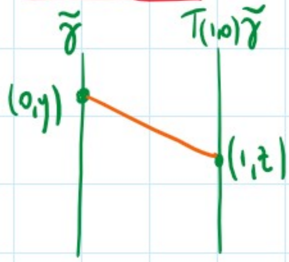
Last time metric g on $\mathbb{R}^2/\mathbb{Z}^2$ $\Rightarrow$ lifted metric $\tilde{g}$ on $\mathbb{R}^2$ with $\tilde{\gamma}(t)=(0,t)$ minimal.
$T_{(q,p)}:\mathbb{R}^2\to\mathbb{R}^2$ transl. by $(q,p)\in\mathbb{Z}^2$



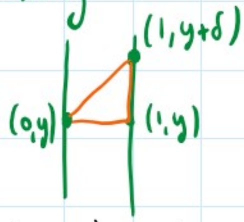
Discrete Lagrangian: $\ell:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $\ell(y,z)=d((0,y),(1,z))$.
Discrete action: $a:\mathbb{R}^{(J,\dots,K)}\to\mathbb{R}$, $a(x_J,\dots,x_K)=\sum_{i=J}^{K-1}\ell(x_i,x_{i+1})$.
Minimal segments $(x_J,\dots,x_K)$: $a(x_J,\dots,x_K)\le a(x_J^*,\dots,x_K^*)$ if $x_J^*=x_J$, $x_K^*=x_K$.
Global minimizers $(x_i)_{i\in\mathbb{Z}}\in\mathcal{M}\subset\mathbb{R}^{\mathbb{Z}}$: $\forall J<K$ $(x_i)_{J\le i\le K}$ is min. segment.

Rmk $\mathbb{R}^{\mathbb{Z}}$ endowed with product topology: $x^{(n)}\to x \Leftrightarrow x_i^{(n)}\to x_i$ $\forall i\in\mathbb{Z}$. Hence, $\mathcal{M}$ is closed.
Tychonoff: $(K_i)_{i\in\mathbb{Z}}$ compact in $\mathbb{R}$ $\Rightarrow \{x\in\mathbb{R}^{\mathbb{Z}} \mid x_i\in K_i\ \forall i\in\mathbb{Z}\}$ is compact. (Proof using diagonal argument)

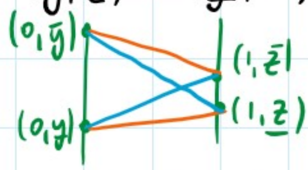
Lemma The function ℓ is a continuous function satisfying the properties:

(ℓ_1) "1-Periodicity" $\ell(y+1,z+1)=\ell(y,z)$ $\forall y,z$; $T_{(0,1)}$ isometry

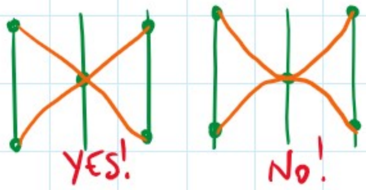
(ℓ_2) "Coercivity" $\lim_{|\delta|\to\infty}\ell(y,y+\delta)=+\infty$ unif. in $y\in\mathbb{R}$;



(ℓ_3) "Ordering" $\forall\, \underline{y}<\bar{y},\ \underline{z}<\bar{z}$, $\ell(\underline{y},\underline{z})+\ell(\bar{y},\bar{z})<\ell(\bar{y},\underline{z})+\ell(\underline{y},\bar{z})$;



(ℓ_4) "Crossing" If $(x_{-1},x_0,x_1)\neq(x_{-1}^*,x_0^*,x_1^*)$ minimal
$x_0=x_0^* \Rightarrow (x_{-1}-x_{-1}^*)\cdot(x_1^*-x_1)<0$.



In the following let $\ell:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be any continuous function satisfying $\ell_1,\ell_2,\ell_3,\ell_4$...
(action a, minimizing segments, global minimizers defined as above).

Rmk 1 "Coercivity" $\Rightarrow \forall J<K$, $x_J,x_K\in\mathbb{R}$ $\exists$ min. segment $(x_J,\dots,x_K)$.

Rmk 2 Suppose that ℓ is C^2 and satisfies ℓ_1,ℓ_2:

(i) (x_{-1},x_0,x_1) is minimal $\Rightarrow D_2\ell(x_{-1},x_0)+D_1\ell(x_0,x_1)=0$; (discrete E-L equation)

(ii) if $D_1D_2\ell<0$, then
- ℓ_3 and ℓ_4 holds
- $\forall x_0,x_1\in\mathbb{R}$ $\exists!$ $x_2(x_0,x_1)$ s.t. $(x_0,x_1,x_2(x_0,x_1))$ is minim.

(ii) If $D_1 D_2 \ell < 0$, then • $\ell 3$ and $\ell 4$ holds
• $\forall x_0, x_1 \in \mathbb{R}\ \exists!\ x_2(x_0,x_1)$ s.t. $(x_0, x_1, x_2(x_0,x_1))$ is minim.

$\Rightarrow$ discrete E-L flow $\varphi: \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$, $\varphi(x_0,x_1) = (x_1, x_2(x_0,x_1))$

(analogously $\exists!\ x_{-1}(x_0,x_1)$: (x_{-1}, x_0, x_1) is a minim. $\Rightarrow \exists\ \varphi^{-1}$).

Example Take $m:\mathbb{R}\to\mathbb{R}$ C^2, coercive and strictly convex, $V:\mathbb{R}\to\mathbb{R}$ 1-periodic. Then $\ell(y,z) := m(z-y) - V(y) - V(z)$ satisfies $\ell 1, \ell 2, \ell 3, \ell 4$.

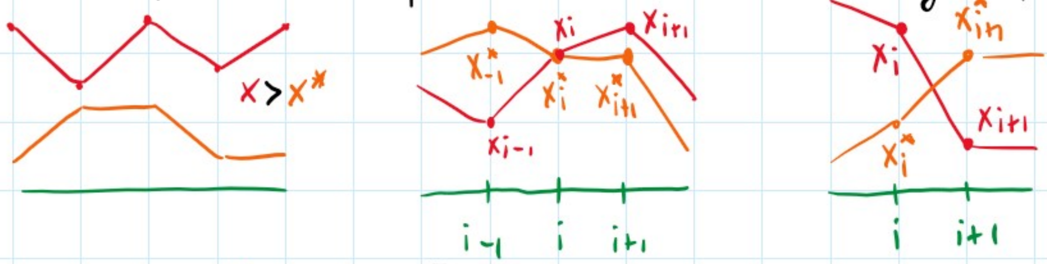
(discrete Frenkel-Kontorova model: chain of atoms in positions $(x_i)_{i\in\mathbb{Z}}$ with interaction potential m and external potential V).

Order and translations

Def There is a partial order on $\mathbb{R}^{\mathbb{Z}}$: $x < x^* \Leftrightarrow x_i < x_i^*\ \forall i \in \mathbb{Z}$. We say that a subset $S \subset \mathbb{R}^{\mathbb{Z}}$ is totally ordered if $\forall x, x^* \in S$, $x < x^*$ or $x = x^*$ or $x > x^*$.

Def x and $x^* \in \mathbb{R}^{\mathbb{Z}}$ are said to cross (a) at $i\in\mathbb{Z}$ if $x_i = x_i^*$ and $(x_{i-1} - x_{i-1}^*)(x_{i+1}^* - x_{i+1}) < 0$;
(b) between i and $i+1$ if $(x_i - x_i^*)(x_{i+1}^* - x_{i+1}) < 0$.

Rmk Useful to depict $x \in \mathbb{R}^{\mathbb{Z}}$ as a piecewise linear curve in $\mathbb{R}^2$ connecting the points $\{(i, x_i) : i \in \mathbb{Z}\}$.



Def "crossing at infinity" $x, x^* \in \mathbb{R}^{\mathbb{Z}}$ are α-asymptotic if $\lim_{i\to-\infty} |x_i - x_i^*| = 0$
ω-asymptotic if $\lim_{i\to+\infty} |x_i - x_i^*| = 0$

Def "translations" Action T of group $\mathbb{Z}^2$ on $\mathbb{R}^{\mathbb{Z}}$ by translation of the graph
$$T_{(q,p)} x = x^*, \quad \text{where } x_i^* = x_{i-q} + p \quad \forall i \in \mathbb{Z}.$$

Notion of crossings and action T similarly defined for finite segments.

Properties of T: • preserves order and crossings;

Notion of crossings and action T similarly defined for finite segments.

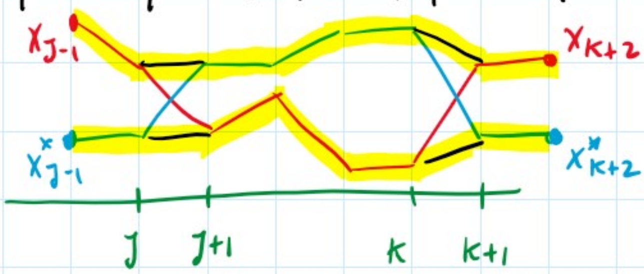
Properties of T:
- preserves order and crossings;
- preserves the discrete action $a \Rightarrow T$ maps minimal segments to minimal segments and global minimizers to global minimizers.

Def "periodic sequences" For $(q,p) \in \mathbb{Z}\setminus\{0\} \times \mathbb{Z}$ set $P_{q,p} := \left\{ x \in \mathbb{R}^{\mathbb{Z}} \mid T_{(q,p)} x = x \right\}$.

Lemma Let $x, x^* \in \mathcal{M}$. Then:
(i) If they coincide at $i \in \mathbb{Z}$ and $x \neq x^*$, they cross at i.
(ii) They cross at most once.
(iii) If they are α- or ω-asymptotic and $\underbrace{|x_i - x_{i-1}| \le C}_{\text{this will be automatic}}$ for $i \to -\infty$ resp. $i \to +\infty$ for some C, then they don't cross.

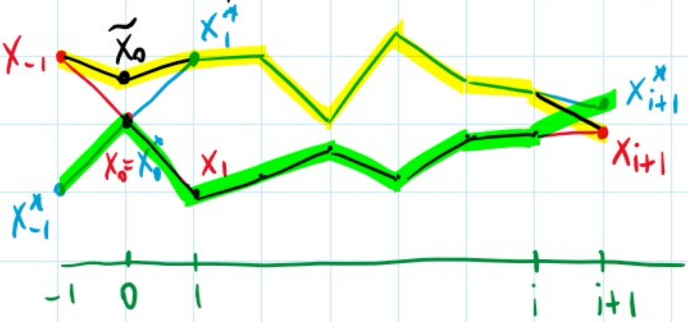
(i) + (ii) $\Rightarrow$ $x, x^* \in \mathcal{M}$ either cross or are comparable ($x < x^*$ or $x = x^*$ or $x > x^*$).

Proof (i) follows from (ℓ4). (ii) follows from (ℓ3) + (ℓ4). For crossing between integers:



either the upper yellow segment has action less than $(x_{J-1}, -, x_{K+2})$ or the lower yellow segment has action less than $(x^*_{J-1}, -, x^*_{K+2})$ by (ℓ_3).

We prove (iii) assuming $|x_i - x_i^*| \xrightarrow{i \to +\infty} 0$ and $x_0 = x_0^*$, $x_i < x_i^*$ $\forall i > 0$ (other cases are similar).
By (ℓ4) one among (x_{-1}, x_0, x_1^*) and (x^*_{-1}, x_0, x_1) is not minimal. Suppose it is the first and take $\tilde{x}_0$: $a(x_{-1}, \tilde{x}_0, x_1^*) - a(x_{-1}, x_0, x_1) =: -\varepsilon$. Call $\tilde{x}$ the



yellow segment and $\tilde{\tilde{x}}$ the green one. Then:

$$a(\tilde{x}) + a(\tilde{\tilde{x}}) - a(x_{-1}, -, x_{i+1}) + a(x^*_{-1}, -, x^*_{i+1})$$
$$= -\varepsilon + \ell(x_i^*, x_{i+1}) + \ell(x_i, x^*_{i+1}) - \ell(x_i, x_{i+1}) - \ell(x_i^*, x^*_{i+1}).$$

If we show $\lim_{i\to+\infty} |\ell(x_i^*, x_{i+1}) - \ell(x_i, x_{i+1})| \overset{\circledast}{=} 0 = \lim_{i \to +\infty} |\ell(x_i, x^*_{i+1}) - \ell(x_i^*, x^*_{i+1})|$
then we are done as one among $(x_{-1}, -, x_{i+1})$ and $(x^*_{-1}, -, x^*_{i+1})$ not minimal.
Take $k_i \in \mathbb{Z}$ s.t. $0 \le x_{i+1} - k_i < 1$, then $-C \le x_i - k_i < 1 + C$ and

$$\ell(x_i^*, x_{i+1}) - \ell(x_i, x_{i+1}) \overset{(\ell 1)}{=} \ell(\underbrace{x_i^* - k_i}_{\text{bounded}}, \underbrace{x_{i+1} - k_i}_{\text{bounded}}) - \ell(\underbrace{x_i - k_i}_{\text{bounded}}, \underbrace{x_{i+1} - k_i}_{\text{bounded}}) \xrightarrow{i \to +\infty} 0 \text{ as}$$

$(x_i^* - k_i) - (x_i - k_i) \to 0$

$(x_i^* - k_i) - (x_i - k_i) \to 0$ bounded bounded bounded bounded

and ℓ uniformly continuous on bounded sets. ☒

Rmk There are cases where x and x^* are both α- and ω-asymptotic.

Corollary (i) $x, x^* \in M \cap P_{q,p}$ don't cross. (ii) If $x \in M \cap P_{kq,kp}$ for some $k \geq 1$, then $x \in P_{q,p}$.

Proof (i) x, x^* cross at $i \in \mathbb{Z} \Rightarrow x, x^*$ cross at $i + q \cdot \mathbb{Z}$. (ii) x and $T_{(q,p)} x$ do not cross.

WLOG $x < T_{(q,p)} x$. T order-preserving: $x < T_{(q,p)} x < T_{2(q,p)} x < \ldots < T_{k(q,p)} x = x$ ↯ ☒

Theorem 1 (analogous to Thm 3 from last time) Let $a_{q,p} : P_{q,p} \to \mathbb{R}$, $a_{q,p}(x) = a(x_0, \_, x_q)$. Then:

$$M_{q,p} := \{ x \in P_{q,p} \mid a_{q,p}(x) = \inf a_{q,p} \} \neq \emptyset \text{ and } M_{q,p} = P_{q,p} \cap M.$$

Moreover: $M_{q,p} = M_{kq,kp}$ and $\inf a_{kq,kp} = k \cdot \inf a_{q,p}$ $\forall k \geq 1$.

Proof $(\ell 1) + (\ell_2) \Rightarrow M_{q,p}$ non-empty. $P_{q,p} \cap M \subset M_{q,p}$ also clear. To prove opposite inclusion we show the following

Claim $x, x^* \in M_{q,p}$ do not cross.

Assume they cross (so $q \geq 2$). Define $x^- := \min\{x, x^*\}$, $x^+ := \max\{x, x^*\}$ in $P_{q,p}$. Then

$$a_{q,p}(x^-) + a_{q,p}(x^+) \leq a_{q,p}(x) + a_{q,p}(x^*) = 2 \inf a_{q,p} \Rightarrow x^-, x^+ \in M_{q,p}.$$

If x, x^* cross between i and $i+1$, then the above inequality is strict ↯.

If x, x^* cross at i, then WLOG $(x_{i-1}^-, x_i^-, x_{i+1}^-)$ not minimal $\Rightarrow \exists\, y \in \mathbb{R}$ s.t. $a(x_{i-1}^-, y, x_{i+1}^-) < a(x_{i-1}^-, x_i^-, x_{i+1}^-)$. Then $\exists\, \tilde{x} \in P_{q,p}$ with $\tilde{x}_i = y$, $\tilde{x}_j = x_j^-$ $j \not\equiv i \pmod q$ $\Rightarrow a_{q,p}(\tilde{x}) < a_{q,p}(x^-)$ ↯

Let us show that $M_{q,p} = M_{kq,kp}$ and $\inf a_{kq,kp} = k \cdot \inf a_{q,p}$.

If $x \in M_{q,p}$, then $a_{kq,kp}(x) = k a_{q,p}(x) = k \cdot \inf a_{q,p}$. Hence, $\inf a_{kq,kp} \leq k \cdot \inf a_{q,p}$.

If $x \in M_{kq,kp}$, then x and $T_{(q,p)} x$ don't cross by the claim. As in the proof of corollary we get $x = T_{(q,p)} x$. Hence $\inf a_{kq,kp} = a_{kq,kp}(x) = k \cdot a_{q,p}(x) \geq k \inf a_{q,p}$.

Hence $\inf a_{kq,kp} = k \cdot \inf a_{q,p}$ and $x \in M_{q,p} \Leftrightarrow x \in M_{kq,kp}$.

Finally, let $x \in M_{q,p}$ and $J^- < J^+$. Let $k > 0$ s.t. $kq \geq J^+ - J^-$, then $x \in M_{kq,kp}$ and so $(x_{J^-}, \_, x_{J^+})$ is a minimal segment (why?) ☒

and so $(x_{j^-}, \dots, x_{j^+})$ is a minimal segment (why?) ☑

Thanks to the theorem above we can show that translates of elements of M do not cross.

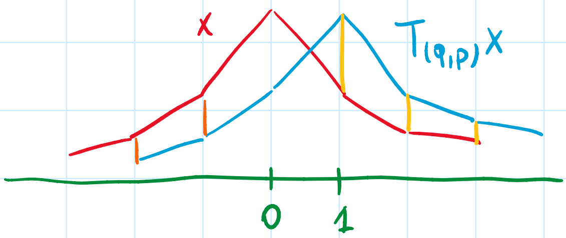
Theorem If $x \in M$, then x and $T_{(q,p)}x$ do not cross $\forall (q,p) \in \mathbb{Z}^2$. In other words: $Tx := \{T_{(q,p)}x \mid (q,p) \in \mathbb{Z}^2\}$ is totally ordered.

Proof Clear if $q=0$ (why?). Assume $q \neq 0$ and, by contradiction, x and $T_{(q,p)}x$ cross at 0 or between 0 and 1. Up to swapping x and $T_{(q,p)}x$:

$$(*) \begin{cases} x_j > (T_{(q,p)}x)_j = x_{j-q} + p & \text{for } j<0 \\ x_j < (T_{(q,p)}x)_j = x_{j-q} + p & \text{for } j>0. \end{cases}$$

We deal with $q>0$ ($q<0$ is similar). Then,

$$(*) \Rightarrow \begin{cases} \forall j<0: & x_k^{(j)} := (T_{(kq,kp)}x)_j = x_{j-kq} + kp \text{ is decreasing in } k \geq 0. \\ \forall j>0: & x_k^{(j)} := (T_{(-kq,-kp)}x)_j = x_{j+kq} - kp \text{ is decreasing in } k \geq 0 \end{cases}$$



Take $y \in M_{q,p}$ and translate it so that $y_0 < x_0$. y cross x at most once, so either $y_j < x_j$ $\forall j<0$ or $y_j < x_j$ $\forall j>0$. We do only first case. By the lemma is enough to prove the following

Claim x and $T_{(q,p)}x$ are α asymptotic and $|x_{i-1} - x_i| \leq C$ for $i<0$.

Then $x_k^{(j)} = (T_{(kq,kp)}x)_j > (T_{(kq,kp)}y)_j = y_j$. So $(x_k^{(j)})_{k\in\mathbb{N}}$ is convergent $\forall j<0$.

Let $i<0$ and take $k\geq 0$, $j \in [-q, \dots, -1]$ s.t. $i = j - kq$, then

$$(T_{(q,p)}x)_i - x_i = (T_{(q,p)}x)_{j-kq} + kp - (x_{j-kq} + kp) = (T_{((k+1)q,(k+1)p)}x)_j - (T_{(kq,kp)}x)_j = x_{k+1}^{(j)} - x_k^{(j)}$$

Since the sequences $(x_k^{(-1)})_{k\in\mathbb{N}}, \dots, (x_k^{(-q)})_{k\in\mathbb{N}}$ are all convergent, we see that $\lim_{i\to-\infty} (T_{(q,p)}x)_i - x_i = 0$. Moreover: $x_{i-1} - x_i = x_{j-1-kq} - x_{j-kq} = x_k^{(j-1)} - x_k^{(j)}$.

Since the sequences $(x_k^{(-1)})_{k\in\mathbb{N}}, \dots, (x_k^{(-q-1)})_{k\in\mathbb{N}}$ are convergent, we see that for i large enough $|x_{i-1} - x_i| < \varepsilon + \max_{j=-1,\dots,-q} \left| \lim_{k\to\infty} x_k^{(j-1)} - \lim_{k\to\infty} x_k^{(j)} \right|$. ☑

Totally ordered subsets of $\mathbb{R}^{\mathbb{Z}}$

Lemma 1 Let S be a T-invariant, totally ordered subset of M. Then $\bar{S} \subset M$ is also T-invariant and totally ordered.

Proof $\bar{S}$ T-invariant is clear (why?). Let $y = \lim y^{(n)}$, $z = \lim z^{(n)}$ in $\bar{S}$. Since $\bar{S} \subset M$

T-invariant and totally ordered.

Proof $\bar S$ T-invariant is clear (why?). Let $y = \lim y^{(n)}$, $z = \lim z^{(n)}$ in $\bar S$. Since $\bar S \subset M$ we just need to show that y and z don't cross. If they would, $y^{(n)} \in S$ and $z^{(n)} \in S$ would cross for n large enough. However, S is ordered ↯ ☒

Dfn $\tilde G_+ := \{ f : \mathbb{R} \to \mathbb{R} \mid f$ homeomorphism, $f(t+1) = f(t)+1 \ \forall t \in \mathbb{R}$.

Lemma Let $S \subset \mathbb{R}^{\mathbb{Z}}$ be a non-empty, closed, T-invariant, totally ordered set. Define $p_i : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ the i-th projection: $p_i(x) = x_i$.

(i) $p_0(S) = p_i(S)$ $\forall i \in \mathbb{Z}$ is a closed set invariant under translations by integers;
(ii) $p_0 : S \to p_0(S)$ is a homeomorphism;
(iii) $\exists f \in \tilde G_+$ s.t. $\forall x \in S, \forall i \in \mathbb{Z}$ $f(x_i) = x_{i+1}$. $\left(f(p_0(S)) = p_0(S) \right)$.

Proof (i) Let $y \in \overline{p_0(S)}$ and take $x^{(n)} \subset S$ s.t. $x_0^{(n)} \to y$. Since S is T-invariant $\exists\, x \in S, p \in \mathbb{N}$ such that $x_0 < y < (T_{(0,p)}x)_0$. For n large: $x_0 < x_0^{(n)} < (T_{(0,p)}x)_0$ and since S is totally ordered $x < x^{(n)} < T_{(0,p)}x$. Thus: $x_i^{(n)} \in [x_i, x_i + p]$ $\forall i$. Since $[x_i, x_i+p]$ is compact, $\exists\, x^{(n_k)} \to x^{\infty} \in S$ by Tychonoff.
Thus $y = x_0^{\infty} \in p_0(S)$. Invariance by translations and $p_0(S) = p_i(S)$ follow from T-inv. of S (why?).

(ii) p_0 is an open map and is injective on S since S is totally ordered.

(iii) We construct f on $p_0(S)$ first putting $f(x_0) = x_1 = p_1 \circ (p_0|_S)^{-1}(x_0)$ $\forall x_0 \in p_0|_S$.
f is strictly increasing since S is totally ordered: $x_0 < x_0^* \Rightarrow x_1 < x_1^*$ and
$f(x_0 + 1) = f((T_{(0,1)}x)_0) = (T_{(0,1)}x)_1 = x_1 + 1$ $\forall x_0 \in p_0(S)$.
The set $\mathbb{R} \setminus p_0(S)$ is a disjoint union of intervals (a_k, b_k) and we take f to be linear there; $f((1-t)a_k + t b_k) = (1-t) f(a_k) + t \cdot f(b_k)$ $(a_k, b_k \in S)$.
f is continuous, strictly increasing and satisfies $f(t+1) = f(t)+1$ $\forall t \in \mathbb{R}$.
Finally: $f(x_i) = f((T_{(-i,0)}x)_0) = (T_{(-i,0)}x)_1 = x_{i+1}$ $\forall x \in S$ $\forall i \in \mathbb{Z}$.
☒

Corollary If $x \in M$, $\exists f \in \tilde G_+$ s.t. $f(x_i) = x_{i+1}$ $\forall i \in \mathbb{Z}$.

Proof Apply Lemma 1 to T_x and Lemma 2 to $\overline{T_x}$. $\boxtimes$