Talk 4a: The Hamilton–Jacobi method

Gabriele Benedetti

May 11, 2020

Introduction

The Hamilton-Jacobi method is a powerful way to find orbits minimizing the action. Two flavours:

- time dependent
  - good for time-fixed (Tonelli) minimizers,
  - we use it to prove Weierstrass' theorem;
- time independent
  - good for time-free minimizers,
  - we use it for the pendulum.

Notation:

- $L : TM \to \mathbb{R}$ Tonelli Lagrangian on manifold M,
- $H : T^*M \to \mathbb{R}$ associated Tonelli Hamiltonian.

Time-dependent subsolutions of HJ-equation

Definition (L-gradient)

Let $S : M \times [a, b] \to \mathbb{R}$ be C^1 and write $S_t := S(\cdot, t)\ \forall\, t \in [a, b]$. The L-gradient of S is the time-dependent vector field on M

$$\mathrm{grad}_L S_t(x) = \mathrm{Leg}^{-1}(\mathrm{d}_x S_t), \qquad \forall\, (x, t) \in M \times [a, b].$$

Definition (Time-dependent subsolutions)

A C^1-function $S : M \times [a, b] \to \mathbb{R}$ is a time-dependent subsolution of the Hamilton-Jacobi equation if

$$H(x, \mathrm{d}_x S_t) + \partial_t S_t(x) \leq 0, \qquad \forall\, (x, t) \in M \times [a, b].$$

We denote by $N_S \subset M \times [a, b]$ the set of pairs (x, t), where equality holds. We say that S is a solution if $N_S = M \times [a, b]$.

Time-dependent subsolutions yield Tonelli minimizers

Theorem (A)

Let $S : M \times [a, b] \to \mathbb{R}$ be a subsolution and $x_0, x_1 \in M$. Then,

$$A_L(\gamma) \geq S_b(x_1) - S_a(x_0), \qquad \forall\, \gamma \in C^{ac}_{x_0,x_1}([a, b], M)$$

with equality iff γ is a flow line of $\mathrm{grad}_L S_t$ with $(t, \gamma(t)) \in N_S$, $\forall\, t$. Each such flow line is a Tonelli minimizer.

Proof.

For all $(x, v) \in TM$ we have by the Fenchel inequality

$$L(x, v) + H(x, \mathrm{d}_x S_t) \geq \mathrm{d}_x S_t \cdot v$$

with equality if and only if $v = \mathrm{grad}_L S_t(x)$. Therefore,

$$\begin{aligned} L(x, v) \geq \mathrm{d}_x S_t \cdot v - H(x, \mathrm{d}_x S_t) &\geq \mathrm{d}_x S_t \cdot v + \partial_t S_t(x) \\ &= \mathrm{d}_{(x,t)} S \cdot (v + \partial_t) \end{aligned}$$

with equality if and only if $v = \mathrm{grad}_L S_t(x)$ and $(x, t) \in N_S$. Thus,

$$\begin{aligned} A_L(\gamma) \geq \int_a^b \mathrm{d}_{(\gamma(t),t)} S \cdot (\dot{\gamma}(t) + \partial_t)\mathrm{d}t &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\Big[S(\gamma(t), t)\Big]\mathrm{d}t \\ &= S(\gamma(b), b) - S(\gamma(a), a) \\ &= S_b(x_1) - S_a(x_0). \end{aligned}$$

□

Reminder of Weierstrass Theorem

Theorem (Part I)

Let L be bounded from below. For all $\tilde{K} \subset TM$ compact there exists $\delta > 0$ such that for all $(x, v) \in \tilde{K}$ the EL-solution

$$\gamma_{(x,v)} : [0, \delta] \to M, \qquad (\gamma_{(x,v)}(0), \dot{\gamma}_{(x,v)}(0)) = (x, v)$$

is well-defined and the unique minimizer in $C^{ac}_{x,\gamma_{(x,v)}(\delta)}([0, \delta], M)$.

Theorem (Part II)

Let L be bounded from below. For all $K \subset M$ compact there exist $C, \delta > 0$ such that for all $x \in K$ and $y \in M$ with $d(x, y) \leq C\delta$ there is a (unique) EL-Solution

$$\gamma : [0, \delta] \to M, \qquad \gamma(0) = x, \ \gamma(\delta) = y$$

which is the unique minimizer in $C^{ac}_{x,y}([0, \delta], M)$.

The proof

Part I $\Rightarrow$ Part II.

By the implicit function theorem there exist $C, \delta > 0$ such that for all $x \in K$

$$\tilde{K}_x := \{v \in T_x M \mid |v|_x \leq 2C\} \to M, \qquad v \mapsto \gamma_{(x,v)}(\delta)$$

is an embedding whose image contains $\bar{B}_{C\delta}(x)$. To deduce Part II, apply Part I to $\tilde{K} = \cup_{x \in K} \tilde{K}_x$. □

To prove Part I we use local existence of HJ-solutions.

Lemma

Let $\tilde{K}$ be a compact set of TM. There are $\delta, \epsilon > 0$ such that for all $(x, v) \in \tilde{K}$ there exists a time-dependent HJ-solution of class C^2 $S : B_\epsilon(x) \times [0, \delta] \to M$ with $v = \mathrm{grad}_L S_0(x)$. □

The proof

Proof of Part I.

Given $\tilde{K} \subset TM$ let δ and ϵ as in the lemma:

$$\forall (x,v) \in \tilde{K},\ \exists S : B_\epsilon(x) \times [0,\delta] \to \mathbb{R},\ C^2 \text{ solution},\ v = \operatorname{grad}_L S_0(x).$$

Theorem (A) $\Rightarrow$ flow line $\gamma_{(x,v)} : [0,\delta] \to B_\epsilon(x)$ of $\operatorname{grad}_L S_t$ through x is unique minimizer in $C^{ac}_{x,y}([0,\delta], B_\epsilon(x))$, $y := \gamma_{(x,v)}(\delta)$.
$\gamma \in C^2 \ \Rightarrow\ \gamma$ is EL-solution with initial condition (x,v).
Left to show: $\gamma_{(x,v)}$ unique minimizer in $C^{ac}_{x,y}([0,\delta], M)$.
Take γ in this set with $\gamma([0,\delta_1)) \subset B_\epsilon(x)$, $\gamma(\delta_1) \in \partial B_\epsilon(x)$ for a δ_1.
WLOG: $L \geq 0$ as L bounded from below. Then:

$$A_L(\gamma) \overset{L \geq 0}{\geq} \int_0^{\delta_1} L(\gamma,\dot\gamma)\mathrm{d}t \overset{L\text{ Tonelli}}{\geq} d(\gamma(0),\gamma(\delta_1)) + B\delta_1 \geq \epsilon - |B|\delta \overset{\delta\text{ small}}{\geq} \epsilon/2,$$

Then: $\tilde{K}$ compact $\Rightarrow L((\gamma_{(x,v)}, \dot\gamma_{(x,v)})) \leq C$ for some C. Thus,

$$A_L(\gamma_{(x,v)}) \leq C\delta \overset{\delta\text{ small}}{<} \epsilon/2.$$

□

k-subsolutions of the HJ-equation

Definition (k-subsolutions)

Let $k \in \mathbb{R}$. A C^1-function $u : M \to \mathbb{R}$ is a k-subsolution of the Hamilton-Jacobi equation if

$$H(x, \mathrm{d}_x u) \leq k, \qquad \forall\, x \in M.$$

We denote by $M_u \subset M$ the set of points x, where equality holds. We say that u is a k-solution if $M_u = M$.

Remark

- *If u is a k-subsolution, $S(x,t) = u(x) - kt$ is a time-dependent subsolution on $M \times \mathbb{R}$.*
- *If M is closed and u_1 is a k_1-solution, u_2 is a k_2 solution, then $k_1 = k_2$ ($\exists x \in M, \mathrm{d}_x u_1 = \mathrm{d}_x u_2$).*
- *If $L(x,v) = \frac{1}{2}|v|_x^2$ then the geodesic radial coordinate $r : B_\epsilon(x) \to (0, \infty)$ is a $\frac{1}{2}$-solution: by Gauss Lemma $|\mathrm{d}r| = 1$.*

k-subsolutions yield time-free minimizers for $L + k$

Theorem (B)

Let $u : M \to \mathbb{R}$ be a k-subsolution and $x_0, x_1 \in M$. Then,

$$A_{L+k}(\gamma) \geq u(x_1) - u(x_0), \qquad \forall\, \gamma \in \bigcup_{T>0} C^{ac}_{x_0,x_1}([0,T], M)$$

with equality iff γ is a flow line of $\mathrm{grad}_L u$ contained in M_u. Each such flow line is a time-free minimizer.
Hence, a flow line $\gamma : \mathbb{R} \to M$ of $\mathrm{grad}_L u$ with $\gamma(\mathbb{R}) \subset M_u$ is a global time-free minimizer for $L + k$.

Proof.

Same ideas as in Theorem (A). □

k-subsolutions with other cohomology classes

Let θ be a closed 1-form on M with $c := [\theta] \in H^1(M; \mathbb{R})$.
Set $L_\theta := L + \theta$. New Hamiltonian is $H_\theta(x, p) = H(x, p - \theta_x)$.
A function $u_\theta : M \to \mathbb{R}$ is a k-subsolution for L_θ iff

$$\forall\, x \in M, \quad H(x, \mathrm{d}_x u_\theta - \theta_x) \leq k.$$

Finding u_θ is equivalent to finding $\tilde{\theta}$ closed 1-form on M with

$$\forall\, x \in M, \quad H(x, \tilde{\theta}_x) \leq k, \qquad [\tilde{\theta}] = -c.$$

Moreover,

$$M_{u_\theta} = \{x \in M \mid H(x, \tilde{\theta}_x) = k\}, \qquad \mathrm{grad}_{L_\theta} u_\theta = \mathrm{Leg}_L^{-1}(\tilde{\theta}).$$

Application to the pendulum

Consider the pendulum $L : TS^1 \to \mathbb{R}$, $L(x,v) = \frac{1}{2}|v|^2 + (1 - \cos x)$.
$\forall\, k \geq 0$, $\exists\, \theta_k^\pm$ two closed forms with $c_k^\pm := [\theta_k^\pm] \in H^1(S^1;\mathbb{R})$ and

$$H(x, -(\theta_k^\pm)_x) = k, \quad \forall\, x \in S^1.$$

$\mathrm{Leg}^{-1}(-\theta_k^\pm)$-flowlines are global time-free minimizers of $L_{\theta_k^\pm} + k$.
For all $r \in [0,1)$ the closed forms $r\theta_0^\pm$ have $[r\theta_0^\pm] = rc_0^\pm$ and satisfy

$$H(x, -(r\theta_0^\pm)_x) \leq 0, \quad \forall\, x \in S^1$$

with equality only at $x = 0$, where $\mathrm{Leg}^{-1}(-r\theta_0^\pm) = 0$. Hence, the constant orbit at $x = 0$ is a global time-free minimizer for $L_{r\theta_0^\pm}$.
It will follow from the general theory that these are the only global time-free minimizers for the pendulum (try direct proof).