Die wunderschöne Welt der Konvexität

Unten findet ihr eine Beschreibung der einzelnen Vorträge. Die Beschreibungen sind auf Englisch, da die meisten Quellen auch auf Englisch sind. Wenn es nicht explizit geschrieben wird, beziehen sich die Kapitel auf das Buch [Web], das in der Bibliothek von Mathematikon im Apparat vom nächsten Semester zu finden ist. Andere interessante Quellen findet ihr unten im Abschnitt "Further Readings". Wenn ihr Probleme habt, Zugang zu einer der Quellen zu haben (zum Beispiel seid ihr nicht in Heidelberg), schreibt mir bitte eine Email.

1 Einführung in die konvexen Mengen: konvexe Hülle und Satz von Gauss-Lucas

In this talk, you will introduce operation on sets and recall some known facts about affine sets and transformations. Then, you will define convex sets and the notion of convex hull. As a nice application you prove the Theorem of Gauss–Lucas about roots of complex polynomials. If you are short on time, you could leave out some of the part referring to Chapter 1 (which should be familiar from Linear Algebra) from the oral presentation and include it only in the handout to distribute beforehand.

[Ch. 1.1] Definition of translate, sum of sets and of multiplication of a set by a scalar. Example in Fig. 1.1. Example that $A + A \neq 2A$. Properties i*, ii*, iii*, vi*, vii*, viii*.

[Ch. 1.2] Definition of flat and segment. Theorem 1.2.1 and Theorem 1.2.3 without proof. Definition of affine hull. Theorem 1.2.4 without proof. Definition of affine combination. Theorem 1.2.8 without proof.

[Ch. 1.3] Definition of affine dependent/independent points. Theorem 1.3.1 and Corollary 1.3.4 with proof.

[Ch. 1.5] Definition of affine transformation. Theorem 1.5.2 without proof. Corollary 1.5.3 with proof.

[Ch. 2.1] Definition of convex sets. Example 2.1.1 and 2.1.2. Theorem 2.1.3 without proof. Theorem 2.1.4 with proof. Definition of convex combination. Theorem 2.1.5 with proof. Affine transformations preserve convex sets: Theorem 2.1.10 without proof.

[Ch. 2.2] Definition of convex hull. Theorem 2.2.2 with proof. Theorem 2.2.9 (Gauss–Lucas Theorem) with proof: The roots of the derivative of a non-constant complex polynomial belong to the convex hull of the set of roots of the polynomial itself. Observe that the theorem is false for real polynomials.

2 Der Satz von Caratheodory und seine Korollare: Radon, Helly, Shapley-Folkman

In the last talk, we saw that every element in the convex hull of a set A in $\mathbb{R}^n$ can be expressed as convex combination of points of A. How many points do we need at most in the convex combination? Caratheodory's theorem says: $n+1$. You will use this fact to prove a number of surprising results for convex sets: Radon's theorem (a set of at least $n+2$ points can be partitioned in two sets whose convex hulls intersect) and Helly's Theorem (a family of convex sets such that any $n+1$ of them intersect has a common intersection point). This has also application to economics via the Shapley–Folkman Theorem: The arithmetic mean of a large number of sets contained in the unit ball is approximately convex.

[**Ch. 2.1**] Theorems 2.2.4, 2.2.5 and 2.2.6 with proofs.

[**Ch. 7.1**] Theorems 7.1.1, 7.1.2 with proofs. Mention the sheep example. If time permits, discuss Theorem 7.1.3.

[**Ber**] Prove Proposition 5.7.1 (Shapley-Folkman Theorem). Deduce the following corollary: Let $Q_1, \ldots, Q_d$ be subsets of the unit ball of $\mathbb{R}^n$ and consider their arithmetic mean $Q := \frac{1}{d}(Q_1 + \ldots + Q_d)$. Then every point in the convex hull of Q is at distance at most n/d to a point in Q. Hence, this distance goes to zero as d goes to infinity.

3 Topologische Eigenschaften von konvexen Mengen

In this talk, you will discuss some important topological properties of convex sets. You will prove that a convex set A always has non-empty interior $\mathrm{ri}(A)$ in the affine space generated by it and give a geometric characterization of $\mathrm{ri}(A)$. You will introduce the notion of distance between sets and show that for every closed convex set C and every point x in $\mathbb{R}^n$ there is a unique point y on C with minimal distance from x. Fun fact: A theorem of Motzkin (which we do not cover in the seminar) says that the closed subsets of $\mathbb{R}^n$ having this property are exactly the convex ones!

[**Page 37**] Definition of relative interior of a set and of relative boundary. Give a couple of examples.

[**Page 15**] Definition of affine basis. Theorem 1.3.9 and Corollary 1.3.10 with proof. Baricentric coordinates.

[**Ch. 2.3**] Theorem 2.3.1 with proof. Lemma 2.3.3 and Theorem 2.3.4 with proof. Theorem 2.3.6 and 2.3.8 with proof. Corollary 2.3.10 without proof.

[**Ch. 1.9**] Definition of distance to a set and proof of Lipschitz condition (page 45). Theorem 1.9.1 and 1.9.4 with proof.

[**Ch. 2.4**] Theorem 2.4.1 with proof.

4 Trennung von konvexen Mengen und Stützebenen

In this talk, you will show that two convex sets can be separated by a hyperplane exactly when their relative interiors are disjoint. Two interesting consequences of this fact: Closed convex sets are the intersection of all the closed half spaces containing them. Every convex set has a non-trivial supporting hyperplane at every point in the relative boundary. This gives us back the intuitive picture that convex regions in $\mathbb{R}^2$ lies on one side of the lines tangent to their boundary. A fact that we do not cover in the seminar: Existence of supporting hyperplanes characterizes convex sets. In other words, if a set with non-empty interior admits a supporting hyperplane at each of its boundary points, then it is convex!

[**Ch. 2.4**] Corollary 2.4.2 with proof. Corollary 2.4.3 without proof (draw a picture). Theorem 2.4.4 with proof. Notions of separation. Theorem 2.4.6 with proof. Corollary 2.4.8 with proof. Lemma 2.4.9 with proof. Theorem 2.4.10 with proof. Corollary 2.4.11 with proof. Definition of (non-trivial) support hyperplane. Theorem 2.4.12 with proof. If there is time, Example 2.4.13.

5 Extrempunkte und der Satz von Krein-Milman

For familiar convex sets like cubes or pyramids, we intuitively know, what a k-dimensional face is. You will generalize this intuition to arbitrary convex set and see how one can reconstruct a closed convex set by taking the convex hull of its primitive faces. You will show then, that primitive faces are either flats or half-flats. Thus, you will derive a very important result of Krein–Milman: Every compact convex set is the convex hull of its 0-dimensional faces.

[**Ch. 2.6**] Definition of face, k-face and extreme points with the formula just above Example 2.6.1. All results from Theorem 2.6.2 to Theorem 2.6.16 with proof but skip Corollary 2.6.9. Explain the example illustrated in Figure 2.10.

6 Konvexe Funktionen einer Variable

In this talk, we make the acquaintance of convex functions $f : \mathbb{R}^n \to \mathbb{R}$. They satisfy

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y), \qquad \forall\, x, y \in \mathbb{R}^n,\ \lambda, \mu \geq 0,\ \lambda + \mu = 1. \tag{6.1}$$

As we will discover in Talk 9, these are exactly the functions with convex epigraph $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq y\}$. In this talk, you will give geometric properties of convex functions on the real line, namely when $n = 1$. You will further discuss analytic properties of these functions. Are they continuous or even differentiable? What can be said in that case, about their derivatives? You will see that a central player in these questions is the support of f at a point x which you can interpret as the support hyperplane of the epigraph (see Talk 4) at the point $(x, f(x))$ under cover.

[**Ch. 5.1**] Present the definitions and theorems (with proof) of Chapter 5.1.

7 Die Jensen-Ungleichung und ihre Verwandten

You will see how many classical inequalities such as the one between the geometric and arithmetic mean can be proved applying (6.1) (or better a swift generalization of it called Jensen inequality) to a given convex function. In this way you get, for instance, Hölder and Minkowski inequality that perhaps you know from analysis. More in general, you will consider the sequence of weighted means of order t and show that they are increasing in t. Finally, you will introduce log-convex functions of which we see an important example in the next talk.

[Ch. 5.2] Present the definitions and theorems (with proof) of Chapter 5.2. Right after the proof of Hölder and Minkowski, give immediately also the proof of the integral version of these inequalities. For Hölder you can see Theorem 5.3.1 in the next section.

[Ch. 5.3] Define log-convex functions and prove that the sum and the product of log-convex functions is log-convex.

8 Die Gamma-Funktion und der Satz von Artin

Euler's gamma function Γ is a famous log-convex function generalizing the factorial of a natural number. It plays a prominent role in number theory and complex analysis. You will show some important properties of the Gamma function and prove a theorem of Artin (called also Bohr-Mollerup Theorem in the literature) saying that Γ is the only function satisfying such properties. In passing, you will establish the Gauß formula for Γ. Artin's theorem enables one to prove some interesting identities about Γ, one of which is the Gaussian integral $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$. This integral plays a central role in the Stirling's formula about the asymptotic expansion of the factorial.

[Ch. 5.3] Present all the results with proofs from the definition of the gamma function on page 208 to the proof of Stirling's formula on page 215.

[Bar] Mention that Γ gives a formula for the volume ω_n of the n-dimensional unit ball: $\omega_n = \pi^{n/2}/\Gamma(n/2+1)$. This will be used in Talk 14. Only if you have time, you can give the proof following [Bar, Lemma 3.4].

9 Konvexe Funktionen mehrerer Variablen I: Epigraph und Stetigkeit

We venture in the realm of convex functions of several variables. You will establish the promised relationship between convex functions and their epigraph and use it to show that the supremum of a family of convex functions is convex. A central result that you will prove is the existence of a support of a convex function at a point using what we saw in Talk 4. As an intermezzo you will discuss the special class of positively homogeneous convex functions and use them to reprove Minkowski's inequality. In the last part, you will show that convex functions are locally Lipschitz and, hence, continuous.

[Ch. 5.4] Present the definition and theorems in Chapter 5.4. Leave the proof of Theorem 5.4.3 and of Theorem 5.4.2 as exercises. When giving the definition of support at a point, observe explicitly that the set of supports is a closed convex subset of the space of affine transformations from $\mathbb{R}^n$ to $\mathbb{R}$.

[Ch. 5.5] Present just Theorem 5.5.1 and its proof. In the statement, change the last sentence to "Then f is locally Lipschitz and, in particular, continuous on X". Observe indeed that the last inequality of the proof shows that f is Lipschitz in the ball $B[x_0; r]$.

10 Konvexe Funktionen mehrerer Variablen II: Differenzierbarkeit

You generalize the result about differentiability that we saw in Talk 6 to the case of convex functions in several variables. The two milestone results that you will meet are that a function with unique support at a point is differentiable at that point and that a twice differentiable function is convex if and only if its Hessian is semipositive definite.

[Ch. 5.5] Present all the definitions and theorems (with proofs) of Chapter 5.5 except Theorem 5.5.1, which has already been covered in the previous talk.

11 Eine Brücke zwischen konvexen Mengen und Funktionen: die Stützfunktion

You will discover a nice connection between compact convex sets $C \subset \mathbb{R}^n$ and positively homogeneous convex functions $h : \mathbb{R}^n \to \mathbb{R}$. Namely, you will associate to C its support function h_C: $h_C(u)$ tells you how "tall" C is in direction $u \in \mathbb{R}^n$. You will see that $C \mapsto h_C$ is a bijection and explore its algebraic properties. In the last part of the talk, you will switch topic and prove a result needed in the next talk about matrix inequalities: Important examples of convex sets C are given by the set of solutions of systems of linear equations and inequalities such as $C = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$ and your task will be to characterize the extreme points of C.

[Ch. 5.6] Present the first part of Chapter 5.6 until Theorem 5.6.5 including proofs.

[Ch. 4.4] Skip the opening paragraph and present Chapter 4.4 up to Theorem 4.4.2 with the included proof.

12 Matrix-Ungleichungen

You will use the theory of convex functions to prove two remarkable inequalities about the determinant of a square matrix of dimension n. Minkowski inequality says that the positively homogeneous function $A \mapsto -\sqrt[n]{\det A}$ is convex on the space of positive definite matrices. Hadamard inequality tries to answer the question: How large can the determinant of a matrix whose entries have absolute value smaller than 1 be? Hadamard inequality yields the upper bound $n^{n/2}$. It is remarkable that for many values of n, it is not known if a matrix with the given property exists whose determinant is exactly $n^{n/2}$.

[Ch. 5.8] Present Chapter 5.8 until Theorem 5.8.6 with proofs. In the statement of Theorem 5.8.4 add that equality holds if and only if $v_1, \ldots, v_n$ are eigenvectors. Present Minkowski inequality before Hadamard inequality. Add to the statement of the first Hadamard inequality that equality holds if and only if the columns of A are pairwise orthogonal. The second inequality is an equality if and only if A is diagonal.

[AZ] Observe that Hadamard's inequality implies the bound $|\det A| \leq n^{n/2}$ where A belongs to the set S of matrices for a matrix whose entries have absolute value smaller than one. Using page 42 in [AZ], state the Hadamard's determinant problem and observe that the maximum of the determinant on S is achieved by a matrix whose entries have values in the set $\{-1, +1\}$. If time permits and you are interested, you can define Hadamard's matrices and present some of their fascinating properties using the discussion starting from equation (6) on page 43 of [AZ].

13 Die Crofton-Formel und ihre Anwendungen

Given two compact, convex regions $A_1 \subset A_2$ in the plane, it seems intuitive that the perimeter of A_1 should be smaller than the perimeter of A_2. You will give a rigorous proof of this fact using Crofton's formula. You will seize this occasion to show two beautiful applications of the formula. Barbier Theorem: There are many convex regions of constant width w in the plane and all of them have perimeter πw (Regions of constant widths answer the question: "What shape can a manhole cover be made so that it cannot fall down through the hole?"). Buffon's Needle Problem: If you have a ruled page and you throw on it a short needle, what is the probability that the needle will hit one of the lines?

[FT, Ch. 19.1] Present this section and include a proof of formula (19.1).

[For, Ch. I §4] Define a continuously differentiable curve in the plane (page 40) and give the example of the circle and of the segment. Define the tangential vector and give its geometrical interpretation (page 41-42). Define the length of rectifiable curves, compute it for circles and segments, and prove Satz 1 (Section Bogenlänge).

[FT, Ch. 19.3-4] Prove Crofton's formula (Theorem 19.2) and present the first three applications in Chapter 19.4. When speaking of curves with constant width give the example of the Realeaux triangle and prove that actually has constant width.

14 Minkowskischer Gitterpunktsatz und Anwendungen in die Zahlentheorie

In this last talk, you will present some applications of the theory of convexity to number theory. The cornerstone is made by Minkowski's theorem asserting that a bounded, symmetric, convex set with enough large volume contains a non-zero lattice point (recall that a lattice point is a vector with integer coordinates). You will use this result to study Farey sequences, the problem of approximating real numbers by rational ones, and the minimum of positive quadratic forms over points with integer coordinates. Finally, you will put the cherry on top of the cake and prove the classical Lagrange's four square theorem saying that every natural number is the sum of four squares.

[Ch. 7.4] Present the content of Chapter 7.4 with proofs.

References

[AZ] Martin Aigner and Günther M. Ziegler, *Proofs from THE BOOK*, Sixth Edition, Springer, 2018.

[Ber] Dimitri P. Bertsekas, *Convex Optimization Theory*, Athena Scientific, 2009. Available at the link: web.mit.edu/dimitrib/www/Convex_Theory_Entire_Book.pdf

[For] Otto Forster, *Analysis 2*, 10. Auflage, Springer Spektrum, 2013.

[FT] Dmitry Fuchs and Serge Tabachnikov, *Ein Schaubild der Mathematik*, Springer, 2011. Originalausgabe auf Englisch: *Mathematical Omnibus*, AMS, 2006.

[Web] Roger Webster, *Convexity*, Oxford University Press, 1994.

Further Readings

[Bar] Alexander Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics 54, AMS, 2002.

[Ber2] Marcel Berger, Convexity, *The American Mathematical Monthly*, Vol. 97, No. 8, 650–678, 1990.

[HW] Daniel Hug and Wolfgang Weil, *Lectures on Convex Geometry*, GTM 286, Springer, 2020.

[MMO] Horst Martini, Louis Montejano and Déborah Oliveros, *Bodies of Constant Width*, Birkhäuser, 2019.