1 Energy

In this section the setting is as in the previous talk, i.e. M is a connected, closed manifold and $L : TM \to \mathbb{R}$ Tonelli.

Definition 1.1. *Let L be a Tonelli Lagrangian on M. Recall the definition of the Hamiltonian*

$$H : T^*M \to \mathbb{R}, H(x,p) := p(Leg^{-1}(x,p)) - L(Leg^{-1}(x,p)).$$

The energy associated to L is the function $E : TM \to \mathbb{R}, E := H \circ Leg$, i.e.

$$E(x,v) := \frac{\partial L}{\partial v}(x,v)(v) - L(x,v)$$

Remark 1.2. If L is Tonelli then its associated energy E is Tonelli as well.

Example 1.3. We consider the electromagnetic Lagrangians

$$L(x,v) = \frac{1}{2}g_x(v,v) + \theta_x(v) - U(x)$$

where g is a Riemannian metric on M, $\theta \in \Omega^1(M)$ a 1-form and $U \in C^\infty(M)$ a function on M. To compute the energy and Hamiltonian we first have to compute the conjugate momentum. For that we choose local coordinates on M and get:

$$\frac{\partial L}{\partial v}(x,v) = \frac{\partial L}{\partial v^i}dx^i = \frac{\partial}{\partial v^i}(\frac{1}{2}g_{jk}v^j v^k + \theta_j(x)v^j - U(x))dx^i = g_x(v,\cdot) + \theta_x$$

Therefore the energy is given by:

$$E(x,v) = g(v,v) + \theta(v) - (\frac{1}{2}g_x(v,v) + \theta_x(v) - U(x)) = \frac{1}{2}g_x(v,v) + U(x).$$

This is just the sum of the kinetic and potential energy, the 1-form θ doesn't affect the energy. Recall that the norm $|\cdot|_x$ on T_xM induces a norm also denoted by $|\cdot|_x$ on T_x^*M, which is given by $|p|_x = \sup_{w \in T_xM, |w|_x \le 1} |p(w)|$. Now setting $(x,p) = Leg(x,v)$ we compute the Hamiltonian:

$$H(x,p) = E(x,v) = \frac{1}{2}|v|_x^2 + U(x) = \frac{1}{2}|g_x(v,\cdot)|_x^2 + U(x) = \frac{1}{2}|p - \theta_x|_x^2 + U(x).$$

Example 1.4. Let L be a Tonelli Lagrangian, θ a 1-form and U a function on M. Let $\tilde{L}(x,v) := L(x,v) + \theta_x(v) - U(x)$. For the associated energies $E, \tilde{E}$ we get:

$$\tilde{E}(x,v) = E(x,v) + U(x).$$

2 Tonelli Theorem and action minimizers

In this section let $a, b \in \mathbb{R}$ with $a < b$ and M a connected manifold.

Theorem 2.1. *Let M be a connected, closed manifold, L a Tonelli Lagrangian. For each $x_0, x_1 \in M$ and homotopy class h of curves connecting x_0 and x_1, there is a $\gamma_h \in C^2_{x_0,x_1}([a,b], M; h)$ minimizing the action in the set $C^2_{x_0,x_1}([a,b], M; h)$.*

Firstly we will consider absolutely continuous curves to get better compactness properties. Secondly, the idea is to lift the problem of finding action minimizers in a fixed homotopy class to the universal cover $\tilde{M}$ of M. On $\tilde{M}$ the task will then be to find action minimizers. So far we have only considered M compact. But we don't know whether the universal cover $\tilde{M}$ is compact as well. We therefore have to consider the non-compact case as well.

Definition 2.2. *Let d be the metric on M obtained by some fixed Riemannian metric on M.*
A curve $\gamma : [a, b] \to M$ is called absolutely continuous if for each $\epsilon > 0$ there exists $\delta > 0$ such that for any familiy of disjoint intervals $(]a_i, b_i[)_{i=1,\dots,n}$ all included in $[a, b]$ and satisfying $\sum_i (b_i - a_i) < \delta$, we have $\sum_i d(\gamma(b_i), \gamma(a_i)) < \epsilon$.
We denote by $C^{ac}([a.b], M)$ the set of absolutely continuous curves $\gamma : [a, b] \to M$.

Remark 2.3. For a curve $\gamma : [a, b] \to M$ the property of being absolutely continuous is independent of the chosen Riemannian metric, see [5, Proposition 3.18].

Remark 2.4. Let $\gamma : [a, b] \to M$ be an absolutely continuous curve. Then: $\dot{\gamma} \in T_\gamma M$ exists almost everywhere on $[a, b]$ and

$$d(\gamma(a), \gamma(b)) \leq \int_a^b ||\dot{\gamma}(s)||_{\gamma(s)} ds$$

Remark 2.5. For a Tonelli Lagrangian L and an absolutely continuous curve $\gamma : [a, b] \to M$ the action $A_L(\gamma)$ is well defined and in $\mathbb{R} \cup \{\infty\}$.(Since L is bounded below)

Definition 2.6. *Let L be Tonelli, $x_0, x_1 \in M$. An absolutely continuous curve $\gamma_L \in C^{ac}_{x_0,x_1}([a, b], M)$ is called Tonelli minimizer if*

$$A_L(\gamma_L) = \min_{\gamma \in C^{ac}_{x_0,x_1}([a,b],M)} A_L(\gamma)$$

Theorem 2.7. *(Tonelli theorem) Let M be a connected manifold, $L : TM \to \mathbb{R}$ a Tonelli Lagrangian bounded below by a complete Riemannian metric on M, i.e. there exist a complete Riemannian metric g on M and some $B \in \mathbb{R}$ such that $L(x, v) \geq |v|_g + B$.*
Then for each $x_0, x_1 \in M$ there exists a Tonelli minimizer.

Remark 2.8. If M is compact, then the assumption of L being Tonelli suffices.

Proof. We only sketch the proof for the Tonelli theorem. The main idea is to show that for each $x_0, x_1 \in M, C \in \mathbb{R}$ the set

$$S_C^{x_0,x_1} := \{\gamma \in C^{ac}_{x_0,x_1}([a,b], M) \mid A_L(\gamma) \leq C\}$$

is a compact subset of $C^{ac}([a,b], M)$ for the topology of uniform convergence. Using this fact one can proceed as follows: Set $C := \inf_{\gamma \in C^{ac}_{x_0,x_1}([a,b],M)} A_L(\gamma)$(exists since L is bounded below). Then the sets $S^{x_0,x_1}_{C+\frac{1}{n}}$ form a decreasing sequence of non-empty compact sets. Therefore the intersection $\bigcap_n S^{x_0,x_1}_{C+\frac{1}{n}}$ is nonempty. Each curve in this intersection is a minimizer. □

We now sketch how the compactness of the sets $S_C^{x_0,x_1}$ in the Tonelli theorem can be proven when M is compact:

1. The sets $S_C = \{\gamma \in C^{ac}([a,b], M) | A_L(\gamma) \leq C\}$ are absolutely equicontinuous, i.e for each $\epsilon > 0$, there exists $\delta > 0$ such that for each disjoint family $(]a_i, b_i[)_{i=1,\dots,n} \subset [a,b]$ with $\sum_{i=1}^{n}(b_i - a_i) < \delta$ we have $\sum_{i=1}^{n} d(\gamma(a_i), \gamma(b_i)) < \epsilon$ for all $\gamma \in S_C$.(Here one needs superlinearity) This implies $cl_{C^0} S_C \subset C^{ac}([a,b], M)$.

2. If $(\gamma_n)_n \subset S_C$ converges uniformly to γ, then $A_L(\gamma) \leq \liminf_{n\to\infty} A_L(\gamma_n)$.

3. Apply the Arzela-Ascoli theorem: By 1. and 2., S_C is closed and equicontinuous. Since M is compact, the sets $\{\gamma(t) | \gamma \in S_C\}$ are precompact for all $t \in [a,b]$. Thus S_C is compact in the C^0 topology. $S_C^{x_0,x_1} \subset S_C$ is compact as a closed subset of a compact set.

In the following we will always assume $L \geq 0$. This is possible by adding a constant, since L is bounded below.

Proof. of 1.:
Set for $r > 0$:

$$K(r) := \inf\{\frac{L(x,v)}{|v|_x} \mid (x,v) \in TM, |v|_x \geq r\}$$

By superlinearity of L

$$\lim_{r\to\infty} K(r) = +\infty.$$

Thus for given $\epsilon > 0$ we can find $r > 0$ with

$$\frac{C}{K(r)} < \frac{\epsilon}{2}.$$

Let $\gamma \in S_C$, $J := \bigcup_{i=1}^{N} [a_i, b_i]$ and $E := J \cap \{|\dot{\gamma}|_\gamma > r\}$. Then

$$\begin{aligned} K(r)\sum_{i=1}^{N} d(\gamma(a_i),\gamma(b_i)) &\leq K(r)\int_E |\dot{\gamma}(s)|_{\gamma(s)}ds + K(r)\int_{J-E} |\dot{\gamma}(s)|_{\gamma(s)}ds \\ &\leq \int_E L(\gamma(s),\dot{\gamma}(s))ds + K(r)r\mu(J) \\ &\leq C + K(r)r\mu(J) \ (L \geq 0). \end{aligned}$$

Dividing by $K(r)$ we obtain:

$$\sum_{i=1}^{N} d(\gamma(a_i),\gamma(b_i)) \leq \frac{\epsilon}{2} + r\mu(J),$$

which proves that the set S_C is absolutely equicontinuous. Here μ denotes the lebesque measure. (This also implies $cl_{C^0}S_C \subset C^{ac}$, since the uniform limit of an absolutely equicontinuous family of absolutely continuous curves is absolutely continuous) □

Proof. of 2.: Let $(\gamma_n) \subset S_C$ converge uniformly to γ. From the above discussion we know $\gamma \in C^{ac}([a,b],M)$ and want to show $A_L(\gamma) \leq \liminf_{n\to\infty} A_L(\gamma_n)$. The main steps to show this are:

- Reduction to the case $im\gamma \subset U$ where (U,ϕ) is a chart on M.
  We cover $im\gamma$ by finitely many charts U_i such that there is a subdivision $a = a_0 < a_1 < ... < a_k = b$ with $\gamma([a_{i-1},a_i]) \subset U_i$. By uniform convergence of γ_n we can assume $\gamma_n([a_{i-1},a_i]) \subset U_i$. If the assertion holds for $im\gamma \subset U$, where U is a chart on M, then:

  $$A_L(\gamma) = \sum_i A_L(\gamma_{|[a_{i-1},a_i]}) \leq \sum_i \liminf_{n\to\infty} A_L(\gamma_{n[a_{i-1},a_i]}) \leq \liminf_{n\to\infty} A_L(\gamma_n).$$

  By the identification $U = \phi(U)$ we can from now on assume that $im\gamma$ is contained in an open subset U of $\mathbb{R}^n$.

- Lemma: Let $K \subset U$ be compact, $r > 0, \epsilon > 0$. There exists $\delta > 0$ such that if $x \in K, y \in K, |x-y| \leq \delta$ and $v,w \in \mathbb{R}^n, |v| \leq r$, then

  $$L(x,v) + \frac{\partial L}{\partial v}(x,v)(w-v) - \epsilon \leq L(y,w).$$

  Proof. We define

  $$C_1 := \sup\{|\frac{\partial L}{\partial v}(x,v)| \mid x \in K, |v| \leq r\},$$
  $$C_2 := \sup\{L(x,v) - \frac{\partial L}{\partial v}(x,v)v \mid x \in K, |v| \leq r\}.$$

Then we choose $s > 0$ such that for all $R \geq s$:

$$K(s) \cdot R \geq C_2 + C_1 \cdot R,$$

where $K(s)$ is from the above proof. If $|w| \geq s$, then

$$L(y,w) \geq K(s)|w| \geq C_2 + C_1|w| \geq L(x,v) + \frac{\partial L}{\partial v}(x,v)(w-v).$$

Hence we only have to find a δ such that the asserted inequality holds if $|w| \leq s$. Since L is convex,

$$L(x,w) \geq L(x,v) + \frac{\partial L}{\partial v}(x,v)(w-v).$$

By compactness of $\{(x,w) \mid x \in K, |w| \leq s\}$ we obtain the desired δ. □

- Apply this lemma with the compact (due to uniform convergence) set $K = im\gamma \cup \bigcup_n im\gamma_n$ and set $E_r := \{|\dot\gamma| \leq r\}$ to get for n big enough:

$$\int_{E_r} [L(\gamma,\dot\gamma) + \frac{\partial L}{\partial v}(\gamma,\dot\gamma)(\dot\gamma_n - \dot\gamma) - \epsilon] ds \leq \int_{E_r} L(\gamma_n,\dot\gamma_n) ds \leq A_L(\gamma_n).$$

- Show
$$\int_{E_r} [\frac{\partial L}{\partial v}(\gamma,\dot\gamma)(\dot\gamma_n - \dot\gamma)] ds \to 0, \text{ as } n \to \infty.$$

  This follows from Lemma 1.3.3 in [1, Maz]. To apply this Lemma note that as in the proof of 1. we see that $(\dot\gamma_n)_n$ is uniformly integrable and therefore $\dot\gamma_n - \dot\gamma$ is uniformly integrable.

- Let $r \to \infty$ and get:

$$A_L(\gamma) - \epsilon|b-a| \leq \liminf_{n\to\infty} A_L(\gamma_n).$$

- Let $\epsilon \to 0$, then $A_L(\gamma) \leq \liminf_{n\to\infty} A_L(\gamma_n)$.

□

This proof doesn't work for the noncompact case, since superlinearity holds only above compact subsets of M. Even if we could show that S_C is equicontinuous, we couldn't apply Arzela-Ascoli, because the sets $\{\gamma(t)|\gamma \in S_C^{x_0,x_1}\}$ aren't necessarily precompact. But we can modify this proof to obtain that for $K \subset M$ compact, the sets $S_{C,K} := \{\gamma \in S_C | im\gamma \subset K\}$ are equicontinuous and therefore compact. Let's see how the fact that L is bounded below by a complete Riemannian metric can be used to show that $S_C^{x_0,x_1}$ is compact. Let $\gamma \in S_C^{x_0,x_1}$, then for each $t \in [a,b]$:

$$d(\gamma(a),\gamma(t)) \leq \int_a^t |\dot\gamma|_\gamma ds \leq A_L(\gamma) - B\cdot(t-a) \leq C + B(b-a) =: R,$$

and hence $S_C^{x_0,x_1} \subset S_{C,\bar{B}_{x_0}(R)}$. Since the Riemannian metric is complete, closed metric balls in M are compact. Thus $S_C^{x_0,x_1}$ is compact, because it is a closed subset of the compact set $S_{C,\bar{B}_{x_0}(R)}$.
In the next talk we will see that for a Tonelli Lagrangian, Tonelli minimizers have the same regularity as the Lagrangian.

Theorem 2.9. *Suppose that L is a Tonelli Lagrangian on M. Let γ_L be a Tonelli minimizer. If L is C^r, then γ_L is C^r as well.*

Let us now return to Theorem 2.1, which stated the existence of action minimizers in a given homotopy class. Let $\pi : \tilde{M} \to M$ be the universal cover. We fix $\tilde{x_0} \in \pi^{-1}(x_0)$. Then we have a bijection $f : [C_{x_0,x_1}] \to \pi^{-1}(x_1)$ between homotopy classes of curves connecting x_0, x_1 and elements of the fiber of x_1. For $[\gamma] \in [C_{x_0,x_1}]$ we choose a lift $\tilde{\gamma}$ of γ and set $f([\gamma]) = \tilde{\gamma}(b)$. For each homotopy class $h \in [C_{x_0,x_1}]$ we have a bijection $h \to C_{\tilde{x_0},f(h)}, \gamma \to \tilde{\gamma}$ where $\tilde{\gamma}$ is the lift of γ with $\gamma(a) = \tilde{x_0}$ and $\tilde{\gamma}(b) = f(h)$.

Proof. of 2.1
The idea is to apply the Tonelli theorem to the universal cover $\tilde{M}$ of M and use the $1:1$ correspondence between curves in h and curves in $\tilde{M}$ with end point $f(h)$.
First we consider the universal cover $\pi : \tilde{M} \to M$ and set $\tilde{g} := \pi^* g$. We now show the the Lagrangian $\tilde{L} := L \circ d\pi$ on $T\tilde{M}$ satifies the assumptions of the Tonelli theorem. $\tilde{L}$ Tonelli can easily be verified.. Since M is compact g is complete. Since π is a Riemannian covering, $\tilde{g}$ is also complete. Since L is superlinear we can find $C \in \mathbb{R}$ such that $L(x,v) \geq |v|_{g,x} + C$ for all $(x,v) \in TM$ and therefore $\tilde{L}(\tilde{x},\tilde{v}) = L(\pi\tilde{x}, d\pi\tilde{v}) \geq |d\pi\tilde{v}|_{g,\pi\tilde{x}} + C = |\tilde{v}|_{\tilde{g},\tilde{x}} + C$.
By the Tonelli theorem there is a $\tilde{\gamma}_h \in C^{ac}_{\tilde{x_0},f(h)}([a,b],\tilde{M})$ such that

$$A_{\tilde{L}}(\tilde{\gamma}_h) = \min_{\tilde{\gamma} \in C^{ac}_{\tilde{x_0},f(h)}([a,b],\tilde{M})} A_{\tilde{L}}(\tilde{\gamma}) = \min_{\gamma \in C^{ac}_{x_0,x_1}([a,b],M;h)} A_L(\gamma).$$

In the second equation we used the bijection $h \to C_{\tilde{x_0},f(h)}, \gamma \to \tilde{\gamma}$ and the following two facts for a lift $\tilde{\gamma}$ of some curve $\gamma \in C([a,b],M)$:
1) $A_{\tilde{L}}(\tilde{\gamma}) = A_L(\gamma)$ if γ is absolutely continuous and
2) γ absolutely continuous iff $\tilde{\gamma}$ absolutely continuous: Let $\tilde{U}_\alpha \subset \tilde{M}, U_\alpha \subset M$ such that $\pi : \tilde{U}_\alpha :\to U_\alpha$ is a diffeomorphism, $\tilde{U}_\alpha$ and U_α are uniformly normal neighborhoods and the $\tilde{U}_\alpha$ covering $im\tilde{\gamma}$. There is a lebesque number $\delta_0 > 0$ such that for $s,t \in [a,b], |s-t| < \delta_0$, we have $\tilde{\delta}([s,t]) \subset \tilde{U}_\alpha$ for some α. For such s,t we have: $d_{\tilde{M}}(\tilde{\gamma}(s),\tilde{\gamma}(t)) = d_{\tilde{U}_\alpha}(\tilde{\gamma}(s),\tilde{\gamma}(t)) = d_{U_\alpha}(\gamma(s),\gamma(t)) = d_M(\gamma(s),\gamma(t))$ where the first and third equality follow from $U_\alpha, \tilde{U_\alpha}$ being uniformly normal neighborhoods and the second from $\pi : \tilde{U}_\alpha :\to U_\alpha$ being a isometry. The stated equivalence now follows if we choose $\delta < \delta_0$. This can be shown more easily if we use the definition of absolute continuity using charts.

Therefore the curve $\gamma_h := \pi \circ \tilde{\gamma}_h$ has the desired property. By the preceeding regularity theorem, γ_h is C^2. □

Remark 2.10. Let M be compact and L Tonelli. Fix $x_0, x_1 \in M$. Then

$$C^2_{x_0,x_1}([a,b],M) = \coprod_{h\in[C_{x_0,x_1}]} C^2_{x_0,x_1}([a,b],M;h).$$

For $h \in [C_{x_0,x_1}]$ there exists a minimizer γ_h for A_L in $C^2_{x_0,x_1}([a,b],M;h)$. Moreover there exists a minimizer γ_L for A_L in $C^2_{x_0,x_1}([a,b],M)$. In particular there exists a homotopy class $h \in [C_{x_0,x_1}]$ such that $\gamma_L = \gamma_{h_L}$ and

$$A_L(\gamma_L) = \min_{h\in[C_{x_0,x_1}]} A_L(\gamma_h).$$

Now consider a closed 1-form θ. By talk 1 γ_h is still a minimizer for $A_{L+\theta}$ in $C^2_{x_0,x_1}([a,b],M;h)$, with $A_{L+\theta}(\gamma_h) = A_L(\gamma_h) + C_h$ for some constant C_h. If θ is exact, then C_h is independent of h and

$$\{\gamma_L \in C^{ac}_{x_0,x_1}([a,b],M)\} = \{\gamma_{L+\theta} \in C^{ac}_{x_0,x_1}([a,b],M)\}.$$

However, if θ is not exact, then it might happen that this is false because

$$A_{L+\theta}(\gamma_{L+\theta}) = \min_{h\in[C_{x_0,x_1}]} (A_L(\gamma_h) + C_h),$$
$$A_{L+\theta}(\gamma_L) = \min_{h\in[C_{x_0,x_1}]} A_L(\gamma_h) + C_{[\gamma_L]}$$

If $c \in H^1_{deRham}(M)$, we can then consider $Ton^c := \{\gamma_{L+\theta} \in C^{ac}_{x_0,x_1}([a,b],M)\}$, where $[\theta] = c$. By the discussion above Ton^c does not depend on the representative θ. We will see the role of $H^1(M;\mathbb{R})$ and of $H_1(M;\mathbb{R})$ in more details when we will consider minimizing measures.

2.1 Appendix

Proposition 2.11. *Let (M,d) be a metric space and (K_n) a family of decreasing nonempty compact subsets of M. Then $\bigcap_{i\in\mathbb{N}} K_i$ is nonempty.*

Proof. Let $x_n \in K_n$. Since $(x_n)_n$ is contained in the compact set K_1, there is a subsequence (x_{n_j}) converging to some $x \in K_1$. For each n and j with $n_j > n$, $x_{n_j} \in K_n$. Since K_n is compact this implies $x \in K_n$. □

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