A REMARK ON THE EXISTENCE OF $K(\pi, 1)$'s OVER THE RATIONALS

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ABSTRACT. Let p be an odd prime number and S a finite set of prime numbers $\equiv 1 \mod p$. In [Lab06], J. Labute showed that there exists set S_0 of prime numbers $\equiv 1 \mod p$ with $|S_0| = |S|$ such that the group $G_{S \cup S_0}(p)$ is a mild pro-p-group. In particular we have $cd(G_{S\cup S_0}(p)) = 2$ and the scheme $\mathrm{Spec}(\mathbb{Z}) \setminus (S \cup S_0)$ is a $K(\pi, 1)$ for p . In [Sch07] and in more generality in [Sch08], A. Schmidt extended this result to arbitrary number fields k. We will show that if $k = \mathbb{Q}$ it is always possible to enlarge a given set S of prime numbers $\equiv 1 \mod p$ by two prime numbers $\equiv 1 \mod p$ such that the resulting group is mild. Moreover in many cases it is sufficient to add one single prime. Finally we give analogous result in the case $p = 2$.

1 Construction of strongly free sequences

We make use of the following general criterion by A. Schmidt, cf. [Sch07], Th. 5.5.

(1.1) Proposition. *Let k be a field and $L(X)$ the free Lie algebra over $X = \{\xi_1, \dots, \xi_d\}$. Furthermore let $\rho_1, \dots, \rho_r \in L(X)$ be given such that*

$$\rho_i = \sum_{1 \le k < l \le d} a_{ikl}[\xi_k, \xi_l]$$

for all $i = 1, \dots, r$ with $a_{ikl} \in k$. Assume that there exists a natural number a, $1 \le a < d$ such that

(i) $a_{ikl} = 0$ for $a < k < l \le d$ and all $i = 1, \dots, r$,

(ii) the $r \times a(d-a)$-matrix

$$(a_{ikl})_{i,(k,l)},\ 1 \le i \le r,\ 1 \le k \le a < l \le d$$

has rank r.

Then the sequence $\rho_1, \dots, \rho_r$ is strongly free.

Applying this criterion we will prove the following lemma:

(1.2) Lemma. *Let k be a field, $n \geq 2$ and $L(X)$ the free Lie algebra over $X = \{\zeta_1, \dots, \zeta_{n+2}\}$. Let $\rho_1, \dots, \rho_{n+2} \in L(X)$ be given such that*

$$\begin{aligned} \rho_i &= \sum_{1\leq k<l\leq n} a_{ikl}[\zeta_k, \zeta_l] + b_i[\zeta_i, \zeta_{n+1}] + c_i[\zeta_i, \zeta_{n+2}],\ i = 1, \dots, n, \\ \rho_{n+1} &= d[\zeta_{n+1}, \zeta_2], \\ \rho_{n+2} &= e[\zeta_{n+2}, \zeta_1], \end{aligned}$$

where $a_{il}, b_i, c_i \in k$, such that

(i) $d, e, b_1, c_2 \neq 0$ and

(ii) $b_i \neq 0$ or $c_i \neq 0$ for $i = 3, \dots, n$.

Then the sequence $\rho_1, \dots, \rho_{n+2}$ is strongly free.

Proof. We apply (1.1) with $a = n$. Since the commutator $[\zeta_{n+1}, \zeta_{n+2}]$ doesn't occur in any of the ρ_i, $i = 1, \dots, n+2$ condition (i) of (1.1) holds. Furthermore the matrix of condition (ii) of (1.1) is of the form

$$\begin{pmatrix} & {\scriptstyle [\xi_1,\xi_{n+1}]} & {\scriptstyle [\xi_2,\xi_{n+1}]} & \cdots & {\scriptstyle [\xi_n,\xi_{n+1}]} & {\scriptstyle [\xi_1,\xi_{n+2}]} & {\scriptstyle [\xi_2,\xi_{n+2}]} & \cdots & {\scriptstyle [\xi_n,\xi_{n+2}]} \\ {\scriptstyle \rho_1} & b_1 & 0 & \dots & 0 & c_1 & 0 & \dots & 0 \\ {\scriptstyle \rho_2} & 0 & b_2 & \dots & 0 & 0 & c_2 & \dots & 0 \\ \vdots & & & \vdots & & & & & \vdots \\ {\scriptstyle \rho_n} & 0 & 0 & \dots & b_n & 0 & 0 & \dots & c_n \\ {\scriptstyle \rho_{n+1}} & 0 & -d & \dots & 0 & 0 & 0 & \dots & 0 \\ {\scriptstyle \rho_{n+2}} & 0 & 0 & \dots & 0 & -e & 0 & \dots & 0 \end{pmatrix}.$$

This matrix has rank $n + 2$ which follows immediately from the assumptions. This completes the proof. □

Roughly spoken, the above lemma states the following: If $n \geq 2$, $L(X)$ is the free Lie Algebra on $X = \{\zeta_1, \dots, \zeta_n\}$ and a sequence $\rho'_1, \dots, \rho'_n$ of homogeneous elements of degree 2 in $L(X)$ is given, then by adding two generators ζ_{n+1}, ζ_{n+2} one can obtain a strongly free sequence $\rho_1, \dots, \rho_{n+2}$ in $L\big(X \cup \{\zeta_{n+1}, \zeta_{n+2}\}\big)$ such that for $i = 1, \dots, n$ we have $\rho_i \equiv \rho'_i \mod \langle\zeta_{n+1}, \zeta_{n+2}\rangle$ (here $\langle\zeta_{n+1}, \zeta_{n+2}\rangle$ denotes the ideal of $L\big(X \cup \{\zeta_{n+1}, \zeta_{n+2}\}\big)$ generated by $\{\zeta_{n+1}, \zeta_{n+2}\}$). Under certain conditions one can obtain a strongly free sequence by adding one single extra generator. The precise statement is the following:

(1.3) Lemma. *Let k be a field, $n \geq 3$ and $L(X)$ the free Lie algebra over $X = \{\xi_1, \dots, \xi_{n+1}\}$. Let $\rho_1, \dots, \rho_{n+1} \in L(X)$ be given by*

$$\begin{aligned} \rho_i &= \sum_{1\leq l\leq n} a_{il}[\xi_i, \xi_l] + b_i[\xi_i, \xi_{n+1}],\ i = 1, \dots, n, \\ \rho_{n+1} &= \sum_{1\leq l\leq n} c_l[\xi_{n+1}, \xi_l], \end{aligned}$$

where $a_{il}, b_i, d \in k$, such that *(i)* $a_{1n}, a_{n2}, c_1 \neq 0,\ c_n = 0$ *and*

(ii) $b_i = 0$ *if and only if* $i \in \{1, n\}$.

Then the sequence $\rho_1, \ldots, \rho_{n+1}$ *is strongly free.*

Proof. We apply (1.1) with $a = n-1$. Since $b_n = c_n = 0$ condition (i) of (1.1) holds and the $(n+1) \times 2(n-1)$-matrix of conditon (ii) is of the form

$$\begin{pmatrix} & [\xi_1,\xi_n] & [\xi_2,\xi_n] & \cdots & [\xi_{n-1},\xi_n] & [\xi_1,\xi_{n+1}] & [\xi_2,\xi_{n+1}] & \cdots & [\xi_{n-1},\xi_{n+1}] \\ \rho_1 & \mathbf{a_{1n}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \rho_2 & 0 & a_{2n} & \cdots & 0 & 0 & \mathbf{b_2} & \cdots & 0 \\ \vdots & & & \ddots & & & & \ddots & \\ \rho_{n-1} & 0 & 0 & \cdots & a_{n-1,n} & 0 & 0 & \cdots & \mathbf{b_{n-1}} \\ \rho_n & -a_{n1} & -\mathbf{a_{n2}} & \cdots & -a_{n,n-1} & 0 & 0 & \cdots & 0 \\ \rho_{n+1} & 0 & 0 & \cdots & 0 & -\mathbf{c_1} & -c_2 & \cdots & -c_{n-1} \end{pmatrix}$$

where the bold coefficients are non-zero by assumption. This matrix clearly has rank $n+1$ and the claim follows. □

2 Arithmetic Results

We want to deduce some arithmetic consequences for p-extensions with tame ramification over $\mathbb{Q}$. We start by recalling some notation:

We fix an odd prime number p. Let $S = \{q_1, \ldots, q_n\}$ be a finite set of prime numbers $\equiv 1 \mod p$. Let $G_S(p)$ denote the Galois group of the maximal pro-p-extension of $\mathbb{Q}$ unramified outside S. Then $\dim_{\mathbb{F}_p} H^1(G_S(p), \mathbb{Z}/p\mathbb{Z}) = \dim_{\mathbb{F}_p} H^2(G_S(p), \mathbb{Z}/p\mathbb{Z}) = n$ and $G_S(p)$ admits a minimal presentation of *Koch type,* i.e. a presentation

$$1 \to R \to F \to G_S(p) \to 1$$

where F is the free pro-p-group on generators $x_1, \ldots, x_n$ and R is generated by $\rho_1, \ldots, \rho_n$ as a normal subgroup of F with

$$\rho_i \equiv x_i^{q_i - 1} \cdot \prod_{j=1,\ldots,n} [x_i, x_j]^{a_{ij}} \mod F_3,\ i = 1, \ldots, n,$$

where $a_{ij} \in \mathbb{F}_p$, $a_{ii} = 0$, for $i \neq j$ we have $a_{ij} = 0$ if and only if q_i is a p-th power modulo q_j (cf. [Koc02], ch. 11.3) and $(F_i)_{i \geq 1}$ denotes the descending p-central series of F. The numbers a_{ij} are called *linking numbers* of S with respect to p. We denote by $\Gamma_S(p)$ the associated *linking diagram* of S with respect to p (cf. [Lab06]).

We make the following notational convention for all upcoming statements in this section:

(2.1) Notation. *A prime* q is a prime number $q \equiv 1 \mod p$.

(2.2) Proposition. *Let $S = \{q_1, \dots, q_n\}$ be a finite set of primes. Then there exists a prime q_{n+1} with the additional edges of $\Gamma_{S\cup\{q_{n+1}\}}(p)$ arbitrarily prescribed. Precisely if $d_i, e_i \in \{0,1\}$, $i = 1,\dots,n$ are given, then there exists a prime q_{n+1} such that for the linking numbers a_{ij} of $S \cup \{q_{n+1}\}$ with respect to p we have $a_{i,n+1} = 0$ if and only if $d_i = 0$ and $a_{n+1,i} = 0$ if and only if $e_i = 0$.*

Proof. See [Lab06], Prop. 6.1. □

We can now proof the following

(2.3) Theorem.

(i) Let $n \geq 3$ and $S = \{q_1, \dots, q_n\}$ a finite set of primes. Suppose there exist pairwise distinct $i_1, i_2, i_3 \in \{1, \dots, n\}$ such that for the linking numbers a_{ij} of S with respect to p we have $a_{i_1 i_2}, a_{i_2 i_3} \neq 0$. Then there exists a prime q_{n+1} such that the group $G_{S\cup\{q_{n+1}\}}(p)$ is mild.

(ii) Let $n \geq 2$ and $S = \{q_1, \dots, q_n\}$ a finite set of primes. Then there exist two primes q_{n+1}, q_{n+2} such that the group $G_{S\cup\{q_{n+1},q_{n+2}\}}(p)$ is mild.

(For the definition of a mild *pro-p-group we refer to [Lab06].)*

Proof. (i) We may assume that $i_1 = 1, i_2 = n, i_3 = 2$. By (2.2) there is a prime q_{n+1} such that $a_{1,n+1} = a_{n,n+1} = a_{n+1,n} = 0$, $a_{i,n+1} \neq 0$ for $i = 2,\dots,n-1$ and $a_{n+1,1} \neq 0$. Now (1.3) (ii) applies and $G_{S\cup\{q_{n+1},q_{n+2}\}}(p)$ is mild.
(ii) If $n \geq 2$ and $S = \{q_1, \dots, q_n\}$ is an arbitrary set of primes, then by (2.2) we can find a prime q_{n+1} such that $a_{1,n+1}, a_{n+1,2}$ are both non-zero. Now the claim follows by applying (i) to the set $S \cup \{q_{n+1}\}$. Note that one can also directly show this claim by applying the more general lemma (1.2). □

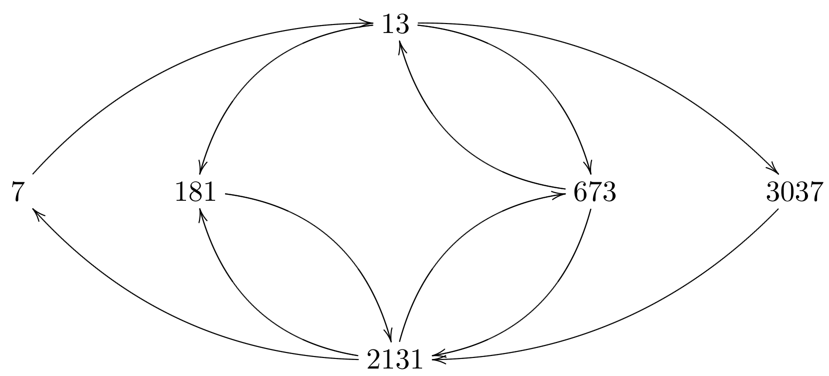
(2.4) Remark. Using arithmetic properties of $G_S(p)$, in [Sch06], Th. 2.1 A. Schmidt proofs the following result: Let S be a finite set of primes and assume there is a subset $T \subseteq S$ such that $\Gamma_T(p)$ is a non-singular circuit and for each $q \in S \setminus T$ there is a directed path in $\Gamma_S(p)$ starting in q and ending with a prime in T. Using (2.2) again, from this one can easily deduce that for a given set $S = \{q_1, \dots, q_n\}$ two primes q_{n+1}, q_{n+2} can be found such that we have $cdG_{S\cup\{q_{n+1},q_{n+2}\}}(p) = 2$. Note that in (2.3) we obtain the slightly stronger statement that $G_{S\cup\{q_{n+1},q_{n+2}\}}(p)$ is a mild pro-p-group.

We want to give an example for $p = 3$.

(2.5) Example. Let $p = 3$ and $S = \{q_1, \dots, q_4\}$ where

$$q_1 = 7,\ q_2 = 181,\ q_3 = 673,\ \text{and}\ q_4 = 3037.$$

The linking diagram of S with respect to p has no edges, i.e. we have $a_{ij} = 0$ for all $i, j \in \{1, \dots, 4\}$. In particular we do not know whether $cd(G_S(3)) = 2$. Now by (2.3) we can find two primes $q_5, q_6 \equiv 1 \mod 3$ such that $G_{S\cup\{q_5,q_6\}}(3)$ is mild. First we are looking for a prime q_5 such that the linking numbers satisfy $a_{15}, a_{52} \neq 0$. For example, such a prime is given by $q = 13$. Finally we'd like to find a prime q_6 such that $a_{16} = a_{56} = a_{65} = 0$ and $a_{26}, a_{36}, a_{46}, a_{61} \neq 0$. We may choose $q_6 = 2131$. More precisely the resulting linking diagram $\Gamma_{S\cup\{q_5,q_6\}}(3)$ is of the form



and (1.3) applies. Note that the primes $\{7, 13, 3037, 2131\}$ form a non-singular circuit and since there are edges starting in 181 and 673 respectively and ending in 2131, [Sch06], Th. 2.1 also applies (cf. (2.4)).

3 The case $p = 2$

Now we consider the case $p = 2$. By quadratic reciprocity clearly proposition (2.2) cannot hold anymore in this case. However we can prove the following analogue of theorem (2.3):

(3.1) Theorem.

(i) Let $n \geq 3$ and $S = \{q_0, q_1, \ldots, q_n\}$ a finite set of odd prime numbers. Suppose that there exist pairwise distinct $i_0, i_1, i_2, i_3 \in \{0, \ldots, n\}$ such that the following holds:

- $q_{i_0}, q_{i_1} \equiv 3 \mod 4$,
- $q_{i_3} \equiv 1 \mod 4$,
- q_{i_3} *is a square mod* q_{i_0} *but is not a square mod* q_{i_1} *and* q_{i_2}.

Then there exists a prime number $q_{n+1} \equiv 1 \mod 4$ such that the group $G_{S\cup\{q_{n+1}\}}(2)$ is a mild pro-2-group.

(ii) Let $n \geq 2$ and $S = \{q_0, q_1, \ldots, q_n\}$ a finite set of odd prime numbers containing at least two prime numbers $\equiv 3 \mod 4$. Then there exist two prime numbers $q_{n+1}, q_{n+2} \equiv 1 \mod 4$ such that $G_{S\cup\{q_{n+1},q_{n+2}\}}(2)$ is a mild pro-2-group.

(3.2) Corollary. *Let S be a finite set of odd prime numbers. We set $S_0 := \{q \in S |\ q \equiv 3 \mod 4\}$ and*

$$\delta \ := \ \begin{cases} 3, & \text{if } S = \varnothing \\ 2, & \text{if } S \neq \varnothing, S_0 = \varnothing, \\ 1, & \text{if } \#S_0 = 1, \\ 0, & \text{else.} \end{cases}$$

Then there exist $k := 2 + \delta$ odd prime numbers $q_1, \ldots, q_k$ such that the group $G_{S\cup\{q_1,\ldots,q_k\}}(2)$ is a mild pro-2-group. In particular if S is non-empty, there are always four odd primes $q_1, \ldots, q_4$, such that $G_{S\cup\{q_1,\ldots,q_4\}}(2)$ is mild.

In order to give a proof of (3.1) we need a slight generalization of (1.1) involving quadratic terms. The desired result is the following:

(3.3) Proposition. *Let G be a pro-2-group that admits a presentation*

$$1 \to R \to F \to G \to 1,$$

where F is the free pro-p-group on generators $x_1, \ldots, x_d$ and as R is generated by $\rho_1, \ldots, \rho_r$ as a normal subgroup of F with

$$\rho_i \equiv \prod_{j=1}^{d} \left(x_j^2\right)^{a_{ij}} \cdot \prod_{1\le k<l\le d} [x_k, x_l]^{a_{ikl}} \quad \bmod F_3,\ i = 1, \ldots, r,$$

where $a_{ij}, a_{ikl} \in \mathbb{F}_2$ and $(F_i)_{i\ge1}$ denotes the lower 2-central-series of F. Moreover, suppose that there is a natural number a satisfying $1 \le a < d$ such that the following conditions hold:

(i) $a_{ij} = 0$ for $a < j \le d$ and all $1 \le i \le r$,

(ii) $a_{ikl} = 0$ for $a < k < l \le d$ and all $i = 1, \ldots, r$,

(iii) the $r \times a(d - a)$-matrix

$$(a_{ikl})_{i,(k,l)},\ 1 \le i \le r,\ 1 \le k \le a < l \le d$$

has rank r.

Then G is a mild pro-2-group with generator rank d and relation rank r. In particular we have cd $G \le 2$.

Proof. This is a direct reformulation of [LM09], Th. 1.1. In order to see this, first note that $\dim_{\mathbb{F}_p} H^1(G) = d$ since $R \subseteq F_2$. Now let $\chi_1, \ldots, \chi_d \in H^1(G)$ be the corresponding dual basis of the images of $x_1, \ldots, x_d$ in G^{ab}/p. Then we have $a_{ij} = \overline{\rho}_i(\chi_j \cup \chi_j)$ and $a_{ikl} = \overline{\rho}_i(\chi_k \cup \chi_l)$. Here $\overline{\rho}_i$ denotes the image of ρ_i under the map

$$R \longrightarrow R/R^2[R, F] \xrightarrow{\psi} H^2(G)^*$$

where ψ is the inverse map of the dual of the transgression isomorphism tg: $H^1(R/R^2[R, F]) \xrightarrow{\sim} H^2(G)$ (cf. [NSW08], Th. 3.9.13). Note that by condition (iii) we have $\dim_{\mathbb{F}_p} H^2(G) = \dim_{\mathbb{F}_p}(R/R^2[R, F]) = r$, so $\{\rho_i,\ i = 1, \ldots, r\}$ is a minimal system of defining relations of G. Let U, V be the subspaces of $H^1(G)$ generated by $\chi_1, \ldots, \chi_a$ and $\chi_{a+1}, \ldots, \chi_d$ respectively. Then by conditions (i) and (ii), the cup product is trivial on $V \times V$ and maps $U \times V$ surjectively onto $H^2(G)$ by condition (iii). Thus G is mild by [LM09], Th. 1.1. (A proof using a different approach from that in [LM09] can also be found in [For10], Cor. 6.5.) □

Proof of (3.1). For the proof of (i) we may assume that $i_0 = 0,\ i_1 = 1,\ i_2 = 2$ and $i_3 = n$. Now we can choose a prime number q_{n+1} such that the following holds:

- $q_{n+1} \equiv 1 \mod 4$,
- q_{n+1} is not a square mod q_0,
- q_{n+1} is a square mod q_1,
- q_{n+1} is not a square mod $q_i,\ i = 2, \dots n$,
- q_{n+1} is a square mod q_n.

Let $G := G_{S\cup\{q_{n+1}\}}(2)$. Since S contains a prime number $\equiv 3 \mod 4$, the group Б_S and hence also the group $\text{Б}_{S\cup\{q_{n+1}\}}$ vanishes and G has the presentation $G = \langle x_0, \dots, x_{n+1} |\ \rho_0, \dots, \rho_{n+1}, \rho\rangle$, where

$$\begin{aligned} \rho_i &\equiv \left(x_i^2\right)^{a_i} \prod_{j=0}^{n+1} [x_i, x_j]^{a'_{ij}} \mod F'',\ i = 0, \dots, n+1, \\ \rho &\equiv \prod_{i=0}^{n+1} x_i^{a_i} \mod F', \end{aligned}$$

with $a_i, a'_{ij} \in \mathbb{F}_2$ such that $a_i = 0$ if and only if $q_i \equiv 1 \mod 4$ and $a'_{ij} = 0$ if and only if q_i is a square mod q_j. Thus since $q_0 \equiv 3 \mod 4$, we may omit the generator x_0. Furthermore, we may omit the relation ρ_0 and obtain a minimal presentation $G = F/R = \langle x_1, \dots, x_{n+1} |\ \rho'_1, \dots, \rho'_{n+1}\rangle$ where

$$\rho_i \equiv \left(x_i^2\right)^{a_i} \prod_{j=1}^{n+2} [x_i, x_j]^{a_{ij}} \mod F'',\ i = 1, \dots, n+2,$$

with

$$a_{ij} = a'_{ij} + a'_{i0} a_j,\ i, j = 1, \dots, n+2$$

(cf. [Koc02], Ex. 11.12). By the assumptions made and by construction of q_{n+1} we have $a_{1,n} = a_{n,1} = a_{2,n} = a_{n,2} = 1$, $a_{1,n+1} = 0$ and $a_{i,n+1} = 1$ for all $i = 2, \dots, n-1$. Furthermore $a_{n+1,1} = 1$ and for $i = 2, \dots, n-1$ we have $a_{n+1,i} = 0$ if and only if $q_i \equiv 3 \mod 4$. We will now apply (3.3) with $a = n-1$. Since $q_n \equiv q_{n+1} \equiv 1 \mod 4$, we have $a_n = a_{n+1} = 0$ and hence condition (i) holds. Furthermore we have $a_{n,n+1} = a_{n+1,n} = 0$, hence also condition (ii) is fulfilled. Finally the $(n+1) \times 2(n-1)$-matrix of condition (iii) is of the form

$$\left(\begin{array}{ccccccccc} & [x_1,x_n] & [x_2,x_n] & \dots & [x_{n-1},x_n] & [x_1,x_{n+1}] & [x_2,x_{n+1}] & \dots & [x_{n-1},x_{n+1}] \\ \rho_1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \rho_2 & 0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & & & & \ddots & \\ \rho_{n-1} & 0 & 0 & \dots & a_{n-1,n} & 0 & 0 & \dots & 1 \\ \rho_n & 1 & 1 & \dots & a_{n,n-1} & 0 & 0 & \dots & 0 \\ \rho_{n+1} & 0 & 0 & \dots & 0 & 1 & a_{n+1,2} & \dots & a_{n+1,n-1} \end{array}\right)$$

and clearly has rank $n+1$. Thus we obtain (i).
For the proof of (ii) let $n \geq 2$ and $S = \{q_0, q_1, \dots, q_n\}$ a finite set of odd prime numbers where we may assume $q_0 \equiv q_1 \equiv 3 \mod 4$. Then we can find a prime $q_{n+1} \equiv 1 \mod 4$ such that q_{n+1} is a square mod q_0 but is not a square mod q_1 and q_2. Now the claim follows by applying (i) to the set $S \cup \{q_{n+1}\}$. $\square$

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