Talk 14: Left-overs

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Before going on vacation one needs to finish the left-overs in the fridge:

A) $c(L)=\alpha(0)$
$$\begin{cases} c(L):=\inf\{k \mid A_{L+k}(\gamma)\geq 0 \ \forall \gamma:[a,b]\to M \ \gamma(a)=\gamma(b)\} \\ \alpha(0):=-\min\{A_L(\mu) \mid \mu \text{ inv. measure}\}\end{cases}$$

B) $\tilde{M}\subset\tilde{A}$
$$\begin{cases} \tilde{M}=\bigcup_{\mu \text{ min.}} \operatorname{supp}\mu \ ; \ \tilde{A}=\{v\in TM \mid \gamma_v \text{ is static}\} \\ \gamma: I\to\mathbb{R} \text{ static if } \forall s\leq t \ \ A^{\gamma}(s,t):=A_{L+c(L)}(\gamma|_{[s,t]})=-\Phi_{c(L)}(\gamma(t),\gamma(s))\end{cases}$$

Mañé potential

$I=\mathbb{R}$

semi

$\Phi_{c(L)}(\gamma(s),\gamma(t))$

C) Global Tonelli minimizers are semi-static
$$\begin{cases} \gamma: I\to M \text{ Tonelli min. if } \forall s\leq t \ \ A^{\gamma}(s,t)=h_{t-s}(\gamma(s),\gamma(t)) \\ h_T: M\times M\to\mathbb{R}, \ \ h_T(x,y)=\min\{A_L(\eta) \mid \eta:[0,T]\to M, \eta(0)=x, \eta(T)=y\}\end{cases}$$

A) $c(L)=\alpha(0)$

Lemma 1 For every $T_0>0$ $\exists C>0$ s.t. $\forall\gamma:[0,T]\to M$ Tonelli min, $T\geq T_0 \Rightarrow \|\dot\gamma\|_\infty\leq C$. ▨

Prop 2 $c(L)\leq\alpha(0)$

Proof Let $k<c(L)$ and pick $x\in M$. Then $\Phi_k(x,x)=-\infty$. Thus, $\exists$ Tonelli min. $\gamma_n:[0,T_n]\to M$ with $\gamma_n(0)=x=\gamma_n(T_n)$ s.t. $A_{L+k}(\gamma_n)\to-\infty$.
L bounded below $\Rightarrow T_n\to+\infty$. By lemma 1, $\dot\gamma_n$ lie in a compact subset of TM.
Thus: $\mu_{\gamma_n}\rightharpoonup\mu$. T_n diverges $\Rightarrow \mu$ invariant. Hence:
$$k-\alpha(0)\leq k+A_L(\mu)=A_{L+k}(\mu)\leq\underline{\lim}\, A_{L+k}(\mu_{\gamma_n})=\underline{\lim}\,\frac{1}{T_n}A_{L+k}(\gamma_n)\leq 0.$$ ▨

Prop 3 $c(L)\geq\alpha(0)$

Proof Take μ min. and ergodic. By Birkhoff's ergodic thm (BET), there is $v\in TM$ s.t.
$\forall k<\alpha(0)$: $\lim_{T\to\infty}\frac{1}{T}A_{L+k}(\gamma_v|_{[0,T]})=A_{L+k}(\mu)=-\alpha(0)+k<0.$

Proof Take μ min. and ergodic. By Birkhoff's ergodic Thm (BET), there is $v \in TM$ s.t.
$$\forall k < \alpha(0): \lim_{T\to\infty} \frac{1}{T} A_{L+k}(\gamma_v|_{[0,T]}) = A_{L+k}(\mu) = -\alpha(0) + k < 0.$$
Take η^T closed curve obtained concatenating $\gamma_v|_{[0,T]}$ with unit-speed geod. Since $T \to \infty$, we have $\lim_{T\to\infty} \frac{1}{T} A_{L+k}(\eta^T) = \lim_{T\to\infty} \frac{1}{T} A_{L+k}(\gamma_v|_{[0,T]})$. Hence, $A_{L+k}(\eta^T) < 0$ for T large. Thus, $k < c(L)$. ▨

Below we take $c(L) = 0 = \alpha(L)$. [observe $c(L+k) = c(L) - k$]

B) $\tilde{M} \subset \tilde{A}$

Lemma 4 (Mañé) (X,d) metric space, $f: X \to X$ Borel map, ν ergodic Borel probability for f. Then: For all integrable $\varphi: X \to \mathbb{R}$, $n_0 \in \mathbb{N}$, $\varepsilon > 0$ $\exists\, n \geq n_0$ s.t. for a.e. $x \in X$:
$$\text{(i)}\ d(x, f^n(x)) < \varepsilon, \quad \text{(ii)}\ \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int_X \varphi\, d\nu \right| < \frac{\varepsilon}{n}. \quad ▨$$

Rmk 5 (i) is just Poincaré recurrence. (ii) with ε instead of ε/n and for all $n \geq n_0$ instead of $\exists\, n \geq n_0$ would be just the BET. So (ii) says that for some n the arithmetic mean gets very close to the integral of φ.

Rmk 6 Applying Lemma 4 simultaneously for $n_0 \to +\infty$ and $\varepsilon = n_0^{-1}$ to φ with $\int_X \varphi\, d\nu = 0$, we get $n_k \to \infty$ s.t. for a.e. $x \in X$: $f^{n_k}(x) \to x$, $\sum_{j=0}^{n_k-1} \varphi(f^j(x)) \to 0$.

Prop 7 $\operatorname{supp} \mu \subset \tilde{A}$ for μ minimizing and ergodic.

Proof Use Lemma 4 with $X = TM$, $f = $ EL-flow, $\nu = \mu$, $\varphi = L$. Since $\int_{TM} L\, d\mu = 0$, Rmk 6 implies: For a.e. $(x,v) \in TM$ $\exists\, T_k \to +\infty$, $S_k \to -\infty$ s.t.
$\dot\gamma_v(S_k) \to v$, $\dot\gamma_v(T_k) \to v$, $A^{\gamma_v}(S_k, T_k) \to 0$. If $a \leq b$, then:
$$\begin{aligned} A^{\gamma_v}(a,b) &= -A^{\gamma_v}(S_k, a) + A^{\gamma_v}(S_k, T_k) - A^{\gamma_v}(b, T_k) \\ &\leq -\phi(\gamma_v(S_k), \gamma_v(a)) + A^{\gamma_v}(S_k, T_k) - \phi(\gamma_v(b), \gamma_v(T_k)) \\ &\longrightarrow -\phi(x, \gamma_v(a)) + 0 - \phi(\gamma_v(b), x) \leq -\phi(\gamma_v(b), \gamma_v(a)). \end{aligned}$$
Thus, $\exists$ full measure set Z inside $\tilde{A}$. By the def. of support and since $\tilde{A}$ is closed $\operatorname{supp} \mu \subset \overline{Z} \subset \tilde{A}$. ▨

Thus, $\exists$ full measure set Z inside A. By the def. of support and since A is closed $\operatorname{supp}\mu \subset \overline{Z} \subset \tilde{A}$. ☒

Rmk 8 Something more is true: Let μ be invariant. Then: μ is min. $\iff$ $\operatorname{supp}\mu \subset \tilde{N}$. If this happens, then $\operatorname{supp}\mu \subset \tilde{A}$.

C) Global Tonelli minimizers (GTM) are semi-static

Def 9 Let $\gamma: I \to M$ be a Tonelli minimizer. If $I = [0,\infty)$, γ is called a positive ray. If $I = (-\infty, 0]$, γ is called a negative ray.

Lemma 10 $\forall x \in M$ $\exists$ positive/negative ray starting/ending at x. There exist GTM.

Proof For all $T>0$ let $\gamma^T : [0,T] \to M$ be a Tonelli min. with $\gamma^T(0) = x$, $\gamma^T(T) = x$. By Lemma 1 $\exists$ γ^{T_n} with $T_n \to \infty$ converging unif. on compact intervals to $\eta : [0,+\infty) \to M$. Check that η is a positive ray starting at x. Analogously we find a negative ray arriving at x. Since $\dot\eta$ is contained in a compact set, $\exists$ $v \in TM$ and $S_n \to \infty$ s.t. $\dot\eta(S_n) \to v$. We claim that γ_v is a GTM. Indeed, $\forall a>0$ $\gamma_v|_{[-a,a]}$ is the uniform limit of $\gamma_{\dot\eta(S_n)}|_{[-a,a]}$ which is a Tonelli min. since it is just the time shift of the Tonelli min. $\eta|_{[S_n-a,\, S_n+a]}$. ☒

Rmk 11 From Leon's talk we know more: $\forall x \in M$ there are semi-static curves $\gamma^+ : [0,+\infty) \to M$, $\gamma^- : (-\infty, 0] \to M$ with $\gamma^+(0) = x = \gamma^-(0)$. Semi-static curves defined on $\mathbb{R}$ exist since $\tilde{N} \neq \emptyset$.

Def 12 For $x \in M$ and $s>0$ let $f_s^x, g_s^x : M \to \mathbb{R}$, $f_s^x(y) := h_s(x,y)$, $g_s^x(y) := -h_s(x,y)$.

Lemma 13 For all $t,s>0$: $T_t^- f_s^x = f_{t+s}^x$, $T_t^+ g_s^x = g_{t+s}^x$.

Proof $T_t^- f_s^x(y) = \inf\limits_{z\in M} \left[f_s^x(z) + h_t(z,y) \right] = \inf\limits_{z \in M} \left[h_s(x,z) + h_t(z,y) \right] = h_{s+t}(x,y) = f_{s+t}^x(y)$. ☒

Proof $T_t^- f_s^\wedge(y) = \inf_{z\in M}\left[f_s^\wedge(z) + h_t(z,y)\right] = \inf_{z\in M}\left[h_s(x,z)+h_t(z,y)\right] = h_{s+t}(x,y) = f_{s+t}^\wedge(y)$. ▨

Lemma 14 For all $f: M\to\mathbb{R}$, $T_t^\pm f$ converges unif. to a Lipschitz function for $t\to\infty$. ▨

Cor/Dfn 15 The Peierls barrier is the Lipschitz function $h: M\times M\to\mathbb{R}$ defined as

[minimal action with arbitrarily large period] $h(x,y) := \lim_{t\to\infty} h_t(x,y)$, where the limit is unif. in $M\times M$.

Thus: $\exists C>0$ s.t. $\forall \gamma: I\to M$ Tonelli min. $|A^\gamma(a,b)|\le C$ for all $a\le b$.

Lemma 16 For all $x,y,z,w\in M$, there holds:

i) $\phi(x,y)\le h(x,y)\le \phi(x,z)+h(z,w)+\phi(w,y)$

ii) $\gamma:\mathbb{R}\to M$ semistatic, $\gamma(0)=x$, $y\in\omega(\gamma) \Rightarrow h(x,y)=\phi(x,y)$

$\gamma:\mathbb{R}\to M$ static, $\gamma(0)=x$, $y\in\omega(\gamma)\Rightarrow h(x,y) = -\phi(y,x)$.

iii) $x\in\pi(\tilde{A}) \Leftrightarrow h(x,x)=0$

iv) $x\in\pi(\tilde{A})$ or $y\in\pi(\tilde{A}) \Rightarrow h(x,y)=\phi(x,y)$.

Proof i) exercise. iv) follows from i) and iii) taking $z=w=x$ or $z=w=y$.

ii) Let $t_n\to\infty$ s.t. $\gamma(t_n)\to y$. $\forall n$: $h_{t_n}(x,\gamma(t_n)) = \begin{cases}\phi(x,\gamma(t_n)) & \text{for } \gamma \text{ semistatic}\\ -\phi(\gamma(t_n),x) & \text{for } \gamma \text{ static}\end{cases}$

Now let $t_n\to\infty$ and use Cor 15. [Try direct proof]

iii) We only show "$\Rightarrow$". Let $\gamma:\mathbb{R}\to M$ static, $\gamma(0)=x$ and take $y\in\omega(\gamma)$.

By i) and ii), $0=\phi(x,x)\le h(x,x)\le h(x,y)+\phi(x,y) = 0$. ▨

Theorem 17 Rays are semi-static.

Proof Let $\gamma:[0,+\infty)\to M$ be a ray. For $0\le s\le t$ set $\delta(s,t) := A^\gamma(s,t) - \phi(\gamma(s),\gamma(t))$.

Since γ is a ray, $\delta(s,t) = h_{t-s}(\gamma(s),\gamma(t)) - \phi(\gamma(s),\gamma(t))$. By Cor 15, $\exists C>0$, $0\le\delta(s,t)\le C$ $\forall s,t$. We need to show $\delta(s,t)\equiv 0$.

Claim 1: $[s_1,t_1]\subset[s_2,t_2] \Rightarrow \delta(s_1,t_1)\le\delta(s_2,t_2)$. [exercise using triang. ineq. for ϕ]

Let $\delta(s,\infty) := \sup_{t\ge s}\delta(s,t) = \lim_{t\to\infty}\delta(s,t)$ and $\delta(\infty,\infty) := \inf_{s\ge 0}\delta(s,\infty) = \lim_{s\to\infty}\delta(s,\infty)$.

Claim 2: For $a\le b\le c$, $A^\gamma(a,b)+\phi(\gamma(b),\gamma(c)) = \phi(\gamma(a),\gamma(c)) + \delta(a,c) - \delta(b,c)$.

Claim 2: For $a \leq b \leq c$, $A^\gamma(a,b) + \phi(\gamma(b),\gamma(c)) = \phi(\gamma(a),\gamma(c)) + \delta(a,c) - \delta(b,c)$.

Proof: $A^\gamma(a,b) + \phi(\gamma(b),\gamma(c)) = A^\gamma(a,b) + A^\gamma(b,c) - \delta(b,c) = \underbrace{A^\gamma(a,c)}_{=\phi(\gamma(a),\gamma(c)) + \delta(a,c)} - \delta(b,c)$. $\boxtimes$

Claim 3: If $v \in \omega(\dot\gamma)$, then $\gamma_v : \mathbb{R} \to M$ is static.

Proof: Take $t_n \to \infty$ with $\dot\gamma(t_n) \to v$. Fix $s > 0$ and let n, m such that $t_n + s < t_m - s$.

Apply Claim 2 with $a = t_n - s$, $b = t_n + s$, $c = t_m - s$:

$$A^\gamma(t_n - s, t_n + s) + \phi(\gamma(t_n+s), \gamma(t_m - s)) = \phi(\gamma(t_n - s), \gamma(t_m - s)) + \delta(t_n - s, t_m - s) - \delta(t_n + s, t_m - s).$$

Take $m \to \infty$ and then $n \to \infty$ to get [Keep in mind $A^\gamma(t_n - s, t_n + s) = A^{\dot\gamma(t_n)}(-s,s)$]

$$A^{\gamma_v}(-s,s) + \phi(\gamma_v(s), \gamma_v(-s)) = \phi(\gamma_v(-s), \gamma_v(-s)) + \delta(\infty,\infty) - \delta(\infty,\infty) = 0. \boxtimes$$

It's over: By Claim 3, $\exists\, t_n \to \infty$ s.t. $\gamma(t_n) \to y \in \pi(\tilde{A})$. Then,

$$\delta(s,\infty) = \lim_{n\to\infty} h_{t_n - s}(\gamma(s), \gamma(t_n)) - \phi(\gamma(s),\gamma(t_n)) = h(\gamma(s), y) - \phi(\gamma(s), y) \overset{\text{Lemma 16.iv}}{=} 0.$$

Thus, for all $t \geq s$ $\quad 0 \leq \delta(s,t) \leq \delta(s,\infty) = 0.$

$\boxtimes$