# Monotone Twist Maps

Raphael Schlarb

June 12, 2020

## 1 Introduction

In the previous two talks we introduced the one-dimensional Aubry-Mather theory and discussed some results. We now want to see how we can apply this to monotone twist maps.

**Definition 1.** *A monotone twist map is a $C^1$-diffeomorphism*

$$\begin{aligned}\phi : \mathbb{R} \times (a,b) &\to \mathbb{R} \times (a,b)\\ (x_0, y_0) &\mapsto (x_1, y_1)\end{aligned}$$

*that satisfies $\phi(x_0 + 1, y_0) = \phi(x_0, y_0) + (1, 0)$ and the following properties:*

*(i)* $\phi^* dx_1 \wedge dy_1 = dx_0 \wedge dy_0$ *($\phi$ preserves the area and orientation),*

*(ii)* $\phi$ *preserves the boundaries in the sense that $y_1 \to a, b$ as $y_0 \to a, b$. If $a, b$ are finite, we additionally require $\phi$ to extend to the boundaries by rotations by some fixed angles $\omega_\pm$,*

*(iii) the monotone twist condition* $\frac{\partial x_1}{\partial y_0} > 0$.

**Example 1.** *Integrable twist maps*
*The most simple examples are maps of the form $\phi(x, y) = (x + f(y), y)$ where $f' > 0$.*

**Example 2.** *Billiard*
*Consider a strictly convex domain $\Omega$ in the Euclidean plane with smooth boundary $\partial\Omega$. Imagine a mass point moving freely inside $\Omega$ and starting at some point on the boundary with some initial direction pointing into $\Omega$. Each time the mass point hits the boundary, its velocity vector is reflected on the boundary's tangent such that the reflection angle equals the incident angle (See Figure 1). By this description we get a map that can be lifted to some map:*

$$\phi : \mathbb{R} \times (0, \pi) \to \mathbb{R} \times (0, \pi), (l, \varphi) \mapsto \phi(l, \varphi).$$

*This doesn't preserve the volume form yet, so we still need to do a slight transformation $(l, \psi) = (l, -\cos(\varphi))$.*

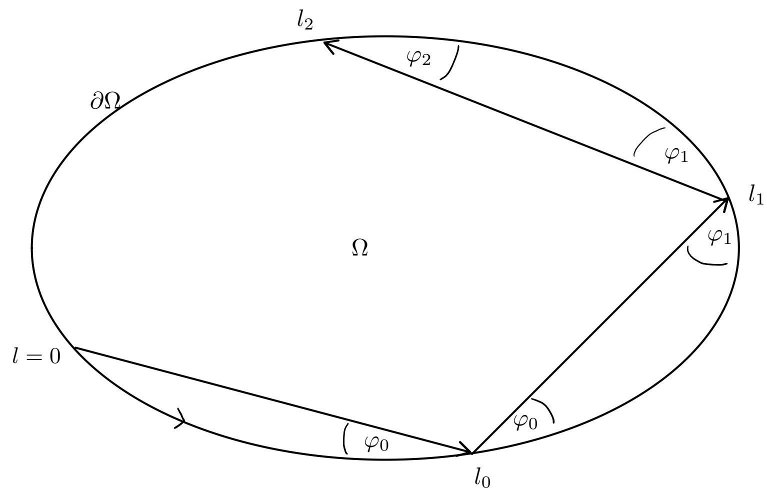


Figure 1: Billiard

# 2 The discrete Lagrangian for monotone twist maps

Because $\frac{\partial x_1}{\partial y_0} > 0$, vertical lines $\{x_0\} \times (a,b) \subset \mathbb{R} \times (a,b)$ are mapped to graphs over the $x$-axis (See figure 2). Thus, for each pair $x_0, x_1$ we can find $y_0, y_1$ such that $\phi(x_0, y_0) = (x_1, y_1)$. This means that any orbit $(x_i, y_i)_{i\in\mathbb{Z}}$ of $\phi$ is already fully determined by the sequence $(x_i)_{i\in\mathbb{Z}}$. We now want to find a discrete Lagrangian that we can apply to such sequences. Because $\phi$ is symplectic, we know that there exists, at least locally, a generating function $H$ such that

$$\begin{aligned} y_0 &= -\partial_1 H(x_0, x_1) \\ y_1 &= \partial_2 H(x_0, x_1). \end{aligned} \tag{1}$$

If we can extend $H$ to $\mathbb{R}^2$ then $\partial_1 H(x_i, x_{i+1}) + \partial_2 H(x_{i-1}, x_i) = 0$ for a sequence $(x_i)_{i\in\mathbb{Z}}$ if and only if it corresponds to an orbit $(x_i, y_i)_{i\in\mathbb{Z}}$ of $\phi$. This means that if we take $H$ to be our discrete Lagrangian, we can interpret our results on minimizing sequences of the discrete action functional $\alpha_H = \sum_{i\in\mathbb{Z}} H(x_i, x_{i+1})$ on statements about certain types of orbits of $\phi$. We now construct a function $H : \mathbb{R}^2 \to \mathbb{R}$ that satisfies (1) and the discrete Lagrangian properties.

## 2.1 Geometrical construction of the discrete Lagrangian

The geometrical definition was taken from ([3], section 9.3). Consider the vertical lines $\{x_0\} \times (a,b)$ as well as $\{x_1\} \times (a,b)$ and its image and preimage under $\phi$ respectively. Without loss of generality, we assume $a = 0$ and define $H(x_0, x_1)$ to be the area of $A_{x_0,x_1}$, which is the domain enclosed by the image of $\{x_0\} \times (a,b)$, the vertical line $\{x_1\} \times (a,b)$ and the horizontal $\mathbb{R} \times \{a\}$ or in other words, the area under the graph of $x_1 \mapsto y_1(x_0, y_0(x_0, x_1))$ (See figure 2). It's now clear that $\frac{\partial H}{\partial x_1} = y_1$. Because $\phi$ preserves the area, $H$ also describes the area of $B_{x_0,x_1}$ which is the area under $y_0$, but on the right side of $\{x_0\} \times (a,b)$, implying $\frac{\partial H}{\partial x_0} = -y_0$. At first, $H$ is only defined on the strip $\{(x_0, x_1) \in \mathbb{R}^2 | x_0 + w_- < x_1 < x_0 + w_+\}$, but we can

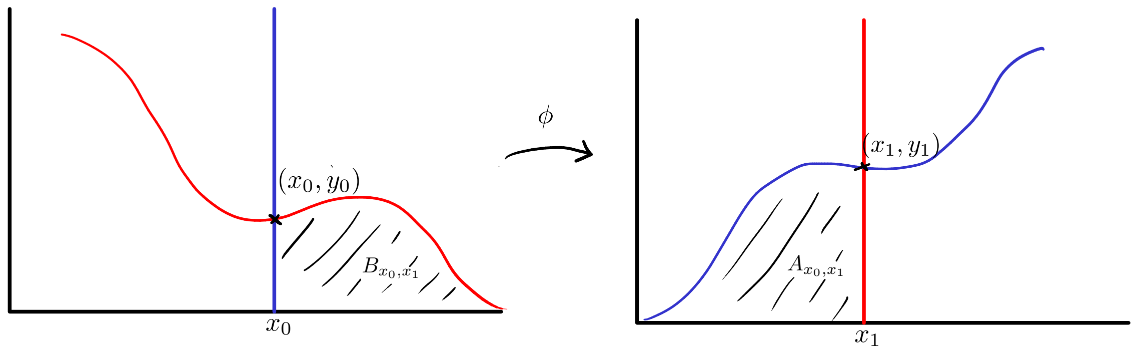


Figure 2: Geometrical definition of H

extend $\phi$ by

$$(x, y) \mapsto (x + \omega_- + (y - a), y) \text{ for } y \leq a \text{ and}$$
$$(x, y) \mapsto (x + \omega_+ + (y - b), y) \text{ for } y \geq b.$$

to get an extension of $H$ that satisfies the discrete Lagrangian properties:

(i) $H(x_0 + 1, x_1 + 1) = H(x_0, x_1)$ follows from the periodicity of $\phi$.

(ii) $H(x_0, x_1 + \xi) \to \infty$ as $\xi \to \infty$ can be seen from the geometrical definition after $H$ has been extended to $\mathbb{R}^2$.

(iii) For any $x_0 < \bar{x}_0, x_1 < \bar{x}_1$, we have $H(x_0, x_1) + H(\bar{x}_0, \bar{x}_1) < H(x_0, \bar{x}_1) + H(\bar{x}_0, x_1)$. This follows from figure 2 after adding another vertical on each side, the corresponding image and preimage and comparing the enclosed areas.

(iv) If $x_0, x_1, x_2$ and $\bar{x_0}, \bar{x_1}, \bar{x_2}$ are minimal segments and $x_1 = \bar{x_1}$, then the segments cross. Assume they don't cross. From figure 2 we can easily see that $\partial_1 H$ and $\partial_2 H$ are monotone, but then $\partial_1 H(x_0, x_1) + \partial_2 H(x_{-1}, x_0)$ and $\partial_1 H(\bar{x}_0, \bar{x}_1) + \partial_2 H(\bar{x_{-1}}, \bar{x_0})$ can't be both zero.

## 2.2 Some properties

The monotone twist condition implies $\partial_2\partial_1 H < 0$. Indeed, remember that $y_0$ is determined by $x_0, x_1$, so we can write $x_1 = x_1(x_0, y_0(x_0, x_1))$, and have $1 = \frac{\partial x_1}{\partial x_1} = \frac{\partial x_1}{\partial y_0}\frac{\partial y_0}{\partial x_1}$, indicating that $\frac{\partial y_0}{\partial x_1} > 0$ because $\frac{\partial x_1}{\partial y_0} > 0$. Thus $\partial_2\partial_1 H = -\partial_1 y_1 < 0$. Recall that $\mathcal{M}_\rho = \{x \in \mathcal{M} | \rho(x) = \rho\}$ with $\rho$ being the rotation number $\rho := \lim_{i\to\infty} \frac{x^i}{i}$. We have previously seen that if $\rho$ is irrational, then there exists $f \in \widetilde{G_+}$, where

$$\widetilde{G_+} := \{f : \mathbb{R} \to \mathbb{R} \text{ homeomorphism } | f(x + 1) = f(x) + 1 \ \forall x \in \mathbb{R}\},$$

such that $\rho(f) = \rho$ and $f(x_i) = x_{i+1}\ \forall x \in \mathcal{M}_\rho, \forall i \in \mathbb{Z}$. For monotone twist maps, $f$ will be bi-Lipschitz. The proof was taken from ([2], 3.19).

**Lemma 1.** *If $\rho$ is irrational and $\partial_2\partial_1 H < 0$, then there exists a bi-Lipschitz function $f \in \widetilde{G_+}$ such that $\rho(f) = \rho$ and $\forall x \in \mathcal{M}_\rho, \forall i \in \mathbb{Z} : f(x_i) = x_{i+1}$.*

*Proof.* We only need to show that the $f \in \widetilde{G_+}$ we got from previous talks is bi-Lipschitz. Fix some $\bar{x} \in \mathcal{M}_\rho$, then $|x_1 - x_0| \leq |\bar{x}_1 - \bar{x}_0| + 1$ for all $x \in \mathcal{M}_\rho$, because $\mathcal{M}_\rho$ is totally ordered by theorem 2 of talk 6. Similarly $|x_{-1} - x_0| \leq |\bar{x}_0 - \bar{x}_0| + 1$, thus $|x_1 - x_0|$ and $|x_{-1} - x_0|$ are uniformly bounded for all $x \in \mathcal{M}_\rho$. Now let $x$ and $\bar{x}$ be arbitrary elements in $\mathcal{M}_\rho$. Because $H(x_0 + 1, x_1 + 1) = H(x_0, x_1)$ and the T-invariance of $\mathcal{M}_\rho$, it suffices to consider the case $0 \leq x_0 \leq \bar{x}_0 < 2$. Then $x_{-1} < \bar{x}_{-1}$, $x_1 < \bar{x}_1$ and $x_{-1}, x_0, x_1, \bar{x}_{-1}, \bar{x}_0, \bar{x}_1$ are all contained in some compact interval $I$. Therefore, there exist $\delta > 0, L > 0$ such that $\partial_2\partial_1 H < -\delta < 0$ on $I \times I$ and $\partial_1 H$, $\partial_2 H$ are Lipschitz on $I \times I$ with Lipschitz constant $L$. Then we can estimate:

$$\begin{aligned}
\delta(\bar{x}_1 - x_1) &\leq \delta(\bar{x}_{-1} - x_{-1}) + \delta(\bar{x}_1 - x_1) \\
\text{mean value theorem} &\leq \partial_2 H(\bar{x}_{-1}, x_0) - \partial_2 H(x_{-1}, x_0) + \partial_1 H(x_0, \bar{x}_1) - \partial_1 H(x_0, x_1) \\
x \text{ and } \bar{x} \text{ are stationary} &= \partial_2 H(\bar{x}_{-1}, x_0) - \partial_2 H(\bar{x}_{-1}, \bar{x}_0) + \partial_1 H(x_0, \bar{x}_1) - \partial_1 H(\bar{x}_0, \bar{x}_1) \\
&\leq 2L(\bar{x}_0 - x_0)
\end{aligned}$$

Hence, $\bar{x}_1 - x_1 \leq \frac{2L}{\delta}(\bar{x}_0 - x_0)$ and similarly $\bar{x}_{-1} - x_{-1} \leq \frac{2L}{\delta}(\bar{x}_0 - x_0)$. $\square$

# 3 Mather sets

This section was mainly taken from ([2], section 7). For $\rho = \frac{q}{p}$ reduced, we get so called Birkhoff orbits of minimum type. If $x \in \mathcal{M}_{q,p} = \mathcal{M} \cap \mathcal{P}_{q,p}$, then the corresponding orbit $(x_i, y_i)_{i \in \mathbb{Z}}$ satisfies $(x_{i+q}, y_{i+q}) = (x_i + p, y_i)$.
For $\rho \in \mathbb{R} \setminus \mathbb{Q}$ we get Mather sets.

**Theorem 1.** *For any irrational $\rho \in (\omega_-, \omega_+)$, there exists a $\phi$-invariant set $M_\rho \subset \mathbb{R} \times (a, b)$ with the following properties.*

*(i) $M_\rho$ is the graph of a Lipschitz function $\psi_\rho : A_\rho \to (a, b)$ defined on a closed set $A_\rho \subset \mathbb{R}$.*

*(ii) $\phi$ has rotation number $\rho$ on $M_\rho$, i.e. there exists $h \in \widetilde{G_+}$ with $\rho(h) = \rho$ such that $h(A_\rho) = A_\rho$ and $\phi(\xi, \psi_\rho(\xi)) = (h(\xi), \psi_\rho(h(\xi)))$ for all $\xi \in A_\rho$.*

*(iii) The set $M_\rho^{rec}$ of recurrent points of $M_\rho$ projects either to a Cantor set in $\mathbb{R}$ or to all of $\mathbb{R}$. In the latter case $M_\rho$ is a $\phi$-invariant Lipschitz curve.*

*Proof.* Define

$$M_\rho := \{(\xi, y) \in \mathbb{R} \times (a, b) | \text{ there exists } x \in \mathcal{M}_\rho \text{ such that } \xi = x_0, y = -\partial_1 H(x_0, x_1)\}.$$

$M_\rho$ is $\phi$-invariant. By Theorem 2 of talk 6 $\mathcal{M}_\rho$ is totally ordered. Hence, $M_\rho$ has a one-to-one projections onto a closed set $A_\rho \subset \mathbb{R}$. Lemma 1 gives us a Lipschitz $h \in \widetilde{G_+}$ such that $h(x_0) = x_1$ for all $x \in \mathcal{M}_\rho$. Thus $\psi_\rho(\xi) := -\partial_1 H(\xi, h(\xi))$ for $\xi \in A_\rho$ is Lipschitz as well. This proves $(i)$ and $(ii)$. In last talk we already discussed that $Rec_h = p_0(\mathcal{M}_\rho^{rec}) = p_x(M_\rho)$ is either $\mathbb{R}$ or a Cantor set in $\mathbb{R}$. $\square$

In the following, when we talk about $\phi$-invariant circles, then we mean the lifts of embedded, homotopically non-trivial, curves in $S^1 \times (a, b)$ to $\mathbb{R} \times (a, b)$ that are $\phi$-invariant.

**Remark 1.** *Consider a perturbed integrable twist map $\varphi(x,y) = (x + f(y) + \epsilon(x,y), y)$. KAM theorem tells us that if the perturbation $\epsilon$ is small enough, then most $\varphi$-invariant circles with sufficient irrational rotation number survive. Theorem 1 tells us that the destroyed circles become Cantor sets.*

**Theorem 2.** *If $C$ is a $\phi$-invariant circle and if $\phi_{|C}$ has rotation number $\rho$, then $C \subseteq M_\rho$ and $C = M_\rho$ if $\rho$ is irrational.*

*Proof.* According to Birkhoff's Theorem (See [1]), C is the graph of a Lipschitz function $\psi : \mathbb{R} \to (a,b)$. Hence, $\phi(\xi, \psi(\xi)) = (h(\xi), \psi(h(\xi)))$ for some $h \in \widetilde{G}_+$ with $\rho(h) = \rho$.
Define $S := \{x \in \mathbb{R}^{\mathbb{Z}} |\, x_{i+1} = \phi_x(x_i, \psi(x_i)) = h(x_i)\ \forall i \in \mathbb{Z}\}$. $S$ is a closed, totally ordered set of stationary trajectories with regard to $H$. We now need to show that $S \subseteq \mathcal{M}$. Let $(x_i, \ldots, x_j)$ be a segment of some $x \in S$. Let $(\bar{x}_i, \ldots, \bar{x}_j)$ be a minimal segment with $\bar{x}_i = x_i$ and $\bar{x}_j = x_j$. We can find a maximal $x^* \in S$ such that $(x^*_i, \ldots, x^*_j) \le (\bar{x}_i, \ldots, \bar{x}_j)$. For every $\xi \in \mathbb{R}$ we can find a $x^{**} \in S$ such that $x^{**}_0 = \xi$, so, if $(x^*_i, \ldots, x^*_j) < (\bar{x}_i, \ldots, \bar{x}_j)$, then $(x^*_i, \ldots, x^*_j)$ can't be maximal. This implies $x^*_k = \bar{x}_k$ for some $k, i \le k \le j$. The argument for the discrete lagrangian property (iv) holds for stationary segments as well. Thus, $x^*_i = \bar{x}_i$ or $x^*_j = \bar{x}_j$ and because $S$ is totally ordered we get $x^* = x$ and therefore, $(x_i, \ldots, x_j) \le (\bar{x}_i, \ldots, \bar{x}_j)$. Similarly $(x_i, \ldots, x_j) \ge (\bar{x}_i, \ldots, \bar{x}_j)$ and we get $(x_i, \ldots, x_j) = (\bar{x}_i, \ldots, \bar{x}_j)$. If $\rho$ is irrational, then $S = \mathcal{M}_\rho$ because $\mathcal{M}_\rho$ is totally ordered. □

We still need to show that the orbits corresponding to trajectories in $\mathcal{M}_\rho$ lie in the domain of $\phi$.

**Lemma 2.** *Suppose $x \in \mathcal{M}_\rho$ and $\rho \in (\omega_-, \omega_+)$ then the orbit $(x_i, y_i)_{i \in \mathbb{Z}}$, $y_i = -\partial_1 H(x_i, x_{i+1})$, is contained in $\mathbb{R} \times (a,b)$.*

*Proof.* By Theorem 2, for $s = a, b$, we have $\{x \in \mathbb{R}^{\mathbb{Z}} |\, x_{i+1} = f_s(x_i)\ \forall i \in \mathbb{Z}\} \subset \mathcal{M}_\rho$ Let $x \in \mathcal{M}_\rho$ and $\omega_- < \rho < \omega+$. Define $\underline{x} \in \mathcal{M}_{\omega_-}$, $\bar{x} \in \mathcal{M}_{\omega_+}$ by $\underline{x}_i = f^i_a(x_0)$, $\bar{x}_i = f^i_b(x_0)$. From talk 5 lemma 2 we know that two elements from $\mathcal{M}$ cross at most once. Since $\underline{x}, x, \bar{x}$ then cross only in 0 and $\omega_- < \rho < \omega_+$ we have $\underline{x}_1 < x_1 < \bar{x}_1$. Hence,

$$a = -\partial_1 H(\underline{x}_0, \underline{x}_1) < -\partial_1 H(x_0, x_1) < -\partial_1 H(\bar{x}_0, \bar{x}_1) = b.$$

Since $y_0 = -\partial_1 M(x_0, x_1)$, we get $a < y_0 < b$

□

# 4 Literature

[1] Siburg, K.F. - A dynamical systems approach to Birkhoff's Theorem, 1998

[2] Bangert - Mather sets for twist maps and geodesics on tori, 1988

[3] Katok, Hasselblatt - Introduction to the modern theory of dynamical systems, 1995