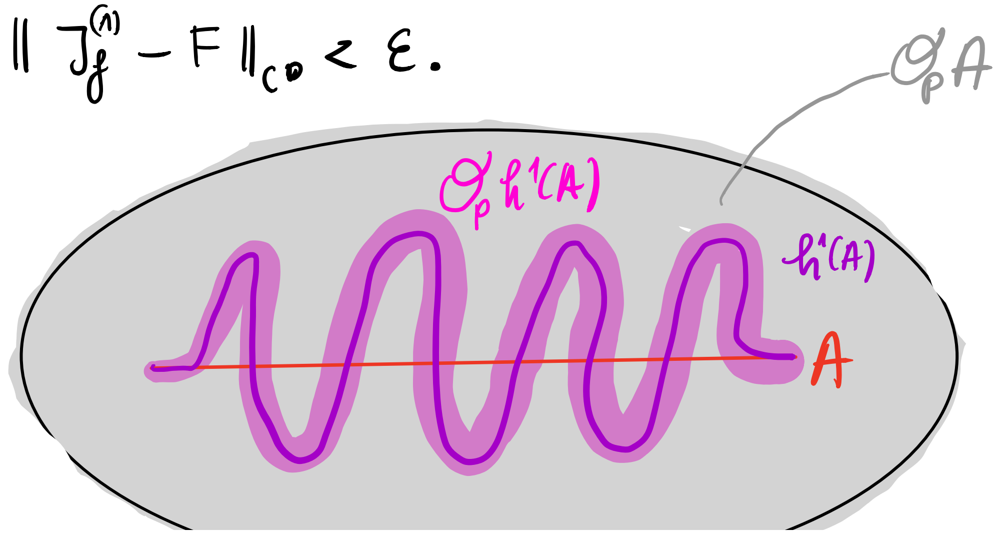
In other words $\| J^1_f - \underset{\substack{\| \\ ((\theta,p),\,0,\,(d\theta,0))}}{F}|_{\mathcal{O}_p h(S^1)} \| < \varepsilon.$

Holonomic Approximation Theorem (Polyhedron version)

Let $X \xrightarrow{\text{smooth Fib.}} V$ and $A \subset V$ a subcomplex of positive codimension and $F : \mathcal{O}_p A \longrightarrow X^{(r)}$ a section. For arbitrarily small $\varepsilon, \delta > 0$ there is a diffeotopy (wiggling transf.) $h^\tau : V \longrightarrow V$, $\tau \in [0,1]$ and a holonomic section $J^{(r)}_f : \mathcal{O}_p h^1(A) \longrightarrow X^{(r)}$ such that

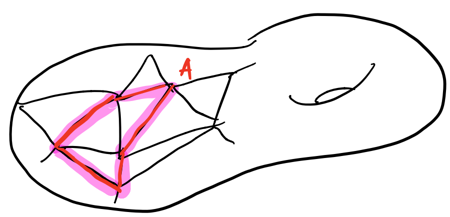
(1) h is δ-small i.e. $\|h - id_V\| < \delta$.

(2) $\| J^{(1)}_f - F \|_{C^0} < \varepsilon.$



Remarks (1) We suppose every thing compact.

(2) Polyhedron in the sense that A is a subcomplex of a smooth triangulation over V

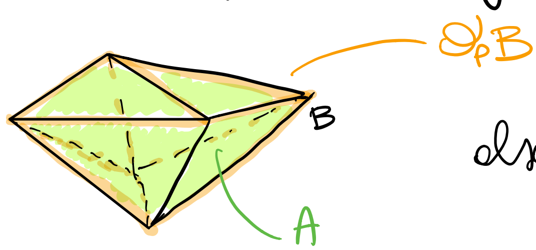


Thank Peter

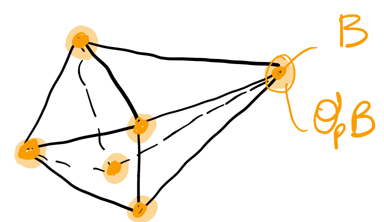
(2) We assume that $\mathcal{O}_p h^1(A_1) \subset \mathcal{O}_p A$ as well as

(3) We assume V endowed with a Riemannian Metric and $X^{(r)}$ is endowed with a euclidean metric in $\mathcal{U} = \mathcal{O}_p F(V) \subset X^{(r)}$

5 We assume F to be holonomic in an open neighborhood of a subpolyhedron of A.

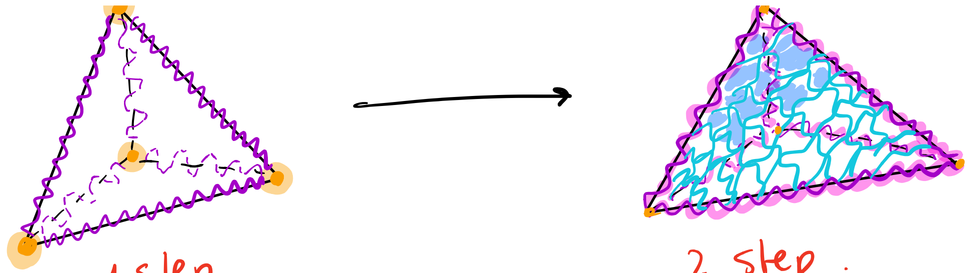


also



6 h^τ is such that fixes $\mathcal{O}_p B$, $(h^\tau(\mathcal{O}_p B) = \mathcal{O}_p B \ \ \forall \tau \in [0,1])$ and $\tilde{F}$ coincides with F over $\mathcal{O}_p B$.

About the proof: The main idea is to an induction method over the skeleton of A



1 step
(wiggle the edges)
from the corners

2 step
(wiggle the faces)
from the edges

Supposing that the fibration $X \longrightarrow V$ is trivial over the simplices we reduce the problem to the euclidean case i.e. considering $I^k \subset \mathbb{R}^k$ and its boundary ∂I^k.

Theorem (HOLONOMIC APPROXIMATION OVER A CUBE)

Let $I^k \subset \mathbb{R}^k$ $(k < m)$ k-cube on the first k-coord.
For any section

$$F : \mathcal{O}_p I^k \longrightarrow J^r(\mathbb{R}^m, \mathbb{R}^q)$$

which is holonomic over $\mathcal{O}_p \partial I^k$ and for arbitrarily small $\varepsilon, \delta > 0$ there exist a δ-small diffeomorphism $h : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ on the form $h(x_1 \dots x_m) = (x_1 \dots x_{m-1}, x_m + f(x_1 \dots x_m))$.
And a holonomic section

$$\tilde{F} : \mathcal{O}_p h(I^k) \longrightarrow J^r(\mathbb{R}^m, \mathbb{R}^q)$$

Such that

(1) $h = \mathrm{Id}$ and $\tilde{F} = F$ over $\mathcal{O}_p \partial I^k$

(2) $\| \tilde{F} - F|_{\mathcal{O}_p h(I^k)} \|_{C^0} < \varepsilon$.

Proof This result will be deduced from the INDUCTIVE LEMMA.

But first we need the notion of a fiberwise holonomic section.

Def Let $\pi : V \longrightarrow B$ a fibration (main ex. $\pi : I^k \to I^{k-\ell}$)

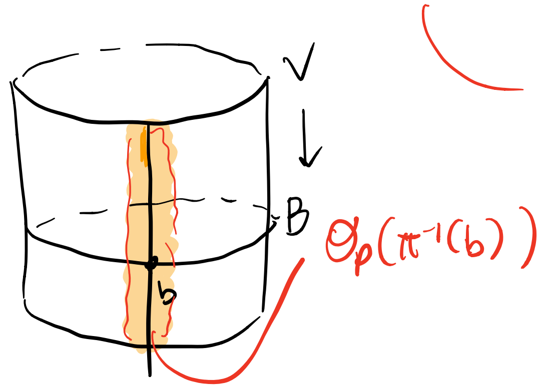
A section $F : V \longrightarrow X^{(r)}$ is said to be fiberwise holonomic if there exist a continuous family of holonomic section $\{\tilde{F}_b\}_{b \in B}$ with

$$\tilde{F}_b : \mathcal{O}_p(\pi^{-1}(b)) \longrightarrow X^{(r)}$$

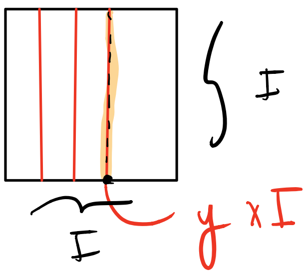
such that

$$\tilde{F}_b |_{\pi^{-1}(b)} = F |_{\pi^{-1}(b)}$$

coincide over the fiber of b.



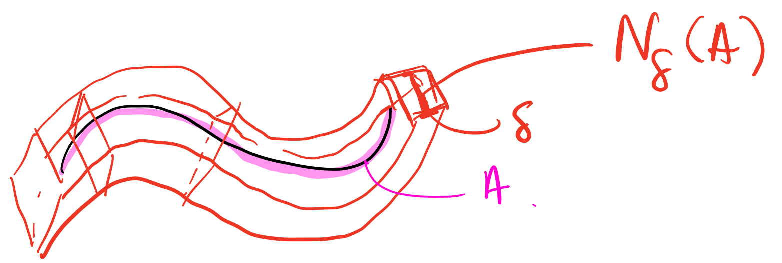
We will consider I^k as $\{y \times I^\ell\}_{y \in I^{k-\ell}}$



CASE $k=2$, $\ell=1$

(Keep in mind the cases: $m=3,\ k=2,\ \ell=1$; $m=2,\ k=1,\ \ell=0$)

Given a subset $A \subset \mathbb{R}^m$ we denote by $N_\delta(A)$ the δ-cubical neighborhood of A.



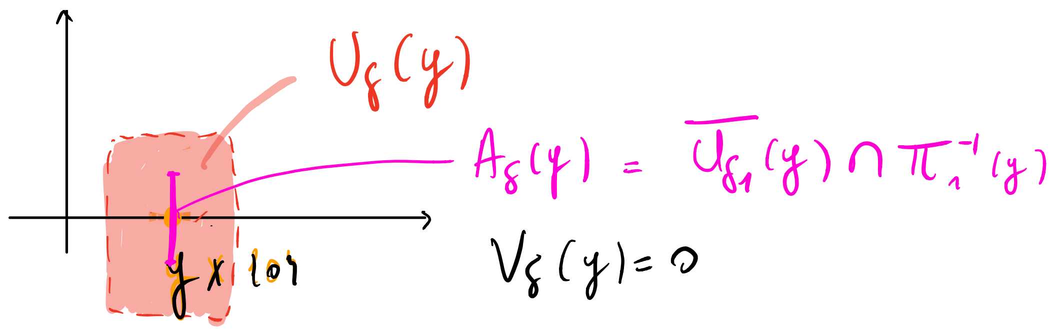
and $\pi_s : \mathbb{R}^n \longrightarrow \mathbb{R}^s$ the projec. over the first s-coordinates.

$$y = (x_1 \cdots x_{k-\ell}) \in I^{k-\ell} \subset \mathbb{R}^m, \quad \delta > \delta_1 > 0 \text{ small.}$$

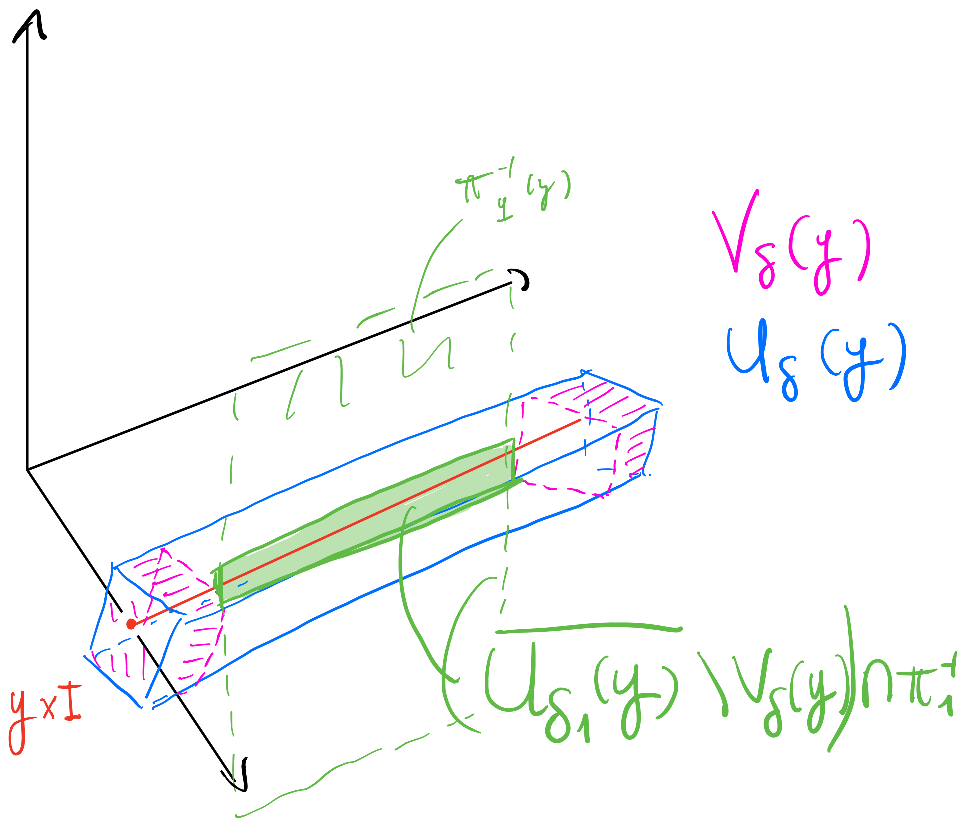
$$U_\delta(y) = N_\delta(y \times I^\ell), \quad V_\delta(y) = N_\delta(y \times \partial I^\ell)$$

$$A_\delta(y) = \left(\overline{U_{\delta_1}} \setminus V_\delta(y)\right) \cap \pi^{-1}_{k-\ell}(y)$$

CASE $m=2,\ k=1,\ \ell=0$



CASE $m=3$, $k=2$, $\ell=1$:



INDUCTIVE LEMMA (PART 1):

Let $I^k \subset \mathbb{R}^m$, $F : \mathcal{O}_p I^k \longrightarrow J^r(\mathbb{R}^m, \mathbb{R}^q)$ a section with the following properties:

- $F|_{\mathcal{O}_p \partial I^k}$ is holonomic

• For a non negative integer $l < k$ F is Fiberwise holonomic respect to the trivial fibration $\pi_{k-l} : I^k \longrightarrow I^{k-l}$ (along the cubes $y \times I^l$, $y = (z,t) \in I^{k-l} = \underset{(z\ ,\ t)}{I^{k-l-1} \times I}$)

The last condition is equivalent to say:
There is a small $\delta > 0$ and a family of holonomic sections $\{F_y\}_{y \in I^{k-l}}$ such that:

$$\cdot\ F_y = J^n_{f_y} : U(\delta) \longrightarrow J^n(\mathbb{R}^n, \mathbb{R}^q)$$

$$\cdot\ F_y|_{(y \times I^l) \cup V_\delta(y)} = F|_{(y \times I^l) \cup V_\delta(y)}$$

$$\cdot\ F_y = F|_{U_\delta(y)} \text{ for } y \in \mathcal{O}_p \partial I^{k-l}$$

Then for arbitrarily small $\varepsilon > 0$ there exist $N > 0$ integer and a family of holonomic sections

$$\tilde{F}_z : \Omega_z \longrightarrow J^n(\mathbb{R}^n, \mathbb{R}^q), \quad z \in I^{k-l-1}$$

where

$$\Omega_z = \mathcal{O}_p\Big(\bigcup_{i=1}^N A_\delta(\underset{\overset{\shortparallel}{y}}{z, c_i}) \cup z \times I^{l+1} \Big) \setminus \bigcup_{i=1}^N A_\delta(z, c_i)$$

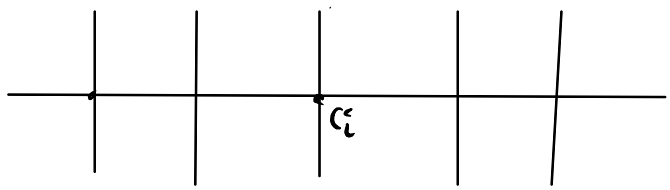
$$c_i = \frac{2i-1}{2N}, \quad i = 1 \dots N.$$

such that

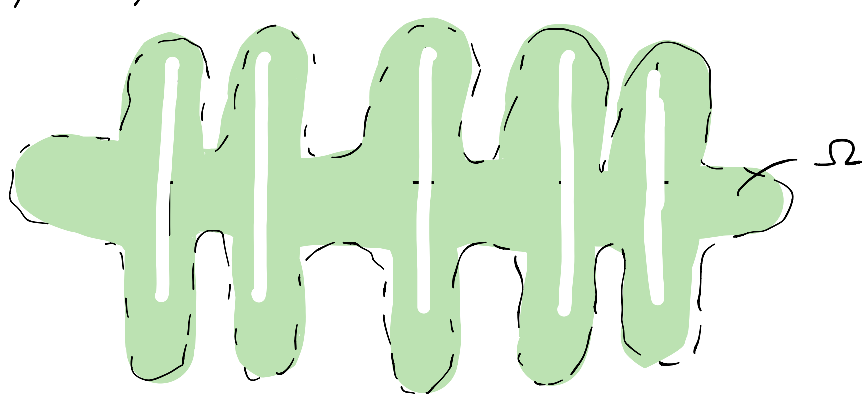
$$\cdot\ \tilde{F}_z = F \text{ on } \Omega_z \cap \mathcal{O}_p \partial I^k$$

- $\| \tilde{F}_z - F|_{\Omega_z} \|_{C^0} < \varepsilon$

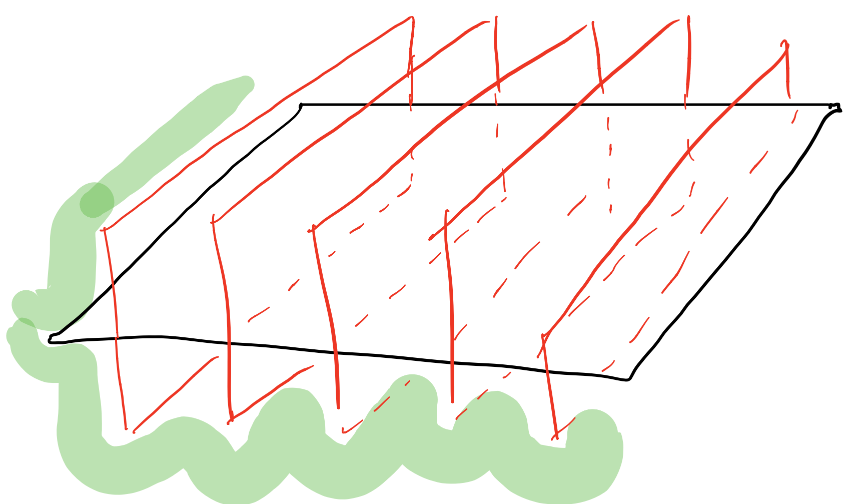
Remark: if $\ell = k-1$ we have $z \in I^\circ$ and eue the family consists in just one $\tilde{F}$ defined over $\Omega_z = \Omega$.



Case $m=2$, $k=1$, $\ell=0$



Case $m=3$, $k=2$, $\ell=1$



If inductive Lemma (First Part) holds then:

INDUCTIVE LEMMA (SECOND PART): Under the conditions of IND. LEMMA P.1 there exist a δ-small diffeomorphism

$$h(x_1, x_2, \dots, x_n) = (x_1 \cdots x_{n-1}, x_n + f(x_1 \cdots x_n))$$

and a section

$$\tilde{F} : \mathcal{O}_p h(I^n) \longrightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$$

such that:

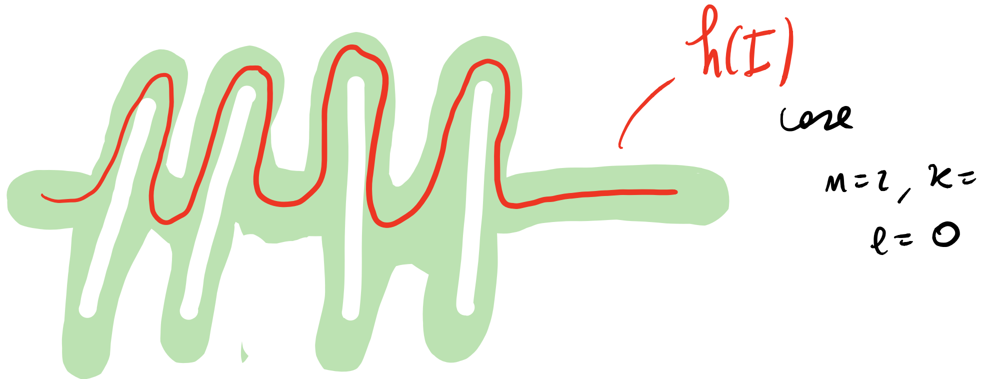
- $h = \mathrm{id}$ and $\tilde{F} = F$ on $\mathcal{O}_p \partial I^n$
- $\| \tilde{F} - F|_{\mathcal{O}_p h(I^k)} \|_{C^0} < \varepsilon$
- the section $\tilde{F}|_{h(I^k)}$ is fiberwise holonomic with respect to the fibration

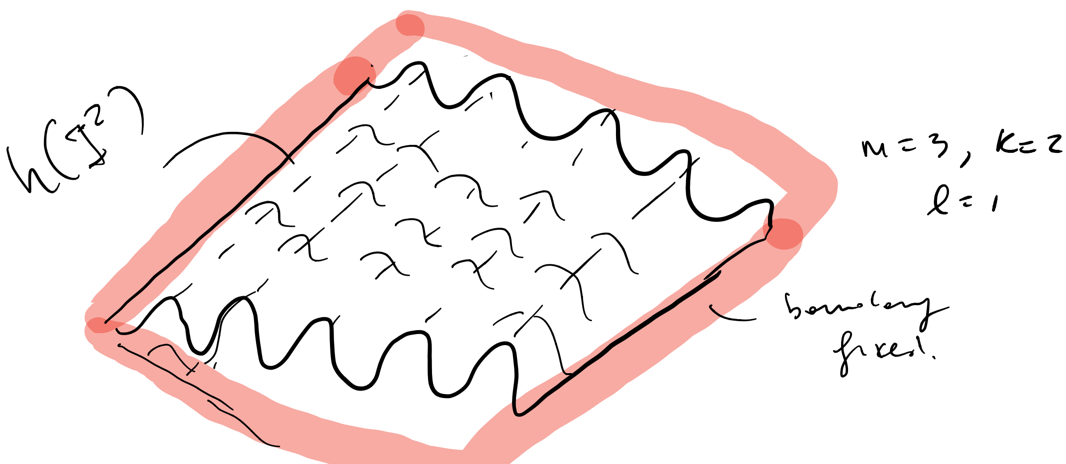
$$\pi_{k-\ell-1} : h(I^k) \longrightarrow I^{k-\ell-1}$$

reduced by 1

(i.e along cube $h(z \times I^{\ell+1})$, $z \in I^{k-\ell-1}$)

Remark In particular for $\ell = k-1$ the new section $\tilde{F}$ is holonomic as a whole section.





Proof The small diff. as in the picture.

$$\tilde{F}(z,t,x) = \begin{cases} \tilde{F}_z(z,t,x) & \text{over } \mathcal{O}_p\, h(I^k)\cap(I^{k-\ell-1}\times\mathbb{R}^{m-k+\ell+1}) \\ F(z,t,x) & \text{over } \mathcal{O}_p(\partial I^k) \end{cases}$$

is fixed. by h.