The Setting

- M is a smooth compact n-dimensional manifold without boundary with a Riemannian metric g
- tangent bundle TM, cotangent bundle T^*M
- We denote points by: $x \in M$, $v \in T_xM$, $p \in T_x^*M$
  $\Rightarrow (x, v) \in TM$, $(x, p) \in T^*M$
- the by g induced vector norms on T_xM and T_x^*M are both denoted by $|| \cdot ||_x$

Langragian and the action functional

Definition

A C^2 function $L : TM \to \mathbb{R}$ is called a Lagrangian on M.

Definition

For $x_0, x_1 \in M$, $a \leq b$ we set

$$C^2_{x_0,x_1}([a,b],M) := \{\gamma \in C^2([a,b],M) |\ \gamma(a) = x_0, \gamma(b) = x_1\}.$$

Additionally given a homotopy class of paths between x_0 and x_1 α we define

$$C^2_{x_0,x_1}([a,b],M;\alpha) := C^2_{x_0,x_1}([a,b],M) \cap \alpha.$$

Then the action $\mathscr{A} : C^2_{x_0,x_1}([a,b],M;\alpha) \to \mathbb{R}$ is given by:

$$\mathscr{A}(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t))\ dt$$

Variation and extremizing curves

Definition

A C^2 variation $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ (also denoted by γ_s) of a C^2 curve $\gamma : [a, b] \to M$ is a C^2 mapping with $\varepsilon > 0$, s.t.

- $\Gamma(0, t) = \gamma(t)$, $\forall t \in [a, b]$ and
- $\Gamma(s, a) = \gamma(a)$ and $\Gamma(s, b) = \gamma(b)$, $\forall s \in (-\varepsilon, \varepsilon)$.

Definition

A C^2 curve is called an extremizer or motion or just extremizing curve of the C^2 Lagrangian L, if

$$\left.\frac{d}{ds}\right|_{s=0} \mathscr{A}(\Gamma(s, \cdot)) = 0$$

for each C^2 variation $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$

Action extremizers and the Euler-Lagrange equation

Is there a different way to characterize those extremizing curves γ? Extremizing curves are exactly the curves, that satisfy the Euler-Lagrange equation in local coordinates:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t),\dot{\gamma}(t)) - \frac{\partial L}{\partial x}(\gamma(t),\dot{\gamma}(t)) = 0 \tag{1}$$

Consider some variation $\Gamma : (-\varepsilon,\varepsilon) \times [a,b] \to M$ of an extremal curve $\gamma : [a,b] \to M$. Consider the subdivision $a = r_0 < \cdots < r_m = b$, such that each $\gamma([r_k, r_{k+1}])$ is contained in some coordinate chart denoted by (U_k, φ_k).
We'll set:

$$\sigma(t) := \frac{\partial \Gamma}{\partial s}(0,t) \text{ and } \sigma_k(t) := d_{\gamma(t)}\varphi_k \cdot \sigma(t) \tag{2}$$

Action minimizers and the Euler-Lagrange equation

$$
\begin{aligned}
0 \overset{!}{=} \frac{d}{ds}\bigg|_{s=0} \mathscr{A}(\Gamma(s,\cdot)) &= \int_a^b \frac{d}{ds}\bigg|_{s=0} L(\gamma(t),\dot\gamma(t))\ dt \\
&= \sum_{k=0}^{m-1} \int_{r_k}^{r_{k+1}} \left(\frac{\partial L}{\partial x}(\gamma(t),\dot\gamma(t))\sigma_k(t) + \underbrace{\frac{\partial L}{\partial v}(\gamma(t),\dot\gamma(t))\dot\sigma_k(t)}_{= -\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma,\dot\gamma)\sigma_k + \frac{d}{dt}\left(\frac{\partial L}{\partial v}(\gamma,\dot\gamma)\sigma_k\right)} \right) dt \\
&= \sum_{k=0}^{m-1} \int_{r_k}^{r_{k+1}} \underbrace{\left(\frac{\partial L}{\partial x}(\gamma(t),\dot\gamma(t)) - \frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t),\dot\gamma(t)) \right)}_{(*)} \sigma_k(t)\ dt \\
&\quad + \underbrace{\sum_{k=0}^{m-1} \left(\frac{\partial L}{\partial v}(\gamma(r_{k+1}),\dot\gamma(r_{k+1}))\sigma_{k+1}(r_{k+1}) - \frac{\partial L}{\partial v}(\gamma(r_k),\dot\gamma(r_k))\sigma_k(r_k) \right)}_{\overset{\text{telescope sum}}{=} \frac{\partial L}{\partial v}(\gamma(b),\dot\gamma(b))\sigma_m(b) - \frac{\partial L}{\partial v}(\gamma(a),\dot\gamma(a))\sigma_0(a) \ = \ 0}
\end{aligned}
$$

Action extremizers and the Euler-Lagrange equation

Theorem

Let L be a C^2 Lagrangian on M and let $\gamma : [a,b] \to M$ be a C^2 curve. Then:

(i) *γ is extremal $\Rightarrow \forall\ [a',b'] \subseteq [a,b]$, s.t. $\gamma([a',b'])$ is contained in a chart (U,ϕ), then $\gamma|_{[a',b']}$ solves the Euler-Lagrange equation.*

(ii) *If for every $t \in [a,b]$ there exists an $[a',b'] \subseteq [a,b]$ containing t, s.t. $\gamma([a',b'])$ lies in an coordinate chart (U,φ) and $\gamma|_{[a',b']}$ solves the Euler-Lagrange equation, then γ is an extremal curve.*

The Action functional and the Euler-Lagrange equation

For now we want to consider what happens to the Euler-Lagrange equation and the action functional, when we add a function $f : TM \to \mathbb{R}$ to our Lagrangian L. How does this change look like?

Euler-Lagrange:

$$\frac{\partial(L+f)}{\partial x}(\gamma,\dot{\gamma}) - \frac{d}{dt}\frac{\partial(L+f)}{\partial v}(\gamma,\dot{\gamma}) = \frac{\partial L}{\partial x}(\gamma,\dot{\gamma}) - \frac{d}{dt}\frac{\partial L}{\partial v}(\gamma,\dot{\gamma}) + \underbrace{\frac{\partial f}{\partial x}(\gamma,\dot{\gamma}) - \frac{d}{dt}\frac{\partial f}{\partial v}(\gamma,\dot{\gamma})}_{(1)}$$

The action $\mathscr{A}$:

$$\tilde{\mathscr{A}}(\gamma) = \int_a^b (L+f)(\gamma,\dot{\gamma})\ dt = \mathscr{A}(\gamma) + \underbrace{\int_a^b f(\gamma,\dot{\gamma})\ dt}_{(2)}$$

The Action functional and the Euler-Lagrange equation

First case: $f : TM \to \mathbb{R}$ is a constant function with $f = C$ for some $C \in \mathbb{R}$.

$\Rightarrow$ (1) equal to 0 and (2) is equal to $C(b-a)$

Second case: consider the function $\tilde{\theta} : TM \to \mathbb{R}$, $\tilde{\theta}(x,v) := \theta_x(v)$, where $\theta \in \Omega^1(M)$ is a 1-form.

$\Rightarrow$ (1) is equal to the exterior derivative $-d\theta_x(\dot{x}, \cdot) = 0$. If θ closed. (2) is a constant and only depends on homotopy class α.

Let Γ be variation of γ:

$$\frac{d}{ds}\int_a^b \tilde{\theta}(\Gamma(t)), \dot{\Gamma}(t))\ dt =$$

$$\sum_{k=0}^{n-1}\int_{r_k}^{r_{k+1}} \underbrace{\left(\frac{\partial\tilde{\theta}}{\partial x}(\gamma(t),\dot{\gamma}(t)) - \frac{d}{dt}\frac{\partial\tilde{\theta}}{\partial v}(\gamma(t),\dot{\gamma}(t))\right)}_{= -d\theta_{\gamma(t)}(\dot{\gamma}(t),\cdot) = 0} \sigma_k(t)\ dt = 0$$

If θ is exact, meaning $\theta = du$, for some function $u \in C^\infty(M)$

$\Rightarrow$ (2) $= u(x_1) - u(x_0)$ is independent of α

The Legendre condition and the Euler-Lagrange vector field X_L

- We now want to study the Euler-Lagrange equation itself a little further. Using chain rule, we can expand the EL to:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t),\dot{\gamma}(t)) = \frac{\partial^2 L}{\partial v \partial x}(\gamma(t),\dot{\gamma}(t))\dot{\gamma}(t) + \frac{\partial^2 L}{\partial v^2}(\gamma(t),\dot{\gamma}(t))\ddot{\gamma}(t)$$

$$\Rightarrow \frac{\partial^2 L}{\partial v^2}(\gamma(t),\dot{\gamma}(t))\ddot{\gamma}(t) = \frac{\partial L}{\partial x}(\gamma(t),\dot{\gamma}(t)) - \frac{\partial^2 L}{\partial v \partial x}(\gamma(t),\dot{\gamma}(t))\dot{\gamma}(t)$$

- Legendre condition:

$$\frac{\partial^2 L}{\partial v^2}(x,v) \text{ is non-degenerate } \forall (x,v) \in TM \Leftrightarrow \det \frac{\partial^2 L}{\partial v^2}(x,v) \neq 0$$

- Legendre-condition is met $\Rightarrow$ we can solve for $\ddot{\gamma}(t)$ and we can define a vector field X_L, *Euler-Lagrange vector field*, on TM

$$X_L(x,v) = (x,v,v,\tilde{X}_L(x,v)) \in T_{(x,v)}TM,$$

  where $\tilde{X}_L$ satisfies the equation above (corresponding to $\ddot{\gamma}$), and (if it exists) ϕ_t^L denotes the *Euler-Lagrange flow*
- Since L is C^2 , X_L is just C^0 $\Rightarrow$ we cannot apply the theorem on existence and uniqueness of solutions of ordinary differential equations (this would require X_L to be locally Lipschitz)

The Legendre transform and Tonelli-Lagrangians

Definition

Let L be a Lagrangian on M. We define the (global) Legendre transform as:

$$\text{Leg} : TM \to T^*M,\ (x, v) \mapsto \frac{\partial L}{\partial v}(x, v) \in T_x^*M \tag{3}$$

Definition

We will call $L : TM \to \mathbb{R}$ a Tonelli-Lagrangian if:

(1) L is C^2

(2) $\forall (x, v) \in TM$: $\frac{\partial^2 L}{\partial v^2}(x, v)$ is positive definite

(3) L is superlinear in each fiber:

$$\forall x \in M : \lim_{||v||_x \to \infty} \frac{L(x, v)}{||v||_x} = +\infty$$

$$\Leftrightarrow \forall x \in M, A \in \mathbb{R}, \exists B \in \mathbb{R} :\ L(x, v) \geq A||v||_x - B.$$

- Since M is compact, the superlinearity is uniform over M and independent of the metric g.

Theorem

The (global) Legendre-transform $Leg : TM \to T^*M$ *is a diffeomorphism if* L *is a Tonelli-Lagrangian.*

Proof: Leg is fiber-preserving $\Rightarrow$ we must only consider the restriction $\text{Leg}\big|_{T_xM} : T_xM \to T_x^*M$. Proof with the following Lemma.

Lemma

Let V *be (finite dimensional) vector space. For* $F : V \to \mathbb{R}$, C^2 *and strictly convex* $(HessF > 0)$ *we have:* F *superlinear* $\Leftrightarrow$ $dF : V \to V^*$ *is a diffeomorphism*

„$\Rightarrow$“:

- Hess$F$ is pos. def. $\Rightarrow$ dF is a local diffeomorphism by the inverse function theorem
- dF is bijective:
  - surjectivity: For some $p_0 \in V^*$ define $F^{p_0} : V \to \mathbb{R}$ by $F^{p_0}(v) = F(v) - p_0(v)$. This function is superlinear, thus it reaches its minimum for some $v_0 \in V \Rightarrow dF^{p_0}(v_0) = 0 \Rightarrow dF(v_0) = p_0$.
  - injectivity: $HessF^{p_0}$ pos. def. $\Rightarrow$ F^{p_0} can at most have one critical point

„⇐“: For some $k > 0$, we define the compact set

$$S_k := \{v \in V \setminus \{0\} | \ |dF(v)| = k\}.$$

Since dF is a diffeomorphism, there exists a unique $v_0 \in V$, such that

$$dF(v_0) = \frac{k}{|v|}\langle v, \cdot \rangle,$$

where $\langle \cdot, \cdot \rangle$ is some inner product on V. We have $dF(v_0) \in S_k$ and $dF(v_0)v = k|v|$. Thus by convexity of F we have:

$$\begin{aligned} F(v) - F(v_0) &\geq dF(v_0)[v - v_0] \\ F(v) &\geq dF(v_0)v + F(v_0) - dF(v_0)v_0 \\ &\geq k|v| + \inf_{w \in S_k} \{F(w) - dF(w)w\} \end{aligned}$$

which shows, that F is superlinear.

Regularity of extremizers

Theorem

Let L be Tonelli. Then every extremizing curve $\gamma : [a, b] \to M$ is 'just as smooth as its Lagrangian L'. That means if L is C^r, $r \geq 2$, then γ will be C^r as well.

Example

The electromagnetic Lagrangian

$$L(x, v) = \frac{1}{2} g_x(v, v) + \theta_x(v) - U(x)$$

where g is the Riemannian metric, $U : M \to \mathbb{R}$ and θ is a 1-form.

- In physics the first term would correspond to the kinetic energy of particle
- the U-term corresponds to electromagnetic potential of the electric field $\vec{E}$
- the 1-form θ corresponds to the 'vector potential' of the magnetic field $\vec{B}$

It's solutions satisfy Newton's equation:

$${}^{x}\nabla_{\partial_t} \dot{x} = -\nabla U(x) - Y_x \cdot \dot{x}$$

where ${}^{x}\nabla_{\partial_t}$ is the Levi-Civita connection. ∇U is the gradient of U with respect to g and the vector field Y is the Lorentz force defined by:

$$g_x(Y_x \cdot u, v) = d\theta_x(u, v), \ \forall x \in M, \ u, v \in T_x M$$

The Hamiltonian

Definition

Let L be Tonelli $\text{Leg} : TM \to T^*M$ the Legendre transform. We define the *Hamiltonian* H by

$$H : T^*M \to \mathbb{R}, \; H(x,p) := \langle p, \text{Leg}^{-1}(x,p) \rangle_x - L(\text{Leg}^{-1}(x,p)) \tag{4}$$

where $\langle \cdot, \cdot \rangle_x$ is the canonical pairing between the tangent and cotangent bundles. We say that H is the Legendre dual of L.

Definition

We say that a Hamiltonian $H : T^*M \to \mathbb{R}$ is a Tonelli-Hamiltonian if:

(1) H is C^2

(2) $\frac{\partial^2 H}{\partial p^2}(\cdot, \cdot) > 0$

(3) H is superlinear in each fiber:

$$\forall x \in M : \lim_{||p||_x \to \infty} \frac{H(x,p)}{||p||_x} = +\infty$$

$$\Leftrightarrow \forall x \in M, A \in \mathbb{R}, \exists B \in \mathbb{R} : \; H(x,p) \geq A||p||_x - B.$$

The Hamiltonian

Properties

Lemma

Let L be Tonelli and H its Legendre dual. Let $x \in M$, $v \in T_xM$, $p \in T_x^*M$, s.t. $p = Leg(x, v)$. Then:

(i) $\frac{\partial H}{\partial p}(x, p) = v$

(ii) $\frac{\partial H}{\partial x}(x, p) = -\frac{\partial L}{\partial x}(x, v)$

(iii) *H is Tonelli*

(iv) *(Fenchel inequality):* $\forall p' \in T_x^*M, v' \in T_xM$:

$$\langle p', v' \rangle_x \leq L(x, v') + H(x, p')$$

with equality if and only if $p' = Leg(x, v')$

(v) $H(x, p) = \sup_{v' \in T_xM} [\langle p, v' \rangle_x - L(x, v')]$

The Hamiltonian vector field X_H

First we define the tautological 1-form or Liouville form λ of T^*M, which is given by

$$\lambda = \sum_{i=1}^{n} p_i dx^i$$

in local coordinates.(This definition is independent from the used coordinates.) The canonical symplectic structure is then defined by $\omega = -d\lambda$, given in local coordinates by

$$\omega = \sum_{i=1}^{n} dx^i \wedge dp_i.$$

This 2-form is closed and non-degenerate.

The Hamiltonian vector field X_H

Definition

The Hamiltonian vector field X_H is the vector field that satisfies the following equation:

$$\omega(X_H(x,p),\cdot) = d_{(x,p)}H$$

This means that the Hamiltonian vector field X_H is given in local coordinates by:

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}$$

where $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$ is a basis for $T_{(x,p)}T^*M$

The Hamiltonian equations and the Hamiltonian flow

- H is $C^2 \Rightarrow X_H$ is C^1 and can be locally integrated and its integral curves satisfy the *Hamiltonian equations*:

$$\frac{\partial H}{\partial p}(x, p_x) = \dot{x}$$
$$\frac{\partial H}{\partial x}(x, p_x) = -\dot{p}_x$$

- We can then define the *Hamiltonian flow* ϕ_t^H.
- It turns out that H is an integral of motion, meaning it is constant along its integral curves, because

$$\frac{d}{dt}H(\phi_t^H) = dH(\dot{\phi}_t^H) \ = dH(X_H(\phi_t^H)) \ = \omega(X_H(\phi_t^H), X_H(\phi_t^H)) \overset{\omega \text{ antisymm.}}{=} 0.$$

- The sets $\{(x,p) \in T^*M | \ H(x,p) = K\}$ for all $K \in \mathbb{R}$ are compact (by superlinearity of H) and invariant by $\phi_t^H \Rightarrow \phi_t^H$ is complete.

Lagrangians and Hamiltonians

- The projection onto M of the solutions (γ, p_γ) solve the Euler-Lagrange equation. And using (i) and (ii) from the Lemma above, we obtain:

$$d_{(x,v)}\text{Leg} \cdot X_L(x,v) = X_H \circ \text{Leg}(x,v), \;\; \forall (x,v) \in TM$$

In other words: The Lagrangian and Hamiltonian flows are conjugated by the Legendre transform:

$$\begin{aligned} \text{Leg} \circ \phi_t^L &= \phi_t^H \circ \text{Leg} \\ \phi_t^L &= \text{Leg}^{-1} \circ \phi_t^H \circ \text{Leg} \end{aligned}$$

- ϕ_t^H is well defined $\Rightarrow$ ϕ_t^L is well defined $\Rightarrow$ solutions to EL are unique
- ϕ_t^H is complete $\Rightarrow$ ϕ_t^L is complete.

Theorem

$\gamma : [a,b] \to M$ is a solution to the Euler-Lagrange equation if and only if $\tilde{\gamma} := Leg(\gamma, \dot{\gamma}) : [a,b] \to T^*M$ is a solution to the Hamiltonian equations

Minimizers of Tonelli-Lagrangians

- Lastly, if L is Tonelli $\Rightarrow$ there exist special extremizers (minimizers), that minimize the action in $C^2_{x_0,x_1}([a,b],M;\alpha)$
- Proof of existence in the next talk

Lagrangian & and Hamiltonian vector fields

- The Euler-Lagrange vector field:

$$\frac{\partial^2 L}{\partial v^2}(x,v)(\tilde{X}_L(x,v),\cdot\,) = \frac{\partial L}{\partial x}(x,v) - \frac{\partial^2 L}{\partial v \partial x}(x,v)(v,\cdot\,)$$

$$\Rightarrow X_L(x,v) = (x,v,v,\tilde{X}_L(x,v)) \in T_{(x,v)}TM$$

- derivation of the identity above:

$$\begin{aligned}
d_{(x,v)}\mathrm{Leg}\cdot X_L(x,v) &= (x,p,v,\frac{\partial\,\mathrm{Leg}}{\partial x}(x,v)(v) + \frac{\partial\,\mathrm{Leg}}{\partial v}(x,v)(\tilde{X}_L(x,v)))\\
&= (x,p,v,\frac{\partial^2 L}{\partial v \partial x}(x,v)(v,\cdot\,) + \frac{\partial^2 L}{\partial v^2}(x,v)(\tilde{X}_L(x,v),\cdot\,))\\
&= (x,p,v,\frac{\partial L}{\partial x}(x,v)) = (x,p,\frac{\partial H}{\partial p}(x,p),-\frac{\partial H}{\partial x}(x,p))\\
&= (x,p,\frac{\partial H}{\partial p}\circ \mathrm{Leg}(x,v),-\frac{\partial H}{\partial x}\circ \mathrm{Leg}(x,v)) = X_H \circ \mathrm{Leg}(x,v)
\end{aligned}$$

Proof of compactness of $\{(x,p) \in T^*M | \ H(x,p) = K\} \ \forall K \in \mathbb{R}$

By uniform superlinearity we have

$$K = H(x,p) \geq ||p||_x - B$$

for some $B \in \mathbb{R}$. Thus we have:

$$\underbrace{\{(x,p) \in T^*M | \ H(x,p) = K\}}_{\text{closed}} \subseteq \underbrace{\{(x,p) \in T^*M | \ ||p||_x \leq K + B\}}_{compact}.$$

Fenchel-inequality

Fix some $x \in M$ and let $v \in T_x M$ and $p \in T_x^* M$ be arbitrary. We have $p = \frac{\partial L}{\partial v}(x, w)$ for some $w \in T_x M$:

$$
\begin{aligned}
L(x,v) + H(x,p) - p_x(v) &= L(x,v) - H(\frac{\partial L}{\partial v}(x,w)) - \frac{\partial L}{\partial v}(x,w)(v) \\
&= L(x,v) - L(x,w) - \frac{\partial L}{\partial v}(x,w)[v-w] \\
&\geq 0,
\end{aligned}
$$

if L is convex. Since L is strictly convex, equality holds if and only if $v = w$.

Superlinearity is uniform over compact subsets of M, because for some $A \in (0, +\infty)$:

$$
\begin{aligned}
L(x,v) &\geq \max_{||p||_x \leq A} \{p_x(v) - H(x,p)\} \\
&\geq \max_{||p||_x \leq A} \{p_x(v)\} - \max_{||p||_x \leq A} \{H(x,p)\} \\
&\geq A||v||_x - \max\{H(x',p')| \ (x',p') \in T^*M, ||p'||_{x'} \leq k\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial\tilde{\theta}}{\partial v^i}(x,\dot{x}) - \frac{\partial\tilde{\theta}}{\partial x^i}(x,\dot{x}) &= \frac{d}{dt}(\theta_x^i) - \partial_{x^i}\theta_x \cdot \dot{x} \\
&= \sum_j \left[\partial_{x^j}\theta_x^i \cdot \dot{x}^j - \partial_{x^i}\theta_x^j \cdot \dot{x}^j\right] \\
&= \sum_j \left[\partial_{x^j}\theta_x^i - \partial_{x^i}\theta_x^j\right] \cdot \dot{x}^j \\
&= d\theta_x(\dot{x}, \cdot)
\end{aligned}
$$