Variational Methods for Convex Hamiltonian Systems

Part I: Basics on Tonelli Lagrangians

Talk 1 (*Alex Arnhold*): The action functional

Setting: M compact manifold without boundary. Definition of Lagrangians $L : TM \to \mathbb{R}$ of class C^2. The action functional for curves between two given points with fixed time interval or fixed homotopy class[1] Euler-Lagrange equations (sketch of proof) and Euler-Lagrange solutions. Addition of constant to the Lagrangian and of closed 1-forms does not change the equations but change the action functional. However, within the curves with fixed time interval or fixed homotopy class the change in the action functional is given by a constant. The Legendre condition and the Euler-Lagrange vector field on TM. Is it complete? The Legendre (Fenchel) transform

$$\mathrm{Leg} : TM \to T^*M, \qquad (x, v) \mapsto \partial_v L(x, v).$$

Definition of Tonelli Lagrangians (see Remark 1.2.1 in [Maz] about uniformity of the superlinearity). Example of electromagnetic Lagrangians

$$L(x, v) = \tfrac{1}{2} g_x(v, v) + \theta_x(v) - U(x)$$

where g is a Riemannian metric, θ is a 1-form and U is a function. The Euler-Lagrange equations in this case turn into Newton's equation

$${}^x\nabla_{\partial_t} \dot{x} = -\nabla U(x) - Y_x \cdot \dot{x},$$

where ∇U is the gradient with respect to g and Y is the Lorentz force defined by

$$g_x(Y_x \cdot u, v) = \mathrm{d}\theta_x(u, v), \qquad \forall\, x \in M,\ u, v \in T_x M.$$

The Legendre transform is a diffeomorphism if the Lagrangian is Tonelli. What are the images $\mathrm{Leg}(x, \dot{x}) = (x, p_x)$ of Euler-Lagrange solutions x? Definition of the Hamiltonian $H : T^*M \to \mathbb{R}$

$$H(x, p) = \langle p, \mathrm{Leg}^{-1}(x, p) \rangle - L(\mathrm{Leg}^{-1}(x, p)).$$

Relation between $\partial_q H$, $\partial_p H$ and $\partial_q L$ and $\partial_p L$. The function H is C^2 and convex ($\partial^2_{pp} H = (\partial^2_{vv} L)^{-1}$) (later we will see that it is also Tonelli). The curves (x, p_x) satisfy the Hamilton equations

$$\dot{x} = \partial_p H(x, p_x), \qquad \dot{p}_x = -\partial_x H(x, p_x).$$

[1] a homotopy class is a subset of curves connecting the two end-points which are homotopic to each other with fixed end-points.

Uniqueness of solutions for Hamilton equations implies uniqueness of solutions for the Euler-Lagrange equations.

Remark. *Please see if the time is enough to discuss all the topics above. Please give only few details of the proofs.*

Talk 2 (*Christian Alber*): Tonelli's Theorem

Definition of the Energy $E : TM \to \mathbb{R}$, $E(x, v) = \langle \partial_v L, v\rangle - L(x, v)$ so that $E = H \circ \mathrm{Leg}$. Energy is invariant under addition of 1-form to L. If $L_U = L - U$, where $U : M \to \mathbb{R}$, then $E_U = E + U$. Energy and Hamiltonian for the example of electromagnetic Lagrangians. The energy (equivalently) the Hamiltonian are constant of motions. The Euler-Lagrange flow is complete for M compact.

Statement of the problem: existence of action minimizers in the set $\{x : [a, b] \to M \mid x(a) = q_0,\ x(b) = q_1\}$. Statement of result: Corollary 3.3.3 in [Fat] (Tonelli's Theorem). Introduce the lift $\tilde{L} : T\tilde{M} \to \mathbb{R}$ of a Lagrangian $L : TM \to \mathbb{R}$ from a compact manifold M to a covering space $\tilde{M} \to M$ of M. Observe that the hypothesis are satisfied for $\tilde{L}$ by lifting a metric on M as well. Deduce that for every $q_0, q_1 \in M$ compact and every homotopy class of paths from q_0 to q_1 there is an Euler-Lagrange orbit minimizing the action. Other references for Tonelli's: Theorem 4.1.1 in [Sor], [CI, Section 3-1], [Maz, Theorem 1.3.1]. Definition of Tonelli minimizers.

Remark. *The proof of Tonelli's theorem should only be sketched.*

Talk 3 (*Camillo Tissot*): Weierstrass Theorem and global minimizers

Statement (without proof) of Weierstrass Theorem [Fat, Theorem 3.6.1 and Theorem 3.6.2]: Euler-Lagrangian solutions defined on short intervals are Tonelli minimizers; two nearby points are connected by a unique Tonelli minimizer defined for short time. Use Weierstrass Theorem to sketch the proof of the regularity of Tonelli minimizers [Fat, Theorem 3.7.1].

Problem: Are there curves $\gamma : \mathbb{R} \to M$ such that $\gamma|_{[a,b]}$ are Tonelli minimizers for all $a < b$. What is their energy? Definition of global Tonelli minimizers [Sor, Definition 4.1.4]. Observe that the answer depends on the addition of closed 1-form η (see Talk 2) to the Lagrangian.

This problem is related to finding free-time minimizers for L connecting two points x_0 and x_1. These are curves minimizing the action $\mathcal{A}_L$ on the set

$$\bigcup_{T>0} C^2_{x_0,x_1}([0, T], M).$$

Observe that free-time minimizers for L and for $L + k$ might a priori be different. Indeed, there holds: free-time minimizers of $L + k$ connecting x_0 and x_1 have energy $E = k$ [CI, 3-3.2 Lemma]. In particular, finding free-time minimizers yields orbits connecting two points with given energy.

We can now define global free-time minimizers of $L+k$ as curves $\gamma : \mathbb{R} \to M$ such that $\gamma|_{[a,b]}$ is a free-time minimizer for $L+k$ connecting $\gamma(a)$ and $\gamma(b)$ [Sor, Definition 4.1.5]. Observation: a global free-time minimizer of $L+k$ is a global Tonelli minimizer. Is the converse true? For which k do we have (global) free-time minimizers of $L+k$?

Later in the seminar we will prove the following statement

THEOREM. *There exists $c(L) \in \mathbb{R}$ such that*

- *For all $k > c(L)$ and for all q_0, q_1 there exists a free-time minimizer of $L+k$ connecting q_0, q_1;*
- *There exists global free-time minimizers of $L + c(L)$.*
- *Global Tonelli minimizers are global free-time minimizers of $L + c(L)$ (in particular, all global free-time minimizers have energy $c(L)$).*

Example of the pendulum: $L : TS^1 \to \mathbb{R}$, $L(x,v) = \frac{1}{2}v^2 - U(x)$, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $U(x) := (\cos x - 1)$. Draw the trajectories in the coordinates (x,v). For $k \geq 0$ define $\theta_k^\pm := \pm\sqrt{2(k-U)}\mathrm{d}x$. We have

- $L + r\theta_0^\pm \geq 0$ for all $r \in [0,1]$. When does the equality sign hold? Determine the global free-time minimizers of $L + r\theta_0^\pm$;
- $L + \theta_k^\pm + k \geq 0$ for all $k \geq 0$. When does the equality sign hold? Determine the global free-time minimizers of $L + \theta_k^\pm + k$.

Observe that for $r \in [0,1]$ and $k \in [0,\infty)$, $r\theta_0^\pm$ and $\theta_k^\pm$ attain all possible cohomology classes in $H^1(S^1;\mathbb{R})$.

Part II: Minimizing orbits in dimension 1 - a toy model

Let us look at the problem of the existence of global Tonelli minimizers in a simple setting.

Talk 4 (Gabriele Benedetti): The variational problem and circle homeomorphisms

Section 1 and 2 and Lemma 3.1, Corollary 3.2 from Section 3 [Ban].

Talk 5 (Gabriele Benedetti): Rotation number of minimal trajectories

Section 3 starting from Theorem 3.3 [Ban].

Talk 6 (Gabriele Benedetti): Structure of the set of minimal trajectories

Section 4 and 5 from [Ban].

Talk 7 (*Raphael Schlarb*): Application to monotone twist maps

Section 7 from [Ban]. Definition of twist map and example of billiards. Invariant circles: Theorem 1.3.3 in [Sib]. Minimal action of a monotone twist map, see Definition 1.3.6 in [Sib] (Please use the notation β instead of α for the minimal action since this is the notation we will use later on in the seminar). State Theorem 1.3.7 in [Sib].

REMARK. *An alternative reference are Sections 9.3 and 13.1-13.3 in [KH].*

Part III: Minimizing orbits for Tonelli Lagrangians – Mañé Theory

Talk 8 (*Maximilian Schmahl*): Aubry and Mañé sets

Finite-time potential $\Phi_k(x, y; T)$ (see [CI, Section 3-4] and Proposition 4.1.11 in [Sor]). Mañé potential

$$\Phi_k(x, y) = \inf_{T>0} \Phi_k(x, y; T)$$

Properties of the Mañé potential (see [CI, Section 2-1] and [Sor, 4.1.9]). Definition of Mañé critical value $c(L)$. Definition of semi-static and static curves and corresponding definition of Aubry and Mañé sets. Observe that they are closed. Prove that the Aubry set contains the non-wandering points of the Mañé set (see the first statement of Proposition 3-5.7 in [CI] and the proof of Proposition 2.1.11 in [Sor]). In particular, if the Mañé set is non-empty so is the Aubry set since non-wandering points always exist (for instance take α or ω limits). Use the separatrix of the pendulum to give an example of a non-wandering point which is not an α- or ω-limit. Observation: the Euler-Lagrange flow restricted to the Mañé set can be as complicated as the one given by a vector field on M. Take X vector field on M and define the Lagrangian $L(x, v) := \frac{1}{2}|v - X(x)|^2$. Then the graph of X is contained in the Mañé set and the Euler-Lagrange flow on the graph project to the flow of X on M.

If time permits: Compute the Mañé potential and the Mañé critical value for the pendulum for all cohomology classes and use it to find the Mañé and Aubry sets (see [Sor, Section 4.3], but the argument using Lemma 4.3.2 is not quite right). Sketch: use the Lagrangians $L_k^\pm := L + \theta_k^\pm$, $k \geq 0$ and $L_r^\pm := L + r\theta_0^\pm$, $r \in [0, 1]$ defined in Talk 3. Show that $c(L_k^\pm) = k$ and $c(L_r\pm) = 0$ by observing that there are periodic global free-time minimizers for $L_k^\pm$ and $L_r^\pm$ with zero action. This shows that for $k > 0$, $\Phi_k^{L_k^\pm}(x, y) = 0$ for all $x, y \in S^1$. Hence, the free-time minimizers are also static curves in this case. In the same way, the unstable critical point is a static curve for $L^\pm_r$. Now show that the separatrix $\gamma^\pm : \mathbb{R} \to S^1$ is a static curve for $L^\pm_r$ only for $r = 1$. Indeed, $\Phi_0^{L^\pm_1}(x, y) = 0$ for all $x, y \in S^1$ (obtain a closed curve by concatenating $\gamma^\pm|_{[-n,n]}$ for n large with a short curve connecting $\gamma^\pm(n)$ and $\gamma^\pm(-n)$). On the other hand $A_{L^\pm_r}(\gamma^\pm|_{(-\infty,+\infty)}) \geq \epsilon$ for $r < 1$ but then we can find a curve with action smaller than ϵ connecting the points $\gamma^\pm(-n)$ and $\gamma^\pm(n)$ for n large.

Talk 9 (*Levin Maier*): Graph property for the Aubry set and generalities on probability measures

Graph theorem: Theorem 4.1.30 and Lemma 4.1.31 in [Sor]. Statement of problem: are the Mañé and Aubry sets non-empty? Via weak KAM-Theorem one shows that the Mañé set is non-empty. Then the Aubry set is non-empty since it contains the α- and ω-limit of the Mañé set. We will follow Mather approach: construct elements in the Aubry set via invariant measures.

Setting: Borel probability measures on compact manifold N. Definition of support. Definition of pushforward of a probability by a map. Example: probability uniformly distributed on a curve. Riesz-representation theorem. Weak*- topology in the space of Borel probabilities and weak*-compactness. Support and weak*-convergence:

$$\mu_n \to \mu, \qquad \Longrightarrow \qquad \mathrm{supp}\mu \subset \bigcap_n \overline{\bigcup_{m\geq n} \mathrm{supp}\mu_m}.$$

Definition of measure invariant by a flow generated by vector field X. Definition of ergodic probability measure. Example: rotation of angle α on a circle. Birkhoff's ergodic theorem. Decomposition of a probability measure μ in ergodic components $\{\mu(s)\}_{s\in S}$:

$$\mu = \int_S \mu(s)\nu$$

where (S, ν) is a probability space. Show that for all $S' \subset S$ with $\mu(S') = 1$, there holds

$$\mathrm{supp}\mu \subset \overline{\bigcup_{s\in S'} \mathrm{supp}\mu(s)}.$$

Definition of rotation vector $\rho(\mu) \in H^1(N; \mathbb{R})^* \cong H_1(N; \mathbb{R})$ with proof of the identity

$$\int_N dh(X)\mu, \qquad \forall\, h \in C^\infty(N).$$

Note: in the literature the rotation vector is also known as the *asymptotic cycle*.

Continuity of the rotation vector in the weak*-topology. Formula for the rotation vector for ergodic measures.

Remark. *For the part on probability measures, you can consult [KH, Section 4.1 (a)-(f)]. For the part on the rotation vector (asymptotic cycle), see [KH, Section 14.7 (b)].*

Part IV: Minimizing orbits for Tonelli Lagrangians – Mather Theory

Talk 10 (*Valerio Assenza*): Mather measures

Section 3.1, 3.2 and 3.3 in [Sor]. For the part on Fenchel duals and the proof that $\beta^* = \alpha$ you can consult [CI, Appendix D].

Talk 11 (*Valerio Assenza*): The Mather set is contained in the Aubry set

Proposition 4.1.24 in [Sor]: Mather's α function coincides with the Mañé critical value. Proposition 4.1.26 in [Sor]: the Mather set is contained in the Aubry set. Proposition 4.1.29: semi-static curves are asymptotic to the Mather set. Example: the Mather set and the α function for the pendulum [Sor, Section 3.5].

Part V: Minimizing orbits for Tonelli Lagrangians – Weak KAM Theory

Talk 12 (*Johanna Bimmermann*): Weak KAM solutions and dynamics

Dominated functions and Proposition 5.1.4 in [Sor]. Dominated functions exist only of $k \geq c(L)$. Definition of critical subsolution.

C^0-characterization of HJ-solutions in Proposition 5.1.9 [Sor]. Definition of calibrated curves. Calibrated curves and differentiability of dominated functions (Proposition 5.1.12 [Sor]). Weak KAM solutions and weak KAM Theorem [Sor, Theorem 5.1.16]. Characterization of the Mather set in terms of weak KAM solutions. Characterization of Aubry set and Mañé set in terms of conjugated solutions, see [Fat, Section 5.2].

Talk 13 (*Leon Happ*): Lax-Oleinik semigroup and weak KAM solutions

For this talk we use the reference [Fat]. Define the Lax-Oleinik semigroup from Section 4.6 and show that its fixed points coincide with weak KAM solutions (Proposition 4.6.7). Sketch the proof of the existence of a fixed point from Section 4.7.

Talk 14 (*Gabriele Benedetti*): The Peierls Barrier and the relationship between Tonelli and time-free minimizers

References

[Arn] Marie-Claude Arnaud, *On a theorem due to Birkhoff*, Geom. Funct. Anal. 20 (2010), no. 6, 1307-1316.

[Ban] Victor Bangert, *Mather sets for twist maps and geodesics on tori*, Dynamics reported, Vol. 1, 1-56, Dynam. Report. Ser. Dynam. Systems Appl., 1, Wiley, Chichester, 1988.

[CI] Gonzalo Contreras and Renato Iturriaga, *Global minimizers of autonomous Lagrangians*, 22^o Coloquio Brasileiro de Matematica, IMPA, 1999. Online version.

[Fat] Albert Fathi, *Weak KAM Theorem in Lagrangian Dynamics*, Online version.

[KH] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.

[Maz] Marco Mazzucchelli, *Critical point theory for Lagrangian systems*, Progress in Mathematics 293, Birkhäuser/Springer Basel AG, Basel, 2012.

[Sib] Karl Friedrich Siburg, *The Principle of Least Action in Geometry and Dynamics*, Lecture notes in Mathematics 1844, Springer, 2004.

[Sor] Alfonso Sorrentino, *Action-minimizing methods in Hamiltonian dynamics. An introduction to Aubry-Mather theory*, Mathematical Notes 50, Princeton University Press, 2015.