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Unitary representations of the Poincaré group

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Chapter I. Representations

1. The groups that we will treat in detail

Let K be a field. The basic groups are

$$\begin{aligned} \mathrm{GL}(n,K) &= \{A \in K^{n\times n};\quad \det A \neq 0\} \quad \text{(general linear group)},\\ \mathrm{SL}(n,K) &= \{A \in K^{n\times n};\quad \det A \neq 0\} \quad \text{(special linear group)}, \end{aligned}$$

in particular

$$\mathrm{SL}(2,\mathbb{R}), \quad \mathrm{SL}(2,\mathbb{C}).$$

We denote by E_p the $p\times p$ unit-matric and by

$$E_{pq} = \begin{pmatrix} -E_q & 0 \\ 0 & E_p \end{pmatrix}.$$

The orthogonal groups are

$$\mathrm{O}(p,q) = \bigl\{A \in \mathrm{GL}(n,\mathbb{R});\quad A'E_{pq}A = E_{pq}\bigr\}, \quad p+q=n$$

and the unitary groups are

$$\mathrm{U}(p,q) = \bigl\{A \in \mathrm{GL}(n,\mathbb{C});\quad \bar{A}'E_{pq}A = E_{pq}\bigr\}, \quad p+q=n$$

Their subgroups of determinant 1 are denoted by $\mathrm{SO}(p,q)$ and $\mathrm{SU}(p,q)$ will be studied in detail. In the case $q=0$ we omit q in the notation, $\mathrm{O}(p) = \mathrm{O}(p,0),\dots$. The main examples we will treat are

$$\mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R}), \quad \mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C}).$$

There will occur some exceptional isomorphisms. Let S^1 be the group of complex numbers of absolute value one. Obviously

$$S^1 = U(1).$$

There is also the isomorphism

$$\mathrm{SO}(2) \xrightarrow{\sim} S^1, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a - \mathrm{i}b.$$

Its inverse is given by

$$\zeta = e^{\mathrm{i}t} \longmapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

The *Lorentz group* is $\mathrm{O}(3,1)$. It contains two subgroups of index two. One is $\mathrm{SO}(3,1)$, the other (the so-called *orthochronous Lorentz group*) can be defined through

$$\mathrm{O}^+(3,1) = \{A \in \mathrm{O}(3,1);\ a_{11} > 0\}.$$

(We will see that this is actually a group). Both subgroups are open and closed. The intersection

$$\mathrm{SO}^+(3,1) = \mathrm{O}^+(3,1) \cap \mathrm{SO}(3,1)$$

is called the *proper orthochronous Lorentz group*. It is a subgroup of index 4 of the Lorentz group. This subgroup is closely related to the group $\mathrm{SL}(2,\mathbb{C})$. We will construct a surjective homomorphism

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{SO}^+(3,1)$$

such that each element of $\mathrm{SO}^+(3,1)$ has two pre-images which differ only by the sign. One says that $\mathrm{SL}(2,\mathbb{C})$ is a twofold covering of $\mathrm{SO}^+(3,1)$ and one calls this the *spin covering* and uses the notation $\mathrm{Spin}(3,1) = \mathrm{SL}(2,\mathbb{C})$.

The group $\mathrm{O}(3)$ can be embedded into the Lorentz group $\mathrm{O}(3,1)$ by means of

$$A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

It is contained in $\mathrm{O}^+(3,1)$, hence $\mathrm{SO}(3)$ occurs as subgroup of $\mathrm{SO}^+(3,1)$. It turns out that the subgroup $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$ maps onto $\mathrm{SO}(3)$. Hence

$$\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$$

is also a surjective homomorphism such that each element of the image has exactly two pre-images which differ by a sign. This should be considered again as a spin covering, so the notation $\mathrm{Spin}(3) = \mathrm{SU}(2)$ looks natural.

The group $\mathrm{O}(3,1)$ is also called the *homogeneous Lorentz group.* The *inhomogeneous Lorentz group* is the set of all transformations of $\mathbb{R}^4$ of the form

$$v \longmapsto A(v) + b$$

where A is a Lorentz transformation and $b \in \mathbb{R}^4$. This group can be identified with the set $\mathrm{O}(3,1) \times \mathbb{R}^4$. The group law then is

$$(g,a)(h,b) = (gh, a + gb).$$

We write for the inhomogeneous Lorentz group simply

$$\mathrm{O}(3,1)\mathbb{R}^4.$$

There is an embedding

$$\mathrm{O}(3,1)\mathbb{R}^4 \longrightarrow \mathrm{GL}(5,\mathbb{R}), \quad (g,a) \longmapsto \begin{pmatrix} g & a \\ 0 & 1 \end{pmatrix}.$$

This defines an isomorphism of the extended Lorentz group onto a closed subgroup of $\mathrm{GL}(5,\mathbb{R})$.

A variant of the inhomogeneous Lorentz group is the *Poincaré group* $P(3)$. As set it is

$$P(3) = \mathrm{SL}(2,\mathbb{C}) \times \mathbb{R}^4$$

and the group law is

$$(g,a)(h,b) = (gh, a + gb).$$

Here one has to use the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{R}^4$ that comes from the projection $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(3,1)$.

There is a natural homomorphism $P(3) \to \mathrm{SO}^+(3,1)\mathbb{R}^4$. It is a twofold covering in the obvious sense.

The group $P(3)$ can be considered as closed subgroup of $\mathrm{SL}(7,\mathbb{C})$, the embedding given by

$$(G,a) \longmapsto \begin{pmatrix} G & 0 & 0 \\ 0 & g & a \\ 0 & 0 & 1 \end{pmatrix},$$

where $g \in \mathrm{SO}(3,1)$ is the image of $G \in \mathrm{SL}(2,\mathbb{C})$.

Table of the important groups

$$\begin{array}{c}
S^1 \cong \mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R}) \\
\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C}) \\
\\
\mathrm{SO}(3) \subset \mathrm{SO}^+(3,1) \\
\uparrow \qquad\qquad \uparrow \\
\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C}) \\
\\
P(3) = \mathrm{SL}(2,\mathbb{C})\mathbb{R}^4
\end{array}$$

The four little groups

There are 4 subgroups of $\mathrm{SL}(2,\mathbb{C})$ which are called *little groups*, namely

$$\boxed{\mathrm{SL}(2,\mathbb{C}), \quad \mathrm{SL}(2,\mathbb{R}), \quad \mathrm{SU}(2), \quad \mathrm{Iso}(2).}$$

Here

$$\mathrm{Iso}(2) := \left\{ \begin{pmatrix} \zeta & z \\ 0 & \zeta^{-1} \end{pmatrix}; \quad \zeta \in S^1,\ z \in \mathbb{C} \right\}$$

is the so-called *isobaric spin group.*

The classification of the irreducible unitary representations of the Poincaré group needs the following steps.

Classify all irreducible unitary representations of the little groups. The easiest case is $\mathrm{SU}(2)$, since this is a compact group which implies that the irreducible unitary representations are finite dimensional. The case $\mathrm{SL}(2,\mathbb{R})$ is already involved. The classification in this case is due to Bargmann, V. (1947). The case $\mathrm{SL}(2,\mathbb{C})$ has been settled be Gelfand, I. and Naimark, M. (1950). The group $\mathrm{Iso}(2)$ is a special case that follows easily from the Mackey theory.

The irreducible unitary representations of the Poincaré group are derived from the representations of the little groups through an induction procedure. Irreducible unitary representations of the little groups are lifted in a certain way to irreducible unitary representations $P(3)$. These are called induced representations. This is due to Wigner, E.P. (1938) and has been extended by Mackey, G. (1978).

In these notes we will touch on all of these topics and some them will be treated in very detail.

2. The Lie algebras that will occur

A (real or complex algebra) $\mathfrak{g}$ is a real or complex vector space $\mathfrak{g}$ together with a ($\mathbb{R}$- or $\mathbb{C}$-) bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Every complex algebra has an underlying real algebra. There is an obvious notion of homomorphism of real or complex algebras and there is an obvious notion of cartesian product of two (real or complex) algebras $\mathfrak{g}_1 \times \mathfrak{g}_2$. The product has to be taken componentwise.

$$[(A_1, B_1), (A_2, B_2)] = ([A_1, A_2], [B_1, B_2]).$$

An algebra $\mathfrak{g}$ is called a real or complex Lie algebra if there exists an injective $\mathbb{R}$- or $\mathbb{C}$-linear map $\mathfrak{g} \to \mathbb{C}^{n\times n}$ such that the multiplication corresponds to the Lie bracket

$$[A, B] = AB - BA.$$

Notice that any complex Lie algebra can be considered as a real Lie algebra.

We recall the exponential function for matrices $A \in \mathbb{C}^{(n,n)}$:

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

It is clear that this series converges absolutely. The rule

$$B^{-1}\exp(A)B = \exp(B^{-1}AB)$$

is trivial. We need also the rule

$$\det \exp(A) = \exp(\operatorname{tr}(A))$$

which can be reduced to diagonal matrices (using the previous rule and the fact that the set of all matrices with n pairwise different eigenvalues is dense in the set of all matrices).

The rule

$$\exp(A + B) = \exp(A) + \exp(B)$$

holds if A, B commute. There are generalizations to the case where A, B do not commute. To get them one needs the matrix logarithm

$$-\log(E - A) = \sum_{n=1}^{\infty} \frac{A^n}{n}$$

that converges for small A. Hence $\log A$ is defined in a small neighbourhood of the unit matrix. One has

$$e^{\log A} = A,\ A \text{ close to } E, \quad \log e^A = A,\ A \text{ close to } 0.$$

For fixed A, B we now consider.

$$\log\bigl(\exp(tA)\exp(tB)\bigr).$$

This is defined when t is sufficiently small It can be expanded into a power series. One can compute

$$\log\bigl(\exp(tA)\exp(tB)\bigr) = tA + tB + \frac{1}{2}t^2[A, B] + \cdots$$

or

$$\exp(tA)\exp(tB) = \exp\Bigl(tA + tB + \frac{1}{2}t^2[A, B] + \cdots\Bigr).$$

This formula is a link between the multiplication in the group and the Lie bracket.

In these lectures we deal with matrix groups. Here we understand by a matrix group a topological group G together with an embedding $G \hookrightarrow \mathrm{GL}(n,\mathbb{C})$. This means an topological isomorphism onto a closed subgroup of some $\mathrm{GL}(n,\mathbb{C})$. So matrix groups are locally compact groups.

Let $G \hookrightarrow \mathrm{GL}(n,\mathbb{C})$ be a matrix group. We consider the set $\mathfrak{g}$ of all matrices A such that $\exp(tA) \in G$ for all $t \in \mathbb{R}$. So we get a map

$$\mathfrak{g} \longrightarrow G, \quad A \longmapsto \exp(A).$$

It can be shown that $\mathfrak{g}$ is a real Lie algebra. (This does not exclude that in some cases it is a complex Lie algebra.) This means that $\mathfrak{g}$ is a real vector space and that $A, B \in \mathfrak{g}$ implies that $[A,B] \in \mathfrak{g}$. There is no need to give a proof, since in all cases that we treat this will be clear.

We give an example. Consider the group $\mathrm{SU}(2)$. Its Lie algebra consists of all 2×2-matrices A such that e^{tA} is unitary for all real t. This means $e^{At}e^{\bar{A}'t} = E$. We differentiate this formula by t and evaluate at $t = 0$. By means of the product formula above we get $A + \bar{A}' = 0$. Conversely this formula implies that e^{tA} is unitary for real t. Hence the Lie algebra of $\mathrm{SU}(2)$ consists of all A with the property $A + \bar{A}' = 0$. This algebra is denoted by $\mathfrak{su}(2)$. Similar notations are used for other groups.

There is another basic fact which we will use in some particular cases where it will be clear. Let $G \subset \mathrm{GL}(n,\mathbb{C})$, $H \subset \mathrm{GL}(m,\mathbb{C})$ be two closed subgroups and let $G \to H$ be a continuous homomorphism. Then there exists a unique real Lie homomorphism $\mathfrak{g} \to \mathfrak{h}$ of the corresponding Lie algebras such that the diagram

$$\begin{array}{ccc} G & \longrightarrow & H \\ \uparrow & & \uparrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{h} \end{array}$$

commutes. If $G \to H$ is a topological isomorphism then $\mathfrak{g} \to \mathfrak{h}$ is an isomorphism too. In particular, the Lie algebra of a matrix group G is the same for all embeddings $G \to \mathrm{GL}(n,\mathbb{C})$. But there is even a better result. If $G \to H$ is locally topological at the origin then $\mathfrak{g} \to \mathfrak{h}$ is an isomorphism of Lie algebras. Assume for example that G is an open subgroup of H, then the canonical inclusion induces an isomorphism of Lie algebras. Please notice that an open subgroup of a topological group is always a closed subgroup. Hence the groups $\mathrm{O}(3,1)$, $\mathrm{SO}(3,1)$, $\mathrm{O}(3,1)$, $\mathrm{SO}^+(3,1)$ have the same Lie algebras.

A similar result states. Let $G \to H$ be a surjective continuous homomorphism with discrete kernel. Then $\mathfrak{g} \to \mathfrak{h}$ is an isomorphism.

Some Lie algebras

We associate to each group G in the list

$\mathrm{GL}(n,\mathbb{C}),\ \mathrm{SL}(n,\mathbb{C}),\ \mathrm{GL}(n,\mathbb{R}),\ \mathrm{SL}(n,\mathbb{R}),\ \mathrm{O}(p,q),\ \mathrm{SO}(p,q),\ \mathrm{U}(p,q),\ \mathrm{SU}(p,q)$

their Lie algebra $\mathfrak{g}$

$$\begin{aligned}
\mathfrak{gl}(n,\mathbb{C}) &= \mathbb{C}^{n\times n},\\
\mathfrak{sl}(n,\mathbb{C}) &= \{A\in\mathfrak{gl}(n,\mathbb{C}),\quad \operatorname{tr}(A)=0\},\\
\mathfrak{gl}(n,\mathbb{R}) &= \mathbb{R}^{n\times n},\\
\mathfrak{sl}(n,\mathbb{R}) &= \{A\in\mathfrak{gl}(n,\mathbb{R}),\quad \operatorname{tr}(A)=0\},\\
\mathfrak{o}(p,q) &= \{A\in\mathfrak{gl}(n,\mathbb{R}),\quad A'E_{p,q}+E_{p,q}A=0\},\\
\mathfrak{so}(p,q) &= \mathfrak{o}(p,q)\cap\mathfrak{sl}(p+q,\mathbb{R})\},\\
\mathfrak{u}(p,q) &= \{A\in\mathfrak{gl}(n,\mathbb{C}),\quad \bar{A}'E_{p,q}+E_{p,q}A=0\},\\
\mathfrak{su}(p,q) &= \mathfrak{u}(p,q)\cap\mathfrak{sl}(p+q,\mathbb{C}).
\end{aligned}$$

We mentioned that we will construct a surjective homomorphism $\mathrm{SL}(2,\mathbb{C})\to\mathrm{SO}^+(3,1)$ whose kernel consists of two elements $\pm E$. Then the induced homomorphism of Lie algebras is an isomorphism,

$$\begin{array}{ccc}
\mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \mathrm{SO}^+(3,1)\\
\uparrow & & \uparrow\\
\mathfrak{sl}(2,\mathbb{C}) & \xrightarrow{\sim} & \mathfrak{so}(3,1)
\end{array}.$$

We will construct this isomorphism explicitly later. The groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SO}(3,1)$ have the same Lie algebra. But these groups are not isomorphic.

Next we investigate the Lie algebra of the extended Lorentz group. In our setting it arises as set of all real 5×5 matrices of the form

$$\begin{pmatrix} A & a\\ 0 & 0\end{pmatrix},\quad A\in\mathfrak{so}(3,1),\ a\in\mathbb{R}^4.$$

This can identified with the vector space

$$\mathfrak{so}(3,1)\times\mathbb{R}^4$$

equipped with the Lie bracket

$$([(A,a),(B,b)]=([A,B],Ab-Ba).$$

In this formula a,b are understood as column vector. Similarly the Lie algebra of the Poincaré group group can identified with the space of all matrices

$$\begin{pmatrix} A & 0 & 0\\ 0 & B & a\\ 0 & 0 & 1\end{pmatrix},\quad A\in\mathfrak{sl}(2,\mathbb{C}),\ B\text{ its image in }\mathfrak{so}(3,1),\ b\in\mathbb{R}^4.$$

This can be also identified with the pairs (A,a). Hence the Lie algebras of the extended Lorentz group and the Poincaré group are isomorphic. We take for the Lie algebra of the Poincaré group the model

$$\mathfrak{p}=\mathfrak{sl}(2,\mathbb{C})\mathbb{R}^4$$

equipped with the Lie bracket

$$([(A,a),(B,b)] = ([A,B], Ab - Ba).$$

In all our cases we described concretely a map

$$\exp : \mathfrak{g} \longrightarrow G, \quad \exp(A) = e^A.$$

We formulate a general fact which is rather clear in our cases.

2.1 Lemma. *In all cases above, the map* $\exp : \mathfrak{g} \to G$ *is locally topological at* 0*, i.e. it maps a suitable small open neighborhood of* $0 \in \mathfrak{g}$ *onto a small open neighborhood of the unit element in* G*.*

Proof. The proof is very easy. One constructs an inverse of the exponential map by means of the matrix logarithm

$$-\log(E - A) = \sum_{n=1}^{\infty} \frac{A^n}{n}$$

which, as we know already, converges in a small neighborhood of $A = 0$.

□

(For readers who know the notion of a Lie group we mention that Lemma 2.1 can be used to equip G with a structure as Lie group such that $\exp : \mathfrak{g} \to G$ is a local diffeomorphism at e. The dimension of G and the (real) dimension of $\mathfrak{g}$ is the same. This makes it easy to compute the dimension of a matrix group.)

3. Generalities about Banach- and Hilbert spaces

Usually, we consider only vector spaces over the field of real or complex numbers. If E, F are two vector spaces, we denote by $\mathrm{Hom}(E,F)$ the space of all (real- or complex-) linear maps. In the case $E = F$ we write $\mathrm{Hom}(E,E) = \mathrm{End}(E)$. If we want to point out to the ground field, we write

$$\mathrm{Hom}_{\mathbb{C}}(E,F), \quad \mathrm{Hom}_{\mathbb{R}}(E,F), \dots$$

The group of all invertible operators in $\mathrm{End}(E)$ is denoted by $\mathrm{GL}(E)$ (or $\mathrm{GL}_{\mathbb{C}}(E)$, $\mathrm{GL}_{\mathbb{R}}(E)$).

A *seminorm* on a vector space E is a real valued function $\|\cdot\|$ on E with the properties $\|a\| \geq 0$ and $\|Ca\| = |C|\|a\|$, $\|a+b\| \leq \|a\| + \|b\|$ $(a, b \in E)$, $C \in \mathbb{C}$ (or $\mathbb{R}$)). It is called a *norm* if $\|a\| = 0$ implies $a = 0$. For a seminorm $\|\cdot\|$ on E the set of all vectors of norm 0 is a subspace $N \subset E$. The seminorm factors to a norm on E/N. If $\|\cdot\|$ is a norm, then hen $\|a-b\|$ is a metric on E. The normed space E is called *complete*, or a *Banach space*, if every Cauchy sequence converges. Every normed space E can be embedded into a Banach space $\bar{E}$ as a dense subspace (with the restricted norm) in an essentially unique manner. One calls $\bar{E}$ the completion of E. Let $F \subset E$ be a linear subspace of a Banach space. It is a closed subspace if and only if it is a Banach space (with respect to the restricted norm). The closure of a linear subspace in a Banach space is a linear subspace and hence a Banach space. It can be identified with its completion. Since any two norms on a finite dimensional vector space are equivalent, every finite dimensional normed vector space is a Banach space. As a consequence, every finite dimensional subspace of a normed vector space is closed.

A linear map $A : E \to F$ between normed vector spaces is called *bounded* if there exists a constant $C \geq 0$ such that $\|Aa\| \leq C\|a\|$ for all $a \in E$. Then there exists a smallest number C with this property. It is called the norm of A and is denoted by $\|A\|$. We mention that A is bounded if and only if it is continuous (at the origin is enough). Any bounded operator $A : E \to F$ extends to a bounded operator of the completions $\bar{E} \to \bar{F}$ which we usually denote by the same letter. For finite dimensional E, F each linear map is bounded. Let E be a normed space and F be a Banach space. The subspace of all bounded operators

$$B(E, F) \subset \mathrm{Hom}(E, F)$$

of $\mathrm{Hom}(E, F)$ is a Banach space (equipped with the operator norm). We use the abbreviation

$$B(E) = B(E, E).$$

If F is the ground field ($\mathbb{R}$ or $\mathbb{C}$) then $E' = B(E, F)$ is the so called dual space.

An important theorem on Banach spaces is the *open mapping theorem.* It states that any linear bounded and surjective operator $f : E \to F$ of Banach spaces is open, i.e. the image of an open subset is open. In particular, a bijective linear bounded operator $f : E \to F$ has the property that its inverse is automatically bounded, hence an invertible element in $B(E)$. We denote the group of invertible elements by $B^*(E)$. A consequence of this is the *closed graph theorem.* It states that a linear map $f : E \to F$ between Banach spaces is bounded if an only of the graph $\{(x, f(x));\ x \in E\}$ is a closed subset of $E \times F$ (equipped with the product topology).

All what we have said so far about Banach spaces can be formulated and is true for real and complex Banach spaces. Now we consider complex vector spaces.

A Hermitian form on a complex vector space E is a function $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{C}$ which is linear in the first variable and which has the property $\langle a, b\rangle = \overline{\langle b, a\rangle}$. It is called semipositive if $\langle a, a\rangle \geq 0$ and positive definite if $\langle a, a\rangle = 0$ implies $a = 0$. For a semipositive Hermitian form $\|a\| := \sqrt{\langle a, a\rangle}$ is a seminorm. It is a norm if the Hermitian form is positive definite. Then we call $(E, \langle\cdot,\cdot\rangle)$ a Hermitian space and a Hilbert space if it is a Banach space with this norm. The completion of a Hermitian space carries not only a structure as Banach space but also as Hilbert space.

We will make use of the theorem of Riesz:

Let $L : H \to \mathbb{C}$ be a continuous linear functional on a Hilbert space H. Then there exists a unique vector $a \in H$ such that $L(x) = \langle x, a\rangle$ (and each linear functional of this kind is continuous and has the norm $\|L\| = \|a\|$).

These special linear forms show that for every vector $a \in H$, $a \neq 0$, there exists a continuous linear functional L with the property $L(a) \neq 0$.

This statement is also true for Banach spaces. From the theorem of *Hahn-Banach* follows the following result:

For each non-zero vector $a \in E$ of a Banach space there exists a continuous linear functional L with the property $L(a) \neq 0$. One can obtain $\|a\| = \|L\|$.

We will make use of another important result about Hilbert spaces. Let $A \subset H$ be a closed linear subspace (hence a Hilbert space). Denote by

$$B = \{b \in H;\ \langle a, b\rangle = 0 \text{ for all } a \in A\}$$

the orthogonal complement of A. This is a closed linear subspace (hence a Hilbet space) and one has $H = A \oplus B$.

A family $(a_i)_{i\in I}$ is called an orthonormal system if any two members with different indices are orthogonal and if the norm of each member is one. A *Hilbert space basis* of a Hilbert space is by definition a maximal orthonormal system. It is easy to show (using Zorn's lemma and the above remark about orthogonal complements) that Hilbert space bases do exist. Even more, every orthonormal system is contained in a maximal one.

A Hilbert space H is called *separable* if it contains a countable dense subset. One can show that this is the case if and only if each (one is enough) Hilbert space basis is finite or countable.

We recall some basics about infinite series. A series $a_1 + a_2 + \cdots$ in a Banach space E is called convergent if there exist a such that

$$\|a - \sum_{\nu=1}^{n} a_\nu\| \longrightarrow 0 \quad \text{for } n \longrightarrow \infty.$$

A sufficient condition is that $\sum \|a_\nu\|$ converges. If this is the case, the series is called absolutely convergent. But this condition is not necessary for convergence.

In the special case that $E = H$ is a Hilbert space and that the a_i are pairwise orthogonal one can show the following. The series converges if and only if $\sum \|a_\nu\|^2$ converges. So in this special case convergent and absolutely convergent are the same.

We give an example of a separable Hilbert space. The space ℓ^2 consists of all sequences $(a_1, a_2, \dots)$ of complex numbers such that $\sum |a_n|^2$ converges. It can be shown that for two $a, b \in \ell^2$ the series

$$\langle a, b\rangle = \sum a_n \bar{b}_n$$

converges absolutely and equips ℓ^2 with the structure as a Hilbert space. The usual unit vectors (1 at one place and 0 at the others) give a Hilbert space basis.

Let now H be any infinite dimensional separable Hilbert space with a Hilbert space basis $e_1, e_2, \dots$. For each $a \in \ell^2$ the series

$$\sum_{n=1}^{\infty} a_n e_n := \lim_{N\to\infty} \sum_{n=1}^{N} a_n e_n$$

then converges in H. This gives a map

$$\ell^2 \xrightarrow{\sim} H.$$

This map is actually an isomorphism of Hilbert spaces (which means that it is an isomorphism of vector spaces which preserves the Hermitian forms). Hence all infinite dimensional separable Hilbert spaces are isomorphic as Hilbert spaces. (The same kind of argument shows a standard result of linear algebra, namely that two finite dimensional Hilbert spaces are isomorphic as Hilbert spaces if and only if their dimensions agree.)

Assume that H_1, $H_2, \dots$ is a sequence of pairwise orthogonal closed subspaces of the Hilbert space H. Assume that their algebraic sum is dense in H. If we choose a Hilbert space basis in each H_i and collect them, we get a Hilbert space basis of H. This shows that every $a \in H$ has a unique representation as convergent series $a = a_1 + a_2 + \cdots$ where $a_i \in H_i$. Recall that this means that $\sum_i \|a_i\|^2$ converges. We write this as

$$H = \widehat{\bigoplus_i} H_i$$

and call this a *direct Hilbert sum*.

There is an abstract version of this. Let H_n be a family of Hilbert spaces. We define H to be the set of all sequences (h_n), $h_n \in H_n$ such that $\sum \|h_n\|^2$ converges. There is a natural imbedding of H_n into H. The image $\tilde{H}_n$ consists of all elements of H such only the nth component can be different from 0. The space H carries a natural structure as Hilbert space and it is the direct Hilbert of the $\tilde{H}_n$. Usually one identifies $\tilde{H}_n$ with H_n and calls H the direct Hilbert sum of the H_n.

4. Generalities about measure theory

All topological spaces that carry measures are assumed to be Hausdorff, locally compact and to have a countable basis of the topology. The latter means that there exists a countable system of open subsets such that each open subset can be written as a union of sets from this system. Every metric space with an countable dense subset (for example $\mathbb{C}^n$) has this property. Every subspace (equipped with the induced topology) keeps this property.

We denote by $\mathcal{C}(X)$ the set of complex valued continuous functions on a locally compact space X and by $\mathcal{C}_c(X)$ the subset of all continuous functions with compact support. A *Radon measure* is a $\mathbb{C}$-linear functional $I : \mathcal{C}_c(X) \to \mathbb{C}$ which is real in the sense $I(\bar{f}) = \overline{I(f)}$ and positive in the sense that $I(f) \geq 0$ for real $f \geq 0$. Usually one writes

$$I(f) = \int_X f(x)dx.$$

We assume that the reader is familiar with some way to extend a Radon measure to the class of integrable functions. We just indicate the steps, how this can be done.

One introduces $\mathbb{R} \cup \{\infty\}$ as ordered set ($x \leq \infty$ for all x). Every non-empty set $M \subset \mathbb{R} \cup \{\infty\}$ has a smallest upper bound $\operatorname{Sup}(M)$ in $\mathbb{R} \cup \{\infty\}$. One extends the addition to $\mathbb{R} \cup \{\infty\}$ by $x + \infty = \infty + x = x$ for all x and similarly the multiplication with a positive $C > 0$ by $C\infty = \infty$.

A function $f : X \to \mathbb{R} \cup \{\infty\}$ is called a *Baire function* if there exists an increasing sequence $f_n \in \mathcal{C}_c(X)$, $f_1 \leq f_2 \leq \dots$ such that $f(x) = \operatorname{Sup}\{f_n(x);\ x \in X\}$. One can show that

$$I_B(f) := \operatorname{Sup}\{I(f_n)\}$$

is independent of the choice of the sequence. We call this the Baire integral of f. Every $f \in \mathcal{C}_c(X)$ is a Baire function and in this case $I_B(f)$ agrees with $I(f)$. We mention that the function "constant ∞" is a Baire function. Hence we can define for an arbitrary nowhere negative function $f : X \to \mathbb{R} \cup \{\infty\}$

$$\bar{I}(f) = \operatorname{Inf}\{I_B(h); \quad f \leq h \text{ Baire function}\}.$$

The general rule $\bar{I}(f + g) \leq \bar{I}(f) + \bar{I}(g)$ holds.

Now one can define integrable functions:

A function $f : X \to \mathbb{R}$ is called integrable if there exists s sequence $f_n \in \mathcal{C}_c(X)$ such that $\bar{I}(|f - f_n|)$ is finite and tends to zero.

One can show that then $(I(f_n))$ converges and that the limit

$$I_L(f) = \lim_{n\to\infty} I(f_n)$$

is independent of the choice of f_n. This is called the (Daniell-Lebesgue) integral of f. One can show even more that Baire functions f with finite $I_B(f)$ (for example elements of $\mathcal{C}_c(X)$) are integrable and that $I_L(f) = I_B(f)$ in this case. Hence we can simply write $I(f) = I_B(f)$ for Baire functions and $I(f) = I_L(f)$ for integrable functions. $I(f) = \bar{I}(f)$ for integrable f. It is easy to see that the space $\mathcal{L}^1(X, dx)$ of all integrable functions is a vector space. It has the property that with f also $|f|$ is integrable. The integral is a linear functional on $\mathcal{L}^1(X, dx)$ with the property $I(f) \geq 0$ for $f \geq 0$.

A function $f : X \to \mathbb{C}$ is called a *zero function* if $\bar{I}(|f|) = 0$. This means that for each $\varepsilon > 0$ there exists a Baire function h with $|f| \leq h$ and $I(h) < \varepsilon$. It is easy to see that zero functions are integrable. A subset of X is called a *zero subset* if its characteristic function is a zero function. A function f is a zero function if an only if $\{x;\ f(x) \neq 0\}$ is a zero set. If f is integrable and g is a function that coincides with f outside a zero set then g is integrable too and $I(f) = I(g)$.

We recall the basic limit theorems:

4.1 Theorem of Beppo Levi. *Assume that $f_1 \leq f_2 \ldots$ is an increasing sequence of integrable functions such that the sequence of their integrals is bounded in $\mathbb{R}$. Then the pointwise limit $f(x) = \lim f_n(x)$ exists outside a zero set. If one defines $f(x)$ arbitrarily for this zero set, one gets an integrable function with the property*

$$\int_X f(x)dx = \lim_{n\to\infty} \int_X f_n(x)dx.$$

4.2 Lebesgue's limit theorem. *Let $f_n(x)$ be a pointwise convergent sequence of integrable functions. Assume that there exists an integrable function h with the property $|f_n(x)| \leq h(x)$ for all n and x. Then $f(x) = \lim f_n(x)$ is integrable and one has*

$$\int_X f(x)dx = \lim_{n\to\infty} \int_X f_n(x)dx.$$

The subset $\mathcal{N} \subset \mathcal{L}^1(X, dx)$ of zero functions is a sub-vector space and the integral factors through the quotient

$$L^1(X, dx) := \mathcal{L}^1(X, dx)/\mathcal{N}.$$

(This defines also a seminorm on $\mathcal{L}^1(X, dx)$ and $\mathcal{N}$ is the nullspace of this seminorm.) From the limit theorems one can deduce that $L^1(X, dx)$ gets a Banach space with the norm

$$\|f\|_1 := \int_X |f(x)|dx.$$

Let (f_n) be a sequence in $\mathcal{L}^1(X, dx)$ and $f \in \mathcal{L}(X, x)$. Assume that $f_n \to f$ in the Banach space $L^1(X, dx)$. (Usually we will denote the class of an element $f \in L^1(X, dx)$ in $\mathcal{L}(X, dx)$ by the same letter f. A more careful notation would be to use a notation like $[f]$ for the class. For sake of simplicity we avoid this as long it is clear whether we talk about f or of its class.) Then one can show that there exists a zero set S and a subsequence of (f_n) that converges pointwise to f. (This is the essential step in the proof that $L^1(X, dx)$ is a Banach space).

Let us assume that the Radon measure is non-trivial in the following sense: Let $f \in \mathcal{C}_c(X)$ be a non-negative function with the property $I(f) = 0$. Then $f = 0$. For such a measure the natural map

$$\mathcal{C}_c(X) \longrightarrow L^1(X, dx)$$

is injective and $L^1(X, dx)$ is the completion of $\mathcal{C}(X)$ with respect to the norm $\|\cdot\|_1$. Hence integration theory can be understood as a concrete realization of the completion.

There is another important notion:

A function $f : X \to \mathbb{C}$ is called ***measurable*** *if for any non-negative function $h \in \mathcal{C}_c(X)$ the function*

$$f_h(x) := \begin{cases} f(x) & \text{if } -h(x) \le f(x) \le h(x), \\ 0 & \text{else} \end{cases}$$

is integrable.

Integrable functions are measurable. All continuous functions are measurable. Measurability is conserved under all kind of standard constructions of functions which are used in analysis as addition and multiplication of functions but also taking pointwise limits and constructions as sup, inf, lim sup, lim inf for sequences of functions. A subset of X is called measurable if its characteristic function is measurable. Open subsets of X are measurable. Complements of measurable sets are measurable. Countable unions and intersections of measurable sets are measurable. Hence all sets which can be constructed from open and closed subsets be taking countable unions and intersections and complements are measurable with respect to each Radon measure. They are called Borel sets. The notion of Borel set makes sense in any topological space A. There is also the notion of a Borel map $f : A \to B$ between topological spaces. This means that the inverse images of Borel sets are Borel sets.

So the statement *"all functions are measurable"* is not really true but nearly true. (Counter examples need sophisticated application of the axiom of choice.)

4.3 Theorem. *A function f is integrable if and only if it is measurable and if $\bar{I}(|f|) < \infty$.*

Together with the previous remark this means that integrability means a kind of boundedness.

Let $p \geq 1$. The spaces $\mathcal{L}^p(X, dx)$ consist of all measurable functions f such that $|f|^p$ is integrable. This is the case for zero functions. One defines

$$\|f\|_p := \sqrt[p]{\int_X |f(x)|^p dx}.$$

This is a seminorm which seems that it satisfies the axioms of a norm besides the definiteness. It induces a norm on the space

$$L^p(X, dx) = \mathcal{L}^p(X, dx)/\mathcal{N}$$

which is a Banach space with this norm. The case $p = 2$ is of special importance. One can consider on $\mathcal{L}^2(X, dx)$ the Hermitian form

$$\langle f, g \rangle := \int_X f(x)\overline{g(x)}dx.$$

This induces a positive definite form on $L^2(X, dx)$ and equips this space with a structure as separable Hilbert space.

As a special example one can take the space $X = \mathbb{N}$ equipped with the discrete topology and the Radon measure $I(a) = \sum_n a_n$. The associated L^2-space is ℓ^2.

A function $f : X \to \mathbb{C}$ is called *essentially bounded* with respect to a Radon measure dx if there exists a zero set S and a constant C such that f is bounded by C outside S. We denote the infimum of all such C by $\|f\|_\infty$. The space of zero functions is contained in $\mathcal{L}^\infty(X, dx)$. The seminorm $\|f\|_\infty$ factors through

$$L^\infty(X, dx) = \mathcal{L}^\infty(X, dx)/\mathcal{N}$$

and equips this space with a structure as Banach space.

There is an extension of measure theory, the Bochner integral. For a Banach space E we can consider the space of compactly supported continuous functions $\mathcal{C}_c(X, E)$ with values in E.

4.4 Lemma. *Let (X, dx) be a Radon measure and E a Banach space. There exists a unique linear map*

$$\mathcal{C}_c(C, E) \longrightarrow E, \quad f \longmapsto \int_X f(x)dx,$$

such that for each continuous linear functional $L : E \to \mathbb{C}$ one has

$$L\Big(\int_X f(x)dx\Big) = \int_X L(f(x))dx.$$

The uniqueness follows directly from the Hahn-Banach theorem. So the existence, but not so quite obvious. Since for our purposes it would be sufficient to treat the case of Hilbert spaces we mention that the existence in this case is a direct consequence of the theorem of Riesz. In the case the defining formula reads

$$\left\langle \int_X f(x)dx, a \right\rangle = \int_X \langle f(x), a\rangle dx.$$

There is also the notion of a measurable function. We only need it in the case where E is *separable* which means that it contains a countable dense subset. Then a function $f : X \to E$ is measurable if and only if its composition with all continuous linear forms is measurable. A function $f : X \to E$ is called integrable if it is measurable and if $\|f\|$ is integrable. A measurable function $f : X \to E$ is called a zero function if $\|f\|$ is a zero function. This means that f is zero outside a zero set. Now the spaces $\mathcal{L}^p(X, E, dx)$ can be defined in the same way as in the case $E = \mathbb{C}$. They contain the space $\mathcal{N}$ of zero functions and the quotients $L^p(X, E, dx)$ are Banach spaces. If $E = H$ is a Hilbert space, the space $L^2(X, H, dx)$ gets a Hilbert space with an obvious inner product.

Finally we mention the notion of the *product measure*. Let (X, dx), (Y, dy) be two locally compact spaces with Radon measures. We consider $X \times Y$ equipped with the product measure. This is also locally compact space. Let $f \in \mathcal{C}_c(X \times Y)$. If we fix y we get a function $f(x, y)$ which is contained in $\mathcal{C}_c(X)$. It is easy to see that the integral $\int f(x, y)dy$ is contained in $\mathcal{C}_c(Y)$. Hence we can define the product measure

$$\int_{X\times Y} f(x, y)dxdy := \int_Y \left[\int_X f(x, y)dx\right] dy.$$

We claim that one can interchange the orders of integration, i.e.

$$\int_Y \left[\int_X f(x, y)dx\right] dy = \int_X \left[\int_Y f(x, y)dy\right] dx.$$

This is trivial for splitting functions $f(x, y) = \alpha(x)\beta(y)$ and follows in general by means of the Weierstrass approximation theorem. The formula

$$\int_{X\times Y} f(x, y)dxdy = \int_Y \left[\int_X f(x, y)dx\right] dy = \int_X \left[\int_Y f(x, y)dy\right] dx$$

extends to a broader class of functions and is then called Fubini's theorem. One has to assume that $f \in \mathcal{L}^1(X \times Y, dxdy)$. But one has to be somewhat cautious with the interpretation of the formula. One only can say that the function $y \mapsto f(x, y)$ is integrable outside a set of measure zero. Inside this exceptional set one can take for $\int_X f(x, y)dy$ an arbitrary value, for example 0.

There is a variant, the theorem of Tonelli. Assume that f is measurable and that the iterated outer integral $\bar{I}_Y \bar{I}_X |f(x, y)|$ is finite. Then f is integrable (and the Fubini formula holds).

5. Generalities about Haar measures

A topological group G is a group which carries also a topology such that the maps
$$G \times G \longrightarrow G,\ (g,h) \longmapsto gh, \quad G \longrightarrow G,\ g \longmapsto g^{-1},$$
are continuous. Here $G \times G$ has been equipped with the product topology. A *locally compact group* is a topological space whose underlying space is locally compact. We always assume that G has a countable basis of the topology.

Examples of locally groups are $\mathrm{GL}(n,\mathbb{C})$. One just takes the induced topology of $\mathbb{C}^{n\times n}$. Closed subgroups of a locally group are locally compact groups as well. Hence $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(p,q)$, $U(p,q)$ are locally compact groups. Also the additive groups $\mathbb{R}^n$, $\mathbb{C}^n$ and the inhomogeneous Lorentz group and the Poincarè group $P(3) = \mathrm{SL}(2,\mathbb{C})\cdot\mathbb{R}^4$ are locally compact groups. (Take the product topology.)

A *Haar measure* on a locally compact group G is a non-zero *left invariant* Radon measure
$$\int_G f(x)dx = \int_G f(gx)dx \qquad (g \in G).$$
We make use of the fact that a non zero Haar measure always exists and is uniquely determined up to a constant factor.

Of course we can work also with right invariant measures.

5.1 Lemma. *If $\int_G f(x)dx$ is a left invariant measure then*
$$\int_G f(x^{-1})dx$$
is a right invariant measure and conversely.

The usual integral on $\mathbb{R}$ is a Haar measure on the additive group $\mathbb{R}$ and a Haar measure on the multiplicative group $\mathbb{R}^*$ is given by
$$\int_{\mathbb{R}^*} f(t)\frac{dt}{t}$$
where dt is the usual measure.

If $f \in \mathcal{C}_c(G)$ is a function with the properties $f \geq 0$ and $I(f) = 0$. Then $f = 0$. Hence we have $\mathcal{C}_c(G) \hookrightarrow L^p(G,dx)$.

Let $g \in G$. Then
$$f \longmapsto \int_G f(xg)dx$$

is also left invariant. Hence there exists a positive real number $\Delta(g) = \Delta_G(g)$ with the property

$$\int_G f(xg^{-1})dx = \Delta(g)\int_G f(x)dx.$$

The function $\Delta : G \to \mathbb{R}_{>0}$ is of course independent of the choice of dx. It is called the *modular function* of G. It is clearly a continuous homomorphism, $\Delta(gh) = \Delta(g)\Delta(h)$.

Here we defined the modular function by means of left invariant measures. But this serves also for right invariant measures. If $\int_G f(x)dx$ is right invariant then

$$\int_G f(gx)dx = \Delta(g)\int_G f(x)dx.$$

We collect.

5.2 Remark. *The modular function Δ of a locally compact group has the following properties.*

1) $\displaystyle\int_G f(xg^{-1})dx = \Delta(g)\int_G f(x)dx$ *for left invariant measure dx,*

2) $\displaystyle\int_G f(gx)dx = \Delta(g)\int_G f(x)dx$ *for right invariant measure dx.*

We add an another interesting formula.

5.3 Lemma. *For every function $f \in \mathcal{L}^1(G, dx)$ the formulas*

1) $\displaystyle\int_G f(x^{-1})\Delta(x)^{-1}dx = \int_G f(x)dx$ *for left invariant measure dx,*

2) $\displaystyle\int_G f(x^{-1})\Delta(x)dx = \int_G f(x)dx$ *for right invariant measure dx,*

hold.

Proof. One can check that the integral on the left hand side is a Haar measure. Hence it agrees with the right hand side up to constant a factor $C > 0$. Applying the formula twice we get $C^2 = 1$ and hence $C = 1$. □

The group G is called *unimodular* if $\Delta(g) = 1$ for all g. There are 4 obvious classes of unimodular groups:

1) Abelian groups are unimodular.
2) A group G is unimodular if its commutator subgroup is dense.
3) Compact groups are unimodular, more generally, for arbitrary G the restriction of Δ_G to any compact subgroup is trivial.
4) Discrete groups are unimodular.

The last statement is true since the only compact subgroup of the multiplicative group of positive reals is $\{1\}$.

We give an example of a group which is not unimodular. Let $P \subset \mathrm{SL}(2,\mathbb{R})$ be the group of all upper triangular matrices of determinant 1 and with positive diagonal entries. Each p can be written in the form

$$p = an, \quad a = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad (\alpha \neq 0).$$

Moreover the map
$$\mathbb{R}_{>0} \times \mathbb{R} \xrightarrow{\sim} P, \quad (\alpha, n) \longmapsto p,$$
is topological. The measures

$$da = \frac{d\alpha}{\alpha}, \quad dn = dx$$

are Haar measures on A and N.

5.4 Lemma. *Let $P \subset \mathrm{SL}(2,\mathbb{R})$ be the group of upper triangular matrices with positive diagonal entries. Let da be a Haar measure on $\mathbb{R}_{>0}$ and dn a Haar measure on $\mathbb{R}$. Then the measure*

$$\int_P f(p)dp := \int_{\mathbb{R}_{>0}} \int_{\mathbb{R}} f(an)da\,dn$$

is a Haar measure. The modular function is

$$\Delta(p) = \alpha^{-2}.$$

(One can also write $\int\int f(an)dadn$ for the right hand side, since orders of integration can be interchanged, but $\int\int f(na)dadn$ would be false.)

Proof. The proof can be given by a simple calculation which rests on the formula
$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha^2 x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}. \qquad \square$$

We also need quotient measures. Let $H \subset G$ be a closed subgroup of a locally compact group G. Then H is also locally compact. We consider the coset space $H\backslash G$ that consists of all right cosets Hg. This is the quotient space of G by the natural action of H (multiplication from the right.) We equip it with the quotient topology with respect to the natural projection $G \to H\backslash G$. Then this projection is continuous and open. We claim that $H\backslash G$ is Hausdorff. Hausdorff means that the diagonal in $H\backslash G \times H\backslash G$ is closed. This means that its inverse image in $G \times G$ is closed. But this inverse image of H with respect to the map $G \times G \to G$, $(x, y) \mapsto xy^{-1}$.

Since $G \to H\backslash G$ is open, the space $H\backslash G$ is locally compact. There is also a natural continuous map

$$(H\backslash G) \times G \longrightarrow H\backslash G, \quad (Hg_1, g_2) \longmapsto Hg_1g_2$$

which as action from the right. A Radon measure dx on $X = H\backslash G$ is called G-invariant if

$$\int_{H\backslash G} f(xg)dx = \int_{H\backslash G} f(x)dx.$$

Similarly we can define G/H and the notion of an invariant measure on G/H (which now means invariance under translation from the left).

5.5 Proposition. *Let $H \subset G$ be a closed subgroup. Assume that $\Delta_G|H = \Delta_H$. Then there exists a non-zero invariant Radon measure dy on G/H and this Radon measure is unique up to a positive constant factor. It has the following property. Let dh be a left invariant measure on H. Then*

$$\int_G f(x)dx = \int_{G/H} \left[\int_H f(yh)dh\right] dy$$

is a left invariant measure on G.

We should mention that the function $x \mapsto \int_H f(xh)dh$ can be considered as a function on G/H. It is continuous and with compact support there.

The following lemma can be proved by means of the technique of partition of unity. We omit the details.

5.6 Lemma. *The map*

$$\mathcal{C}_c(G) \longrightarrow \mathcal{C}_c(G/H), \quad f \longmapsto f', \quad f'(y) = \int_H f(yh)dh,$$

is surjective.

Proof of Proposition 5.5. A function $f' \in \mathcal{C}_c(G/H)$ can be written in the form

$$f'(y) = \int_H f(yh)dh.$$

We want to define its integral through

$$\int_{G/H} f'(y)dy = \int_G f(x)dx$$

where dx is a left invariant measure on G. There is a problem. The function f' does not determine f uniquely. Hence one has to prove a lemma.

5.7 Lemma. *Let $f \in \mathcal{C}_c(G)$. Then*

$$\int_H f(xh)dh = 0 \Longrightarrow \int_G f(x)dx = 0.$$

It is a good exercise to do this for a finite group G. The integrals then just are finite sums. In the general case the condition $\Delta_G|H = \Delta_H$ will play a role.

Proof of Lemma 5.7. From the assumption follows

$$\int_G g(x) \int_H f(xh)dhdx = 0 \quad \text{for } g \in \mathcal{C}_c(G).$$

We interchange the integrations.

$$\int_H \int_G g(x)f(xh)dxdh = 0.$$

In the inner integral we replace $x \mapsto xh^{-1}$. Since dx is left invariant, a factor $\Delta_G(h)$ arises (Remark 5.2).

$$\int_H \int_G f(xh)g(x)dxdh = \int_H \Delta_G(h)^{-1} \int_G f(x)g(xh^{-1})f(x)dxdh$$

By means of the assumption about the modular functions we have $\Delta_G(h) = \Delta_H(h)$. Finally we transform h by h^{-1} applying Lemma 5.3. There occurs a factor $\Delta_H(h)$. The two Delta-factors cancel. □

In this way we get the existence of a measure on G/H such the claimed formula holds. The invariance of this measure is trivial. The proof of the uniqueness of the quotient measure is the same as the proof of the uniqueness of the Haar measure. This finishes the proof of Proposition 5.5. □

We also mention that the formula in Proposition 5.5 holds for all $f \in \mathcal{L}^1(G, dx)$ with the usual caution: the inner integral exists outside of a set of measure zero and gives – extended arbitrarily – an integrable function on $H\backslash G$.

Instead of $G\backslash H$ one can also consider the space of right cosets $H\backslash G$ and G acts by multiplication from the right. Proposition 5.5 remains true if one replaces “right” by “left”.

5.8 Proposition. *Let $H \subset G$ be a closed subgroup. Assume that $\Delta_G|H = \Delta_H$. Then there exists a non-zero invariant Radon measure dy on $H\backslash G$ and this Radon measure is unique up to a positive constant factor. It has the following property. Let dh be a left invariant measure on H. Then*

$$\int_G f(x)dx = \int_{H\backslash G} \left[\int_H f(h^{-1}y)dh \right] dy$$

is a right invariant measure on G.

6. Generalities about representations

A representation π of a group G on a complex vector space is a homomorphism $\pi : G \to \mathrm{GL}(V)$ of G into the group of $\mathbb{C}$-linear automorphisms of V. Frequently we will write $g(a)$ or even simply ga instead of $\pi(g)(a)$. The map

$$G \times V \longrightarrow V, \quad (g, a) \longmapsto ga,$$

then has the properties:

1) $ea = a$ for all $a \in V$ (e denotes the unit element of G).
2) $(gh)a = g(ha)$ for all $g, h \in G$, $a \in V$.
3) $g(a + b) = g(a) + g(b)$, $g(Ca) = Cga$ ($C \in \mathbb{C}$).

Conversely, a map with the properties 1)-3) comes from a unique representation π.

Left and Right

Let G be a group and V simply a set. A map

$$G \times V \longrightarrow V, \quad (g, a) \longmapsto ga,$$

with the properties 1)-2) is also called an action of G *from the left* on V. If one replaces in 2) the condition by $(gh)a = h(g(a))$ one gets the notion of an action *from the right.* This looks better if one uses the notation ag instead of ga since then the rule takes the better looking form $a(gh) = (ag)h$. If ga is an action from the left then $g^{-1}a$ is an action from the right, and conversely. Hence there is no essential difference between the two. Keep in mind that due to our definition representations are actions from the left.

Continuous representations

There are several equivalent ways to define when a representation of a locally compact group on a Banach space is continuous. A natural way is a follows.

6.1 Definition. *A representation of a locally compact group G on a Banach space E is called continuous if the corresponding map*

$$G \times E \longrightarrow E$$

is continuous.

Here $G \times E$ of course carries the product topology. For a continuous representation the operators $\pi(g) : E \to E$ are continuous (hence bounded) and the map $G \to E$, $g \mapsto g(a)$, is continuous for each $a \in E$.

6.2 Proposition. *A representation π of a locally compact group G on a Banach space E is continuous if all operators $\pi(g) : E \to E$ are bounded and if the map*

$$G \longrightarrow E, \quad g \longmapsto \pi(g)(a),$$

is continuous for all $a \in E$.

Hence we can write also $G \to B^*(E)$ for a continuous representation. The proof rests on the *theorem of uniform boundedness*:

6.3 Theorem. *Let E be a Banach space and let $\mathcal{M} \subset B(E)$ be a set of bounded operators such that $\{Aa,\ a \in E\}$ is bounded for each $a \in E$. Then $\mathcal{M}$ is a bounded subset of $B(E)$.*

We omit the prove. □

For the proof of Proposition 6.2 we need another observation.

6.4 Lemma. *Let $\pi : G \to \mathrm{GL}(E)$ be a continuous representation and $K \subset G$ a compact subset. Then the set $\pi(K)$ is bounded in $B(E)$.*

Proof. Since $\pi(K)a$ is compact and hence bounded for all a, the theorem of uniform boundedness gives the claim. □

Proof of Proposition 6.2. It is sufficient to prove the $\pi : G \times E \to E$ is continuous at a point (e, a). The proof follows from the lemma and the estimate

$$\|g(x) - a\| \le \|g(x) - g(a)\| + \|g(a) - a\|. \qquad \square$$

The condition of continuity in the definition of a representation can be further weakened.

6.5 Lemma. *Let $\pi : G \to \mathrm{GL}(E)$ be a homomorphism with the following properties:*

1) *all $\pi(g)$ are bounded.*
2) *There is a neighborhood of the identity whose image in $B(E)$ is bounded.*
3) *There is a dense subset of vectors $a \in E$ such that $g \mapsto \pi(g)(a)$ is continuous.*

Then π is a continuous representation.

Proof. We have to show that for fixed a the function $x \mapsto \pi(x)a$ is continuous. It is obviously enough to proof this at the unit element $x = e$. Hence we have to estimate $\|\pi(x)a - a\|$. For some b in the dense subset we use the estimate

$$\|\pi(x)a - a\| \le \|\pi(x)a - \pi(x)b\| + \|\pi(x)b - b\|\|b - a\|.$$

If we choose b close enough to a we obtain the desired result. □

A natural question is whether there exists a topology on $B(E)$ such that the representation $\pi : G \to B^*(X)$ is continuous if and only if it is a continuous map with respect to the topology on $B^*(E)$ induced from this topology. The answer is "'yes"'. But the topology in question is not the topology induced by the operator norm. It is the *strong operator topology*, also called the SOT-topology that is defined as follows.

6.6 Definition. *Let E by a Banach space. The* SOT*-topology on $B(E)$ is the weakest topology such that the functions*

$$B(E) \longrightarrow \mathbb{R}, \quad A \longmapsto \|A(a)\|$$

are continuous for all $a \in E$

Clearly this topology is weaker than the norm topology. A sequence A_n of operators converges to A in the SOT topology if and only if $A_n a$ converges to Aa for all $a \in E$ (with respect to the norm topology of E).

6.7 Remark. *A representation $\pi : G \to B^*(H)$ is continuous if and only if it is a continuous map with respect to the* SOT*-topology.*

Algebraic irreducibility

Let $\pi : G \to \mathrm{GL}(V)$ be a representation. A subspace $W \subset V$ is called *invariant* if $g \in G$ and $a \in W$ implies $ga \in W$. Then we obtain a representation $\pi' : G \to \mathrm{GL}(W)$. A representation $\pi : G \to \mathrm{GL}(V)$ is called *algebraically irreducible* if $V \neq 0$ and if besides $\{0\}$ and V there are no invariant subspaces. Let W_1, W_2 be two invariant subspaces of V. Then $W_1 + W_2$ and $W_1 \cap W_2$ are also invariant. If W_1 and W_2 are irreducible then either they are equal or their intersection is zero.

Topological Irreducibility

Let now $\pi : G \to \mathrm{GL}(V)$ be a *continuous* representation. It is called *topologically irreducible* if there is no *closed* invariant subspace different from $\{0\}$ and V.

For finite dimensional representations (this means that V is finite dimensional) algebraic and topological irreducibility is the same.

A representation of a topological group on a Hilbert space H is called *unitary* if it is continuous and if all operators $\pi(g)$ are unitary operators. This means concretely

$$\langle ga, gb \rangle = \langle a, b \rangle$$

for $a, b \in H$ and $g \in G$. It is enough to demand this for $a = b$. If we talk about an irreducible unitary representation, we always mean that it is topologically irreducible.

We describe a fundamental example of a unitary representation. Let G be a locally compact group. We consider a closed subgroup $H \subset G$. For sake of simplicity we assume that both are unimodular. Then dx is left- and right invariant. We consider the space of right cosets $H\backslash G$. The group G acts on $H\backslash G$ by multiplication from the right. This is an action from the right. Let $f : H\backslash G \to \mathbb{C}$ be a function and $g \in G$. We define the translate $R_g f$ of f by

$(R_g f)(x) = f(xg)$. This is an action from the left of G on the set of function on $H\backslash G$. This defines a map

$$R : G \longrightarrow \mathrm{GL}(L^2(H\backslash G, dx)).$$

By means of Theorem 6.3 one can show that this representation is continuous. It is obviously a unitary representation. In the special case $H = \{e\}$ one obtains the so-called regular representation of G on $L^2(G)$.

One of the basic problems of harmonic analysis is the investigation of this representation and to describe its spectral decomposition. This problem has been studied for the regular representation of semi simple groups G (for example $\mathrm{SL}(n, \mathbb{R})$) by Harish Chandra. In the theory of automorphic forms one studies the case where $H = \Gamma$ is a discrete subgroup such that $\Gamma\backslash G$ has finite volume.

What means "spectral decomposition"? This is not so easy to explain and not the goal of these notes. Nevertheless it is useful to get an idea of it. We give two examples. The first example is the group S^1 of complex numbers of absolute value one (circle group). The functions f on S^1 correspond to the periodic functions (period 2π) F on $\mathbb{R}$ through

$$F(t) = f(\exp(2\pi \mathrm{i} t)).$$

From the theory of Fourier series one knows that $L^2(S^1)$ is the direct Hilbert sum of the one dimensional subspaces $H(n)$ spanned by $f(\zeta) = \zeta^n$ $(n \in \mathbb{Z})$. These are invariant subspaces which are pairwise orthogonal. The spectral decomposition of the regular representation of S^1 is

$$L^2(S^1) = \widehat{\bigoplus_{n\in\mathbb{Z}}} H(n).$$

The second example deals with the regular representation of $\mathbb{R}$. There are also one dimensional spaces $H(t)$ generated by the function $x \mapsto e^{2\pi \mathrm{i} t x}$ which are invariant under translations $t \mapsto t + a$. Now t can be an arbitrary real number. But the difference is that now $H(t)$ is not contained in $L^2(\mathbb{R})$. Nevertheless the theory of Fourier transformation shows that all f in a certain dense subspace of $L^2(\mathbb{R})$ can be written in a unique way in the form

$$f(t) = \int_{-\infty}^{\infty} g(t) e^{2\pi \mathrm{i} t} dt.$$

Hence one is tempted to say that $L^2(\mathbb{R})$ is the direct integral of the spaces $H(t)$ and to write this in the form

$$L^2(\mathbb{R}) = \int_{\mathbb{R}}^{\oplus} H(t) dt.$$

For general G the spectral decomposition will include both types (discrete and continuous spectra) and the constituents will not be one-dimensional but irreducible unitary representations (often infinite dimensional).

Intertwining Operators

A morphism between two continuous representations $\pi_i : G \to \mathrm{GL}(E_i)$ on Banach spaces is a continuous linear map $E_1 \to E_2$ which is compatible with the action of G in an obvious sense. Such morphisms are also called "intertwining operators". It is clear what it means that an intertwining operator is an isomorphism. If $F \subset E$ is a closed G-invariant subspace then the natural inclusion $F \hookrightarrow E$ is a morphism. We call (G, F) a sub-representation of (G, E).

For unitary representations we will make use of a more restrictive notion of isomorphy. An isomorphism $H_1 \to H_2$ between two unitary representations $\pi : G \to \mathrm{GL}(H_i)$ is called a *unitary isomorphism,* or an isomorphism of unitary representations if the isomorphism $H_1 \to H_2$ is an isomorphism of Hilbert spaces. This means that it preserves the scalar products.

7. The convolution algebra

let G be a locally compact group with a chosen Haar measure. The convolution of two functions $f, g \in \mathcal{C}_c(G)$ is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy.$$

The convolution defines an associative product on $\mathcal{C}_c(G)$. We leave the proof of the associativity as an exercise. Hence $\mathcal{C}_c(G)$ has the structure of an associative $\mathbb{C}$-algebra.

Let $\pi : G \to \mathrm{GL}(H)$ be a continuous representation on a Banach space. For any $f \in \mathcal{C}_c(G)$ and any $h \in H$ we can consider the function

$$G \longrightarrow H, \quad x \longmapsto f(x)\pi(x)h.$$

It is continuous and with compact support. Hence we can define the integral

$$\int_G f(x)\pi(x)hdx.$$

If we vary h, we get an operator $H \to H$. One can check that it is linear and continuous.

We denote this operator by

$$\pi(f) = \int_G f(x)\pi(x)dx.$$

One verifies

$$\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2).$$

What we obtain is an algebra homomorphism

$$\pi : \mathcal{C}_c(G) \longrightarrow \mathrm{End}(H).$$

The image of π consists of continuous linear operators $T : H \to H$.

Now we assume that H is a Hilbert space. We denote the adjoint of an operator $T \in \mathrm{End}(H)$ by T^*. It is defined by the formula $\langle Tx, y\rangle = \langle x, T^*y\rangle$. The existence of T^* follows from the Riesz lemma. Of course T^* is continuous as T, and moreover both have the same norm.

We define

$$f^*(x) := \Delta(x^{-1})\overline{f(x^{-1})}.$$

We now assume that π is unitary. It is easy to check in this case that the map π has the property that

$$\pi(f^*) = \pi(f)^*.$$

What we obtained is a $*$-algebra representation. We describe briefly what this means. An algebra A is a vector space (in our case over $\mathbb{C}$) together with a bilinear map

$$A \times A \longrightarrow A, \quad (a, b) \longmapsto ab.$$

We assume that this is associative but we do not assume that A contains a unit element. An involution on A is a map

$$A \longrightarrow A, \quad a \longmapsto a^*,$$

with the properties

a) $(a + b)^* = a^* + b^*, \quad (Ca)^* = \bar{C}a^*,$
b) $(ab)^* = b^*a^*.$
c) $a^{**} = a.$

7.1 Definition. *A $*$-algebra $(A, *)$ is an associative algebra (not necessarily with unit) together with a distinguished involution*

An example of a $*$-algebra is the convolution algebra $\mathcal{C}_c(G)$ with the involution defined above. Please notice that the star has a double meaning in this example. It denotes the product of the algebra and also the involution of the algebra.

Another example of a $*$-algebra is the space $B(H)$ of continuous linear operators on a Hilbert space H. Multiplication is the composition of operators and the $*$-operator is given by the adjoint.

By a representation of an algebra A on a vector space V one understands a linear map $A \to \mathrm{End}(V)$ which is compatible with multiplication. By a $*$-algebra representation of a $*$-algebra A on a Hilbert space H we understand a homomorphism

$$A \longrightarrow B(H)$$

that is also compatible with the star operators. We have seen that a unitary representation $\pi : G \to \mathrm{U}(H)$ induces a $*$-algebra representation $\pi : \mathcal{C}_c(G) \to B(H)$.

There are obvious notions of irreducibility:

A representation $A \to \mathrm{End}(V)$ of an algebra is called algebraically irreducible if the image of A is not zero and if there is no invariant subspace of V different form 0 and V.

A $*$-algebra representation $A \to B(H)$ is called topologically irreducible if the image if A is non zero and if there is no closed invariant subspace of H different from 0 and H.

An example of a finite dimensional algebra representation is the tautological representation of $A = \mathrm{End}(V)$ on V. It is just the identity map $\mathrm{End}(V) \to \mathrm{End}(V)$. It is clear that this representation is irreducible. A special case of a fundamental structure theorem of Wedderburn states (in the case of the ground field $\mathbb{C}$):

7.2 Theorem. *Let $\pi : A \to \mathrm{End}(V)$ be an irreducible representation of an algebra A on a finite dimensional vector space V. Then π is surjective.*

We don't give the proof here and refer to the text book of S. Lang on algebra. To be honest, we mention that Lang treats only the case where A contains a unit element. The general case can be reduced by the technique of adjoining a unit element.

A trivial consequence of Theorem 7.2 is as follows. Let $T : V \to V$ be a linear operator that commutes with all $\pi(a)$, $a \in A$. Then T is a multiple of the identity. A basic result states that this carries over to the infinite dimensional case.

7.3 Theorem (Schur's lemma for algebra representations). *Let π be a topologically irreducible $*$-representation of a $*$-algebra $\mathcal{A}$ on a Hilbert space H. Assume that $T : H \to H$ is a linear and continuous operator that commutes with all $A = \pi(a)$, $a \in \mathcal{A}$. Then T is a constant multiple of the identity.*

Corollary. *If A is abelian then H is one-dimensional.*

Proof. The proof rests on the spectral theorem for self adjoint operators. This is explained in the Appendices, Sect.1 and 2. One has to use Lemma VI.2.10. We give the details. First one can assume that T is self adjoint, since one can use the decomposition $2T = (T + T^*) - \mathrm{i}(\mathrm{i}(T - T^*))$. So we can assume that T is self adjoint and commutes with all $A = \pi(a)$. We assume that T is not a multiple of the identity. Then, by Lemma VI.2.10, there exists a B in the bi-commutant of T whose kernel is different from 0 and H. Since B commutes with all $A = \pi(a)$, its kernel is invariant under all A. This contradicts the irreducibility. $\square$

The same theorem is true for irreducible unitary representations of locally compact groups. Actually it is a consequence of Theorem 7.3 as we shall point out. The argument would be very easy if there exists for $g \in G$ a Dirac function $\delta_g \in \mathcal{C}_c(G)$ which means

$$\delta_g(x) = 0 \text{ for } x \neq g \quad \text{and} \quad \int_G \delta_g(x)dx = 1.$$

Such a situation is of course rare, but it occurs, namely for finite groups. A simple computation then gives $\pi(\delta_g) = \pi(g)$. From this one can deduce that a subspace of H is invariant under all $\pi(g)$, $g \in G$, if and only if it is invariant under all $\pi(f)$, $f \in \mathcal{C}_c(G)$. Actually there is a weak variant of Dirac functions.

7.4 Lemma. *For each locally compact group G there exists a sequence of functions $\delta_n \in \mathcal{C}_c(G)$ with the following properties.*

1) $\mathrm{supp}(\delta_{n+1}) \subset \mathrm{supp}(\delta_n)$.
2) *For each neighborhood U of the identity there exists an n such that* $\mathrm{supp}(\delta_n) \subset U$.
3) $\delta_n(x^{-1}) = \delta_n(x)$.
4) $\delta_n(x) \geq 0$ *and* $\int_G \delta_n(x)dx = 1$.

We call (δ_n) a *Dirac sequence.*

7.5 Lemma. *Let (δ_n) be a Dirac sequence. Then $\pi(\delta_n)$ converges to the identity in the sense*

$$\lim_{n\to\infty} \|\pi(\delta_n)h - h\| = 0.$$

(This means pointwise convergence.)

Proof. We have

$$\|\pi(\delta_n)h - h\| \leq \int_G \delta_n(x)\|\pi(x)h - h\|.$$

Let $\varepsilon > 0$. For n big enough we have $\|\pi(x)h - h\| < \varepsilon$ for all $x \in U_n$. We obtain $\|\pi(\delta_n)h - h\| < \varepsilon$. □

There is an obvious generalization. Let $g \in G$ then from Lemma 7.5 we see that $\pi(f_n) \circ \pi(g) \to \pi(g)$ (pointwise) A simple calculation shows

$$\pi(f) \circ \pi(g) = \pi(\tilde{f}) \quad \text{where} \quad \tilde{f}(x) = \Delta(g)f(xg^{-1}).$$

This shows the following result.

7.6 Lemma. *Let $G \to \mathrm{GL}(H)$ be a unitary representation and let $W \subset H$ be a closed subspace. Assume that there exists a subalgebra $A \subset \mathcal{C}_c(G)$ that contains a Dirac sequence and that is invariant under translation $f(x) \mapsto f(xg)$ for all $g \in G$ and such that W is invariant under A. Then W is invariant under G.*

As an application of Lemma 7.6 we get the following lemma.

7.7 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation. A closed subspace $W \subset H$ is invariant under G if and only if it is invariant under $\mathcal{C}_c(G)$.*

Schur's lemma now can be formulated also for group representations.

7.8 Theorem (Schur's lemma for group representations). *Let $\pi : G \to \mathrm{GL}(H)$ be an irreducible unitary representation of a locally compact group. Every linear and continuous operator $T : H \to H$ which commutes with all $\pi(g)$, $g \in G$, is a multiple of the identity.*

Corollary. *If G is abelian then H is one-dimensional.*

Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation. We say that another unitary representation of G occurs in π if it is isomorphic (in the unitary sense) to a sub-representation of π.

7.9 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representations and A, B be two invariant closed subspaces. Assume that the restriction of π to A is (topologically) irreducible. Then either A is orthogonal to B or the representation $\pi|A$ occurs in $\pi|B$.*

Corollary. *If both A and B are irreducible then either they are orthogonal or isomorphic (as G-representations).*

Proof. We assume that A, B are not orthogonal. We consider the pairing $\langle \cdot, \cdot \rangle : A \times B \to \mathbb{C}$. We first notice that it is non degenerate in the following sense. For each $a \in A$ their exists a $b \in B$ such that $\langle a, b \rangle \neq 0$ and conversely. This is clear since the orthogonal complement of B intersected with A is a closed invariant subspace. Next we construct a linear map $f : A \to B$. By the Lemma of Riesz there exists for each $a \in A$ a unique $f(a)$ in B such that $\langle a, b \rangle = \langle f(a), b \rangle$ for all $b \in B$. One easily checks that this is an intertwining operator. □

7.10 Definition. *A unitary representation $\pi : G \to \mathrm{GL}(H)$ is called* ***completely reducible*** *if H can be written as the direct Hilbert sum of pairwise orthogonal closed invariant subspaces*

$$H = \widehat{\bigoplus_i} H_i$$

which are irreducible as G-representations.

In general we denote by $\hat{G}$ the set of all isomorphy classes of irreducible unitary representations of G and call it the *unitary dual* of G. Recall that each irreducible unitary representation $\pi : G \to \mathrm{GL}(H)$ is one dimensional if G is abelian. Hence it is of the form $\pi(g)(h) = \chi(g)h$ where χ is a character of G. By definition, this is a continuous homomorphism from G into the group of complex numbers of absolute value 1. Unitary isomorphic representations give the same character. This gives a bijection with $\hat{G}$ and the set of all unitary characters. Characters can be multiplied in an obvious way. Hence, for abelian G, the set $\hat{G}$ is a group as well. One can show that it carries a structure as locally compact group.

7.11 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ a unitary representation which is completely reducible,*

$$H = \widehat{\bigoplus_{i \in I}} H_i, \qquad H_i \subset H.$$

Let $\tau \in \hat{G}$. Then

$$H(\tau) = \widehat{\bigoplus_{i \in I,\ \pi_i \in \tau}} H_i$$

is the closure of the sum of all irreducible closed invariant subspaces of H that are of type τ. In particular, it is independent of the choice of the decomposition.

This follows immediately from Lemma 7.9. □

We call $H(\tau)$ the τ-isotypic component of π. This is well-defined. The irreducible components H_i are usually not well-defined. Look at the example of the group G that consists only of the unit element. Nevertheless the so-called multiplicity

$$m(\tau) := \#\{i \in I;\ \pi_i \in \tau\} \le \infty$$

is independent on the choice of the decomposition. This can be seen as follows. Let (H_τ, τ) be a realization of τ. We consider the vector space of all intertwining operators $H_\tau \to H(\tau)$. The space of intertwining operators $H_i \to H_\tau$ is zero if π_i is not in τ and – by Schur's lemma – one dimensional otherwise. From this follows easily the space of intertwining operators $H_\tau \to H(\tau)$ has dimension $m(\tau)$. This shows the invariance of $m(\tau)$.

This gives us the following result.

7.12 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a completely reducible unitary representation. The multiplicities*

$$m(\tau) := \#\{i \in I;\ \pi_i \in \tau\} \le \infty$$

(in the notation of Proposition 7.11 are well-defined). Two completely reducible representations are unitary isomorphic if and only of their multiplicities agree.

8. Generalities about compact groups

In this section we treat some general facts about representations of compact groups. Readers who are mainly interested in the classification of the irreducible unitary representations of the group $\mathrm{SL}(2,\mathbb{R})$ can skip this section, since the only compact group which occurs in this context is the group $\mathrm{SO}(2)$. This group is not only compact but also abelian which makes the theory rather trivial.

We need some results of functional analysis. We recall the notion of equicontinuity:

8.1 Definition. *A set $\mathcal{M}$ of functions on a topological space X is called equicontinuous at a point $a \in X$ if for any point $\varepsilon > 0$ their exists a neighborhood U of a such that*

$$|f(x) - f(a)| < \varepsilon \quad \textit{for all} \quad x \in U,\ f \in \mathcal{M}.$$

The set is called equicontinuous if this is the case at all $a \in X$.

(The point is the independence of the neighborhood U from f.) We recall a basic result from functional analysis.

8.2 Theorem (theorem of Arzela-Ascoli). *Let X be a locally compact space with countable basis of the topology. Let $\mathcal{M}$ be an equicontinuous set of functions on X such that the set of numbers $f(x)$, $f \in \mathcal{M}$, is bounded for every $x \in X$. Then each sequence of $\mathcal{M}$ admits a subsequence that converges locally uniformly on X.*

There are variants of this theorem in which equicontinuity does not appear. Let for example $X \subset \mathbb{R}^n$ be an open subset and assume that $\mathcal{M}$ is a set of *differentiable* functions such that there exists a constant C such that

$$|f(x)| \le C \quad \text{and} \quad |(\partial f/\partial x)(x)| \le C \quad \text{for all } x \in X.$$

Then the mean value theorem of calculus shows that this set is equicontinuous.

Another main tool will be the spectral theorem for compact operators on Hilbert spaces. Let H be a Hilbert space. A linear and continuous operator $T : H \to H$ is called *compact* if the image any bounded set is contained in a compact set. For example this is the case if the image of T is finite dimensional. The identity is compact if and only if H is finite dimensional. The set of all compact operators is closed under the operator norm. So, let $T_1, T_2, \dots$ be a sequence of compact operator and T another bounded operator such that $\|T_n - T\|$ tends to 0. Then T is compact. We will not give a proof here.

Recall that an operator T is called normal if it commutes with its adjoint, $T \circ T^* = T^* \circ T$.

8.3 Theorem (Spectral theorem for compact operators). *Let $T : H \to H$ be a compact and normal operator. The set of eigenvalues is either finite or it is countable and 0 is the only accumulation point of it. The eigenspaces $H(T, \lambda)$ are pairwise orthogonal and for $\lambda \neq 0$ they are finite dimensional. The sum of all eigenspaces is dense in H. Hence we have a Hilbert space decomposition*

$$H = \widehat{\bigoplus}_{\lambda} H(T, \lambda).$$

A proof can be found in the Appendices.

We give an example of a compact operator.

8.4 Proposition. *Let X be a compact topological space (with countable basis) and dx a Radon measure. Let $K \in \mathcal{C}(X, X)$ be a continuous function. The operator*

$$L_K : L^2(X, dx) \longrightarrow L^2(X, dx), \quad L_K(f)(x) := \int_X K(x, y) f(y) dy.$$

is a compact (continuous and linear) operator.

We mention that every square integrable function f on a compact space is integrable (since one can write $f = 1 \cdot f$ as product of two square integrable functions). Since $K(x, y)$ for fixed x is an L^2-function the existence of the integral in Proposition 8.4 is clear. Clearly the functions $L_K f$ are continuous. Even more we have

$$|L_X(f)(x)| \leq c \|f\|_2$$

with some constant c by the Cauchy-Schwarz inequality. This also implies that $L_X f \in L^2(X, dx)$ and moreover

$$\|L_K f\|_2 \leq C \|f\|_2$$

with some constant C. Hence the operator is linear and also continuous.

But we have a stronger property. It is easy to show that the set of functions

$$\{L_K f; \quad f \in L^2(X, dx), \quad \|f\|_2 \leq 1\}$$

is equicontinuous. This implies that L_K is a compact operator. For this we have to prove the following. Let $f_n \in L^2(X, dx)$ be a sequence of functions such that $\|f_n\|_2 \leq 1$. We have to show that $L_K f_n$ has a sub-sequence that converges in $L^2(X, dx)$. The theorem of Arzela-Ascoli shows that $L_K f_n$ converges uniformly. Hence it converges point-wise and all functions are bounded by a joint constant. Since X is compact, constant functions are integrable and we can apply the Lebesgue limit theorem to obtain convergence in $L^2(X, dx)$. $\square$

8.5 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation of a locally compact group G on a Hilbert space H. Assume that there exists a Dirac sequence $\delta_n \in \mathcal{C}_c(G)$ such that all $\pi(\delta_n)$ are compact operators. Then the representation decomposes into irreducibles with finite multiplicities.*

Proof. We consider pairs that consist of a closed invariant subspace $H' \subset H$ such the restriction of π to H' is completely reducible and a distinguished decomposition $H' = \widehat{\bigoplus}_{i\in I} H'_i$ into irreducibles. We define an ordering for such pairs. The pair $H' = \widehat{\bigoplus}_{i\in I} H'_i$ is less or equal than the pair $H'' = \widehat{\bigoplus}_{j\in J} H''_j$ if each space H'_i equals some H''_j. (Especially $H' \subset H''$). From Zorn's lemma easily follows that there exists a maximal member. We call its orthogonal complement U. This space cannot contain any irreducible subspace since this could be used to enlarge the maximal element. Hence we have to show:

let π be a representation as in the proposition which is not zero. Then there exists at least one irreducible closed subspace.

To prove this we choose an element f of the Dirac sequence such that $\pi(f)$ is not identically zero. This element will kept fixed during the proof. We also choose an eigenvalue $\lambda \neq 0$ of $\pi(f)$ Let $H(f,\lambda) \subset H$ the eigenspace. This is a finite dimensional vector space.

There may be invariant closed subspaces which have a non-zero intersection with $H(f,\lambda)$. We choose a closed subspace E such that the dimension of its intersection with $H(f,\lambda)$ is non-zero and minimal. Then we set $W = E \cap H(f,\lambda)$. There still may exist several closed invariant subspaces H that share with E the property $W = F \cap H(f,\lambda)$. We take the intersection of all these F and get in this way a smallest closed invariant subspace $F \subset E$ with $W = F \cap H(f,\lambda)$. We claim that this F is irreducible. For this we take any orthogonal decomposition $F = A \oplus B$. The eigenvalue λ must occur as eigenvalue of λ in one of the spaces A, B. (The restriction of a compact operator to a closed invariant subspace remains compact and hence decomposes into eigen spaces.) Let us assume that it occurs in A. Then $A \cap H(f,\lambda)$ is not zero. It must agree with W because of the minimality property of $\dim W$. Moreover it must agree with F because of the minimality property of F. This shows the irreducibility.

It remains to prove that the multiplicities are finite. Let $\tau \in \hat{G}$. Let $H_1, \dots, H_m$ be pairwise orthogonal invariant closed subspaces of type τ. We claim that m is bounded. There exists an element $f = \delta_n$ from the Dirac sequence such that $\pi(f)$ is not zero on H_1. There exists a non-zero eigenvalue λ. This eigenvalue then occurs in all H_i since they are all isomorphic (as representations). Since the multiplicity of the eigenvalue is finite the number m must be bounded. $\square$

A special case of Proposition 8.5 gives the following basic result.

8.6 Theorem. *Let K be a compact group. The regular representation of*

K on $L^2(K)$ (translation from the right) is completely reducible with finite multiplicities.

Proof. Let $f \in \mathcal{C}(K)$. We have to show that the operator R_f is compact. Recall that R_f is defined as Bochner integral

$$R_f(h) = \int_K f(x)R_x(h)dx, \qquad (R_x h)(y) = h(yx).$$

It looks natural to get this as function by interchanging the evaluation if this function with integration, i.e. one should expect

$$R_f(h)(y) = \int_K f(x)h(yx)dx.$$

This is actually true but one has to be careful with the argument since the evaluation map $h \mapsto h(y)$ is not a continuous linear functional on the Hilbert space $L^2(K)$. Instead of this one uses the following argument. Two elements of a Hilbert space are equal if and only if their scalar products with an arbitrary vector are equal. Taking scalar product with a vector is a continuous linear functional which can be exchanged with the Bochner integral. In this way one obtains the desired formula. We can rewrite the formula as

$$R_f(h)(x) = \int_K f(x^{-1}y)h(y)dy.$$

This is the integral operator with kernel $K(x,y) = f(x^{-1}y)$. $\square$

There is a more general result.

8.7 Theorem. *Every irreducible unitary representation of a compact group on a Hilbert space is finite dimensional.*

Every unitary representation of a compact group K is completely reducible.

Proof. We choose $h \in H$ such that $\|h\| = 1$. Let $x \in H$. Then we consider the function

$$K \longrightarrow H, \quad k \longmapsto \langle x, \pi(k)h\rangle\pi(k)h.$$

It is continuous, so we can integrate it to get an operator $T \in B(H)$,

$$Tx = \int_K \langle x, \pi(k)h\rangle\pi(k)h.$$

This operator has the following remarkable properties.

1) $\langle Tx, x\rangle \geq 0$,
2) $\langle Th, h\rangle > 0$,
3) T commutes with all $\pi(x)$,
4) T is a compact operator.

First we assume that 1)-4) have been proved. Since T is non zero (use 2)), there exists a non-zero eigenvalue λ. Its eigenspace H_λ is finite dimensional. From 3) follows that H_λ is invariant under π. So we have shown that every unitary representation of a compact group has an finite dimensional invariant subspace. This proves the first part of the theorem, since the irreducibility implies $H = H_\lambda$. But also the second part follows as the following argument shows.

Consider invariant subspaces $H_0 \subset H$ which are completely reducible. A "Zorn's lemma argument" shows that there exists a maximal one. We have to show $H = H_0$. If this would not be the case we could find in the orthogonal complement an irreducible subspace. This contradicts to the maximality of H_0.

We still have to prove 1)-4). 1) follows from

$$\langle Tx, x\rangle = \int_K |\langle x, \pi(k)h\rangle|^2 dk \geq 0.$$

In the case $x = h$ the integrand is positive at $k = e$ and hence in a full neighbourhood of e. This implies 2). The proof of 3) follows from the calculation

$$\begin{aligned}\pi(x)(Ty) &= \int_K \langle y, \pi(k)h\rangle \pi(xk)h dk = \int_K \langle y, \pi(x^{-1}k)h\rangle \pi(x)h dx \\ &= \int_K \langle \pi(x)y, \pi(x)h\rangle \pi(k)h dk = T(\pi(x)y).\end{aligned}$$

It remains to prove 4). For this we show that T is the limit (with respect to the norm-topology on $B(H)$) of operators with finite dimensional range. Let $\varepsilon > 0$. We want to construct an operator T_ε with finite dimensional range such that $\|T - T_\varepsilon\| < \varepsilon$. We make use of the continuity of the function $k \mapsto \pi(k)h$. Since K is compact, every continuous function is uniformly continuous. Hence we find a finite open covering $U_1, \dots, U_n$ of K and points $k_i \in U_i$ such that

$$\|\pi(k)h - \pi(k_i)h\| < \frac{1}{2}\varepsilon \text{ for } k \in U_i.$$

We want to replace the U_i by disjoint sets $E_i \subset U_i$ which still cover K. Of course this can be not done with open sets but Borel sets, in particular by measurable sets. Now we can define

$$T_\varepsilon x = \sum_{i=1}^{n} \mathrm{vol}(E_i)\langle x, \pi(k_i)h\rangle \pi(k_i)h.$$

This is clearly an operator with finite dimensional range and the property $\|T - T_\varepsilon\| a\varepsilon$. This finishes the proof Theorem 8.7. □

8.8 Proposition. *Let $\pi : K \to \mathrm{GL}(H)$ be a Banach representation of a compact group on a Hilbert space H. There exists a Hermitian product on H whose norm is equivalent to the original one and such that π is unitary.*

The proof is easy. One replaces the original Hermitian product $\langle\cdot,\cdot\rangle$ by the new scalar product

$$\int_K \langle \pi(k)(x), \pi(k)(y)\rangle.$$

(This is called Weyl's unitary trick.) □

There is a broad structure theory for representations of compact groups, in particular of compact Lie groups. We need only little of it. Basic for this theory is the notion of the *character* of a finite dimensional representation $\pi : G \to \mathrm{GL}(H)$. It is the following function on G:

$$\chi_\pi : G \longrightarrow \mathbb{C}, \quad \chi_\pi(x) = \mathrm{tr}(\pi(x)).$$

For a one-dimensional representation this is the usual the underlying character. The character is a class function. This means

$$\chi(yxy^{-1}) = \chi(x).$$

(But in this context characters are usually not homomorphisms.)

8.9 Peter-Weyl theorem. *Let $\pi : K \to \mathrm{GL}(H)$ be an unitary representation of a compact group and let σ be an irreducible unitary representation of K. We denote by $H(\sigma)$ the σ-isotypic component of H. We denote by $P : H \to H(\sigma)$ the projection operator. Then*

$$P = \dim(\sigma) \int_K \overline{\chi(k)}\pi(k).$$

Proof. A proof can be found in the appendices.

Theorem 8.6 admits the following generalization:

8.10 Theorem. *Let G be a unimodular locally compact group and $\Gamma \subset G$ a discrete subgroup such that $\Gamma\backslash G$ is compact. Then the representation of G on $L^2(\Gamma\backslash G)$ (translation from the right) is completely irreducible with finite multiplicities.*

Proof. As in the proof of Theorem 8.6 we can rewrite the operator R_f as an integral operator

$$\int_G f(y)h(xy)dy = \int_G f(x^{-1}y)h(y)dy = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)dy.$$

This is an integral operator with kernel

$$K(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).$$

Since f has compact support, this sum is locally finite and K is a continuous function on $X \times X$ where X is the compact space $\Gamma\backslash G$. So we can apply Proposition 8.4. $\square$

This theorem is of great importance for the theory of automorphic forms and is one reason to study the irreducible representations of G.

Chapter II. The real special linear group of degree two

1. The simplest compact group

We study the group

$$K = \mathrm{SO}(2).$$

So K consists of all real 2×2 matrices k of determinant 1 with the property

$$k'k = e.$$

Because of

$$k^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \left(k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

this means that k is of the form

$$k = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a^2 + b^2 = 1.$$

For $k \in K$ the complex number $\zeta = a + \mathrm{i}b$ is of absolute value 1. Recall that the set of all complex numbers of absolute value 1 is a group under multiplication. One easily checks that the map

$$\mathrm{SO}(2) \xrightarrow{\sim} S^1, \quad k \longmapsto \zeta,$$

is an isomorphisms of locally compact groups. So we see that K is a compact and abelian group. Hence we know that each irreducible unitary representation is one-dimensional and corresponds to a character. The characters of S^1 are easy. They correspond to the integers $\mathbb{Z}$. For each integer n we can define

$$\chi_n(k) = \chi_n(\zeta) := \zeta^n.$$

For an arbitrary unitary representation $\pi : K \to \mathrm{GL}(H)$ we can consider the corresponding isotypic component

$$H(n) := \{h \in H; \quad \pi(g)(h) = \chi_n(g)h\}.$$

Another way to write the elements of $\mathrm{SL}(2, \mathbb{R})$ is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Here θ is determined mod $2\pi\mathrm{i}\mathbb{Z}$. The character χ_n in this presentation is given by

$$\chi_n(k) = e^{2\pi\mathrm{i}n\theta}.$$

2. The Haar measure of the real special linear group of degree two

We use the following notations:

$$
\begin{aligned}
G &= \mathrm{SL}(2,\mathbb{R}),\\
A &= \left\{ a = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}; \quad t \in \mathbb{R} \right\},\ \alpha > 0,\\
N &= \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; \quad x \in \mathbb{R} \right\},\\
K &= \left\{ k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \quad \theta \in \mathbb{R} \right\}.
\end{aligned}
$$

2.1 Lemma (Iwasawa decomposition). *The map*

$$A \times N \times K \longrightarrow G, \quad (a,n,k) \longmapsto ank,$$

is topological.

Proof. The elements of K act as rotations on $\mathbb{R}^2$. To any $g \in G$ one can find a rotation k such that gk fixes the x-axis. Then gk is triangular matrix. This gives the prove of the lemma. □

One can write the decomposition explicitly (which leads to a new proof):

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & 0 \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} \cdot \begin{pmatrix} 1 & ac+bd \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{c^2+d^2}} \begin{pmatrix} d & -c \\ c & d \end{pmatrix}.}$$

We denote the Haar measures on A, N, K by da, dn, dk. Recall

$$da = \frac{d\alpha}{\alpha}, \quad dn = dx.$$

The measure dk is normalized such that the volume of K is 1.

We first consider the group $P = AN$ of upper triangular matrices in $\mathrm{SL}(2,\mathbb{R})$ with positive diagonal. The map $A \times N \to P$ is topological (but not a group isomorphism). Recall that

$$\int_P f(p)dp := \int_A \int_N f(an)dnda$$

is a Haar measure (Lemma I.5.4).

2.2 Proposition. *A Haar measure on $G = \mathrm{SL}(2,\mathbb{R})$ can be obtained as follows*

$$\int_G f(x)dx = \int_A \int_N \int_K f(ank)\, dk\, dn\, da.$$

Proof. Since K is compact we have $\Delta_G|K = \Delta_K$. Hence the invariant quotient measure on $K\backslash G$ exists. There is a natural topological map $P \to K\backslash G$. The quotient measure gives a Haar measure on P. The rest comes from defining properties of a quotient measure (Proposition I.5.5). □

3. Principal series for the real special linear group of degree two

We will use the concept of "induced representation". The basic idea of induced representations is easy to explain. Let $P \subset G$ be a subgroup of a group and $\sigma : P \to \mathrm{GL}(H)$ a representation of the subgroup. We consider the space $\mathrm{Ind}(\sigma)$ of all functions $f : G \to H$ with the property

$$f(px) = \sigma(p)f(x) \quad \text{for} \quad p \in P,\ x \in G.$$

Then G acts by right translation on $\mathrm{Ind}(\sigma)$.

An important special case is obtained if one takes for σ a one dimensional representation. It is given by a quasi character (homomorhism $\chi : P \to \mathbb{C}^*$). Then $\mathrm{Ind}(\sigma) = \mathrm{Ind}(\chi)$ consists of functions $f : G \to \mathbb{C}$ with the property

$$f(px) = \chi(p)f(x) \quad \text{for} \quad p \in P,\ x \in G.$$

In the case that χ is trivial we obtain the standard representation of G on the space of all functions on $P\backslash G$ (translation from the right).

Assume that G is a locally compact group and that P is a closed subgroup. We want to modify this construction in such a way that we get – for unitary σ – a *unitary* induced representation.

The easiest case is when G and P are both unimodular. Then there exists an invariant measure dx on $P\backslash G$. We consider the one dimensional trivial representation of P. Then the induced representations consists of all functions on $P\backslash G$. We modify this and take $L^2(P\backslash G, dg)$. The right translation is a unitary representation as we realized earlier.

Now we consider $G = \mathrm{SL}(2,\mathbb{R})$ and for P we take the group of upper triangular matrices with positive diagonal. We consider the space of all functions

$$f(pg) = \alpha^{1+s} f(g), \qquad p \in P,\ g \in G.$$

Here s can be an arbitrary complex number. The group G acts on the set of these functions by translation from the right. The Iwasawa decomposition shows that such a function is determined by its restriction to K and every function on K is the restriction of such a function. Hence we defined an action of G (depending on s) on functions on K. We denote this action by $\pi_s(g)f$

Now we introduce the space $H^\infty(s)$ of all differentiable functions on G with the property

$$f(pg) = a^{1+s}f(g), \qquad p \in P,\ g \in G$$

We have an isomorphism

$$H^\infty(s) \xrightarrow{\sim} \mathcal{C}^\infty(K)$$

We transport the Hermitian product of $L^2(K, dk)$ to $H^\infty(s)$ and we denote the completion by $H(s)$. So we have an isomorphism

$$L^2(K, dk) \xrightarrow{\sim} H(s)$$

We have a representation of G on $\mathcal{H}^\infty(s)$ which preserves the scalar product. This extends to a unitary representation on the completion $H(s)$. A little later it will be clear that this representation is continuous.

3.1 Proposition (Principal Series Representations). *For each complex s there is a Banach-representation of $G = \mathrm{SL}(2, \mathbb{R})$ on the space $L^2(K, dk)$ which can be defined as follows. Take a differentiable function f on K and extend it to a function on G with the property*

$$f(px) = \alpha^{1+s}f(x) \qquad (x \in G).$$

Consider the translation of G from the right and then take the completion.

We will see that the these representations play a fundamental role. They are not irreducible. We can consider the subspaces $H^{\mathrm{even}}(s)$, $H^{\mathrm{odd}}(s)$ of $H(s)$ that are defined through $f(-g) = \pm f(g)$.

Under the natural isomorphism

$$H(s) \xrightarrow{\sim} L^2(K, dk)$$

the image of $H^{\mathrm{even}}(s)$ is the Hilbert space with the basis e^{inx}, n even, and similarly the image of $H^{\mathrm{odd}}(s)$ is the Hilbert space with the basis e^{inx}, n odd.

3.2 Remark. *The principal series is the orthogonal direct sum of the even and the odd principal series*

$$H(s) = H^{\mathrm{even}}(s) \oplus H^{\mathrm{odd}}(s)$$

which are defined through $f(-g) = \pm f(g)$.

We are not interested in Banach representations but in unitary representations. So we have to investigate for which s the princicpal series gives unitary representations.

Recall the notations $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2)$ and also the group P of upper triangular matrices

$$p = \begin{pmatrix} \alpha & * \\ 0 & \alpha^{-1} \end{pmatrix}$$

with positive diagonal elements. The Iwasawa decomposition gives a natural bijection

$$P\backslash G \longrightarrow K, \quad pk \longmapsto k.$$

Since G acts on $P\backslash G$ from the right, we get an action of G on K. This can be described as follows.

Let $k \in K$, $g \in G$. We write the Iwasawa decomposition of kg in the form

$$kg = \tilde{p}_g(k)\tilde{k}_g(k), \quad \tilde{p}_g(k) \in P,\ \tilde{k}_g(k) \in K.$$

Then the action of G on K is given by

$$K \times G \longrightarrow K, \quad (k,g) \longmapsto \tilde{k}_g(k).$$

Let f be a function on K and $g \in G$. Then the transformed function can be written as

$$\pi_s(g)f(k) = \Delta(\tilde{p}_g(k))^{-(1+s)/2} f(\tilde{k}_g(k)).$$

Let dk be a Haar measure of K. It is not invariant under the action of G. Otherwise we would get an invariant measure on $P\backslash G$ which cannot exist since G but not P is unimodular. Instead of this the following transformation formula holds. Recall that $\Delta(p) = a^{-2}$ is the modular function of P.

3.3 Lemma. *Let $g \in G$. We consider the (continuous) maps $\tilde{k}_g : K \to K$ and $\tilde{p}_g : K \to P$ which are defined by $kg = \tilde{p}_g(k)\tilde{k}_g(k)$. Then for each $f \in \mathcal{C}_c(K)$ the formula*

$$\int_K f(\tilde{k}_g(k))\Delta(\tilde{p}_g(k))^{-1}dk = \int_K f(k)dk$$

holds.

(If the Δ-factor were absent, the measure dk on K would be G-invariant.)

Proof. Since G and K are unimodular we can consider on G/K the invariant quotient measure and this gives a (left invariant) Haar measure dp on P which we can identify with G/K. We choose an arbitrary function $\varphi \in \mathcal{C}_c(P)$ with the property

$$\int_P \varphi(p)dp = 1.$$

Then we consider the function $F(pk) = \varphi(p)f(k)$. This is a function in $\mathcal{C}_c(G)$. The defining formula for the quotient measure on G/K is

$$\int_G F(x)dx = \int_P \int_K F(pk)dkdp = \int_K f(k)dk.$$

We use the right invariance of the Haar measure on G to obtain

$$\int_K f(k)dk = \int_G F(xg)dx = \int_P \int_K F(p\tilde{p}_g(k)\tilde{k}_g(k))dkdp.$$

We first integrate over p. Since the factor $\tilde{p}_g(k) \in P$ is on the right from p we get

$$\int_K f(k)dk = \int_K \int_P F(p\tilde{k}_g(k))\Delta(\tilde{p}_g(k))^{-1}dkdp = \int_K f(\tilde{k}_g(k)\Delta(\tilde{p}_g(k))^{-1}dk.$$

□

We derive a corollary from Lemma 3.3.

3.4 Corollary of Lemma 3.3. *Let $f : G \to \mathbb{C}$ be a function such that $f|K$ is integrable and such that*

$$f(pg) = \alpha^2 f(p) = \Delta(p)^{-1}f(p) \quad \textit{for all } p \in P,\ g \in G.$$

Then for each $g \in G$ the function

$$\tilde{f}(x) = f(xg), \quad x \in G,$$

has the same property and we have

$$\int_K f(x)dx = \int_K \tilde{f}(x)dx.$$

Proof. We have

$$\int_K f(kg)dk = \int_K f(\tilde{p}_g(k)\tilde{k}_g(k)dk = \int_K \Delta(\tilde{p}_g(k))^{-1}f(\tilde{k}_g(k))dk.$$

Now Lemma 3.3 applies. □

Let $f, g \in \mathcal{H}(s)$. Then $f(x)\overline{g(x)}$ transforms under right translation with the factor $\alpha^{1+s}\alpha^{1+\bar{s}}$. In the special case that s is purely imaginary this is α^2. Then the Corollary of Lemma 3.3 shows that $\int_G f(x)\overline{g(x)}$ is invariant under π_s. This gives us the following result.

3.5 Proposition (unitary principal series). *For $s \in \mathrm{i}\mathbb{R}$ the principal series representation is a unitary representation of G. In these cases the decomposition*

$$H(s) = H^{\mathrm{even}}(s) \oplus H^{\mathrm{odd}}(s)$$

is an orthogonal decomposition.

We will see later that $H^{\mathrm{even}}(s)$, $s \in \mathrm{i}\mathbb{R}$, is always irreducible and that $H^{\mathrm{odd}}(s)$, $s \in \mathrm{i}\mathbb{R}$, is irreducible for $s \neq 0$. The odd case $s = 0$ is exceptional. Here the representation breaks into an orthogonal sum

$$H^{\mathrm{odd}}(0) = H^{\mathrm{odd}}(0)_+ \oplus H^{\mathrm{odd}}(0)_-$$

of two unitary representations. Inside $L^2(K, dk)$ they are generated (as Hilbert spaces) by $e^{in\theta}$ where n is odd and $n > 0$ resp. $n < 0$. Later we will see that these two exceptional unitary representations are irreducible. They are called the *Mock discrete* series for reasons we will see.

So far we obtained three series of unitary representations which will turn out to be irreducible.

even principal series	$H^{\mathrm{even}}(s)$	$(s \in \mathrm{i}\mathbb{R})$
odd principal series	$H^{\mathrm{odd}}(s)$	$(s \in \mathrm{i}\mathbb{R},\ s \neq 0)$
mock discrete series	$H^{\mathrm{odd}}(0)_\pm$	(2 representations) derived from the odd principal series in the case $s = 0$

4. The intertwining operator

We go back to the principal series representation for arbitrary complex s.

In what follows it is convenient to introduce the subspace $\mathcal{H}^\infty(s) \subset \mathcal{H}(s)$ of differentiable functions in $\mathcal{H}(s)$. This space corresponds to $\mathcal{C}^\infty(K)$. Clearly the group G acts on $\mathcal{H}^\infty(s)$. We want to construct an intertwining operator

$$M(s) : \mathcal{H}^\infty(s) \xrightarrow{\sim} \mathcal{H}^\infty(-s),$$

in the sense that it is an isomorphism of vector spaces, compatible with the action of G.

4.1 Lemma. *Let $f \in \mathcal{H}(s)$ be a continuous function. Then the integral*

$$\int_N f(wn)dn, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

exists for $\mathrm{Re}\, s > 0$.

Proof. We have to make use of the Iwasawa decomposition

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+x^2}} & 0 \\ 0 & \sqrt{1+x^2} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{x}{\sqrt{1+x^2}} & \frac{-1}{\sqrt{1+x^2}} \\ \frac{1}{\sqrt{1+x^2}} & \frac{x}{\sqrt{1+x^2}} \end{pmatrix}.$$

It shows

$$f(wn) = \frac{1}{\sqrt{1+x^2}^{1+s}} f\begin{pmatrix} \frac{x}{\sqrt{1+x^2}} & \frac{-1}{\sqrt{1+x^2}} \\ \frac{1}{\sqrt{1+x^2}} & \frac{x}{\sqrt{1+x^2}} \end{pmatrix}, \qquad n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Since the function f is bounded on K, we can compare the integral with

$$\int_1^\infty \frac{1}{x^{1+s}}dx.$$

This converges for $\operatorname{Re} s > 0$. □

We can consider the integral in Lemma 4.1 for $f(xg)$ instead of $f(x)$,

$$\int_N f(wng)$$

and consider it as a function on G. For trivial reason we have:

4.2 Remark. *Let* $\operatorname{Re} s > 0$. *The operator*

$$(M(s)f)(x) = \int_N f(wnx)dn$$

is compatible with the action of G.

As we know, $L^2(K, dk)$ is a Hilbert space. The functions $e^{\mathrm{i}n\theta}$, $n \in \mathbb{Z}$, define an orthonormal basis. This follows for example from the fact that every function in $\mathcal{C}^\infty(K)$ admits a Fourier expansion

$$\sum_{m=0}^{\infty} a_m e^{\mathrm{i}m\theta}.$$

Such a Fourier series occurs if and only if (a_m) is tempered, i.e. rapidly decaying which means that $a_m P(m)$ is bounded for all polynomials P. The Fourier series and all its derivatives then converge uniformly. Hence they converge also in $L^2(K, dk)$. We denote by

$$\varepsilon(s, m), \qquad pk \longmapsto a^{1+s}e^{\mathrm{i}m\theta}$$

the corresponding functions in $\mathcal{H}(s)$. Considered in $H(s)$ they build an orthonormal basis. (Recall that we consider at the moment the Hilbert space structure on $H(s)$ which is obtained by transportation from $L^2(K, dk)$).

4.3 Proposition. *Let* $\operatorname{Re} s > 0$. *Then*

$$M(s)\varepsilon(s,m) = c(s,m)\varepsilon(-s,m)$$

where

$$c(s,m) = \frac{2^{1-s}\pi\Gamma(s)}{\Gamma\big(\frac{s+1+m}{2}\big)\Gamma\big(\frac{s+1-m}{2}\big)}.$$

An inductive formula for $c(s,m)$ *is*

$$c(s,0) = \frac{\Gamma\big(\frac{1}{2}\big)\Gamma\big(\frac{s}{2}\big)}{\Gamma\big(\frac{s+1}{2}\big)}, \quad c(s,\pm 1) = \pm \mathrm{i}\frac{\Gamma\big(\frac{1}{2}\big)\Gamma\big(\frac{s+1}{2}\big)}{\Gamma\big(\frac{s+2}{2}\big)}.$$

$$c(s,m+2) = -\frac{s-(m+1)}{s+(m+1)}c(s,m), \quad c(s,m-2) = -\frac{s+(m-1)}{s-(m-1)}c(s,m).$$

Proof. We have to prove

$$c(s,m) = \int_N \varepsilon(s,m)(wn)dn.$$

Using the Iwasawa decomposition this means

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^2}^{1+s}}\left(\frac{x+\mathrm{i}}{\sqrt{1+x^2}}\right)^m dx = \frac{2^{1-s}\pi\Gamma(s)}{\Gamma\big(\frac{s+1+m}{2}\big)\Gamma\big(\frac{s+1-m}{2}\big)}.}$$

This formula is proved and commented in (as far as I know unpublished) notes of Garret, P. “Irreducibles as kernels of intertwinings among principals” (2009) and later also in Casselman, B. “Representations” (2020 so far last version). Both papers can by found on the internet. Casselman says that the proof in his notes is due to Garret but that this formula already had been known to Cauchy, who published it 1825 without proof. For the detailed proof we refer to the two papers above. □

The Gamma function has a meromorphic extension to the whole plane. The poles are in $0, -1, -2, \dots$. Hence the functions $c(s,m)$ can be holomorphically extended to the complement of $\mathbb{Z}$ in $\mathbb{C}$. It is easy to verify that $c(s,m)$ has moderate growth for fixed $s \notin \mathbb{Z}$ and for $m \to \pm\infty$. We can use this extension to define the intertwining operator $M(s)$ for all $s \notin \mathbb{Z}$. This means that

$$|c(s,m)| \le c|m|^K$$

for suitable constants C, K which may depend on s. This implies that $(a_m c(m,s))$ is rapidly decaying if (a_m) is so. This implies that $M(s)$ can be defined th rough

$$M(s)\sum_m a_m\varepsilon(s,m) = \sum_n a_m c(s,m)\varepsilon(s,m).$$

So we see that $M(s)$ maps $\mathcal{H}^\infty(s)$ to $\mathcal{H}^\infty(-s)$ Hence we proved the following proposition.

4.4 Proposition. *The intertwining operator $M(s) : \mathcal{H}^\infty(s) \to \mathcal{H}^\infty(-s)$ can be extended from* $\operatorname{Re} s > 0$ *to all $s \notin \mathbb{Z}$ by means of the formula*

$$M(s) \sum_m a_m \varepsilon(s, m) = \sum_n a_m c(s, m)\varepsilon(s, m).$$

It is compatible with the action of G. For pure imaginary s ist is unitary.

5. Complementary series for the rea lspecial linear group of degree two

Let $f, g \in \mathcal{H}^\infty(s)$. We consider the function $h = fM(s)\bar{g}$. It is easy to check that

$$h \in \mathcal{H}^\infty(2 + s - \bar{s}).$$

Now we assume that s is real. Now we can consider a pairing on $\mathcal{H}^\infty(s)$.

$$\mathcal{H}^\infty(s) \times \mathcal{H}^\infty(s) \longrightarrow \mathbb{C}, \quad \langle f, g\rangle = \int_K f(x)M(s)\overline{g(x)}dx.$$

Corollary 3.4 shows that this pairing is compatible with the action of G. The constants $c(s, m)$ are real for real s. Hence this pairing is a Hermitian form. If all $c(s, m)$ were positive this form would be positive definit. From the recursion formula one can deduce that

$$c(s, m) > 0 \quad \text{for } s \in (-1, 1),\ s \neq 0,\ m \text{ even}.$$

We can consider also the subspaces $\mathcal{H}^{\infty,\mathrm{even}}(s) = \mathcal{H}^\infty(s) \cap \mathcal{H}^{\mathrm{even}}(s)$ and similarly for odd. The formula in Proposition 4.4 shows that theses subspaces are G-invariant. We obtain the following lemma.

5.1 Lemma. *Assume $s \in (-1, 1)$, $s \neq 0$. The Hermitian form $\langle f, g\rangle$ on $\mathcal{H}^{\infty,\mathrm{even}}(s)$ is positive definit. The action of G on $\mathcal{H}^{\infty,\mathrm{even}}(s)$ is unitary (compatible with the action of G.*

Assume $s \in (-1, 1)$. We denote by $\tilde{H}^{\mathrm{even}}(s)$ the Hilbert space which is obtained from $\mathcal{H}^{\infty,\mathrm{even}}(s)$ through completion.

5.2 Proposition (Complementary series). *For each $s \in (-1, 1)$, $s \neq 0$, the representation of G on $\tilde{H}^{\mathrm{even}}(s)$ is a (continuous) unitary representation. The functions $\varepsilon(s, n)$, n even, define an orthonormal basis. Two representations for s and $-s$ are isomorphic.*

6. The discrete series

Möbius transformations

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$$

be a complex invertible 2×2-matrix. Denote by $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. The transformation

$$g(z) = \frac{az + b}{cz + d}$$

is defined first outside a finite set of $\bar{\mathbb{C}}$ but can be extended in a natural way to a bijection $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$. This is an action of $\mathrm{GL}(2, \mathbb{C})$ on $\bar{\mathbb{C}}$ from the left. It is well known and easy to check that the subgroup $\mathrm{SL}(2, \mathbb{R})$ acts on the upper half plane $\mathcal{H}$. It is also well known that the *Cayley transformation*

$$\sigma = \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}$$

maps the upper half plane unto the unit disk. The inverse transformation is given by

$$\sigma^{-1} = \frac{1}{2\mathrm{i}} \begin{pmatrix} \mathrm{i} & \mathrm{i} \\ -1 & 1 \end{pmatrix}.$$

As a consequence the group

$$\sigma \, \mathrm{SL}(2, \mathbb{R}) \sigma^{-1}$$

acts on the unit disk $\mathcal{E}$. This group is also a well-known classical group. We denote by $\mathrm{U}(1, 1)$ the unitary group of signature $(1, 1)$ that is defined through

$$\bar{g}' J g = J \quad \text{where} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The special unitary group of signature $(1, 1)$ is

$$\mathrm{SU}(1, 1) = \mathrm{U}(1, 1) \cap \mathrm{SL}(2, \mathbb{C}).$$

One can check

$$\mathrm{SU}(1, 1) = \sigma \, \mathrm{SL}(2, \mathbb{R}) \sigma^{-1}.$$

A very quick proof rests on the fact that a 2×2-matrix g has determinant 1 if and only if

$$g' I g = I \quad \text{where} \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hilbert spaces of holomorphic functions

6.1 Lemma. *Let $U \subset \mathbb{C}$ be an open subset and $K \subset U$ a compact subset. There exists a constant C such that every holomorphic function satisfies the inequality*

$$|f(a)|^2 \le C \int_U |f(z)|^2 dz \quad \textit{for } a \in K.$$

Here $dz = dxdy$ denotes the standard Lebesgue measure.

Proof. There exists $r > 0$ such that for each $a \in K$ the closed disk of radius r around a is contained in U. We consider the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

By means of

$$\int_{x^2+y^2 \le r^2} z^m \bar{z}^n = 0 \quad \text{for } m \ne n$$

we obtain

$$\int_U |f(z)|^2 dz \ge \int_{|z-a| \le r} |f(z)|^2 dz = \sum_{n=0}^{\infty} \int_{|z-a|^2 \le r^2} |a_n z^n| dz \ge \pi r^2 |f(a)|^2.$$

We can take $C = \pi^{-1} r^{-2}$. □

6.2 Proposition. *Let $U \subset \mathbb{C}$ be an open subset and $h : U \to \mathbb{R}$ an everywhere positive continuous function. Consider the measure $d\omega = h(z)dz$. Then*

$$L^2_{\text{hol}}(U, d\omega) = \{f \in L^2(U, d\omega), \quad f \textit{ holomorphic}\}$$

is a closed subspace of $L^2(U, d\omega)$ and hence a Hilbert space.

Proof. Let (f_n) be a sequence in $L^2_{\text{hol}}(U, d\omega$ that converges to f in the Hilbert space $L^2(U, d\omega)$. We have to show that f is holomorphic. This is true since Lemma 6.1 shows that the sequence converges locally uniformly. □

We denote by $\mathcal{H}$ the upper half plane in the complex plane. Recall that the group $G = \mathrm{SL}(2, \mathbb{R})$ acts on $\mathcal{H}$ through $(az+b)(cz+d)^{-1}$. The measure $dxdy/y^2$ is invariant under the action of G. We consider more generally for integers n the measures

$$d\omega_n = y^n \frac{dxdy}{y^2}.$$

Then we consider the space

$$H_n = L^2_{\text{hol}}(\mathcal{H}, d\omega_n)$$

of all *holomorphic* functions which are square integrable with respect to this measure. We know that this is a Hilbert space. We define an action π_n of $G = \mathrm{SL}(2, \mathbb{R})$ on function on H_n by means of the formula

$$(\pi_n(g)f)(z) = f(g^{-1}z)(cz+d)^{-n}.$$

This defines a unitary representation of G.

6.3 Proposition. *The Hilbert space $H_n = L^2_{\mathrm{hol}}(\mathcal{H}, d\omega_n)$ is different from* 0 *if $n \geq 2$. The formula*

$$(\pi_n(g)f)(z) = f(g^{-1}z)(cz+d)^{-n}$$

defines a unitary representation of $G = \mathrm{SL}(2,\mathbb{R})$ on H_n.

Proof. It remains to show that H_n is not zero if $n \geq 2$. For this transform the measure $d\omega_n$ to the unit disk $\mathcal{E}$ by means of the Caylay transformation $w = (z-\mathrm{i})(z+\mathrm{i})^{-1}$. Its inverse is $z = \mathrm{i}(1+w)(1-w)^{-1}$. The imaginary part y of z transforms as

$$y = \frac{1-|w|^2}{|1-w|^2}.$$

The formula

$$\frac{dz}{dw} = \frac{-2\mathrm{i}}{(1-w)^2}$$

shows that the Euclidian measure $dxdy$ transforms as

$$d\nu = \frac{4}{|1-w|^4}dudv$$

where $dudv$ is the Euclidean measure of $\mathcal{E}$. This means that H_n can be identified with the space of all holomorphic functions

$$f : \mathcal{E} \longrightarrow \mathbb{C}, \qquad \int_{\mathcal{E}} |f(w)|^2 \left(\frac{1-|w|^2}{|1-w|^2}\right)^{n-2} d\nu < \infty \qquad \square$$

This series is called the holomorphic discrete series. If one considers antiholomorphic instead of holomorphic functions one obtains the antiholomorphic discrete series. Both together are the so-called discrete series.

7. The space $S_{m,n}$

We consider the groups

$$G = \mathrm{SL}(2,\mathbb{R}) \quad \text{and} \quad K = \mathrm{SO}(2).$$

Making use of the Iwasawa decomposition, we can write any function $f : G \to \mathbb{R}$ as functions of the variables a, n, θ

$$f(g) = g(a, n, \theta).$$

Since g can be considered as a function on $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$, it makes sense to talk about differentiable g and in this way of differentiable f. We denote the subspace of differentiable functions of $\mathcal{C}_c(G)$ by $\mathcal{C}_c^\infty(G)$.

7.1 Definition. *The space $S_{m,n}$ consists of all $f \in \mathcal{C}_c^\infty(G)$ with the property*

$$f(k_\theta x k_{\theta'}) = f(x)e^{-\mathrm{i}m\theta}e^{-\mathrm{i}n\theta'} \qquad (x \in G).$$

Let be $f \in \mathcal{C}_c^\infty(G)$. Then the Fourier coefficient

$$f_{m,n}(x) = \int_0^{2\pi}\int_0^{2\pi} f(k_\theta x k_{\theta'})e^{-\mathrm{i}m\theta}e^{-\mathrm{i}n\theta'}d\theta d\theta'$$

is contained in $S_{m,n}$. From the theory of Fourier series we obtain

$$f(k_\theta x k_{\theta'}) = \sum_{m,n} f_{m,n}(x)e^{\mathrm{i}m\theta}e^{\mathrm{i}n\theta'}$$

where the convergence is absolute and locally uniform in x. We specialize $\theta = \theta'$ to obtain

$$f(x) = \sum_{m,n} f_{m,n}(x)$$

Let $\mathrm{supp}(f)$ be the support of f. Then $K\,\mathrm{supp}(f)K$ contains the support of $f_{m,n}$. We have proved the following result:

7.2 Lemma. *Let be $f \in \mathcal{C}_c^\infty(G)$ and let be $\varepsilon > 0$. There exists a function g which is a finite linear combination from functions contained in $S_{m,n}$ and with the following property:*

a) $\mathrm{supp}(g) \subset K\,\mathrm{supp}(f)K$,
b) $|f(x) - g(x)| < \varepsilon$ *for* $x \in G$.

Corollary. *The algebraic sum $\sum_{m,n} S_{m,n}$ is dense in the space $L^1(G, dx)$ with respect to the norm $\|\cdot\|_1$.*

Here dx of course is a Haar measure. Recall that G is a unimodular group, hence we have to define

$$f^*(x) = \overline{f(x^{-1})}.$$

We study the convolution.

7.3 Lemma. *We have*

a) $S_{m,n} * S_{p,q} = 0$ *if* $n \neq p$.
b) $S_{m,n}^* = S_{n,m}$.
c) $S_{m,n} * S_{n,q} \subset S_{m,q}$.

The proof can be given by an easy calculation. We restrict to the case a). In the convolution integral

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy$$

we replace y by yk_θ which doesn't change the integral. Now we use the transformation properties of f and g and obtain that $(f * g)(x)$ remains unchanged if one multiplies it by $e^{2\pi(p-n)\theta}$, This proves a). □

From Lemma 7.3 we see that $S_{n,n}$ is a star algebra.

7.4 Proposition. *The algebra $S_{n,n}$ is commutative.*

Proof. There is a very general principle behind this statement. It depends on the fact that $G = \mathrm{SL}(2, \mathbb{R})$ admits two involutions

$$x^\tau = x' \quad \text{(transpose of } x\text{)}$$
$$x^\sigma = \gamma x \gamma \quad \text{where} \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We collect the properties of the two involutions that are needed in the proof.

1) σ is an automorphism ($(xy)^\sigma = x^\sigma y^\sigma$) and τ is an anti-automorphism ($(xy)^\sigma = y^\sigma x^\sigma$)
2) $k^\tau = k^\sigma = k^{-1}$ for $k \in K$.
3) Every element of G can be written as product sk of a symmetric matrix ($s = s^\tau$) and an element $k \in K$.
4) For every symmetric $s = s^\tau$ there exist $k \in K$ such that

$$s^\sigma = ksk^{-1}.$$

1) and 2) are clear. To prove 4) we use that any real symmetric matrix s can be transformed by means of an orthogonal matrix into a diagonal matrix

$$k_1 s k_1^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Here λ_1, λ_2 are the eigen values of s. Since we can replace k_1 by γk_1 we can assume that the determinant of k_1 is 1. The matrix s^σ is also symmetric and has the same eigen values as s. Hence we find an orthogonal matrix k_2 of determinant 1 such that

$$k_2 s^\sigma k_2^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

We obtain $ksk^{-1} = k^\sigma$ where $k = k_2^{-1} k_1$. Finally we prove 3). So, let $x \in \mathrm{SL}(2, \mathbb{R})$. We consider xx'. This is a symmetric positive definite matrix. Transformation to a diagonal matrix by means of an orthogonal matrix gives a symmetric positive matrix s with the property $xx' = s^2$. Then $k = s^{-1}x$ is orthogonal and has the desired property. This finishes the proof of 1)-4).

We also mention that the Haar measure on G is invariant under the two involutions. We give the argument for the anti-automorphism σ. The integral $\int_G f(x^\sigma)dx$ is right invariant. Since G is unimodular it agrees with $\int_G f(x)dx$ up to a positive constant factor C. Since σ is involutive we get $C^2 = 1$ and hence $C = 1$.

Now we can give the proof of Proposition 7.4. We extend the involutions to functions on G by

$$f^\sigma(x) = f(x^\sigma), \quad f^\tau(x) = f(x^\tau).$$

We claim the following two formulae:

$$(f*g)^\sigma = f^\sigma * g^\sigma, \quad (f*g)^\tau = g^\tau * f^\tau.$$

We prove the second formula (the first one is similar). We have

$$(f*g)^\tau(x) = \int_G f(y)g(y^{-1}x^\tau)dy \quad \text{and} \quad (g^\tau * f^\tau)(x) = \int_G g(y^\tau)f((y^{-1})^\tau x)dy.$$

In the first integral we replace y by y^τ, then y by xy and after that y by y^{-1}. This transformations don't change the integrals and proves the claimed identity.

Now we assume that $f \in S_{m,m}$. In this case we claim $f^\tau = f^\sigma$. To prove this we write $x \in G$ in the form $x = sk$. Then we get

$$f^\tau(x) = f(k^\tau s) = \varrho(k)f(s) \qquad (\varrho(k) = e^{\mathrm{i}m\theta})$$

and

$$f^\sigma(x) = f(s^\sigma k^{-1}) = \varrho(k)f(s^\sigma) = \varrho(k)f(\gamma s\gamma^{-1}) = \varrho(k)f(s).$$

Now let f, g be both in $S_{m,m}$. Then $f*g$ is in $S_{m,m}$ too and we get $(f*g)^\tau = (f*g)^\sigma$. This gives

$$g^\tau * f^\tau = f^\sigma * g^\sigma.$$

Since $f, g \in S_{m,m}$ implies that $f^\sigma, g^\sigma \in S_{m,m}$ we can replace f, g by f^σ, g^σ to obtain the final formula $f*g = g*f$. □

Now we consider a Banach representation of $G = \mathrm{SL}(2,\mathbb{R})$,

$$\pi : G \longrightarrow \mathrm{GL}(H).$$

We assume that H is a Hilbert space. But it is not necessary to assume that it is unitary. We restrict this representation to K. Without loss of generality we can assume that the restriction to K is unitary (use Proposition I.8.8). We consider the (closed) subspace

$$H(m) := \{h \in H; \quad \pi(k_\theta)(h) = e^{\mathrm{i}m\theta}h\}.$$

The spaces $H(n)$ are pairwise orthogonal and that H is the direct Hilbert sum of the $H(n)$. For an element h in the algebraic sum, we denote by h_n the component in $H(n)$.

7.5 Lemma. *The space $S_{m,n}$ maps $H(n)$ into $H(m)$. It maps $H(n)$ to zero if $n \neq m$.*

The proof is very easy and can be omitted. □

7.6 Proposition. *Assume that $\pi : G \to \mathrm{GL}(H)$ is an irreducible representation on a Hilbert space. The algebra $S_{m,m}$ acts topologically irreducibly on $H(m)$ if this space is not zero.*

Proof. Let $h \in H(m)$ be a non-zero element. We want to show that $S_{m,m}h$ is dense in $H(m)$. (This means that $S_{m,m}$ acts topologically irreducible on $H(m)$.) We consider the space $\mathcal{A}h$. We know that $\mathcal{A}h$ is a dense subspace of H. It is contained in the algebraic sum $\sum H(n)$. We consider the projection of $\sum H(n)$ to $H(m)$. The image of $\mathcal{A}h$ under this projection is dense in $H(m)$. Lemma 7.5 shows that this image equals $S_{m,m}h$. This shows that $S_{m,m}$ acts topologically irreducible on $H(m)$. $\square$

Since $S_{m,m}$ is abelian, we now obtain the following theorem.

7.7 Theorem. *Let $\pi : G \to \mathrm{GL}(H)$ be an irreducible representation on a Hilbert space. We assume that the restriction to K is unitary. Then H is the direct Hilbert sum of the spaces $H(n)$. Assume that $H(n)$ is finite dimensional. Then $\dim H(n) \le 1$. This is always the case if π is unitary.*

We just mention that this a special case of a more general result that holds for any semi simple Lie group G and a maximal compact subgroup. Examples are $G = \mathrm{SL}(n,\mathbb{R})$, $K = \mathrm{SO}(n,\mathbb{R})$. For every irreducible unitary representation of G the K-isotypic components are finite dimensional. In other words: each irreducible unitary representation of K (which is always finite dimensional) occurs with finite multiplicity in $\pi|K$. The proof more involved, mainly since K is not commutative in general.

A vector $h \in H$ is called K-finite, if the space generated by all $\pi(k)h$ is finite dimensional. The space of K-finite vectors is denoted by H_K. The elements of $H(m,n)$ are K-finite. Since every finite dimensional representation of a compact group is completely reducible, we obtain the following description.

7.8 Lemma. *Let π be a Banach representation of G on a Hilbert space H such that the restriction to K is unitary. Then*

$$H_K = \sum_{m\in\mathbb{Z}} H(m) \qquad \textit{(algebraic sum).}$$

It is important to describe for a given irreducible unitary representation π the set of all n sich such that $H(n)$ is different from zero (and then one-dimensional). For this we look for operators that shift $H(n)$ which means that $H(n)$ is mapped into another $H(m)$. We will find such operators in the Lie algebra. Finally we mention another result which is important in this connection.

7.9 Lemma. *The group $G = \mathrm{SL}(2,\mathbb{R})$ is generated by any neighborhood of the unit element.*

The following proof works for every connected group. (That $\mathrm{SL}(2,\mathbb{R})$ is connected follows from the Iwasawa decomposition). Let U be a an open neighborhood of the identity. By continuity the set of all a such that a and a^{-1} is contained in U is also an open neighborhood. Hence we can assume that $a \in U$ implies $a^{-1} \in U$. We consider the image $U(n)$ of

$$U^n \longrightarrow G, \quad (a_1, \dots, a_n) \longmapsto a_1 \cdots a_n.$$

The union G_0 of all $U(n)$ is an open subgroup of G. Since G is the disjoint union of cosets of G_0, the complement of G_0 in G is also open. Hence G_0 is open and closed in G and hence $G_0 = G$ since G is connected. □

8. The derived representation

Differential calculus usually is defined for maps $U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is an open subset. There is a straight forward generalization where $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$ are replaced by Banach spaces, where in this context they are understood as Banach spaces over the field of real numbers. It is clear what this means. A map $f : U \to F$ in this context is called differentiable at $a \in U$ if there exists a continuous (real) linear map $L_a : E \to F$ such that

$$f(x) - f(a) = L_a(x - a) + r(x) \quad \text{where} \quad \lim_{x \to a} \frac{\|r(x)\|}{\|x - a\|} = 0.$$

If this is true for every $a \in U$ we call f differentiable. Then we can consider the derivative

$$Df : U \le B(E, F), \quad df(a) = L_a.$$

Since the subspace of bounded operators of $\mathrm{Hom}(E, F)$ is a Banach space too, we can ask for differentiability of df. In this way one can define the space of infinite differentiable functions $\mathcal{C}^\infty(U, F)$. As in the finite dimensional case, the chain rule holds for (infinitely often) differentiable functions. We also mention that a continuous linear map is differentiable for trivial reasons.

We want apply this to functions $G \to H$ where H is a Banach space (as usual over the complex numbers). Assume that $\pi : G \longrightarrow \mathrm{GL}(H)$ be a continuous representation. We associate to an arbitrary vector $h \in H$ a function

$$G \longrightarrow H, \quad x \longmapsto \pi(x)h.$$

We call the vector h differentiable if this function is infinitely often differentiable. We denote the space of differentiable vectors by H^∞. These is a sub-vector space. It depends of course on π. Hence, for example, H_π^∞ is a more careful notation.

We give examples of a differentiable vector.

8.1 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation and $f \in \mathcal{C}_c^\infty(G)$. Then the image of $\pi(f)$ is contained in H^∞. As a consequence, the space H^∞ is a dense subspace of H.*

Corollary. *Assume that H is a Hilbert space and that the restriction of π to K is unitary. Let m be an integer such that $\dim H(m) < \infty$. Then the elements of $H(m)$ are differentiable. (This applies if π is an irreducible unitary representation.)*

Proof. The first part follows from the formula

$$\pi(x)\pi(f)v = \int_G f(y)\pi(x)\pi(y)v\,dy = \int_G f(x^{-1}y)\pi(y)dy$$

by means of the Leibniz rule that allows to interchange integration and differentiation. (Of course we need a Banach valued version of the rule. We omit a proof of this, since it can be done as in the usual case.)

To prove the corollary we observe that $\pi(S_{m,m})H(m)$ is dense in $H(m)$ by Lemma 7.5. In the case that $H(m)$ is finite dimensional it is the whole of $H(m)$. Now we can apply the first part of the proof. □

Let $X \in \mathfrak{g}$ and $h \in H^\infty$. The map

$$\mathbb{R} \longrightarrow H, \quad t \longmapsto \pi(\exp(tX))h$$

is differentiable, since it is the composition of two differentiable maps. Hence we can define the operator $d\pi(X) : H^\infty \to H$:

$$d\pi(X)h := \frac{d}{dt}\pi(\exp(tX)h)\Big|_{t=0}.$$

This is related to a another construction, the Lie derivative (from the left). This is for each $X \in \mathfrak{g}$ a map

$$\mathcal{L}_X : \mathcal{C}^\infty(G,H) \longrightarrow \mathcal{C}^\infty(G,H)$$

which is defined by

$$\mathcal{L}_X f(a) = \frac{d}{dt} f(a\exp(tX))\Big|_{t=0}.$$

(It is easy to show that $\mathcal{L}_X f$ is differentiable.) The Lie derivative has nothing to do with the representation π. But we get a link to the derived representation if we apply it to functions of the type $x \mapsto \pi(x)h$.

8.2 Lemma. *Let $X \in \mathfrak{g}$ and $h \in H^\infty$. We consider the differentiable function $f(x) = \pi(x)h$ on G. Then the formula*

$$\pi(a)d\pi(X)h = (\mathcal{L}_X f)(a)$$

holds, in particular

$$d\pi(X)h = (\mathcal{L}_X f)(e) \in H_\pi^\infty.$$

Proof. The second formula is just true by definition. The first one can be obtained if one applies $\pi(a)$ to the second one. One just has to observe that $\pi(a)$ commutes by continuity with the limit

$$\lim_{t\to 0} \frac{\pi(\exp(tX))h - h}{t}. \qquad \square$$

The Lie derivatives satisfy a basic commutation rule.

8.3 Proposition. *For $X, Y \in \mathfrak{g}$ the formula*

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X \qquad ([X,Y] = XY - YX)$$

holds.

Proof. The formula states

$$\begin{aligned}&\frac{d}{dt} f(\exp(t[X,Y]))\big|_{t=0} = \\ &\frac{d}{dt}\frac{d}{ds}\big(f(\exp(tX)\exp(sY)) - f(\exp(tY)\exp(sX))\big)\big|_{t=s=0}.\end{aligned}$$

Here f is a $\mathcal{C}^\infty$ function on some open neighborhood of the unit element of $G = \mathrm{SL}(2,\mathbb{R})$. It is easy to show that f is the restriction of a $\mathcal{C}^\infty$-function on some open neighborhood of the unit element of $\mathrm{GL}(2,\mathbb{R})$ (which can be considered as an open subset of $\mathbb{R}^4$). Hence it is sufficient to prove the formula for $G = \mathrm{GL}(2,\mathbb{R})$ and $\mathfrak{g}$ can be replaces we the space of all real 2×2-matrices. Using Taylor's formula one can reduce the proof to the case where f is a polynomial. The product rule shows that the formula is true for fg if it is true for f and g. Hence it is sufficient to prove it for linear functions. So we reduced the statement to the formula

$$\begin{aligned}&\frac{d}{dt} \exp(t[X,Y])\big|_{t=0} = \\ &\frac{d}{dt}\frac{d}{ds}\big(\exp(tX)\exp(sY)) - \exp(tY)\exp(sX)\big)\big|_{t=s=0}.\end{aligned}$$

This is equivalent to the formula $[X,Y] = XY - YX$. $\square$

As a consequence of the commutation rule of the Lie derivative we obtain the following rule for the derived representation.

8.4 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation. Then the following rule*

$$d\pi([X,Y]) = d\pi(X) \circ d\pi(Y) - d\pi(Y) \circ d\pi(X)$$

holds.

Propositions 8.3 and 8.4 provide special cases of the following definition.

8.5 Definition. *Let $\mathcal{A}$ be a an associative algebra (over the field of real numbers is enough). A map $\varphi : \mathfrak{g} \to \mathcal{A}$, $A \longmapsto \mathbf{A}$ is called a Lie homomorphism if it is $\mathbb{R}$-linear and if*

$$\varphi([A,B]) = \varphi(A)\varphi(B) - \varphi(B)\varphi(A)$$

holds.

Hence Proposition 8.3 provides a Lie homomorphism

$$\mathfrak{g} \longmapsto \mathrm{End}(\mathcal{C}^\infty(G,H))$$

and Proposition 8.4 a Lie homomorphism

$$\mathfrak{g} \longmapsto \mathrm{End}(H^\infty).$$

In both cases the algebra on the right-hand side is a *complex* algebra (since H is a complex vector space and since we understand by End complex linear endomorphisms. In such a case we can extend φ to the complexification $\mathfrak{g}_\mathbb{C}$ by means of the formula

$$\mathfrak{g}_\mathbb{C} \longrightarrow \mathcal{A}, \quad \varphi(A) = \varphi(\mathrm{Re}(A)) + \mathrm{i}\varphi(\mathrm{Im}(A)).$$

It is easy to check that the formula

$$\varphi([A,B]) = \varphi(A)\varphi(B) - \varphi(B)\varphi(A)$$

remains true where the bracket in $\mathfrak{g}_\mathbb{C}$ is of course defined by the formula $[A,B] = AB - BA$.

9. Explicit formulae for the Lie derivatives

In the following we use the elements

$$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{g}$. They give a basis of $\mathfrak{g}$. We also need

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We also will consider the complexification $\mathfrak{g}_{\mathbb{C}}$. Here we use the (complex) basis

$$W, \quad E^- = H - \mathrm{i}V, \quad E^+ = H + \mathrm{i}V.$$

So we have

$$E^- = \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix}.$$

Recall that the Lie derivatives to $\mathfrak{g}_{\mathbb{C}}$ can be extended by $\mathbb{C}$-linearity:

$$\mathcal{L}_{A+\mathrm{i}B} = \mathcal{L}_A + \mathrm{i}\mathcal{L}_B,$$

since H and hence $\mathcal{C}^\infty(G, H)$ is a complex vector space.

From the Iwasawa decomposition we know that we can write $g \in G$ in the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sqrt{y} & \sqrt{y}^{-1}\, x \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with unique x and $y > 0$. The angle θ is determined mod 2π. We need the expressions for x, y, θ in terms of a, b, c, d. To get them it is useful to use complex numbers. Let τ be a complex number in the upper half plane, $\operatorname{Im}\tau > 0$. Since c, d are real but not both zero, the number $c\tau + d$ is different from zero. Hence we can define

$$g(\tau) = \frac{a\tau + b}{c\tau + d}.$$

Let h be a second matrix from G. A direct computation which we omit shows

$$(gh)(\tau) = g(h(\tau)).$$

We also notice

$$k_\theta(\mathrm{i}) = \mathrm{i}.$$

Hence we obtain

$$\frac{a\mathrm{i} + b}{c\mathrm{i} + d} = x + \mathrm{i}y.$$

This gives us x and y in terms of a, b, c, d.

$$y = \frac{1}{c^2 + d^2}, \quad x = \frac{ac + bd}{c^2 + d^2}.$$

Looking at the second row of the Iwasawa decomposition we get

$$c\sqrt{y} = -\sin\theta, \quad d\sqrt{y} = \cos\theta.$$

This shows

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta = \frac{d - \mathrm{i}c}{\sqrt{c^2 + d^2}}.$$

This gives as

$$\theta = \operatorname{Arg}\frac{d - \mathrm{i}c}{\sqrt{c^2 + d^2}}.$$

Since θ is only determined mod 2π, we have to say a word about the choice of the argument Arg. All what we need is that for a given $g_0 \in G$ one can make the choice of Arg such it depends differentiably on g for all g in a small open neighborhood of g_0.

In the following we will fix $g \in G$ and $X \in \mathfrak{g}$ and consider

$$g(t) = g\exp(tX)$$

for small t. We write $x(t), y(t), \theta(t)$ in this case. As we mentioned the function $\theta(t)$ can be chosen for small t such that it depends differentially on t. If we insert $t = 0$ we get the original x, y, θ.

For the Lie derivative we have to consider a differentiable function f on G. We can write it as function f of three variables. We get

$$f(g(t)) = F(x(t), y(t), \theta(t)).$$

By means of the chain rule we get

$$\frac{d}{dt}f(g(t)) = \frac{\partial F}{\partial x}\dot{x}(t) + \frac{\partial F}{\partial y}\dot{y}(t) + \frac{\partial F}{\partial \theta}\dot{\theta}(t).$$

Recall that we have to evaluate this expression at $t = 0$ to get the Lie derivative.

As an example we take

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e^{tX} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$g(t) = \begin{pmatrix} a & b + ta \\ c & d + tc \end{pmatrix}.$$

We obtain
$$z(t) = \frac{a\mathrm{i} + at + b}{c\mathrm{i} + ct + d}.$$
Differentiation and evaluating at $t = 0$ gives
$$\dot{z}(0) = \frac{1}{(c\mathrm{i} + d)^2}.$$
Using the formulae
$$y = \frac{1}{c^2 + d^2}, \quad e^{2\mathrm{i}\theta} = \frac{(d - \mathrm{i}c)^2}{c^2 + d^2}$$
we obtain
$$\dot{z}(0) = ye^{2\mathrm{i}\theta} \quad \text{or} \quad \dot{x}(0) = y\cos 2\theta,\ \dot{y}(0) = y\sin 2\theta.$$
Finally, to compute $\dot{\theta}(0)$, we use the formula
$$\cos\theta(t) = (d + ct)\sqrt{y(t)}.$$
Differentiation gives
$$-\dot{\theta}(t)\sin\theta(t) = (d + ct)\frac{\dot{y}(t)}{2\sqrt{y(t)}} + c\sqrt{y(t)}.$$
Evaluating by $t = 0$ we get
$$\dot{\theta}(0)\sin\theta = \frac{d\dot{y}(0)}{2\sqrt{y}} - c\sqrt{y}.$$
We insert $-c\sqrt{y} = \sin\theta$ and $\dot{y}(0) = y\sin 2\theta = 2y\sin\theta\cos\theta$ to obtain
$$\dot{\theta}(0) = -d\sqrt{y}\cos\theta + 1 = -\cos^2\theta + 1 = \sin^2\theta.$$
Another – even easier example – is $\mathcal{L}_W$. A simple computation gives
$$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \exp tW = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$
Hence we obtain that $\mathcal{L}_W$ is given by the operator $\partial/\partial\theta$. In a similar way other elements of the Lie algebra can be treated. Since $V = 2X - W$ we get $\mathcal{L}_V$. We omit the computation for
$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^{tH} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$
and just collect the formulae together.

9.1 Proposition. *Let $f \in \mathcal{C}^\infty$ and $A \in \mathfrak{g}$. We denote by $F(x,y,\theta)$ the corresponding function in the coordinates and similarly $G(x,y,\theta)$ for $g = \mathcal{L}_A f$. The operator $F \mapsto G$ can be described explicitly as follows:*

$$\begin{aligned}
\mathcal{L}_X &= y\cos 2\theta\frac{\partial}{\partial x} + y\sin 2\theta\frac{\partial}{\partial y} + \sin^2\theta\frac{\partial}{\partial\theta},\\
\mathcal{L}_W &= \frac{\partial}{\partial\theta},\\
\mathcal{L}_V &= 2y\cos 2\theta\frac{\partial}{\partial x} + 2y\sin 2\theta\frac{\partial}{\partial y} - \cos 2\theta\frac{\partial}{\partial\theta},\\
\mathcal{L}_H &= -2y\sin 2\theta\frac{\partial}{\partial x} + 2y\cos 2\theta\frac{\partial}{\partial y} + \sin 2\theta\frac{\partial}{\partial\theta},
\end{aligned}$$

and, as a consequence,

$$\mathcal{L}_{E^-} = -2\mathrm{i}ye^{-2\mathrm{i}\theta}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big) + \mathrm{i}e^{-2\mathrm{i}\theta}\frac{\partial}{\partial\theta}.$$

10. Analytic vectors

Let E, F be Banach spaces over the field of real numbers and let $U \subset E$ be an open subset. We introduced the notion of a differentiable map $U \to F$ In the case that E is finite dimensional (but F may be not) we can also define the notion of an analytic map. In the case $E = \mathbb{R}^n$ this means as usual that for each $a \in U$ there exists a small neighborhood in which there exists an absolutely convergent expansion as power series

$$f(x) = \sum_{\nu\in\mathbb{N}_0^n} a_\nu(x_1 - a_1)^{\nu_1}\cdots(x_n - a_n)^{\nu_n} \qquad (a_\nu \in F).$$

This notion is invariant under linear transformation of the coordinates, hence it carries over to arbitrary E. We denote by $\mathcal{C}^\omega(U,F)$ the space of all analytic functions. This is a subspace of $\mathcal{C}^\infty(U,F)$. The basic property of analytic functions is the principle of analytic continuation. Assume that U is connected and that $a \in U$ a point that all derivatives of f or arbitrary order vanish (this is understood to include $f(a) = 0$). Then f is identically zero.

Using the standard coordinates of G, we can define the notion of analytic function $G \to H$ into any Banach space. If $\pi : G \to \mathrm{GL}(H)$ is a representation we can define the notion of an analytic vector $h \in H$. By definition this means that the function $\pi(x)h$ on G is analytic. The set H^ω of all analytic vectors is a sub-vector space of H^∞.

We recall the formula for the Lie derivative

$$(\mathcal{L}_X f)(y) = \frac{d}{dt} f(y\exp(tX))\Big|_{t=0}.$$

We replace y by $y\exp(uX)$ and obtain

$$(\mathcal{L}_X f)(y\exp(uX)) = \frac{d}{dt} f(y\exp((u+t)X))\Big|_{t=0} = \frac{d}{du} f(y\exp(uX)).$$

By induction follows

$$(\mathcal{L}_X^n f)(y\exp(uX)) = \frac{d^n}{du^n} f(y\exp(uX)).$$

The Taylor expansion of the function $t \mapsto f(y\exp(tX))$ is given by

$$\begin{aligned} f(y\exp(tX)) &= \sum_{n=0}^{\infty} \frac{1}{n!} \, \frac{d^n}{dt^n} f(y\exp(tX))\Big|_{t=0} \, t^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \, (\mathcal{L}_X^n f)(y) \, t^n. \end{aligned}$$

This formula is true for given X, y if $|t|$ is sufficiently small, $|t| < \varepsilon$. For a real constant the formula $\mathcal{L}_{cX} = c\mathcal{L}_X$ can be checked. This shows that (for fixed y) the Taylor formula holds if X is in a sufficiently small neighborhood of the origin. We specialize the Taylor expansion to the function $f(x) = \pi(x)h$ and to $t = 1$.

10.1 Proposition. *Let $h \in H$ be an analytic vector. For sufficiently small X the formula*

$$\pi(\exp(X))h = \sum_{n=0}^{\infty} \frac{1}{n!} d\pi(X)^n h$$

holds.

Making use of Lemma I.2.1 and Lemma 7.9 we now obtain the following important result.

10.2 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be a Banach representation and let $V \subset H$ be a linear subspace consisting of analytic vectors that is invariant under $d\pi(\mathfrak{g})$. Then the closure of V is invariant under G.*

In the next section we will prove the existence of analytic vectors.

11. The Casimir operator

In the following we will make use of the basic commutation rules in $\mathfrak{g}$:

$$[E^+, E^-] = -4\mathrm{i}W, \quad [W, E^+] = 2\mathrm{i}E^+, \quad [W, E^-] = -2\mathrm{i}E^-.$$

They can be verified by direct computation.

Let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra and

$$\varrho : \mathfrak{g} \longrightarrow \mathcal{A}$$

be a Lie homomorphism, i.e. a linear map with the property

$$\varrho([A, B]) = \varrho(A)\varrho(B) - \varrho(B)\varrho(A).$$

We also can consider its $\mathbb{C}$-linear extension $\mathfrak{g}_{\mathbb{C}} \to \mathcal{A}$. Our typical example is that $\mathcal{A}$ is the algebra of (algebraic) endomorphisms of an abstract (complex) vector space $\mathcal{H}$. In this case we talk about a Lie algebra representation of $\mathfrak{g}$ on $\mathcal{H}$. We denote the image of element $A \in \mathfrak{g}_{\mathbb{C}}$ by the corresponding bold letter $\boldsymbol{A}$. We define the *Casimir element* by

$$\omega = \boldsymbol{H}^2 + \boldsymbol{V}^2 - \boldsymbol{W}^2.$$

Using the above commutation rules we can check by a simple computation

$$\omega = \boldsymbol{H}^2 + \boldsymbol{V}^2 - \boldsymbol{W}^2 = \boldsymbol{E}^+\boldsymbol{E}^- + 2\mathrm{i}\,\boldsymbol{W} - \boldsymbol{W}^2.$$

The basic property of the Casimir element is that it commutes with the image of $\mathfrak{g}_{\mathbb{C}}$.

11.1 Lemma. *The Casimir element ω (with respect to $\varrho : \mathfrak{g} \to \mathcal{A}$) commutes with all $\boldsymbol{A}$ for $A \in \mathfrak{g}$.*

Proof. One uses the second formula for the Casimir operator and applies the above commutation rules. □

The Lie algebra $\mathfrak{g}$ acts on the space $\mathcal{C}^\infty(G, H)$. Hence we can consider the Casimir operator ω acting on this space. Using the formulae on Proposition 8.2 we get the explicit expression

$$\omega = 4y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - 4y\frac{\partial^2}{\partial x \partial \theta}.$$

We are especially interested on its action of functions of the type

$$f(xk) = e^{\mathrm{i}n\theta} f(x).$$

Since $\partial f/\partial\theta = \mathrm{i}nf$ we see that the operator

$$4y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)-4y\mathrm{i}n\frac{\partial}{\partial x}$$

has the same effect on f as the Casimir operator. The advantage of the latter operator is that it is – like the Laplace operator – an *elliptic differential operator.* We make use of a basis result that $\mathcal{C}^\infty$-eigen functions $Df=\lambda f$ of an elliptic differential operator D are analytic functions. Usually this theorem is formulated for scalar valued function. But it is also true for Banach valued functions. For this one can use for example the following general result.

11.2 Proposition. *A function $f:G\to H$ is analytic if an only if $L\circ f$ is analytic for every continuous linear function L.*

We do not give a proof. □

Let now $\pi:G\to\mathrm{End}(H)$ be a Banach representation and let $h\in H$ by a differentiable vector. Then there is the Casimir operator acting on H^∞. Assume that h is an eigen vector and that $h\in H(m)$. We claim that h is an analytic vector. We have to show that the function $f_h(x)=\pi(x)h$ is analytic. The condition $h\in H(m)$ implies

$$f_h(xk)=e^{\mathrm{i}n\theta}f_h(x).$$

For any $A\in\mathfrak{g}$ we have

$$\mathcal{L}_Af_h=f_{d\pi(A)h}.$$

This carries over to the Casimir operator. So we can write

$$\omega f_h=f_{\omega h}.$$

By assumption h is an eigen vector of the Casimir operator. This implies that f_h is an eigen function. So we get that f_h is analytic. By definition this means that h is analytic. This gives the following result.

11.3 Proposition. *Let $\pi:G\to\mathrm{GL}(H)$ be a Banach representation and let $h\in H(m)$ be a differentiable vector which is an eigen vector of the Casimir operator. Then h is analytic.*

This gives us the possibility to identify many analytic vectors. For this we have to study the action of the generators of $\mathfrak{g}$ on the spaces $H(m)$ in more detail.

11.4 Lemma. *Let $\pi:G\to\mathrm{GL}(H)$ be a Banach representation on a Hilbert space H. We assume that the restriction to K is unitary. We also assume that the elements of $H(m)$ are differentiable. Then $d\pi(W)$ acts on $H(m)$ by multiplication by $\mathrm{i}m$. The operator $d\pi(E^+)$ maps $H(m)$ to $H(m+2)$ and $d\pi(E^-)$ maps $H(m)$ to $H(m-2)$.*

Proof. Recall that the space H is the direct Hilbert sum of the K-isotypic components, which are $H(m)$. A direct computation gives

$$\exp(tW) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

This gives

$$\pi(\exp(tW))h = e^{\mathrm{i}mt}h \quad \text{for} \quad h \in H(m).$$

If we differentiate by t and evaluate than by $t = 0$ we get the desired result for the action of W.

To get the statement for the action of E^+ we use the rule

$$d\pi(W)d\pi(E^+) = d\pi[W, E^+] + d\pi(E^+)d\pi(W)$$

and the commutation rule $[W, E^+] = 2\mathrm{i}E^+$. For a vector $a \in H(m)$ we get

$$d\pi(W)d\pi(E^+)h = \mathrm{i}(m + 2)d\pi(E^+)h.$$

Hence $d\pi(E^+)h$ is an eigen value of $d\pi(W)$ with eigenvalue $\mathrm{i}(m + 2)$. Hence it must lie in $H(m + 2)$. The argument for E^- is similar. □

From the second formula for the Casimir we see that $H(m)$ is mapped into itself. This gives the following basic result.

11.5 Theorem. *Let $\pi : G \to \mathrm{GL}(H)$ be an unitary representation such that all $H(m)$ have dimension ≤ 1. (This is the case if π is irreducible). Then the vectors from $H_K = \sum H(m)$ (algebraic sum) are analytic and this space is invariant under $\mathfrak{g}$.*

Proof. Since the spaces $H(m)$ have dimension ≤ 1 they consist of differentiable vectors. The elements of $H(m)$ are eigen elements of the Casimir operator. □

By a representation of the Lie algebra $\mathfrak{g}$ on the abstract vector space E we understand a Lie homomorphism map $\pi : \mathfrak{g} \to \mathrm{End}(E)$, i.e. a linear map with the property

$$\pi([X, Y]) = \pi(X) \circ \pi(X) - \pi(Y) \circ \pi(X).$$

For a unitary representation $\pi : G \to \mathrm{GL}(H)$ we can consider the derived representation

$$d\pi : \mathfrak{g} \longrightarrow \mathrm{End}(H_K).$$

11.6 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$, be a unitary representation such that all $H(m)$ have dimension ≤ 1. Then π is irreducible if and only if the derived representation*

$$d\pi : \mathfrak{g} \longrightarrow \mathrm{End}(H_K)$$

has the following property. Let $\mathcal{A}$ be the algebra of operators that is generated by the image of $\mathfrak{g}$ and by the identity. For each non-zero h which is contained in some $H(m)$ we have $\mathcal{A}(h) = H_K$.

Proof. We notice that the set $\mathcal{A}(h) = \{A(h),\ A \in \mathcal{A}\}$ is a vector space. This vector space can also be described as follows. Let $X = X_1 \cdots X_n$ be an operator such that each X_i is one of the $d\pi(E^+)$, $d\pi(E^-)$, $d(W)$. Then $\mathcal{A}(h)$ is the vector space generated by all $X(h)$.

For a X as above the space $X(\mathbb{C}h)$ is either 0 or it is one of the $H(n)$. Hence we see that $\mathcal{A}(h)$ is generated by certain spaces $H(n)$.

Now we assume that π is irreducible. We prove that $\mathcal{A}(h)$ is the full H_K. We argue indirectly. So we can assume that there exist an $H(n) \neq 0$ which is not contained in $\mathcal{A}(h)$. We recall that the spaces $H(k)$ are pairwise orthogonal. Hence the preceding remark shows that $H(n)$ is orthogonal to $\mathcal{A}(h)$. But then $H(n)$ is orthogonal to the closure of $\mathcal{A}(h)$. We know that this space is invariant under G. But this is not possible since we assumed that π is not the trivial one dimensional representation.

Assume now that $\mathcal{A}(h) = H_K$ for all nonzero $h \in H(m)$. We claim that π is irreducible. Again we argue indirectly. We find a proper closed invariant subspace H'. We can take a non-zero isotypic component $H'(m)$. Consider a non-zero element $h \in H'(m)$. By assumption then $\mathcal{A}(h)$ is H_K. This shows $H_K \subset H'$ and hence $H = H'$. □

Since the Casimir operator commutes with all elements of $\mathfrak{g}$ we obtain the following kind of a Schur lemma.

11.7 Proposition. *Let $\pi : G \to \mathrm{GL}(H)$ be an irreducible unitary representation. Then the Casimir operator acts on H_K by multiplication by some constant.*

This constant is a basic invariant of π.

12. Admissible representations

Let $\pi : G \to \mathrm{U}(H)$ be a unitary representation of $G = \mathrm{SL}(2, \mathbb{R})$. Then we can consider the space of K-finite vectors $\mathcal{H} = H_K$. They consist of differentiable vectors. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ acts on them The action of K and $\mathfrak{g}$ are tied together.

12.1 Remark. *Let $\pi : G \to H$ be a unitary representations of G. Then on H_K the following formula holds,*

$$\pi(k) \circ d\pi(X) = \pi(kXk^{-1}) \circ \pi(k), \quad X \in \mathfrak{g},\ k \in K.$$

Corollary. *This formula shows that $\mathfrak{g}$ acts on H_K.*

The action of $\mathfrak{g}$ extends $\mathbb{C}$-linearly to the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$. We are led to consider the following algebraic objects.

12.2 Definition. *A $\mathfrak{g}$-K-module $\mathcal{H}$ is a complex vector space $\mathcal{H}$ together with a homomorphism*

$$\pi : K \longrightarrow \mathrm{GL}(\mathcal{H})$$

and a Lie homomorphism

$$d\pi : \mathfrak{g} \longrightarrow \mathrm{End}(\mathcal{H})$$

such the compatibility relation

$$(d\pi)(A)h = \frac{d}{dt}\pi(e^{tA})h\Big|_{t=0}$$

for $A \in \mathfrak{k}$ is valid. Furthermore, let $H(m)$ be the eigenspace

$$H(m) = \{h \in H; \quad \pi(k_\theta) = e^{\mathrm{i}m\theta}h\},$$

then $\mathcal{H}$ is the algebraic direct sum of all $H(m)$.

So each unitary irreducible representation induces a $\mathfrak{g}$-K- module. Even more, it is admissible in the following sense.

12.3 Definition. *A $\mathfrak{g}$-K-module $\mathcal{H}$ is called admissible if the eigenspaces $H(m)$ are finite dimensional.*

12.4 Definition. *An admissible $\mathfrak{g}$-K-module $\mathcal{H}$ is called irreducible, if it is not the zero representation and if the following condition is satisfied. Let $\mathcal{A}$ be the $\mathbb{C}$-algebra of operators that is generated by the image of $\mathfrak{g}$. For each non-zero h which is contained in some $H(m)$, we have $\mathcal{A}(h) = \mathcal{H}$.*

It is clear what an isomorphism of admissible representation means. We emphasize that this is understood in a pure algebraic way. As we have seen, every irreducible unitary representation $G \to \mathrm{GL}(H)$ has an underlying irreducible admissible $\mathfrak{g}$-K-module. We also recall that the Lie homomorphism can be extended $\mathbb{C}$-linearly to $\mathfrak{g}_\mathbb{C}$.

We study in detail admissible representations. For this we will use the basis E^+, E^+, W for $\mathfrak{g}_\mathbb{C}$. We recall

$$[E^+, E^-] = -4\mathrm{i}W, \quad [W, E^+] = 2\mathrm{i}E^+, \quad [W, E^-] = -2\mathrm{i}E^-.$$

We mention that W acts by multiplication with in on $H(n)$. The same argument as in Lemma 11.4 shows

$$E^+(H(n)) \subset H(n+2), \quad E^-(H(n)) \subset H(n-2).$$

We notice that the operators E^+E^- act on $H(n)$. We consider the $\mathbb{C}$-algebra generated by all operators from $\mathrm{End}(H(n))$ generated by E^+E^- and by multiplications with scalars. It is rather clear that this acts irreducible on $H(n)$. These operators commute with the Casimir operator. Hence the Casimir operator $\omega = E^+E^- + 2\mathrm{i}W - W^2$ acts on $H(n)$ by multiplication with a scalar. It follows that E^+E^- acts as scalar. This shows the following result.

12.5 Theorem. *Let $\mathcal{H}$ be an irreducible admissible $\mathfrak{g}$-K-module. Then* $\dim H(n) \le 1$.

Choose some $h \in H(n)$, $h \ne 0$ for suitable n. We know that the Casimir operator ω acts as scalar on h. Then also E^+h, E^-h, Wh are eigenvectors of ω with the same eigenvalue. The irreducibility shows that ω acts with the same scalar on the whole $\mathcal{H}$.

12.6 Theorem. *Let $\mathcal{H}$ be an irreducible admissible $\mathfrak{g}$-K-module. Then the Casimir operator acts by multiplication with a constant on $\mathcal{H}$.*

Now we study in detail the irreducible unitary representations. We see that the spaces

$$H^{\text{even}} = \sum_{n \text{ even}} H(n), \quad H^{\text{odd}} = \sum_{n \text{ odd}} H(n)$$

are invariant subspaces. Hence we have to distinguish between an even case (all $H(2n+1)$ are zero) and an odd case (all $H(2n)$ are zero).

Let S be a set of all integers which are all odd or all zero. We call S an interval if for $m, n \in S$ each number of the same parity between m and n is contained in S. We claim now that the set S of all n such that $H(n) \ne 0$ is an interval. To prove this we consider an $n \in S$ such that $H(n)$ is different from zero. Recall that $H(n)$ is one-dimensional. We choose a generator h. The space H is generated by all $A_1 \dots A_m h$ where $A_i \in \mathfrak{g}_{\mathbb{C}}$. From the relations between the generators we see that H is generated by $E_+^m h$ and $E_-^m h$. Let for example $H(n+2k) = 0$, $k > 0$. Then $E_+^{n+2k} h = 0$ and hence all $H(m)$, $m > n+2k$, are zero. Hence S is an interval.

12.7 Proposition. *For the set S of integers m with the property $H(m) \ne 0$ of an admissible representation there are the following possibilities:*

1) *S is the set of all even integers.*
2) *S is the set of all odd integers.*
3) *There exists $m \in S$ such that S consists of all $x \ge m$ with the same parity.*
4) *There exists $n \in S$ such that S consists of all $x \le n$ with the same parity.*
5) *There exist integers $m \le n$ of the same parity such that S consists of all $x \in \mathbb{Z}$, $m \le x \le n$, of the same parity.*

In the cases 3)-5) we call m the lowest weight and the non-zero elements of $H(m)$ the lowest weight vectors. Similarly we call n the highest weight.

We study case 1) in more detail. We choose a non-zero vector $h \in H(0)$. We know that E^+ is non zero on all $H(n)$ (n even). Hence we can define for all even n a uniquely determined $h_n \in H(n)$ such that

$$h_0 = h, \quad E^+ h_n = h_{n+2}.$$

Then we define the number $c_n \ne 0$ by

$$E^- h_n = c_n h_{n-2}.$$

The system of numbers $(c_n)_{n \text{ even}}$ is independent of the choice of h. It is clear that the action of $\mathfrak{g}_{\mathbb{C}}$ is determined by this system of numbers and it is also clear that isomorphic representations lead to the same system. A much better result is true. The relation $[E^+, E^-] = -4\mathrm{i}W$ shows

$$c_n - c_{n+2} = 4n, \quad c_0 = \lambda$$

Hence all c_n are determined by one, for example by c_0. This equals the eigenvalue λ of the Casimir operator, actually

$$\omega h = (E^+E^- + 2\mathrm{i}W - W^2)h = \lambda h = c_0 h \quad \text{for} \quad h \in H(0).$$

Hence we obtain in the case 1)

$$\boxed{c_{2n} = \lambda - 4(n-1)n}$$

(and $c_n = 0$ for odd n). So the representation is determined up to isomorphism by λ. What can we say about the existence? We start with some complex number λ We can take for each even n a one dimensional vector space $\mathbb{C}h_n$ and then define the vector space $\mathcal{H} = \bigoplus \mathbb{C}h_{2n}$. The we can take the above formulas to define E^+, E^-, W. It is easy to check that this gives a representation. Obviously this is an admissible representation if all c_{2n} are different from zero. In this way we obtain the following result.

12.8 Proposition. *An irreducible admissible representation of type 1) (in Proposition 12.7) is determined up to isomorphism by the eigenvalue λ of the Casimir operator. An eigenvalue λ occurs if and only if it is different from $4(n-1)n$ for all integers n.*

The case 2) is very similar. Here we choose a non-zero vector $h \in H(1)$. Then we define for all odd n h_n such that $h_1 = h$ and $E^+h_n = h_{n+2}$. Then we define c_n through $E^-h_n = c_n h_{n-2}$. Then one gets

$$c_1 = \lambda + 1, \quad c_n - c_{n+2} = 4n.$$

Here the solution is

$$\boxed{c_{2n+1} = \lambda - 4(n-1)n + 1}$$

(and $c_n = 0$ for even n).

12.9 Proposition. *An admissible representation of type 2) (in Proposition 12.7) is determined up to isomorphism by the eigenvalue λ of the Casimir operator. An eigenvalue λ occurs if and only if it is different from $4(n-1)n+1$ for all integers n.*

Assume now that there is a lowest weight m. Now we choose a non-zero $h \in H(m)$ and define $h_m = h$ and $E^+h_k = h_{k+2}$ for $k \geq n$ (same parity as n). They are all different from 0. Then we define the constants c_k, $k \leq n$, through $E^-h_{k+2} = c_k h_k$ for $k \geq m$. We have $E^-E^+h_m = c_m h_m$ and$E^-h_m = 0$. Hence the relation $[E^+, E^-] = -4\mathrm{i}W$ gives

$$c_m = -4m \quad \text{and} \quad c_{k-2} - c_k = 4k \text{ for } k > m.$$

The only solution is

$$c_k = -4 \sum_{\substack{m \leq \nu \leq k \\ \nu \equiv m \bmod 2}} \nu \qquad (k \geq m).$$

12.10 Proposition. *An admissible representation with a lowest weight vector but no highest weight vector is determined up to isomorphism by its lowest weight m. An integer occurs as lowest weight if and only if $m > 0$.*

We also mention that the eigenvalue of the Casimir operator is

$$\lambda = m^2 - 2m.$$

The same argument works if there is a highest weight vector.

12.11 Proposition. *An admissible representation with a highest weight vector but no lowest weight vector is determined up to isomorphism by its highest weight n. An integer occurs as highest weight if and only if $n < 0$.*

In this case the eigenvalue of the Casimir operator is

$$\lambda = n^2 + 2n.$$

It remains to treat the case where a lowest weight m and a highest weight n exist. The are the only finite dimensional cases. In this case we get for the eigenvalue of the Casimir operator

$$\lambda = n^2 + 2n = m^2 - 2m.$$

This (and $m \leq n$) imply $m = -n$ and $n \geq 0$. Hence we get the following result.

12.12 Proposition. *An irreducible admissible $\mathfrak{g}$-K-module is finite dimensional if and only if it has a lowest weight m and a highest weight n. These have the property $m = -n$, $n \geq 0$. It is determined by n up to isomorphism. Its dimension is $n + 1$. Hence it is determined also by its dimension Every integer $n \geq 0$ occurs. The eigen value of the Casimir operator is $n^2 - n$.*

We mention that the trivial representation occurs here ($m = n = 0$).

We can describe the finite dimensional irreducible admissible $\mathfrak{g}$-K modules in slightly modified form as follows. We write $n = 2l$. Here l is a nonnegative integer or half integer. Then the dimension is $2l + 1$. The above description shows that there exists a basis

$$e_{l,-l}, e_{l,-l+1}, \dots, e_{l,l-1}, e_{ll}$$

such that

$$k_\theta e_{lk} = e^{2\mathrm{i}k\theta} e_k.$$

This implies

$$We_{lk} = 2\mathrm{i}ke_{lk}.$$

Moreover

$$E^+ e_{lk} = e_{k+1}, \quad -l \leq k < l$$

and

$$E^- e_{l,k+1} = c_{lk} e_{lk}, -l \leq k < l, \quad c_{lk} = -8 \sum_{\substack{-l \leq \nu \leq k \\ \nu - l \in \mathbb{Z}}} \nu.$$

We collect the main result in a table.

Irreducible admissible representations

Tpye	determined by	condition
even, no highest or lowest weight	$\lambda \in \mathbb{C}$	$\lambda \neq 4(n-1)n$ $(n \in \mathbb{Z})$.
odd, no highest or lowest weight	$\lambda \in \mathbb{C}$	$\lambda \neq 4(n-1)n + 1$.
lowest but no highest weight	weight m	$m > 0$
highest but no lowest weight	weight n	$n < 0$
finite dimensional	weights $m < n$	$m = -n, \ n \geq 0$

For later purpose we look at the finite dimensional representations of $\mathfrak{g}_{\mathbb{C}}$ in some more detail. Here the assumptions can be weakened.

12.13 Proposition. *For each integer $n \geq 0$ there exists one and up to isomorphism only one irreducible admissible $\mathfrak{g}$-K-module of dimension* $\dim H = n + 1$. *We write $n = 2l$ where l is integral or half integral, $l \geq 0$ There exists* *a basis* $e_{l,-l}, e_{l,-l+1}, \dots, e_{l,l-1}, e_{ll}$ *with the following properties. The group* K *acts through* $k_\theta e_{lk} = e^{2ik\theta} e_k$. *The Lie algebra* $\mathfrak{g}_{\mathbb{C}}$ *acts through*

$$\begin{aligned} We_{lk} &= 2ike_{lk}, \\ E^+ e_{lk} &= e_{l,k+1}, \quad -l \le k < l, \\ E^- e_{l,k+1} &= c_{lk} e_{lk}, -l \le k < l, \quad c_k = -8 \sum_{\substack{-l \le \nu \le k \\ \nu - l \in \mathbb{Z}}} \nu \qquad (k \ge l). \end{aligned}$$

13. The Bargmann classification

Let $\pi : G \to \mathrm{GL}(H)$ be an irreducible unitary representations. Then the derived representation $\mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(H_K)$ has the property that the operators $d\pi(A)$ for $A \in \mathfrak{g}$ are skew symmetric,

$$\langle Ah, h' \rangle = -\langle h, Ah' \rangle.$$

This follows immediately from the definition of the derived representation. It is clear that this formula extends to all $A \in \mathfrak{g}_{\mathbb{C}}$ in the following way

$$\langle Ah, h' \rangle = -\langle h, \bar{A}h' \rangle.$$

Hence it is natural to ask for an admissible representation $\mathfrak{g}_{\mathbb{C}}$ on $\mathcal{H}$ whether there exists a Hermitian scalar product on $\mathcal{H}$ such that the elements of $\mathfrak{g}$ act skew-symmetrically.

13.1 Definition. *An admissible representation is called unitarizible if there exists a Hermitian scalar product on* $\mathcal{H}$ *such that*

$$\langle Ah, h' \rangle = -\langle h, \bar{A}h' \rangle$$

for $A \in \mathfrak{g}_{\mathbb{C}}$.

We want to pick out in the list of all admissible representations the unitary ones.

13.2 Proposition. *An admissible representation is unitarizible if and of one of the following conditions is satisfied.*

1) *Even case without lowest or highest weight: The eigenvalue* λ *of the Casimir operator is real and* $-\lambda \ge 0$.
2) *Odd case without lowest or highest weight: The eigenvalue* λ *of the Casimir operator is real and* $-\lambda \ge 1$.

3) *All representations with a lowest but no highest weight m, $m > 0$, are unitarizible.*
4) *All representations with a highest but no lowest weight n, $n < 0$, are unitarizible.*
5) *A finite dimensional admissible representation is unitarizible if and only if it is trivial.*

Proof. We treat the case 1) since all other cases are similar. There is a non zero element $h \in H(0)$ and $E^+E^-h = \lambda h$. From the condition that the real elements $A \in \mathfrak{g}$ act skew Hermitian we obtain the rule $\langle E^+h, h'\rangle = -\langle h, E^-h'\rangle$ and hence

$$\langle E^-h, E^-h\rangle = -\langle h, -E^+E^-h\rangle = -\langle h, \lambda h\rangle = -\bar{\lambda}\langle h, h\rangle.$$

It follows that λ is real and negative. Assume conversely that this is the case. Then we define a scalar product on $\mathcal{H}$ such that the $H(n)$ are pairwise orthogonal and such that $\langle h, h\rangle = 1$. Then we define $\langle E^+h, E^+h\rangle = 1$ and so on. The proof no should be clear. □

13.3 Proposition. *Two unitary representations $\pi : G \to \mathrm{GL}(H)$, $\pi' : G \to \mathrm{GL}(H')$ of $G = \mathrm{SL}(2, \mathbb{R})$ are unitary isomorphic if and only if the underlying admissible representations are (algebraically) isomorphic.*

Proof. Let $T : H_K \to H'_K$ be an isomorphism of the admissible representations. We choose scalar products such that $\mathfrak{g}$ acts skew symmetric. We choose a non zero $h \in H$ which is contained in some $H(m)$. We normalize h such that $\langle h, h\rangle = 1$. We set $h' = Th$. Without loss of generality we may assume that $\langle h', h'\rangle = 1$ since we can replace the scalar product of H' by a multiple. Now we claim that T preserves the scalar products. For the proof we Definition 12.4. It implies that H_K is generated by all $A_1 \cdots A_n h$, where $A_i \in \mathfrak{g}$. So we have to show that T preserves the scalar products for such elements. This is done by induction. We just explain the beginning to the induction to give the idea. Since $H(m)$ is one-dimensional and since the spaces $H(n)$ are pairwise orthogonal, we know all scalar products $\langle h, x\rangle$. Let $A \in \mathfrak{g}$. The formula $\langle Ah, x\rangle = -\langle h, Ax\rangle$ gives all scalar products $\langle Ah, x\rangle$. Proceeding in this way we get that all scalar products are determined (from $\langle h, h\rangle = 1$). The same calculation can be done in H'. In this way we can see that T preserves the scalar products. Now we can extend to an isomorphism of Hilbert spaces $T : H \to H'$. From Proposition 10.1 in connection with the Lemmas 7.9 and I.2.1 we obtain that T preserves the action of G. □

The question arises whether each admissible representation can be realized by unitary representations of G (in the sense that it is isomorphic to its derived representation).

13.4 Theorem. *Each unitarizible admissible representation can be realized by an irreducible unitary representation.*

Proof. In Sect. 3–5 of this chapter we gave several examples of unitary representations. It can be checked we will see this below that we obtain all unitarizible admissible representations. This gives the proof. We consider the representations $H(s)$ which have been described in Chap. I, Sect. 7.

13.5 Lemma. *Let $f \in H(s)$ be an element that is $\mathcal{C}^\infty$ considered as function on the group G. Then the image of f in $H(s)$ is a $\mathcal{C}^\infty$-vector of the representation π_s and we have*

$$\mathcal{L}_X f = d\pi_s(X) f.$$

Proof. It is easy to check that $\mathcal{L}_X f$ has the transformation properties of functions from $H(s)$. Since it is continuous it is contained in $H(s)$. We have to show that

$$\lim_{t\to 0}\int_K \Bigl|\frac{f(k\exp(tX)) - f(k)}{t} - \mathcal{L}_X f(k)\Bigr|^2 dk = 0.$$

by definition of $\mathcal{L}$ the integrand tends pointwise to 0. Using the mean-value theorem it is easy to show that the integrand is bounded for small t. Hence the Lebesgue limit theorem can be applied. □

The space $H(n,s)$ of all K-eigenfunctions which pick up the nth power of the standard character is one dimensional and generated by the function

$$\varphi\left(\begin{pmatrix}\sqrt{y} & * \\ 0 & \sqrt{y}^{-1}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\right) = y^{(s+1)/2} e^{in\theta}.$$

We can use the formula in Proposition 9.1 to compute the derived representation. The result is

$$\boxed{\begin{aligned} d\pi_s(W)\varphi_n &= in\varphi_n, \\ d\pi_s(E^-)\varphi_n &= (s+1-n)\varphi_{n-2} \\ d\pi_s(E^+)\varphi_n &= (s+1+n)\varphi_{n+2}\end{aligned}}$$

From this description, it is easy to find invariant subspaces, namely

$$H(s)^{\text{even}} = \widehat{\bigoplus_{n\ \text{even}}}\, \mathbb{C}\varphi_n, \quad H(s)^{\text{odd}} = \widehat{\bigoplus_{n\ \text{odd}}}\, \mathbb{C}\varphi_n.$$

We first treat the even case. Then the parameter c from the previous section computes as $c = (s+1)(s-1)$. This was the reason that we introduced already somewhat the parameter s as solution of this equation. We recall that the representation π_s is unitary if $\operatorname{Re} s = 0$. We see that the corresponding derived representation is the even principal series. But one cannot realize the complementary series in this way, since this would demand $s \in (-1, 1)$, $s \neq 0$. But in this case π_s is only a Banach representation. Hence one needs for the complementary series a different kind of realization. We indicated it in Chapt. I, Sect. 7. We will give more details.

The odd principal series is obtained completely from π_s. Here the parameter c is computed as $c = s^2$ which again explains the conventions from the previous section. Hence in the case that s is purely imaginary but $s \neq 0$ In this way we get realizations of the two principal series where $s = 0$ has been excluded.

In the case $s = 0$ the odd space can be decomposed into subspaces again. We obtain in the case $s = 0$ two irreducible subspaces of the odd space,

$$\widehat{\bigoplus_{n\geq 1 \text{ odd}}} \mathbb{C}\varphi_n, \qquad \widehat{\bigoplus_{n\leq 1 \text{ odd}}} \mathbb{C}\varphi_n.$$

Obviously they are realizations of the two mock discrete representations. So in this sense the mock discrete representations are simply degenerations of the principal series.

Hence we have found realizations of the principal series and the two mock discrete representations and we mention that the complementary series can also be realized by concrete unitary representations.

It remains to realize the discrete series. The holomorphic discrete series gives the discrete series with a lowest weight and the antiholomorphic discrete series gives the discrete series with a highest weight. □

Collecting together, we get the classification of the irreducible unitary representations of G. First we introduce some notations. Recall that in case 1) and case 2) the representation is determined by a single parameter λ, the eigenvalue of the Casimir operator. Instead of λ we will use a new parameter s. It is defined through

$$\lambda = (s + 1)(s - 1)$$

and is determined up to its sign.

1) λ is real and $\lambda \leq -1$ if and only if s is purely imaginary.
2) λ is real and $-1 < \lambda < 1$ if and only if s is real and $-1 < s < 1$, $s \neq 0$.

We use the following notations for irreducible unitary representations $\pi : G \to \mathrm{GL}(H)$.

13.6 Definition. *The* ***even principal series*** *consists of all representations of even type without highest or lowest weight and with the property that s is purely imaginary. (Then $\lambda \leq -1$)*

The ***complementary series*** *consists of all representations of even type without highest or lowest weight and with the property that $s \in (-1, 1)$ but $s \neq 0$. (Then $-1 < \lambda < 0$)*

The ***odd principal series*** *consists of all representations of odd type without highest or lowest weight and with the property that s is different from zero and purely imaginary (Here $\lambda \le -1$.)*

The ***holomorphic discrete series*** *consists of all representations with a highest weight $n < -1$ and now lowest weight.*

The ***antiholomorphic discrete series*** *consists of all representations with a lowest weight $m > 1$ and now lowest weight.*

The ***mock discrete series*** *consists only of two representations, namely those with highest weight $m = -1$ (and no lowest weight) and conversely $n = 1$.*

The border cases with highest weight -1 or lowest weight 1 have some special properties. Hence they are separated from the other representations with a highest or lowest weight vector. Those with $m \le -2$ or $n \ge 2$ define the *discrete series* and the two with $m = -1$ or $n = 1$ define the mock discrete series.

Collecting together we obtain Bargmann's classification of all irreducible representations π of G. If this representation is not the trivial one dimensional representation then $\mathfrak{g}$ acts non identically zero (Proposition 2.10.1) and then the derived representation is a unitarizible admissible representation. The above discussion gives now the main result.

13.7 Theorem. *Each unitary irreducible unitary representation of $G = \mathrm{SL}(2, \mathbb{R})$ is either the trivial one-dimensional representation or it is unitary isomorphic to a representation of the following list.*

1) *The even principal series, $s \in \mathrm{i}\mathbb{R}$,*
2) *the odd principal series, $s \in \mathrm{i}\mathbb{R} - \{0\}$,*
3) *the complementary series, $s \in (-1, 1) - \{0\}$,*
4) *the discrete series with highest weight $m \le -2$ or lowest weight $n \ge 2$.*
5) *the mock discrete series (two representations, (highest weight -1 or lowest weight 1).*

In the first three cases, s is determined up to its sign. In the last two cases the weight is uniquely determined.

Why has the mock discrete series been separated from the discrete series? If G is an arbitrary locally compact group, one has a general notion of a discrete series representation. An irreducible unitary representation is called a discrete series representation of it occurs (as unitary representation) in the regular representation $L^2(G)$. It can be shown that the discrete series representations of $G = \mathrm{SL}(2, \mathbb{R})$ in this sense consist of all representations with a higher or lower weight vector with two exceptions, the weights 1 and -1 do not occur. Hence these play a special role. Since they look similar as the discrete series representations they are called "mock discrete".

14. Automorphic forms

We consider a discrete subgroup $\Gamma \subset G$ with compact quotient $\Gamma\backslash G$. We recall that the representation of G on $L^2(\Gamma\backslash G)$ is completely reducible with finite multiplicities. One can ask which representations of the Bargmann list occur and for their multiplicity. For this we make a simple remark.

14.1 Lemma. *Let $f : \mathbb{H} \to \mathbb{C}$ be a function on the upper half plane and let m be an integer. We consider the function*

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = f\Big(\frac{a\mathrm{i}+b}{c\mathrm{i}+d}\Big)(c\mathrm{i}+d)^{-m}.$$

then we have

$$F(gk_\theta) = e^{im\theta}F(g)$$

and every function on G with this transformation property comes from a function f on $\mathbb{H}$. Moreover we can write F as

$$F\left(\begin{pmatrix} \sqrt{y} & \sqrt{y}^{-1}x \\ 0 & \sqrt{y}^{-1} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right) = f(x+\mathrm{i}y)e^{\mathrm{i}m\theta}\sqrt{y}^m.$$

The function F is right-invariant under Γ if and only if f satisfies

$$f((az+b)(cz+d)^{-1}) = (cz+d)^m f(z)$$

for all elements in Γ.

Proof. The proof is straight forward. □

Now we assume that their is an irreducible closed subspace $H \subset L^2(\Gamma\backslash G)$ which belongs to the holomorphic discrete series with lowest weight $m \geq 2$. Wir consider a non-zero lowest weight vector h. From Lemma 14.1 we know that h comes from a function $f : \mathbb{H} \to \mathbb{C}$ with the transformation property

$$f(az+b)(cz+d)^{-1} = (cz+d)^m f(z), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Since h is a lowest weight vector we have $E^-h = 0$. Using the explicit formula for E^- we obtain that

$$\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big)f = 0.$$

This means that f is holomorphic. This means that f is a holomorphic automorphic form. Conversely it can be shown that every holomorphic automorphic form occurs in this way.

But als other representations of the Bargmann list may occur. For example assume that an even principal series representation with parameter s occurs. We can now consider a non zero vector h of weight 0. This is invariant under K and corresponds to a function f on the upper half plane. Recall that h is an eigen form of the Casimir operator with eigen value $\lambda = (s+1)(s-1)$. Looking at the explicit expressions for $E^{\pm}$ we see that this means

$$y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) f = \lambda f.$$

What we have found is so-called wave form in the sense of Maass. Maass gave a generalization of the theory of modular forms replacing holomorphicity by certain differential equation. All these Maass forms can be recovered in the following way: Consider an irreducible sub representation $H \subset L^2(\Gamma\backslash G)$. Take a vector $h \in H(m)$ for an arbitrary m. By Lemma 14.1 this corresponds to function f on $\mathbb{H}$ with a certain transformation property. Make use of the fact that h is an eigen form of the Casimir operator. This produces a differential equation for f. In this way on recovers precisely the differential equations that Maass has introduced.

15. Some comments on the Casimir operator

Let $U \subset \mathbb{R}^n$ be an open subset. We are interested in maps

$$D : \mathcal{C}^\infty(U) \longrightarrow \mathcal{C}^\infty(U)$$

which can be written as finite sum

$$Df = \sum h_{i_1\dots i_m} \, \frac{\partial^{i_1+\cdots+i_m} f}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}$$

with differentiable coefficients $h_{\dots}$. Clearly they are uniquely determined. We call D a *linear differential operator*. This notation is due to the fact that obviously $D(f+g) = Df + Dg$ and $D(Cf) = CDf$. When D is non-zero there exists a maximal m such that $h_{i_1,\dots,i_n}$ is non-zero for some index with $i_1 + \cdots + i_n = m$. We call m the *degree* of this operator and the function on $U \times \mathbb{R}^n$

$$P(x_1,\dots,x_n,X_1,\dots,X_n) = \sum_{i_1+\cdots+i_n=m} h_{i_1,\dots,i_m}(x) X_1^{i_1} \dots X_n^{i_n}$$

is called the *symbol* of D. This is a homogenous polynomial of degree m for fixed x. The operator D is called *elliptic,* if it is not zero and if

$$P(x,X) \neq 0 \quad \text{for all} \quad X \neq (0,\dots,0).$$

There are two simple observations:

a) Let $V \subset U \subset \mathbb{R}^n$ be open subsets and let D be a linear differential operator on U. Then there is a natural restriction to a linear differential operator on V.

b) Let $\varphi : U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^n$ and let D be a linear differentiable operator on U. Then the transported operator to V is a linear differential operator as well. Ellipticity is preserved.

It is clear how to define the notion of a linear differential operator on $G = \mathrm{SL}(2,\mathbb{R})$ (use Iwasawa coordinates): We have seen that elements of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ acts on differentiable functions on G through

$$Af(x) = \frac{d}{dt} f(x \exp(tA))\Big|_{t=0}.$$

It is clear that this is a linear differential operator on G This is a Lie homomorphism

$$\mathfrak{g} \longrightarrow \mathrm{End}(\mathcal{C}^\infty(G)).$$

Here $\mathcal{C}^\infty(G)$ denotes the space of all differentiable functions on G with values in some complex Banach space. So the space is a complex vector space and the Lie homomorphism can be extended $\mathbb{C}$-linearly to $\mathfrak{g}_\mathbb{C}$. From construction it is clear that the images of $\mathfrak{g}$ are left invariant operators, i.e. they commute with the operator "translation from the left"

$$L_y : \mathcal{C}^\infty(G) \longrightarrow \mathcal{C}^\infty(G), \quad (L_y f)(x) = f(y^{-1}x).$$

We also can consider translation from the right

$$R_y : \mathcal{C}^\infty(G) \longrightarrow \mathcal{C}^\infty(G), \quad (R_y f)(x) = f(xy).$$

Both L_y and R_y are actions from the left.

15.1 Definition. *We denote by* $\mathbf{D}(G)$ *the smallest subalgebra of* $\mathrm{End}(\mathcal{C}^\infty(G))$ *that contains the image of* $\mathfrak{g}$.

It can be shown that $\mathbf{D}(G)$ equals the algebra of all left invariant linear differential operators: We don't need this.

15.2 Definition. *A Casimir operator is an element of* $\mathbf{D}(G)$ *that commutes with the image of* $\mathfrak{g}$.

15.3 Lemma. *Casimir operators* C *have the property*

$$R_g \circ C = C \circ R_g.$$

Proof. Since the Casimir operators commute with $X \in \mathfrak{g}$ they also commute with e^X. But we know that G is generated by the image of $\mathfrak{g}$. □

Hence Casimir operators are left and right invariant linear differential operators.

Chapter III. The complex special linear group of degree two

The classification of irreducible unitary representations of $\mathrm{SL}(2,\mathbb{C})$ is partly similar to that of $\mathrm{SL}(2,\mathbb{R})$. But there arise extra difficulties due to the fact that the compact subgroup that now comes instead of $\mathrm{SO}(2)$ into the game is not abelian. In the following we will keep short when the arguments are the same as in the case $\mathrm{SL}(2,\mathbb{R})$. But we treat the differences in great detail.

1. Unitary representations of some compact groups.

In this section we mention some results about the representation theory of the compact groups $\mathrm{U}(n)$ and $\mathrm{SU}(n)$. Here $\mathrm{U}(n)$ denotes the group of all $n\times n$-matrices A with the property $\bar{A}'A = E$ and $\mathrm{SU}(n)$ denotes the subgroup of elements of determinant one. We want to describe the unitary irreducible representations of them. We formulate the general results, but we give proves only the case $n = 2$. Here we apply the $\mathrm{SL}(2,\mathbb{R})$-theory.

We recall some facts for compact groups K (see Chapt. I, Sect. 8):

1) Each irreducible unitary representation of a compact group is finite dimensional.
2) If $K \to \mathrm{GL}(V)$ is a finite dimensional (continuous) representation of K, then there exists a Hermitian scalar product on V such that the representation is unitary.
3) Let $K \to \mathrm{GL}(V_i)$ be two finite dimensional irreducible unitary representations. They are unitary isomorphic if and only of the isomorphic in the usual sense.

Hence the classification of irreducible unitary representation of a compact group and the classification of finite dimensional irreducible representations is the same. So we can forget about the scalar products.

The representation theory of the group $\mathrm{SU}(n)$ is closely related to the theory of *rational* representations of $\mathrm{GL}(n,\mathbb{C})$. We have to consider polynomial functions f on $\mathrm{GL}(n,\mathbb{C})$. These are function which can be written as polynomials in the n^2 variables a_{ik}. Moreover, a function f on $\mathrm{GL}(n,\mathbb{C})$ is called *rational* if there exists a natural number k such that $(\det A)^k f(A)$ is polynomial. We mention the following result (which we will prove in the case $n = 2$).

1.1 Proposition. *Every finite dimensional (continuous) representations $\pi : \mathrm{U}(n) \to \mathrm{GL}(V)$ extends to a rational representation of $\mathrm{GL}(n,\mathbb{C})$. Two* *finite dimensional representations of* $\mathrm{U}(n)$ *are isomorphic if and only if their rational extensions are isomorphic. The representation* π *is irreducible if an only if its rational extension is irreducible.*

Hence the classification of irreducible unitary representations of $\mathrm{U}(n)$ is the same as the classification of irreducible rational representations of $\mathrm{GL}(n, \mathbb{C})$. The classification of the irreducible rational representations can be given by their highest weight.

1.2 Theorem. *Let* $\pi : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(V)$ *be an irreducible rational representation. There exists a one-dimensional subspace* $W \subset V$ *that is invariant under all upper triangular matrices and* W *is unique with this property. There exist integers*

$$r_1 \geq r_2 \geq \cdots \geq r_n$$

such that the action of diagonal matrices A *with diagonal* $a_1, \ldots, a_n$ *is given by*

$$\pi(A)w = a_1^{r_1} \cdots a_n^{r_n} w \quad (w \in W).$$

This gives a bijection between the set of isomorphy classes of irreducible rational representations of $\mathrm{GL}(n, \mathbb{C})$ *and the set of increasing tuples* $r_1 \geq \cdots \geq r_n$ *of integers.*

We will not prove this result general but we will prove it in the case $n = 2$.

□

The tuple $(r_1, \ldots, r_n)$ is called the *highest weight* and the elements of W are called highest weight vectors.

This theorem does not tell, how the representations of a given highest weight can be constructed and, in particular, it does not tell the dimensions of the representations.

For us, the case $n = 2$ is of special importance. Let

$$l \in \{0, 1/2, 1, 3/2, \ldots\}$$

be a non negative integral or half integral number. We consider the space V_l of all polynomial functions $P : \mathbb{C}^2 \to \mathbb{C}$ which are homogenous and of degree $2l$. So the dimension of V_l is $2l + 1$. We define a representation

$$\varrho_l : \mathrm{GL}(2, \mathbb{C}) \longrightarrow \mathrm{GL}(V_l)$$

by

$$(\varrho(g)P)(x) = P(g'x).$$

(We use the standard action of $\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{C}^2$ of linear algebra which is an action from the left.) The subspace W_l that is generated by the polynomial $P(x, y) = y$ is invariant under upper triangular matrices and it is the only one dimensional subspace with this property. From this one can deduce that ϱ_l is irreducible. The highest weight is $(0, 2l)$. More generally, we can consider the representation $\det(g)^k \varrho_l(g)$. Its highest weight is $(k, k+2r)$. These pairs exhaust all highest weights. Hence we have found all representations of $\mathrm{GL}(2,\mathbb{C})$ and as a consequence also of $\mathrm{U}(2)$.

We are more interested in $\mathrm{SU}(2)$. It is not difficult to show that every finite dimensional irreducible representation of $\mathrm{SU}(2)$ is the restriction of a representation of $\mathrm{U}(2)$. In this way one can prove the following result.

1.3 Theorem. *The restriction of ϱ_l to $\mathrm{SU}(2)$ is irreducible. Every finite dimensional irreducible representation is isomorphic to one and only one representation (V_l, ϱ_l).*

Proofs in the case n=2

We study the group $\mathrm{SU}(2)$. It consists of all complex matrices

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1.$$

This can be identified with the 3-dimensional sphere S_3. We parameterize an open part of $\mathrm{SU}(2)$ through

$$\begin{aligned} &\{x = (x_1, x_2, x_3);\ x_1^2 + x_2^2 + x_3^2 < 1\} \longrightarrow \mathrm{SU}(2), \quad x \longmapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \\ &a = x_1 + \mathrm{i}x_2,\ b = x_3 + \mathrm{i}\sqrt{1 - x_1^2 - x_2^2 - x_3^2} \end{aligned}.$$

Using this one can define when a function on this open subset of S^3 is differentiable or analytic. The functions are allowed to be Banach valued. Using similar maps we can introduce differentiable or analytic functions on any open subset of $\mathrm{SU}(2)$. (The reader who is familiar with manifolds will se that there is a natural structure of a real analytic manifold on $\mathrm{SU}(2)$.)

Let $\pi : \mathrm{SU}(2) \to \mathrm{GL}(H)$ be a continuous finite dimensional irreducible representation (on the complex vector space H). As in Chap. II, Sect. 8 we can define the subspace H^∞ of differentiable vectors and we can define a derived representation

$$d\pi : \mathfrak{su}(2) \longrightarrow \mathrm{End}(H^\infty).$$

This is a Lie homomorphism in the sense of Definition II.8.5. The same proof also shows that H^∞ is dense. Since H is finite dimensional, this means that every vector of H is differentiable. We will see more, namely that every vector is analytic. This is more difficult and needs some insight into the structure of $\mathfrak{g} = \mathfrak{su}(2)$. Therefore we consider $\mathfrak{su}(2)$ as subset of $\mathfrak{sl}(2,\mathbb{C})$. This is a real subspace. One checks

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus \mathrm{i}\mathfrak{su}(2).$$

And one checks that the Lie multiplication on $\mathfrak{sl}(2,\mathbb{C})$ is just the $\mathbb{C}$-bilinear extension of the Lie multiplication on $\mathfrak{su}(2)$. (In the notions of the next section this says that $\mathfrak{sl}(2,\mathbb{C})$ is the complexification of $\mathfrak{su}(2)$. In the same sense $\mathfrak{sl}(2,\mathbb{C})$ is also the complexification of $\mathfrak{sl}(2,\mathbb{R})$. So $\mathfrak{sl}(2,\mathbb{C})$ arises as complexification of two different Lie algebras.)

The representation extends $\mathbb{C}$-linearly to a Lie representation

$$\pi : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathrm{End}(H).$$

We studied such representations in detail. In particular, we constructed a Casimir operator ω which commutes with all $A \in \mathfrak{sl}(2,\mathbb{C})$. Then it commutes also with e^A and hence with all g in a small neighbourhood of the unit element. Since $\mathrm{SU}(2)$ is connected this neighbourhood generates the full group. Hence ω commutes with the full group $\mathrm{SU}(2)$. Since the action is assumed to be irreducible, ω acts by a scalar on H.

We choose a basis of $\mathfrak{su}(2)$ and compute the exponentials

$$\begin{aligned}
X_1 &= \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} & \qquad e^{2tX_1} &= \begin{pmatrix} e^{\mathrm{i}t} & 0 \\ 0 & e^{-\mathrm{i}t} \end{pmatrix} \\
X_2 &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \qquad e^{2tX_2} &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \\
X_3 &= \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} & \qquad e^{2tX_3} &= \begin{pmatrix} \cos t & \mathrm{i}\sin t \\ \mathrm{i}\sin t & \cos t \end{pmatrix}
\end{aligned}$$

Notice that $2X_2 = W$ which we introduced earlier (Chap. II, Sect. 8). The commutation relations are

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

(This means that $\mathfrak{su}(2)$ is isomorphic to $\mathbb{R}^3$ with the cross product.) The expressions of $E^{\pm}$ in the X_i are

$$E^{\pm} = 2(\pm X_3 - \mathrm{i}X_1).$$

For the Casimir operator ω (Chap. II, Sect. 10) we get the expression

$$\boxed{\omega = E^+E^- + 2\mathrm{i}W - W^2 = -4(X_1^2 + X_2^2 + X_3^3).}$$

We compute the Lie derivatives

$$(\mathcal{L}_X(f))(x) = \frac{d}{dt} f\big(xe^{tX}\big)\big|_{t=0} \qquad (X \in \mathfrak{su}(2))$$

which transforms functions on SU(2) (may be Banach valued) to functions of the same kind. We restrict the function f to a function on the open ball

$$\{(x_1,x_2,x_3);\ x_1^2+x_2^3+x_3^2<1\}$$

through

$$f_0(x_1,x_2,x_3)=f\begin{pmatrix} a & b\\ -\bar b & \bar a\end{pmatrix},\quad a=x_1+\mathrm{i}x_2,\ b=x_3+\mathrm{i}x_4,$$
$$x_4=\sqrt{1-x_1^2-x_2^2-x_3^2}.$$

We denote by $(\mathcal{L}_X f)_0$ the restriction of $\mathcal{L}_X f$ to this ball. Then we can consider the operator

$$f_0\longmapsto(\mathcal{L}_X f)_0.$$

This is the transformed operator $\mathcal{L}$ to the ball. For simplicity of notation, we denote this operator also by $\mathcal{L}_X$. We compute it for $X=X_1$

$$\mathcal{L}_{X_1}f\begin{pmatrix} a & b\\ -\bar b & \bar a\end{pmatrix}=\frac{d}{dt}f\left(\begin{pmatrix} a & b\\ -\bar b & \bar a\end{pmatrix}\begin{pmatrix} e^{\mathrm{i}t/2} & 0\\ 0 & e^{-\mathrm{i}t/2}\end{pmatrix}\right)\Big|_{t=0}=\frac{d}{dt}f_0(y_1,y_2,y_3)$$

where

$$\begin{aligned} y_1&=\cos(t/2)-x_2\sin(t/2),\\ y_2&=x_1\sin(t/2)+x_2\cos(t/2),\\ y_2&=x_3\cos(t/2)+x_4\sin(t/2).\end{aligned}$$

The chain rule gives

$$\mathcal{L}_{X_1}f_0=\frac12\Bigl(-x_2\frac{\partial f_0}{\partial x_1}+x_1\frac{\partial f_0}{\partial x_2}+x_4\frac{\partial f_0}{\partial x_3}\Bigr).$$

In the same way we get the following formulas.

$$\begin{aligned}\mathcal{L}_{X_1}&=\frac12\Bigl[-x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2}+x_4\frac{\partial}{\partial x_3}\Bigr],\\ \mathcal{L}_{X_2}&=\frac12\Bigl[-x_3\frac{\partial}{\partial x_1}-x_4\frac{\partial}{\partial x_2}+x_1\frac{\partial}{\partial x_3}\Bigr],\\ \mathcal{L}_{X_3}&=\frac12\Bigl[-x_4\frac{\partial}{\partial x_1}+x_3\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_3}\Bigr].\end{aligned}$$

Now we can compute the action of the Casimir operator. We have to square the single $\mathcal{L}_{X_i}$. Then terms

$$\Bigl(x_i\frac{\partial}{\partial x_k}\Bigr)\Bigl(x_\alpha\frac{\partial}{\partial x_\beta}\Bigr)$$

occur. The operators $x_i\partial/\partial x_k$ do not commute. But we have

$$\Big(x_i\frac{\partial}{\partial x_k}\Big)\Big(x_\alpha\frac{\partial}{\partial x_\beta}\Big)=x_ix_\alpha\frac{\partial^2}{\partial x_k\partial x_\beta}\quad \text{if } \alpha\neq k$$

and

$$\Big(x_i\frac{\partial}{\partial x_k}\Big)\Big(x_k\frac{\partial}{\partial x_\beta}\Big)=x_ix_k\frac{\partial^2}{\partial x_k\partial x_\beta}+x_i\frac{\partial}{\partial x_\beta}.$$

Now we can expand and then add the three squares of $\mathcal{L}_{X_i}$. If one does this patiently one will see that many terms vanish and one gets

$$\begin{aligned}\omega=&-(1-x_1^2)\frac{\partial^2}{\partial x_1^2}-(1-x_2^2)\frac{\partial^2}{\partial x_2^2}-(1-x_3^2)\frac{\partial^2}{\partial x_3^2}\\&+2x_1x_2\frac{\partial^2}{\partial x_1\partial x_2}+2x_1x_3\frac{\partial^2}{\partial x_1\partial x_3}+2x_2x_3\frac{\partial^2}{\partial x_2\partial x_3}\\&+2x_1\frac{\partial}{\partial x_3}+2x_2\frac{\partial}{\partial x_2}+2x_3\frac{\partial}{\partial x_3}.\end{aligned}$$

Readers who know the notion of the Laplace operator on a Riemannian manifold will realize that this operator describes just the Laplace operator on the 3-dimensional sphere. It is a linear differential operator of degree two. Its symbol is a quadratic form. Quadratic forms can be described through symmetric matrices. In this case it is the negative of the matrix

$$\begin{pmatrix}1-x_1^2&-x_1x_2&-x_1x_3\\-x_1x_2&1-x_2^2&-x_2x_3\\-x_1x_3&-x_2x_3&1-x_3^2\end{pmatrix}.$$

This matrix is positive definite for all x with $x_1^2+x_2^2+x_3^2$. For example, its determinant is $1-x_1^2-x_2^2-x_3^2$. So ω is an elliptic differential operator.

We have proved that every vector of H is analytic. This implies, as in the case $\mathrm{SL}(2,\mathbb{R})$ the following result.

1.4 Proposition. *Let $\pi:\mathrm{SU}(2)\to\mathrm{GL}(V)$ be a finite dimensional irreducible representation. Then the derived representation $d\pi:\mathfrak{su}(2)\to\mathrm{End}(V)$ is irreducible in the sense that every $\mathfrak{su}$-invariant subspace is V or 0.*

The representation $d\pi$ extends $\mathbb{C}$-linearly to the complexification of $\mathfrak{su}(2)$ which is $\mathfrak{sl}(2,\mathbb{C})$. This can be restricted to a Lie homomorphism $\mathfrak{sl}(2,\mathbb{R})\to\mathrm{End}(\mathcal{H})$. Also $\mathrm{SO}(2)$ acts on H, since it is a subgroup of $\mathrm{U}(2)$. This means that $\mathcal{H}$ is a $\mathfrak{sl}(2,\mathbb{R})$-$\mathrm{SO}(2)$-module. For trivial reason it is admissible. Clearly it is irreducible. Such modules have been determined in Proposition II.12.12. From the description that follows that there is exactly one such module for each dimension n. This finishes the proof of Theorem 1.3. □

1.5 Theorem. *For each integer or half integer $l \geq 0$ there exists a unique irreducible unitary representation of* $\mathrm{SU}(2)$ *on a Hilbert space V_l of dimension $2l+1$. There exists a basis $e_{l,-l}, e_{l,-l+1}, \dots, e_{l,l-1}, e_{ll}$ such that for $-l \leq m \leq l$ we have*

$$k_\theta e_{lm} = e^{\mathrm{i}m\theta} e_{lm},$$

$$\begin{aligned} X_1 e_{lm} &= \frac{\mathrm{i}}{4}(e_{l,m+1} + c_{l,m-1} e_{l,m-1}), \\ X_2 e_{lm} &= \mathrm{i}m e_{lm}, \\ X_3 e_{lm} &= \frac{1}{4}(e_{l,m+1} - c_{l,m-1} e_{l,m-1}), \end{aligned}$$

where

$$c_{lk} = -8 \sum_{\substack{-l \leq \nu \leq k \\ \nu - l \in \mathbb{Z}}} \nu \qquad (k \geq l).$$

Here we have to set $e_{lm} = 0$ if it is outside the range.

2. The Lie algebra of the complex linear group of degree two

We need the notion of the complexification of a real vector space V. By definition this is $V_\mathbb{C} = V \times V$ as real vector space. The multiplication by i is given by

$$\mathrm{i}(a, b) = (-b, a).$$

This extends to an action of $\mathbb{C}$ on $V_\mathbb{C}$ through

$$(\alpha + \mathrm{i}\beta)x = \alpha x + \mathrm{i}\beta x \qquad (x \in V_\mathbb{C})$$

and this equips $V_\mathbb{C}$ with a structure as complex vector space. We can embed V into $V_\mathbb{C}$ by $a \mapsto (a, 0)$ and if we identify V with its image than $V_\mathbb{C} = V \oplus \mathrm{i}V$. The following universal property holds. Let $f : V \to W$ be an $\mathbb{R}$-linear map into a complex vector space W. Then there exist a unique $\mathbb{C}$-linear extension $V_\mathbb{C} \to W$. Just map (a, b) to $f(a) + \mathrm{i}f(b)$.

We must give a warning. The vector space V might be a complex vector space in advance. Of course we can consider V as real vector space and then take its complexification. But on $V \times V$ we can also consider the complex product structure. So we have two different complex structures on the vector space $V \times V$. This might lead to confusion. To avoid this we denote the new multiplication by i by

$$J(a, b) = (-b, a)$$

and the old one by

$$\mathrm{i}(a, b) = (\mathrm{i}a, \mathrm{i}b).$$

2.1 Remark. *Let V be a complex vector space. On $V \times V$ we have two complex structures.*

1) *Internal multiplication with* i *is defined through* $\mathrm{i}(a,b) = (\mathrm{i}a, \mathrm{i}b)$
2) *External multiplication with* i *is defined through* $J(a,b) = (-b,a)$.

Obviously $\mathrm{i}J(a,b) = J\mathrm{i}(a,b)$. The complexification of V is $V \times V$ together with the external J. We write this complex vector space as $V_{\mathbb{C}} = V \times V$ (with external multiplication J). In simplified notation we can write

$$V_{\mathbb{C}} = V + JV, \qquad J(a+Jb) = -b + Ja.$$

We apply this construction to the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of all *complex* 2×2-matrices with trace zero. We have to consider its complexification $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$. We define a Lie bracket on $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ by means of the formula

$$[A_1 + JA_2, B_1 + JB_2)] := [A_1, B_1] - [A_2, B_2] + J([A_1, B_2] + [A_2, B_1]).$$

Embed $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ by $A \mapsto A + J0$. Then the Lie bracket that we introduced on $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ is just the $\mathbb{C}$-linear extension of the Lie bracket on $\mathfrak{sl}(2,\mathbb{C})$. This bracket is $\mathbb{C}$-bilinear (where multiplication by i is given by J). This means

$$[J(A_1 + JA_2), B_1 + JB_2] = J[A_1 + JA_2, B_1 + JB_2] = [A_1 + JA_2, J(B_1 + JB_2)]$$

which is easy to check. In this sense we can call $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ the complexified Lie algebra of $\mathfrak{sl}(2,\mathbb{C})$ considered as real Lie algebra.

We also can consider $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ as complex vector space (with the internal multiplication by i and the Lie bracket

$$[(A_1, B_1], [A_2, B_2]) = ([A_1, B_1], [A_2, B_2]).$$

We call this the product Lie algebra. A priori this is different from $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$. Nevertheless we will see that both are isomorphic.

2.2 Lemma. *The maps*

$$\begin{aligned} \mathfrak{sl}(2,\mathbb{C}) &\longrightarrow \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}, \quad A \longrightarrow A^{\pm}, \\ A^{+} &= \frac{1}{2}(\bar{A}, \mathrm{i}\bar{A}), \\ A^{-} &= \frac{1}{2}(A, -\mathrm{i}A), \end{aligned}$$

are $\mathbb{C}$-linear homomorphisms of Lie algebras.

Proof. The $\mathbb{C}$-linearity means of course for example $(\mathrm{i}A)^{\pm} = JA^{\pm}$. This and the compatibility with the Lie bracket is easy to check. □

We simplify the notation and write for $(A, B) \in \mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}}$

$$A + JB := (A, B)$$

Then we have

$$A^{+} = \frac{1}{2}(\bar{A} + J\mathrm{i}\bar{A}), \quad A^{-} = \frac{1}{2}(A - J\mathrm{i}A).$$

Recall that in $\mathfrak{sl}(2, \mathbb{C})$ we considered the basis E^{+}, E^{-}, W. (Here the signs in the exponent have nothing to do with Lemma 2.2.) We consider their images in the complexification and get 6 elements of $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}}$

$$W^{+}, E^{++}, E^{-+}; \quad W^{-}, E^{+-}, E^{--}$$

which give a complex basis. Now we can proof a structure result.

2.3 Lemma. *The map*

$$\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}}, \quad (A, B) \longmapsto A^{+} + B^{-}$$

is an isomorphism of complex Lie algebras where on the left hand side i *acts componentwise (and the Lie product comes from the product structure) and on the right hand side via J.*

Proof. One has to use Lemma 2.2 and one has to check $[A^{+}, B^{-}] = 0$. □

The both sides in Lemma 2.3 are equal as vector spaces. But they have different structures as Lie algebras. Nevertheless they are isomorphic.

Now we consider a (complex associative) algebra $\mathcal{A}$ and a *real* linear Lie algebra homomorphism

$$\varphi : \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathcal{A}.$$

"Lie homomorphism" means that $\varphi([A, B]) = \varphi(A)\varphi(B) - \varphi(B)\varphi(A)$. We can extend it to a $\mathbb{C}$-linear Lie homomorphism

$$\varphi : \mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}} \to \mathcal{A}.$$

Here $\mathbb{C}$-linear refers of course to the complexification complex structure of $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}}$ where multiplication by i is given by J. This means that we have to define

$$\varphi(A + JB) = \varphi(A) + \mathrm{i}\varphi(B).$$

The $\mathbb{C}$-linearity means

$$\varphi(J(A + JB)) = \mathrm{i}\varphi(A + JB) = \mathrm{i}(\varphi(A) + \mathrm{i}\varphi(B)) = -\varphi(B) + \mathrm{i}\varphi(A).$$

2.4 Remark. *Let $\varphi : \mathfrak{sl}(2, \mathbb{C}) \to \mathcal{A}$ a real linear Lie homomorphism into an associative complex algebra $\mathcal{A}$. Extending it $\mathbb{C}$-linear to $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}}$ and then restricting to $\mathfrak{sl}(2, \mathbb{C})$ by means of $A \mapsto A_{+}$ or (A_{-}), one gets two complex linear Lie homomorphisms*

$$\varphi_{\pm} : \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathcal{A}.$$

We have to start with a real Lie homomorphism

$$\varphi : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathcal{A}$$

where $\mathcal{A}$ is a complex assoziative algebra. First we have to extend φ to a $\mathbb{C}$-linear map

$$\varphi : \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} \longrightarrow \mathcal{A}.$$

To extend φ in the $\mathbb{C}$-linear way we have to define

$$\varphi(A + JB) = \varphi(A) + \mathrm{i}\varphi(B).$$

Now we make use of the two embeddings

$$\mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}, \quad A \longmapsto A^{\pm}.$$

They induce the two $\mathbb{C}$-linear homomorphisms

$$\begin{aligned} \varphi^{\pm} : \mathfrak{sl}(2,\mathbb{C}) \to \mathcal{A}, \quad \varphi^{+}(A) &= \frac{1}{2}(\varphi(\bar{A}) + \mathrm{i}\varphi(\mathrm{i}\bar{A})), \\ \varphi^{-}(A) &= \frac{1}{2}(\varphi(A) - \mathrm{i}\varphi(\mathrm{i}A)). \end{aligned}$$

The point is that these are $\mathbb{C}$-linear (but φ needs not). Hence each of the both gives a Casimir operator.

3. Casimir operators

Recall that any $\mathbb{C}$-linear Lie homomorphism of $\mathfrak{sl}(2,\mathbb{C})$ into an associative complex algebra introduces a certain Casimir element in $\mathcal{A}$. It has the property that it commutes with the image of $\mathfrak{sl}(2,\mathbb{C})$. For trivial reason it also commutes with the image of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$. By means of Remark 2.4 we get now two Casimir elements

$$\boxed{\begin{aligned} \omega_{+} &= \varphi(E^{++})\varphi(E^{-+}) + 2\mathrm{i}\varphi(W^{+}) - \varphi(W^{+})^2, \\ \omega_{-} &= \varphi(E^{+-})\varphi(E^{--}) + 2\mathrm{i}\varphi(W^{-}) - \varphi(W^{-})^2, \end{aligned}}$$

3.1 Proposition. *Let $\mathfrak{sl}(2,\mathbb{C}) \to \mathcal{A}$ be a real linear Lie homomorphism. We extend it by complex linearity to $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ and restrict it in two ways to complex linear Lie homomorphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathcal{A}$. These Lie homomorphisms produce Casimir elements $\omega_{\pm} \in \mathcal{A}$. They commute with the image of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$.*

We want to express these Casimir operators by means of a different real basis of $\mathrm{SL}(2,\mathbb{C})$. We choose an explicit $\mathbb{R}$ basis of $\mathfrak{sl}(2,\mathbb{C})$,

$$X_1, X_2, X_2, \quad \mathrm{i}X_1, \mathrm{i}X_2, \mathrm{i}X_3.$$

We know how the elements X_1, X_2, X_3 act on $\mathcal{H}$. But we do not know so far how $\mathrm{i}X_1, \mathrm{i}X_2, \mathrm{i}X_3$ act. Recall the module structure of $\mathcal{H}$ is given through *real* linear Lie homomorphism

$$\varphi : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathrm{End}(\mathcal{H}).$$

This has been extended $\mathbb{C}$-linear to $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$. This means $\varphi(JA) = \mathrm{i}\varphi(A)$. But $\varphi(\mathrm{i}A)$ and $\mathrm{i}\varphi(A)$ are different.

The basis $X_1, \dots, \mathrm{i}X_3$ is also a $\mathbb{C}$-basis of the complexification $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$. Since the extension of $d\pi$ to this complexification is $\mathbb{C}$-linear, it seems to be natural to work in the complexification with a $\mathbb{C}$-basis. There is another natural $\mathbb{C}$-basis of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$. For this we recall the two $\mathbb{C}$-linear embeddings

$$\mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}, \quad A \longmapsto A^{\pm},$$
$$A^+ = \frac{1}{2}(\bar{A} + J(\mathrm{i}\bar{A})), \quad A^- = \frac{1}{2}(A - J(\mathrm{i}\bar{A})).$$

We choose in $\mathfrak{sl}(2,\mathbb{C})$ the elements W, E^+, E^- and take there images under $A \mapsto A^{\pm}$.

$$W^+, E^{++}, E^{-+}; \quad W^-, E^{+-}, E^{--}.$$

This is also an $\mathbb{C}$-basis of the complexification.

We have

$$\begin{aligned} E^{++} &= \frac{1+J\mathrm{i}}{2}E^-, \\ E^{+-} &= \frac{1-J\mathrm{i}}{2}E^+, \\ E^{-+} &= \frac{1+J\mathrm{i}}{2}E^+, \\ E^{--} &= \frac{1-J\mathrm{i}}{2}E^-,, \\ W^+ &= \frac{1+J\mathrm{i}}{2}W, \\ W^- &= \frac{1-J\mathrm{i}}{2}W. \end{aligned}$$

Making use of $W = 2X_2$ and $E^{\pm} = 2(\pm X_3 - \mathrm{i}X_1)$ we obtain

$$\begin{aligned} E^{++} &= -(1+J\mathrm{i})(X_3 + \mathrm{i}X_1), \\ E^{-+} &= (1+J\mathrm{i})(X_3 - \mathrm{i}X_1), \\ W^+ &= (1+J\mathrm{i})X_2, \\ E^{+-} &= (1-J\mathrm{i})(X_3 - \mathrm{i}X_1), \\ E^{--} &= -(1-J\mathrm{i})(X_3 + \mathrm{i}X_1), \\ W^- &= (1-J\mathrm{i})X_2. \end{aligned}$$

We apply φ which is $\mathbb{C}$-linear with respect to J (but not to i and obtain

$$\begin{aligned}
\varphi^+(E^+) &= -\varphi(X_3 + \mathrm{i}X_1) + \mathrm{i}\varphi(X_1 - \mathrm{i}X_3),\\
\varphi^+(E^-) &= \varphi(X_3 - \mathrm{i}X_1) + \mathrm{i}\varphi(X_1 + \mathrm{i}X_3),\\
\varphi^+(W) &= \varphi(X_2) + \mathrm{i}\varphi(\mathrm{i}X_2),\\
\varphi^-(E^+) &= \varphi(X_3 - \mathrm{i}X_1) - \mathrm{i}\varphi(X_1 + \mathrm{i}X_3),\\
\varphi^-(E^-) &= -\varphi(X_3 + \mathrm{i}X_1) + \mathrm{i}\varphi(-X_1 + \mathrm{i}X_3),\\
\varphi^-(W) &= \varphi(X_2) - \mathrm{i}\varphi(\mathrm{i}X_2).
\end{aligned}$$

For the Casimir operators we get

$$\begin{aligned}
\omega_+ =& \varphi^+(E^+)\varphi^+(E^-) + 2\mathrm{i}\varphi^+(W) - (\varphi^+(W))^2,\\
=& -(\varphi(X_1)^2 + \varphi(X_2)^2 + \varphi(X_3)^2)\\
&+ \varphi(\mathrm{i}X_1)^2 + \varphi(\mathrm{i}X_2)^2 + \varphi(\mathrm{i}X_3)^2\\
&- 2\mathrm{i}(\varphi(X_1)\varphi(\mathrm{i}X_1) + \varphi(X_2)\varphi(\mathrm{i}X_2) + \varphi(X_3)\varphi(\mathrm{i}X_3)\\
\omega_- =& \varphi^-(E^+)\varphi^-(E^-) + 2\mathrm{i}\varphi^-(W) - (\varphi^-(W))^2,\\
=& -(\varphi(X_1)^2 + \varphi(X_2)^2 + \varphi(X_3)^2)\\
&+ \varphi(\mathrm{i}X_1)^2 + \varphi(\mathrm{i}X_2)^2 + \varphi(\mathrm{i}X_3)^2\\
&+ 2(\mathrm{i}(\varphi(X_1)\varphi(\mathrm{i}X_1) + \varphi(X_2)\varphi(\mathrm{i}X_2) + \varphi(X_3)\varphi(\mathrm{i}X_3))
\end{aligned}$$

Instead of ω_+, ω_- we will use also

$$\Box_+ = -\frac{\omega_+ + \omega_-}{2}, \quad \Box_- = \mathrm{i}\frac{\omega_+ - \omega_-}{2}.$$

$$\boxed{\begin{aligned}
\Box_+ =& \varphi(X_1)^2 + \varphi(X_2)^2 + \varphi(X_3)^2 - \varphi(\mathrm{i}X_1)^2 - \varphi(\mathrm{i}X_2)^2 - \varphi(\mathrm{i}X_3)^2\\
\Box_- =& \varphi(X_1)\varphi(\mathrm{i}X_1) + \varphi(X_2)\varphi(\mathrm{i}X_2) + \varphi(X_3)\varphi(\mathrm{i}X_3)
\end{aligned}}$$

4. The Casimir and explicit formulae

We consider a special φ,

$$\varphi : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathrm{End}(\mathcal{C}^\infty(G))$$

that is defined through the Lie derivative.

$$(\varphi(A)(f)) = \frac{d}{dt} f(x \exp(tA))\big|_{t=0}.$$

This map is only real linear (but $\mathcal{C}^\infty(G)$ means the space of complex valued functions. Even more general, one can consider differentiable functions on G with values in a complex Banach space). We use sometimes the notations

$$Af = \varphi(A)f$$

if the context clearly indicates what is meant. In the case of the group $\mathrm{SL}(2,\mathbb{R})$ we got explicit formulas for the Casimir operator. The case $\mathrm{SL}(2,\mathbb{C})$ is more involved. We must be satisfied with a weaker result. For this we consider a finite dimensional representation $\sigma : K \to \mathrm{GL}(H)$ of the compact group $K = \mathrm{SU}(2)$. Then we consider differentiable functions

$$f : \mathrm{SL}(2,\mathbb{C}) \to H, \quad f(xk) = \sigma(k)f(x) \qquad (x \in \mathrm{SL}(2,\mathbb{C}),\ k \in K).$$

Such a function is determined by its restriction f_0 to AN, since

$$f(ank) = \sigma(k)f_0(an)$$

and each differentiable function f_0 on AN extends to a function f.

We consider now a Casimir operator C. Since C is right invariant, we get the following result.

4.1 Lemma. *Let C be a Casimir operator for $G = \mathrm{SL}(2,\mathbb{C})$ and $f : G \to H$ be a differentiable function with the transformation property $f(xk) = \sigma(k)f(x)$. Then $g = C(f)$ has the same transformation property.*

Hence there exists an operator (depending on σ)

$$C_0 : \mathcal{C}^\infty(AN) \longrightarrow \mathcal{C}^\infty(AN)$$

such that $Cf = g$ means $C_0 f_0 = g_0$.

4.2 Lemma. *Let C be a linear combination of $\omega_\pm$ and let $\sigma : K \to \mathrm{GL}(H)$ be a finite dimensional representation. The corresponding operator C_0 on AN is a linear differential operator of degree ≤ 2. Its part of order two is independent of σ.*

Proof. We introduced the notion of a linear differential operator on open subsets of $\mathbb{R}^n$. We can identify AN with the hyperbolic 3-space $\mathcal{H}_3$ and so we can apply this notion. We also know from the Iwasawa decomposition that $\mathcal{H}_3 \times \mathcal{B}_3$, $\mathcal{B}_3$ denotes the 3-ball, parameterises some open part of G. It is clear that the Lie derivative is a linear differential operator of degree ≤ 1 on $\mathcal{H}_3 \times \mathcal{B}_3 \subset \mathbb{R}^6$. The composition of two of them gives a linear differential operator of degree ≤ 2. The same is true for C_0. □

In the case of the group $\mathrm{SL}(2,\mathbb{R})$ we got explicit formulas for a concrete basis of the Lie algebra. The case $\mathrm{SL}(2,\mathbb{C})$ is more involved. We must be satisfied with a weaker result. For this we consider a finite dimensional unitary representation $\sigma : K \to \mathrm{GL}(H)$ of the compact group $K = \mathrm{SU}(2)$. Then we consider differentiable functions

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Such a function is determined by its restriction f_0 to AN, since

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such that $Cf = g$ means $C_0 f_0 = g_0$.

4.4 Lemma. *Let C be a linear combination of $\omega_\pm$ and let $\sigma : K \to \mathrm{GL}(H)$ be a finite dimensional representation. The corresponding operator C_0 on AN is a linear differential operator of degree ≤ 2. Its part of order two is independent of σ.*

Proof. We introduced the notion of a linear differential operator on open subsets of $\mathbb{R}^n$. We can identify AN with the hyperbolic 3-space $\mathcal{H}_3$ and so we can apply this notion. We also know from the Iwasawa decomposition that $\mathcal{H}_3 \times \mathcal{B}_3$, $\mathcal{B}_3$ denotes the 3-ball, parameterises some open part of G. It is clear that the Lie derivative is a linear differential operator of degree ≤ 1 on $\mathcal{H}_3 \times \mathcal{B}_3 \subset \mathbb{R}^6$. The composition of two of them gives a linear differential operator of degree ≤ 2. The same is true for C_0. □

We recall (Lemma 4.4) that for example $\Box_+$ induces a linear differential operators on AN of order ≤ 2. It depends on the choice of a $\sigma \in \hat{K}$. But the part of order two is independent of this choice. We want to prove that this operator is elliptic. For this we need only the part of order two. Hence we can assume that σ is trivial. So we have to consider a differentiable function f on G with the property

$$f(pk) = f(p),$$

We have to consider $h = \Box_+ f$. We know that this has also the property $h(pk) = h(p)$. We have to extract from h the terms of order 2. We want to work with the coordinates of the hyperbolic 3-space. So we have to use the identification

$$P \xrightarrow{\sim} \mathcal{H}_3, \quad p \longmapsto z = p(\mathrm{j}).$$

We write

$$p(j) = x + \mathrm{i}y + \mathrm{j}v, \quad v > 0$$

Hence the coordinates of the hyperbolic 3-space are (x, y, v). We switch freely between the notations

$$(x, y, v), \quad (z, v)\ (z = x + \mathrm{i}x),\ x + \mathrm{i}y + \mathrm{j}v.$$

And we write

$$f_0(z, v) = f(an); \quad a = \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix}, \ n = \begin{pmatrix} 1 & v^{-1}z \\ 0 & 1 \end{pmatrix}.$$

(similarly for h). If X is in the Liealgebra of SU(2), then $e^{\mathrm{i}X}$ is unitary. We apply X to f,

$$\frac{d}{dt} f(anke^{tX})\Big|_{t=0} = \frac{d}{dt} f(an) = 0.$$

So we only have to consider the operators that correspond to $\mathrm{i}X_1, \mathrm{i}X_2, \mathrm{i}X_3$. We know their exponentials.

$$\begin{aligned} e^{2t\mathrm{i}X_1} &= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} \\ e^{2t\mathrm{i}X_2} &= \begin{pmatrix} \cos \mathrm{i}t & \sin \mathrm{i}t \\ -\sin \mathrm{i}t & \cos \mathrm{i}t \end{pmatrix} \\ e^{2t\mathrm{i}X_3} &= \begin{pmatrix} \cos \mathrm{i}t & \mathrm{i}\sin \mathrm{i}t \\ \mathrm{i}\sin \mathrm{i}t & \cos \mathrm{i}t \end{pmatrix} \end{aligned}$$

We start with X_1^2. For this we have to evaluate

$$\frac{\partial^2}{\partial s \partial t} f(g e^{t\mathrm{i}X_1} e^{s\mathrm{i}X_1})$$

at $t = s = 0$. Since we want to apply the formula $f(g) = f_0(g(\mathrm{j}))$ we consider

$$\begin{aligned} Z(t, s) &= x(s, t) + \mathrm{i}y(s, t) + \mathrm{j}v(s, t) = g e^{t\mathrm{i}X_1} e^{s\mathrm{i}X_1}(j) = z + vk(e^{-(t+s)}\mathrm{j}) \\ &= z + vK(s, t) \quad \text{where } K(s, t) = k(e^{-(t+s)}\mathrm{j}). \end{aligned}$$

We have to compute

$$\frac{\partial^2}{\partial s \partial t} f(Z(s, t)) = \frac{\partial}{\partial s}\left[\frac{\partial f_0}{\partial t}(\cdot)\right]$$

$$
\begin{aligned}
= \frac{\partial}{\partial s}\Big[&\frac{\partial f_0}{\partial x}(\cdot)\frac{\partial x(s,t)}{\partial t}+\\
&\frac{\partial f_0}{\partial y}(\cdot)\frac{\partial y(s,t)}{\partial t}+\\
&\frac{\partial f_0}{\partial v}(\cdot)\frac{\partial v(s,t)}{\partial t}\Big]
\end{aligned}
$$

Here the dot stands for $z(s,t), v(s,t)$. Recall that we only intend to compute the terms of second order. Hence the essential part is

$$
\left[\frac{\partial}{\partial s}\frac{\partial f_0}{\partial x}(\cdot)\right]\frac{\partial x(s,t)}{\partial t} + \left[\frac{\partial}{\partial s}\frac{\partial f_0}{\partial y}(\cdot)\right]\frac{\partial y(s,t)}{\partial t} + \left[\frac{\partial}{\partial s}\frac{\partial f_0}{\partial v}(\cdot)\right]\frac{\partial v(s,t)}{\partial t}
$$

The bracket has to be evaluated by means of the chain rule again.

$$
\begin{aligned}
&\left[\frac{\partial^2 f_0}{\partial x^2}(\cdot)\frac{\partial x(s,t)}{\partial s} + \frac{\partial^2 f_0}{\partial x\partial y}(\cdot)\frac{\partial y(s,t)}{\partial s} + \frac{\partial^2 f_0}{\partial x\partial v}(\cdot)\frac{\partial v(s,t)}{\partial s}\right]\frac{\partial x(s,t)}{\partial t}\\
+&\left[\frac{\partial^2 f_0}{\partial x\partial y}(\cdot)\frac{\partial x(s,t)}{\partial s} + \frac{\partial^2 f_0}{\partial y^2}(\cdot)\frac{\partial y(s,t)}{\partial s} + \frac{\partial^2 f_0}{\partial y\partial v}(\cdot)\frac{\partial v(s,t)}{\partial s}\right]\frac{\partial y(s,t)}{\partial t}\\
+&\left[\frac{\partial^2 f_0}{\partial x\partial v}(\cdot)\frac{\partial x(s,t)}{\partial s} + \frac{\partial^2 f_0}{\partial y\partial v}(\cdot)\frac{\partial y(s,t)}{\partial s} + \frac{\partial^2 f_0}{\partial v^2}(\cdot)\frac{\partial v(s,t)}{\partial s}\right]\frac{\partial v(s,t)}{\partial t}
\end{aligned}
$$

We write this in the form of a table

$$
\begin{aligned}
&\frac{\partial^2 f_0}{\partial x^2}(\cdot): &&\frac{\partial x(s,t)}{\partial s}\frac{\partial x(s,t)}{\partial t}\\
&\frac{\partial^2 f_0}{\partial y^2}(\cdot): &&\frac{\partial y(s,t)}{\partial s}\frac{\partial y(s,t)}{\partial t}\\
&\frac{\partial^2 f_0}{\partial v^2}(\cdot): &&\frac{\partial v(s,t)}{\partial s}\frac{\partial v(s,t)}{\partial t}\\
&\frac{\partial^2 f_0}{\partial x\partial y}(\cdot): &&\frac{\partial x(s,t)}{\partial t}\frac{\partial y(s,t)}{\partial s} + \frac{\partial x(s,t)}{\partial s}\frac{\partial y(s,t)}{\partial t}\\
&\frac{\partial^2 f_0}{\partial x\partial v}(\cdot): &&\frac{\partial x(s,t)}{\partial t}\frac{\partial v(s,t)}{\partial s} + \frac{\partial x(s,t)}{\partial s}\frac{\partial v(s,t)}{\partial t}\\
&\frac{\partial^2 f_0}{\partial y\partial v}(\cdot): &&\frac{\partial y(s,t)}{\partial t}\frac{\partial v(s,t)}{\partial s} + \frac{\partial y(s,t)}{\partial s}\frac{\partial v(s,t)}{\partial t}
\end{aligned}
$$

We have to differentiate $Z(s,t)$ by t and s and then evaluate at $s = t = 0$. This is same as to differentiate $Z(0,t)$ by t and then to evaluate at $t = 0$. We abbreviate $Z(t) = Z(0,t)$ and $K(t) = K(0,t)$ and similarly for $x(s,t), \dots$

We make use of some simple rules for differentiating functions $a(t)$ of one real variable with values in the skew field of quaternions (which can be identified with $\mathbb{R}^4$). We use the abbreviation $\dot{a} = da(t)/dt$. Then one has the rules

$$
(ab)^{\cdot} = a\dot{b} + \dot{a}b, \quad (a^{-1})^{\cdot} = -a^{-1}\dot{a}a^{-1}, \quad (ab^{-1})^{\cdot} = (\dot{a} - ab^{-1}\dot{b})b^{-1}.
$$

Now we can compute $\dot Z = v\dot K$. We have

$$K(t) = a(t)b(t)^{-1}, \quad a(t) = k_1 e^{-t}\mathrm{j} - k_2, \quad b(t) = k_2 e^{-t}\mathrm{j} + k_1$$
$$\text{where } k = \begin{pmatrix} k_1 & -k_2 \\ \bar k_2 & \bar k_1 \end{pmatrix}.$$

We have

$$a(0) = k_1\mathrm{j} - k_2,\ \dot a(0) = -k_1\mathrm{j},\ b(0) = \bar k_2\mathrm{j} + \bar k_1,\ \dot b(0) = -\bar k_2\mathrm{j}.$$

Now we apply the quotient rule. Using the rules

$$aja = |a|^2 j \text{ for } a \in \mathbb{C} \text{ and } j\bar a j b = -ab \text{ for } a, b \in \mathbb{C}$$

we obtain

$$\dot K(0) = -\dot k_1 \dot k_2 - \mathrm{j}.$$

Here we use the notation

$$k_1 = \dot k_1 + \ddot k_1 \mathrm{i}, \quad k_2 = \dot k_2 + \ddot k_2 \mathrm{i}$$

(Here the dots have nothing to do with derivations.) Similar calculations have to be done for X_2^2, X_3^2. To distinguish them from X_1^2, we now write $Z_1(s,t), \dots$ instead of $Z(s,t), \dots$ and $Z_2(s,t), \dots$ and $Z_3(s,t), \dots$ in the two other cases.

Next we treat X_2^2. A straight forward calculation gives

$$\begin{pmatrix} \cos(\mathrm{i}t/2) & \sin(\mathrm{i}t/2 \\ -\sin(\mathrm{i}t/2) & \cos(\mathrm{i}t/2) \end{pmatrix}(\mathrm{j}) = \frac{\sin(\mathrm{i}t) + \mathrm{j}}{\cos(\mathrm{i}t)}.$$

This gives

$$\begin{aligned} K_2(t) &= a_2(t) b_2(t)^{-1}, \\ a_2(t) &= k_1(\sin(\mathrm{i}t) + \mathrm{j}) - k_2 \cos(\mathrm{i}t), \\ b_2(t) &= \bar k_2(\sin(\mathrm{i}t) + \mathrm{j}) + \bar k_1 \cos(\mathrm{i}t). \end{aligned}$$

From

$$a_2(0) = k_1\mathrm{j} - k_2,\ \dot a_2(0) = k_1\mathrm{i},\ b_2(0) = \bar k_2\mathrm{j} + \bar k_1,\ \dot b_2(0) = \bar k_2\mathrm{i}$$

we deduce

$$\dot K_2(0) = -2(\dot k_1 \ddot k_1 + \dot k_2 \ddot k_2) + (\dot k_1^2 + \dot k_2^2 - \ddot k_1^2 - \ddot k_2^2)\mathrm{i} + 2(\ddot k_1 \dot k_2 - \dot k_1 \ddot k_2)\mathrm{j}$$

Finally we treat X_3^2. We start with

$$\begin{pmatrix} \cos(\mathrm{i}t/2) & \mathrm{i}\sin(\mathrm{i}t/2) \\ \mathrm{i}\sin(\mathrm{i}t/2) & \cos(\mathrm{i}t/2) \end{pmatrix}(\mathrm{j}) = \frac{\sin(\mathrm{i}t)\mathrm{i} + \mathrm{j}}{\cos(\mathrm{i}t)}$$

From this we get

$$\begin{aligned}K_3(t) &= a_3(t)b_3(t)^{-1},\\ a_3(t) &= k_1(\sin(\mathrm{i}t)\mathrm{i} + \mathrm{j}) - k_2\cos(\mathrm{i}t),\\ b_3(t) &= \bar{k}_2(\mathrm{i}\sin(\mathrm{i}t) + \mathrm{j}) + \bar{k}_1\cos(\mathrm{i}t).\end{aligned}$$

where

$$a_3(0) = k_1\mathrm{j} - k_2,\ \dot{a}_3(0) = -k_2\mathrm{i},\ b_3(0) = \bar{k}_1\mathrm{j} + \bar{k}_2,\ \dot{b}_3(0) = -\bar{k}_2\mathrm{i}$$

We collect in a table

$$\begin{aligned}\dot{K}_1(0) &= 2(\ddot{k}_1\ddot{k}_2 - \dot{k}_1\dot{k}_2) - 2(\dot{k}_1\ddot{k}_2 + \ddot{k}_1\dot{k}_2)\mathrm{i} + (\dot{k}_2^2 + \ddot{k}_2^2 - \dot{k}_1^2 - \ddot{k}_2^2)\mathrm{j},\\ \dot{K}_2(0) &= -2(\dot{k}_1\ddot{k}_1 + \dot{k}_2\ddot{k}_2) + (\dot{k}_1^2 + \dot{k}_2^2 - \ddot{k}_1^2 - \ddot{k}_2^2)\mathrm{i} + 2(\ddot{k}_1\dot{k}_2 - \dot{k}_1\ddot{k}_2)\mathrm{j}\\ \dot{K}_3(0) &= -\dot{k}_1^2 + \ddot{k}_1^2 + \dot{k}_2^2 - \ddot{k}_2^2 + 2(-\dot{k}_1\ddot{k}_1 + \dot{k}_2\ddot{k}_2)\mathrm{i} + 2(\dot{k}_1\dot{k}_2 + \ddot{k}_1\ddot{k}_2)\mathrm{j}\end{aligned}$$

Now get the decisive information about the Casimir operator $C := \square_+$. Recall that we want to act on differentiable functions $f : G \to \mathbb{C}$ of the form $f(pk) = f_0(p)\sigma(k)$ for some finite dimensional representation of $K = \mathrm{SU}(2)$. As we mentioned already, the function Cf is of the same kind. Hence we obtain an operator $C_0 : \mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P)$. We use for P the hyperbolic coordinates (x, y, v).

4.5 Proposition. *The action of the Casimir operator $C = \square_+$ on differentiable functions of the type $f(pk) = f_0(p)\sigma(k)$ is described through a linear differential operator $C_0 : \mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P)$ of order ≤ 2. Its part of order 2 is independent of σ and is given by*

$$v^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial v^2}\right)$$

Hence C_0 is an elliptic differential operator.

Proof. We restrict to a typical example and compute the coefficient of $\partial^2/\partial x^2$. The above table shows that we have to evaluate

$$\frac{\partial x(s,t)}{\partial s}\frac{\partial x(s,t)}{\partial t}$$

at $s = t = 0$ for each of the X_1, X_2, X_3 and to sum them up. This is the same as

$$\sum_{i=1}^{3}\dot{x}_i(0)^2.$$

Recall that $\dot{Z}_i(t) = v\dot{K}_i(t)$. Hence the sum equals v^2 times

$$4(\ddot{k}_1\ddot{k}_2 - \dot{k}_1\dot{k}_2)^2 + 4(\dot{k}_1\ddot{k}_1 + \dot{k}_2\ddot{k}_2)^2 + (-\dot{k}_1^2 + \ddot{k}_1^2 + \dot{k}_2^2 - \ddot{k}_2^2)^2$$

But this is the same as

$$(\dot{k}_1^2 + \ddot{k}_1^2 + \dot{k}_2^2 + \ddot{k}_2^2)^2 = 1.$$

The other coefficients are similar. □

5. Structure of the complex special linear group of degree two

We need a generalization of the upper half plane. The *hyperbolic space* is defined through

$$\mathcal{H}_n = \mathbb{R}^{n-1} \times \mathbb{R}_{>0}.$$

We can identify $\mathcal{H}_2$ with the usual upper half plane. Now we need the three dimensional hyperbolic space. We identify it with $\mathbb{C} \times \mathbb{R}_{>0}$. We write its coordinates in the form (z, p) where z is a complex number and $p > 0$. An elegant way to describe the action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathcal{H}_3$ is to use the skew-field of quaternions $\mathbb{R} + \mathrm{i}\mathbb{R} + \mathrm{j}\mathbb{R} + \mathrm{k}\mathbb{R}$. Multiplication is defined by $\mathbb{R}$-bilinear extension of

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \quad \mathrm{ij} = \mathrm{k},\ \mathrm{jk} = \mathrm{i},\ \mathrm{ki} = \mathrm{j}.$$

This multiplication is associative and without zero divisors. The center is $\mathbb{R}$ and $1 \in \mathbb{R}$ is the unit element. The conjugate of a quaternion is

$$\overline{a_1 + \mathrm{i}a_2 + \mathrm{j}a_2 + \mathrm{k}a_3} = a_1 - \mathrm{i}a_2 - \mathrm{j}a_2 - \mathrm{k}a_3.$$

The rule $\overline{ab} = \bar{b}\bar{a}$ holds. The absolute value of a quaternion is $|a| = \sqrt{a\bar{a}} = \sqrt{a_1^1 + a_2^2 + a_3^3 + a_4^2}$. The inverse of a non zero quaternion is

$$a^{-1} = \frac{\bar{a}}{|a|^2}.$$

Each complex number can be considered as a quaternion whose j- and k-component is 0.

A quaternion is called *pure* if its k-component is zero. We identify the elements $(z, v) \in \mathcal{H}_3$ with the pure quaternion $P = z + jv$. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$. On can show that the quaternion $cP + d$ is different from zero and define then $M(P) = (aP + b)(cP + d)^{-1}$. On also can check that $M(P)$ is in $\mathcal{H}_3$ again and that this defines an action of $\mathrm{SL}(2, \mathbb{C})$ from the left. We leave this as an exercise for the reader (see [EGM], 1.1). The action in the coordinates (z, v) can be calculated.

$$(z^*, v^*) = M(z, v), \quad z^* = \frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}v^2}{|cz + d|^2 + |c|^2 v^2}$$
$$v^* = \frac{v}{|cz + d| + |c|^2 r^2}.$$

We consider the distinguished point $(0, 1)$ (which corresponds to j). One checks that the stbilizer of this point is $\mathrm{SU}(2)$ and one checks

$$\begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix} \begin{pmatrix} 1 & v^{-1}z \\ 0 & 1 \end{pmatrix} (0, 1) = (z, v).$$

In the rest of this section, we use the following notations:

$$\begin{aligned}
G &= \mathrm{SL}(2,\mathbb{C}),\\
A &= \left\{ a = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}; \quad \alpha > 0 \right\},\\
N &= \left\{ n = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}; \quad z \in \mathbb{C} \right\},\\
K &= \mathrm{SU}(2).
\end{aligned}$$

Similar to the real case there is an Iwasawa decomposition.

5.1 Lemma (Iwasawa decomposition). *The map*

$$A \times N \times K \longrightarrow G, \quad (a,n,k) \longmapsto ank,$$

is topological.

Proof. Let $g \in G$. We consider $g(0,1) = (z,v)$. We have

$$p(0,1) = (z,v) \quad \text{where} \quad p = \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix} \begin{pmatrix} 1 & v^{-1}z \\ 0 & 1 \end{pmatrix}.$$

Then $k = p^{-1}g$ stabilizes $(0,1)$ and is hence in K. So $g = pk$ is the Iwasawa decomposition. □

It can be made explicit:

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{c\bar c + d\bar d}} & 0 \\ 0 & \sqrt{c\bar c + d\bar d} \end{pmatrix} \begin{pmatrix} 1 & a\bar c + b\bar d \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{c\bar c + d\bar d}} \begin{pmatrix} \bar d & -\bar c \\ c & d \end{pmatrix}.}$$

We choose the Haar measure dk of K such that the volume of K is one. We denote by da the Haar measure on A. (Recall that it is $d\alpha/\alpha$.) We denote by $dz = dxdy$ the usual Lebesgue measure. It corresponds to the Haar measure dn on N. on $\mathbb{C}$.

5.2 Proposition. *A Haar measure on $G = \mathrm{SL}(2,\mathbb{C})$ can be obtained as follows:*

$$\int_G f(x)dx = \int_A \int_N \int_K f(ank)\, dk\, dn\, da.$$

The proof is the same as that in the case $\mathrm{SL}(2,\mathbb{R})$.

Similar to $\mathrm{SL}(2,\mathbb{R})$ the group P of upper triangular matrices with positive real diagonal plays an important role. The multiplication map

$$A \times N \xrightarrow{\sim} P$$

is a topological map but not a group isomorphism. Nevertheless

$$dp = dnda$$

is a Haar measure on P. he modular function computes as

$$\Delta(p) = \alpha^{-4}, \quad p = an.$$

The proof is the same as in the $\mathrm{SL}(2,\mathbb{R})$-case.r

6. Principal series for the complex special linear group of degree two

Next we construct for each $s \in \mathrm{i}\mathbb{R}$ a unitary representation of $G = \mathrm{SL}(2,\mathbb{C})$. The construction is the same as in the case $\mathrm{SL}(2,\mathbb{R})$. First we define the space

$$H^\infty(s) = \bigl\{f \in \mathcal{C}^\infty(G), \quad f(px) = \alpha^{2+s} f(x)\bigr\}.$$

The group G acts on this space through translation from the right. A function $f \in H^\infty(s)$ is determined by its restriction to K. This gives an identification of $H^\infty(s)$ and $\mathcal{C}^\infty(K)$. We consider on this space the scalar product coming from the Haar measure of K. As in the $\mathrm{SL}(2,\mathbb{R})$-case, the action of G is unitary with respect to this scalar product if $s \in \mathrm{i}\mathbb{R}$. We denote the completion of $H^\infty(s)$ by $H(s)$. This can be identified with $L^2(K)$. In this way we get a unitary representation of G for $s \in \mathrm{i}\mathbb{R}$.

So far we proceeded as in the case $\mathrm{SL}(2,\mathbb{R})$. In this case we considered the even and odd parts of $H(s)$. This is now a little bit more involved. Instead of ± 1 we have to consider now the group

$$K_0 = \left\{ m = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}; \quad |\zeta| = 1 \right\}$$

This is a subgroup of $\mathrm{SU}(2)$ which is isomorphic to $\mathrm{SO}(2)$. This group acts on $H^\infty(s)$ through

$$f(x) \longrightarrow f(mx), \ m \in K_0.$$

We decompose the space into eigenspaces of this action. The characters of K_0 are of the form $m \mapsto \zeta^n$ for arbitrary integers n. The eigenspace of such a character is

$$H^\infty(n,s) = \bigl\{f \in H^\infty(s); \quad f(mx) = \zeta^n f(x)\bigr\}.$$

Since K_0 is a compact abelian group the space $H(s)$ is the direct Hilbert sum of the spaces $H(n,s)$ which are the completions of the $H^\infty(n,s)$.

6.1 Definition. *The principal series of unitary representations of $G = \mathrm{SL}(2,\mathbb{C})$ consists of the representations $\pi_{n,s}$ on the spaces $H(n,s)$ defined above*

A basic result states that this representation is irreducible. The proof that we have in mind uses the derived representation which we will study in Sect. ?

For this it is important to study the decomposition of the restriction of $\pi_{n,s}$ to $K = \mathrm{SU}(2)$ into irreducibles. Recall that $H(n,s)$ has been embedded into $L^2(K)$. In corresponds in this picture to all $f \in L^2(K)$ with the property

$$f(mk) = \zeta^n f(k)$$

The action of K on this space is given by translation from the right.

Now we have to use the structure theory of compact groups K. To describe the basic result we use the representation of $K \times K$ on $L^2(K, dk)$ given by

$$f(x) \longmapsto f(k_1^{-1} x k_2)$$

We describe its irreducible constituents. Let $\sigma : K \to \mathrm{U}(H)$ be a (finite dimensional) irreducible representation. Consider the space $\mathrm{End}(H)$. There is a natural action of $K \times K$ on $\mathrm{End}\, H$ given by

$$A \longmapsto N, \quad B(v) = \sigma(k_1^{-1}) B(k_2 v).$$

One knows that this representation of K is irreducible. As a consequence it carries an essentially unique structure as irreducible representations. The following result holds:

For each irreducible representation $\sigma : K \to \mathrm{GL}(H)$ there exists a $K \times K$-sub representation of $L^2(K)$ which is isomorphic to $\mathrm{End}(H)$. The multiplicity is one. The representation of $K \times K$ on $L^2(K)$ is the direct Hilbert sum of these components.

Now we go back to $K = \mathrm{SU}(2)$. We have to exhibit the subspace of $L^2(K)$ with the property

$$f(mk) = \zeta^{-n} f(k).$$

This means that for each irreducible representation $\sigma : K \to \mathrm{GL}(H)$ we have to compute the subspace of $\mathrm{End}(H)$ the subspace of all A with the property

$$A(\sigma(m)h) = \zeta^{-n} A(h).$$

We take for H the representation V_l. Recall that this is an irreducible representation of K of dimension $m = 2l + 1$. Here n is an arbitrary nonnegative integer. Concretely V_l is the space of all homogeneous polynomials of two variables x_1, x_2 of degree m. The action of K is given by

$$P(k^{-1}x)$$

where x has to be written as column. So we have to compute the space of all linear maps $A : V_l \to V_l$ with the property

$$A(P(\zeta x_1 \zeta^{-1} x_2) = \zeta^n P(x_1, x_2).$$

We take for P a monomial $x_1^{\nu_1} x_2^{\nu_2}$, $\nu_1 + \nu_2 = m$. Then the equation reads as

$$A(\zeta^{\nu_1 - \nu_2} P) = \zeta^n A(P).$$

In the case $A(P) \neq 0$ this means

$$\nu_1 - \nu_2 = n.$$

Together with the relation $\nu_1 + \nu_2 = m$ we see that there is at most one solution (ν_1, ν_2) for given n, m, namely

$$\nu_1 = \frac{m+n}{2}, \quad \nu_2 = \frac{m-n}{2}.$$

Both must be non negative. Hence a solution exists if and only if $m \geq |n|$. Hence the subspace of $\mathrm{Hom}(V_l, V_l)$ we are just considering is 0 if $m < |n|$ and it is $\mathrm{Hom}(\mathbb{C}, V_l)$ in the case $m \geq 0$. But this space is isomorphic to V_l (and this isomorphism is compatible with the action of K. This gives the following result.

6.2 Theorem. *The principal series $\pi_{n,s}$ splits under the group $K = \mathrm{U}(2)$ as follows. The representation V_l of dimension $m = 2l + 1$ occurs if and only if $m \geq |n|$. In this case it occurs only once.*

We discuss another realization of the principle series. For this we derive an explicit formula for the Haar measure on $K_0 \backslash K$. For this we consider a function f on K that is left invariant under K_0 and hence can be considered as function on $K_0 \backslash K$. We extend f to a function on G with the property

$$f\left(\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} g\right) = |a|^{-2} f(g).$$

This is possible due to the assumption on f. Then we define

$$f_0(c, d) = f\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b are arbitrarily chosen such that $ad - bc = 1$. This definition is independent of the choice of a, b, since for another choice a', b' we have

$$\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

The matrix on the right hand side is in the group N and, by assumption f is left invariant under N. Finally we set

$$f_1(z) = f_0(z, 1).$$

6.3 Lemma. *For each differentiable function f on $K_0 \backslash K$ the measure*

$$\int_K f(k) dk := \int_{\mathbb{C}} f_1(z) dx dy$$

is a Haar measure on $K_0 \backslash K$.

A little more generally we can associate to each function $f \in H(n,s)$ the function $f_0 : \mathbb{C} \times \mathbb{C} - \{(0,0)\} \to \mathbb{C}$. We want to define

$$f_0(c,d) = f\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b are chosen such that $ad - bc = 1$. This defines an isomorphism

$$H^\infty(n,s) \xrightarrow{\sim} H'^\infty(n,s)$$

where $H'^\infty(n,s)$ denotes the space of all differentiable functions f_0 on $\mathbb{C} \times \mathbb{C} - \{(0,0)\}$ with the transformation property

$$f_0(Cc, Cd) = |C|^{-s-2}\left(\frac{C}{|C|}\right)^n f_0(c,d).$$

The action $\pi_{n,s}$ of G on $H^\infty(n,s)$ corresponds to the action

$$\pi'_{n,s}(g)f_0(c,d) = f_0((c,d)g).$$

The function f_0 is determined by its restriction

$$f_1(z) = f_0(z,1).$$

We denote by $H''^\infty(n,s)$ the space of all function f_1 that occur in this way.

6.4 Lemma. *The space $H''^\infty(n,s)$ has the property*

$$\mathcal{C}_c^\infty(\mathbb{C}) \subset H''^\infty(n,s) \subset L^2(\mathbb{C}, dxdy)$$

where $dxdy$ is the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$.

Proof. Let $f_1 \in \mathcal{C}_c^\infty(\mathbb{C})$. We have to reconstruct $f_0(z,w)$ from $f_1(z)$. We set

$$f_0(z,w) = \begin{cases} |w|^{2+s} f_1(z/w) & \text{for } w \neq 0, \\ 0 & \text{else.} \end{cases}$$

It is easy to see that this function is differentiable on $\mathbb{C} \times \mathbb{C} - \{(0,0)\}$ and that it has the desired transformation property. Let $f_1 \in H''(n,s)$. We have to show that f_1 is square integrable. It is enough to show that

$$\int_{|z|\geq 1} |f_1(z)|^2 dxdy$$

converges. We transform the integral by means of $z \mapsto 1/z$. The complex derivative is $-1/z^2$. Hence the real functional determinant is $1/|z|^2$. Hence the integral equals

$$\int_{|z|\leq 1} f_1(1/z)/|z|^2 dxdy = \int_{|z|\leq 1} f_0(1/z,1)/|z|^2 dxdy = \int_{|z|\leq 1} f_0(1,z) dxdy.$$

This shows the existence of the integral.

It is easy to compute the action of G on $\mathcal{H}''(n,s)$. The formula is

$$\pi''\begin{pmatrix} a & b \\ c & d \end{pmatrix} f_1(z) = |bz+d|^{-s-2}\Big(\frac{bz+d}{|bz+d|}\Big)^n f\Big(\frac{az+c}{bz+d}\Big).$$

To be precise. This formula holds for all z such that $bz+d\neq 0$. But it extends to the whole of $\mathbb{C}$ by continuity. Of course this action extends to the completion $L^2(\mathbb{C})$. To get an explicit formula we must extend π'' to a continuous action of G on $L^2(\mathbb{C},dxdy)$. The result is as follows. Let $f_1\in L^2(\mathbb{C},dxdy)$. We take a representative $F\in\mathcal{L}^2(\mathbb{C},dxdy)$. Then we define

$$G = |bz+d|^{-s-2}\Big(\frac{bz+d}{|bz+d|}\Big)^n F\Big(\frac{az+c}{bz+d}\Big)$$

first outside the finite set $bz+d=0$ and then we extend it by 0 to the whole of $\mathbb{C}$. This function is square integrable and we can take its class in $L^2(\mathbb{C},dxdy)$. It is easy to show that this defines a continuous action of G on $L^2(\mathbb{C},dxdy)$. It coincides with the action defined on $\mathcal{H}''(s,n)$. Since this space is dense in $L^2(\mathbb{C},dxdy)$, it is the unique continuous extension.

6.5 Remark. *The principal series with parameters (s,n) is isomorphic to the representation of $G=\mathrm{SL}(2,\mathbb{C})$ on $L^2(\mathbb{C},dxdy)$ where the action is given by the formula*

$$|bz+d|^{-s-2}\Big(\frac{bz+d}{|bz+d|}\Big)^n f\Big(\frac{az+c}{bz+d}\Big).$$

7. Complementary series for the complex special linear group of degree two

In the following we use three complex variables

$$z=x+\mathrm{i}y,\quad w=u+\mathrm{i}v,\quad \zeta=\xi+\mathrm{i}\eta.$$

We also will use polar coordinates

$$z=re^{\mathrm{i}\varphi},\quad \zeta=\varrho e^{\mathrm{i}\psi}.$$

The complementary series is a variant of the principal series in the model described in Remark 7.2. This variant depends on the parameters $n=0$ and $s\in(-1,1)$, $s\neq 0$. We start with the space $\mathcal{C}_c^\infty(\mathbb{C})$ and define on this space

$$\langle f,g\rangle = \int_{\mathbb{C}\times\mathbb{C}} |z-w|^{-2-s} f(z)\overline{f(w)}dxdydudv$$

The existence(as Lebesgue integral on $\mathbb{C}^2 = \mathbb{R}^4$) follows from the existence of

$$\int_K |z-w|^{-2-s} dxdydudv$$

where $K \subset \mathbb{C} \times \mathbb{C}$ is a compact subset. We transform $(z,w) \mapsto (z+w,w)$. It remains to show that the integral

$$\int_0^R |z|^{-2-s} dxdy = 2\pi \int_0^R r^{-1-s} dr$$

converges. This is the case if $s > 0$.

We consider the Fourier transform $\hat{f}$ of $f \in \mathcal{C}_c^\infty(\mathbb{C})$,

$$\hat{f}(w) = \frac{1}{2\pi} \int_{\mathbb{C}} f(w) e^{-\mathrm{i}(xu+yv)} dxdy = \frac{1}{2\pi} \int_{\mathbb{C}} f(w) e^{-\mathrm{i}\,\mathrm{Re}(z\bar{w})} dxdy.$$

In Sect. 1 of the appendices we introduced the Fourier transformation and explained that the Fourier transformation of tempered functions is tempered. In particular, the Fourier transform of a differentiable function with compact support is tempered.

7.1 Proposition. *For $f \in \mathcal{C}_c^\infty(\mathbb{C})$ and $0 < s < 2$ the formula*

$$\langle f, f \rangle = 2^s \pi \frac{\Gamma(s/2)}{\Gamma(1-s/2)} \int_{\mathbb{C}} |\hat{f}(z)|^2 |z|^{-s} dxdy$$

holds.

Proof. We have to make use of the *Bessel function*

$$J_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\mathrm{i}x\cos\varphi} d\varphi.$$

It is a differentiable function on the real axis and for big x it is asymptotically to

$$\sqrt{\frac{2}{\pi x}} \cos(x - \pi/4).$$

We need a well known formula (a special case of a Hankel transform)

$$\int_0^\infty r^{-1+s} J_0(rx) dr = 2^{-1+s} \frac{\Gamma(s/2)}{\Gamma(1-s/2)} x^{-s}, \quad 0 < s < 1/2.$$

The existence of the integral for the given parameters follows from the asymptotic behaviour of J_0.

Now we can compute the integral $\langle f,f\rangle$. We integrate first along z and have to consider the integral

$$\int_{\mathbb{C}} |z-w|^{-2+s} f(z)dxdy.$$

We apply the Fourier inversion formula and express f by means of $\hat{f}$.

$$f(z)=\frac{1}{2\pi}\int_{\mathbb{C}} \hat{f}(\zeta)e^{\mathrm{i}\,\mathrm{Re}(z\bar{\zeta})}d\xi d\eta.$$

We insert this to obtain

$$\begin{aligned}&\int_{\mathbb{C}} |z-w|^{-2+s} f(z)dxdy\\ &=\frac{1}{2\pi}\int_{\mathbb{C}} |z-w|^{-2+s}\int_{\mathbb{C}} \hat{f}(\zeta)e^{\mathrm{i}\,\mathrm{Re}(z\bar{\zeta})}d\xi d\eta dxdy\\ &=\frac{1}{2\pi}\int_{\mathbb{C}} |z|^{-2+s}\int_{\mathbb{C}} \hat{f}(\zeta)e^{\mathrm{i}\,\mathrm{Re}((z+w)\bar{\zeta})}d\xi d\eta dxdy.\end{aligned}$$

Now we go over to polar coordinates $z=re^{\mathrm{i}\varphi}$.

$$\frac{1}{2\pi}\int_0^\infty\int_{-\pi}^{\pi} r^{-2+s}\int_{\mathbb{C}} \hat{f}(\zeta)e^{\mathrm{i}\,\mathrm{Re}(re^{\mathrm{i}\varphi}\bar{\zeta})}e^{\mathrm{i}\,\mathrm{Re}(w\bar{\zeta})}d\xi d\eta r dr d\varphi.$$

We replace one of the ζ by $\varrho e^{\mathrm{i}\psi}$ and get

$$\frac{1}{2\pi}\int_0^\infty\int_{-\pi}^{\pi} r^{-1+s}\int_{\mathbb{C}} \hat{f}(\zeta)e^{\mathrm{i}\,\mathrm{Re}(r\rho e^{\mathrm{i}(\varphi-\psi)})}e^{\mathrm{i}\,\mathrm{Re}(w\bar{\zeta})}d\xi d\eta dr d\varphi.$$

We first integrate along φ to obtain

$$\frac{1}{2\pi}\int_0^\infty r^{-1+s}\int_{\mathbb{C}} \hat{f}(\zeta)J_0(\varrho r)e^{\mathrm{i}\,\mathrm{Re}(w\bar{\zeta})}d\xi d\eta dr.$$

Now we can insert the mentioned integral formula for J_0 to finish the proof of Proposition 7.1 in the case $0<s<1/2$. But both sides of the formula are defined for $0<s<2$ and give analytic functions there. Hence the formula holds in this bigger range. □

Due to the Proposition $\langle f,g\rangle$ is a positive definite Hermitian form on $\mathcal{C}_c(\mathbb{C})$. We denote the completion by $H(0,s)$ where $s\in(-1,1)$, $s\neq 0$. This is a Hilbert space. We have to define the action of $G=\mathrm{SL}(2,\mathbb{C})$ on this Hilbert space. For this we take $g=\left(\begin{smallmatrix}a b\\ c d\end{smallmatrix}\right)\in G$. We want to define the action with help of the formula

$$f\Big(\frac{az+c}{bz+d}\Big)|cz+d|^s.$$

There exists a finite set $S\subset\mathbb{C}$ that this transformation defines a map

$$\mathcal{C}_c^\infty(\mathbb{C}-S)\longrightarrow\mathcal{C}_c^\infty(\mathbb{C}).$$

For $c=0$ one can take $S=\emptyset$ and $S=\{a/c\}$ else. This map is norm preserving and hence extends to the completions. But both completions are $H(0,s)$. Hence we get a unitary representation of G on $H(0,s)$.

7.2 Remark. *The complementary series with parameters $(s, 0)$, $s \in (-1, 1)$, $s \neq 0$, is isomorphic to the representation of $G = \mathrm{SL}(2, \mathbb{C})$ on $H(0, s)$ where the action on a certain dense subspace is given by the formula*

$$|bz + d|^{-s-2} \left(\frac{bz + d}{|bz + d|}\right)^n f\left(\frac{az + c}{bz + d}\right).$$

8. Multiplicity one

In this section we generalize the multiplicity one theorem (Theorem II.7.7) to $G = \mathrm{SL}(2, \mathbb{C})$, $K = \mathrm{SU}(2)$.

8.1 Theorem. *Let $\pi : G \to \mathrm{GL}(H)$ by an irreducible unitary representation. In the restriction of π to K each irreducible representation of K occurs wit such multiplicity ≤ 1.*

Proof. The proof is different from the proof in the $\mathrm{SL}(2, \mathbb{R})$. There we made use of the commutativity of the algebra $\mathcal{S}_{n,n}$ which now is not available.

We explain the analogue of $\mathcal{S}_{n,n}$. For this we need a generalization of the convolution product. Let G be a locally compact unimodular group and let $K \subset G$ be a compact subgroup and let dx, dk be their Haar measures. Then we have convolution products $\alpha * \beta$ on $\mathcal{C}(K)$ and $f * g$ on $\mathcal{C}_c(G)$ and, in addition

$$\mathcal{C}(K) \times \mathcal{C}_c(G) \longrightarrow \mathcal{C}_c(G), \quad (\alpha * f)(x) = \int_K \alpha(k) f(k^{-1}x) dk,$$
$$\mathcal{C}_c(G) \times \mathcal{C}(K) \longrightarrow \mathcal{C}_c(G), \quad (f * \alpha)(x) = \int_K f(xk) \alpha(k^{-1}) dk.$$

Notice that in the case $G = K$ this is the usual convolution product. The associative law remains valid, for example $f * (\alpha * \beta) = (f * \alpha) * \beta$.

Now we consider a unitary representation $\pi : G \to \mathrm{U}(H)$. We want to study its restriction to K. Therefore we consider some $\sigma \in \hat{K}$. We recall the element e_σ that is an idempotent in the convolution algebra $\mathcal{C}(K)$. For any $f \in \mathcal{C}_c(G)$ we consider $e_\sigma * f * e_\sigma$. From the Peter Weyl theorem follows that $\pi(e_\sigma * f * \sigma)$ maps $H(\sigma)$ into itself. Therefore it looks natural to consider

$$\mathcal{C}_{c,\sigma}(G) = \{e_\sigma * f * e_\sigma; \quad f \in \mathcal{C}_c(G)\}.$$

8.2 Theorem. *Let G be a locally compact group and let $K \subset G$ be a compact subgroup. Let $\sigma \in \hat{K}$. Then $\mathcal{C}_{c,\sigma}(G)$ is a star algebra (sub algebra of $\mathcal{C}_c(G)$). Let $\pi : G \to \mathrm{U}(H)$ be a unitary representation. Then $\mathcal{C}_{c,\sigma}(G)$ acts on the isotypic component $H(\sigma)$.*

In the following we need the von-Neumann density theorem. It is explained in Sect. 3 from the Appendices (Chap. VI). It uses the SOT-topology on $B(H)$ where H is a Hilbert space. We have to use Theorem VI.3.3. Let π be irreducible. Then the image of $\mathcal{C}_c(G)$ in $B(H)$ is SOT-dense. An easy consequence is that the image of $\mathcal{C}_{c,\sigma}$ is SOT-dense in $B(H(\sigma))$.

8.3 Definition. *An associative algebra $\mathcal{A}$ admits many finite dimensional representations, bounded by n, if for every $A \in \mathcal{A}$, $A \neq 0$, there exists a homomorphism $\pi : \mathcal{A} \to \mathrm{End}(H)$, $\dim(H) \leq n$, $\pi(A) \neq 0$.*

8.4 Theorem (Kaplansky-Godement). *Let $\mathcal{A}$ be an associative algebra that admits many finite dimensional representations, bounded by n. Let H be a Hilbert space and $\mathcal{A} \to B(H)$ a homomorphism such that the image of $\mathcal{A}$ is SOT-dense, then $\dim(H) \leq n$.*

Wie apply the theorem of Kaplansky-Godement to $\mathcal{A} = \mathcal{C}_{c,\sigma}$ and to $H = H(\sigma)$. We make use of a result which we will formulate and prove a little later. It concerns the construction of the principal series of $\mathrm{SL}(2, \mathbb{C})$. The corresponding representation of $\mathcal{C}_{c,\sigma}$ will turn out to be isomorphic to σ. This means that $\mathcal{A}$ admits many finite dimensional representations, bounded by $\dim(H(\sigma))$. So we obtain that $H(\sigma)$ is irreducible. This completes the proof of Theorem 8.1.

□

9. Differentiable and analytic vectors

As in the case of $\mathrm{SL}(2, \mathbb{R})$ we can define the notion of differentiable and analytic functions (may be Banach space valued) on G. Just use the Iwasawa coordinates. We consider irreducible unitary representations $\pi : G \to \mathrm{GL}(H)$. A vector $h \in H$ is called differentiable (analytic) if the function $\pi(x)h$ is differentiable (analytic). This map is only real linear (but $\mathcal{C}^\infty(G)$ means the space of complex valued functions. We denote by H^∞ the subspace of differentiable vectors and by H^ω the subspace of analytic vectors and we denote by H_K the space of the K-finite vectors. Recall that this is the algebraic sum of all (finite dimensional) subspaces which are invariant under K. The spaces of differentiable (analytic) elements are invariant under the action of G. As in the case $\mathrm{SL}(2, \mathbb{R})$ it is easy to prove that the space of differentiable functions is dense.

We consider the restriction of K and decompose the representation into isotypics with respect to K.

$$H = \widehat{\bigoplus}_{\sigma \in \hat{K}} H(\sigma).$$

Then H_K is the algebraic sum of the $H(\sigma)$.

9.1 Lemma. *Let $\pi : G \to \mathrm{GL}(H)$ be a unitary representation such that the isotypic components $H(\sigma)$ are finite dimensional. Then is all elements of H_K are differentiable.*

Proof. As in the case of $\mathrm{SL}(2,\mathbb{R})$ we know that the elements

$$\pi(f)h = \int_G f(x)\pi(x)dx, \quad f \in \mathcal{C}_c^\infty(G),$$

are differentiable. They generate a dense subspace of H. Hence their orthogonal projection to $H(\sigma)$ generates a dense subspace of $H(\sigma)$. Since this space is finite dimensional, all its elements are in the projection It remains to show that the projections are differentiable. This follows from the Peter Weyl theorem.

□

In the case $\mathrm{SL}(2,\mathbb{R})$ we proved more, namely that the elements of H_K are analytic. We can now prove the same result for $G = \mathrm{SL}(2,\mathbb{C})$.

Let $\pi : G \to \mathrm{U}(H)$ be an irreducible unitary representation of $G = \mathrm{SL}(2,\mathbb{C})$. The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ acts on H_K and the formula

$$\pi(k) \circ d\pi(A) = d\pi(kAk^{-1}) \circ d\pi(A), \quad k \in K,\ A \in \mathfrak{g},$$

holds. Let C be a Casimir operator. Then C acts on H_K. We have seen already more, namely that C acts on the isotypic components $H(\sigma)$. Since this irreducible with respect to $\mathfrak{su}(2)$, the Casimir operator acts by a scalar on f. From ? follows that the elements of $H(\sigma)$ are analytic. Hence H_K consists of analytic vectors. Similar to the $\mathrm{SL}(2,\mathbb{R})$-case we are lead to the notion of a $\mathfrak{g}$-K-module. Before we give the definition, we make a simple remark. We have to consider representations $\pi : K \to \mathrm{GL}(V)$ on a complex vector space such that all vectors of V are K-finite. We call such a representation continuous if the representation of K on any finite dimensional subspace of V is continuous in the usual sense. Then one can define for each $\sigma \in \hat{K}$ the isotypic component $V(\sigma)$ and V is the (algebraic) direct sum of all $H(\sigma)$.

9.2 Definition. *Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ and $K = \mathrm{SU}(2)$. A $\mathfrak{g}$-K-module is a complex vector space $\mathcal{H}$ together with a representation*

$$\pi : K \to \mathrm{GL}(\mathcal{H})$$

and a real linear Lie homomorphism

$$d\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H})$$

such that the following conditions hold.

1) *$\mathcal{H}$ consists of K-finite vectors and π is continuous.*

2) *The formula*

$$\pi \circ d\pi(A) = d\pi(kAk^{-1}) \circ d\pi(A), \quad k \in K,\ A \in \mathfrak{g}$$

holds.

*Such a module is called **admissible** if the K-isotypics are finite dimensional and **irreducibly admissible** if in addition for each $h \in H(\sigma)$, $h \neq 0$, one has $\mathcal{A}(h) = \mathcal{H}$. Here $\mathcal{A} \subset \mathrm{End}(\mathcal{H})$ is the $\mathbb{C}$-algebra generated by the image of $\mathfrak{g}$.*

9.3 Remark. *Let $\mathcal{H}$ be an irreducible admisible $\mathfrak{g}$-K-module ($\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, $K = \mathrm{SU}(2)$). Then the Casimir operators act by scalars on $\mathcal{H}$.*

So we have seen that, as in the case $\mathrm{SL}(2,\mathbb{R})$, every unitary irreducible representation of $G = \mathrm{SL}(2,\mathbb{C})$ on a Hilbert space H induces a structure as irreducible admissible $\mathfrak{g}$-K-module on $\mathcal{H} = H_K$. It should be clear what it means that two $\mathfrak{g}$-K modules are isomorphic. Then we have, as in the case $\mathrm{SL}(2,\mathbb{R})$.

Two irreducible unitary representations of $G = \mathrm{SL}(2,\mathbb{C})$ are isomorphic if and only of the associated $\mathfrak{g}$-K-modules are isomorphic.

Now we have the following two tasks.

1) Classify all irreducible admissible $\mathfrak{g}$-K-modules.
2) Exhibit those which come from an irreducible unitary representation of $G = \mathrm{SL}(2,\mathbb{C})$.

In the case $\mathrm{SL}(2,\mathbb{R})$ we solved both problems. Now we are content with a slightly weaker argument.

A $\mathfrak{g}$-K-module is called unitarizible if there exists a (Hermitian positive definit) scalar product on $\mathcal{H}$ such that the elements $A \in \mathfrak{g}$ act skew symmetric, $\langle Ax, y\rangle = -\langle x, Ay\rangle$ and if the operators $\pi(k)$ are unitary.

As in the $\mathrm{SL}(2,\mathbb{R})$ it is rather clear that the $\mathfrak{g}$-K-module associated to an irreducible unitary representation of $G = \mathrm{SL}(2,\mathbb{C})$ is unitarizible.

We have to classify unitarizible irreducible admissible $\mathfrak{g}$-K-modules.

These representations have the following remarkable property.

9.4 Theorem. *For every unitarizable irreducible admissible $\mathfrak{g}$-K-module ($\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, $K = SU(2)$ the multiplicities of the irreducible unitary components are one.*

We will not prove this result, since for our final purpose (classification of the irreducible unitary representations of $\mathrm{GL}(2,\mathbb{C})$ this is result is not necessary, since for admissible modules that come from such representations it is true (Theorem 8.1). If somebody is interested in proof, we recommend to read first the following section. It is possible to modifiy this a little such that it includes such a proof.

10. Unitary dual of the complex special linear group of degree two

In this section we consider $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ as real Lie algebra and $K = \mathrm{SU}(2)$. Let $\mathcal{H}$ be a unitarizable irreducible admissible $\mathfrak{g} \times K$-module H,

$$\pi : \mathfrak{g} \longrightarrow \mathrm{End}(\mathcal{H}), \quad \pi : K \longrightarrow \mathrm{GL}(\mathcal{H}).$$

Recall that we assume in addition that the isotypic components $H(\sigma)$, $\sigma \in \hat{K}$, are irreducible. This means that each irreducible unitary representation occurs at most once in $\mathcal{H}$. Recall that the irreducible representations can be parameterised by an integral or half integral $l \geq 0$. Surely there is a smallest l_0 such that the corresponding σ occurs in $\mathcal{H}$. This is the first important invariant of the module π. Here we have to consider $\mathfrak{sl}(2, \mathbb{C})$ as real Lie algebra.

We can extend this representation $\mathbb{C}$-linearly to the complexification

$$\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) + J\mathfrak{sl}(2, \mathbb{C}).$$

Then we can consider the Casimir operators $\omega^{\pm} \in \mathrm{End}(\mathcal{H})$. Similar to the $\mathrm{SL}(2, \mathbb{R})$-case it can be shown that both act by multiplication with constants $\mu^{\pm}$. These are two other basic invariants for the representation.

10.1 Theorem. *A unitarizible irreducible admissible* $\mathfrak{sl}(2, \mathbb{C})$*-*$\mathrm{SU}(2)$*-module is determined by the parameters* l_0, μ^+, μ^- *up to unitary isomorphism. The parameters* $\mu^{\pm}$ *are real and satisfy the relations*

$$(\mu^-)^2 = 32 l_0^2(\mu^+ + 8l_0^2 - 8), \quad 32(l_0 + 1)^2(\mu^+ + 8l_0^2 + 16l_0) - (\mu^-)^2 > 0.$$

Proof. We have a decomposition

$$\mathcal{H} = \bigoplus_{l \in S} H(l)$$

where S is a certain set of non-negative integral or half-integral non-negative numbers and where $H(l)$ is of dimension $2l+1$ for $l \in S$. Each $H(l)$ is invariant and irreducible under $\mathrm{SU}(2)$. As in Theorem 1.5 we use a basis of $H(l)$ of the form

$$e_{l,-l}, e_{l,-l+1}, \dots, e_{l,-l-1}, e_{ll}$$

and we set

$$H(l, m) = \mathbb{C} e_{lm} \quad \text{for } -l \leq m \leq l, \ l - m \in \mathbb{Z}.$$

So we have

$$\mathcal{H} = \bigoplus_{\substack{l \in S \\ -l \leq m \leq l}} H(l, m).$$

The bases have the following properties.

$$\begin{aligned} X_1 e_{lm} &= \frac{\mathrm{i}}{4}(e_{l,m+1} + c_{l,m-1} e_{l,m-1}), \\ X_2 e_{lm} &= \mathrm{i} m e_{lm}, \\ X_3 e_{lm} &= \frac{1}{4}(e_{l,m+1} - c_{l,m-1} e_{l,m-1}). \end{aligned}$$

$$c_{lm} = -8 \sum_{\substack{-l\le\nu\le m\\ \nu-l\in\mathbb{Z}}} \nu \qquad (\ge l).$$

Recall that in this formuala e_{lm} has to be set to zero if it is outside the range. We introduce a new basis.

$$\boxed{W = 2X_2, \quad R^+ = 2(X_3 - JX_1), \quad R^- = 2(-X_3 - JX_1)}$$

Its advantage is that $R^\pm$ are lowering resp. raising operators

$$We_{l,m} = 2\mathrm{i}me_{lm}, \quad R^+e_{lm} = e_{l,m+1}, \quad R^-e_{lm} = c_{l,m-1}e_{l,m-1}.$$

The generated complex sub-vector space of $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$ (complex structure coming from J) is

$$\mathfrak{u}(2) + J\mathfrak{u}(2).$$

This is isomorphic to $\mathrm{SL}(2,\mathbb{C})$ (complex structure coming from i). The isomorphism comes from the correspondence

$$W \leftrightarrow W, \quad E^+ \leftrightarrow R^+, \quad E^- \leftrightarrow R^-.$$

We consider also the elements

$$W' = \mathrm{i}W, \quad R'^+ = \mathrm{i}R^+, \quad R'^- = \mathrm{i}R^-.$$

Then the 6 elements

$$W, R_+, R_-, \quad W', R'_+, R'_-$$

give a complex basis (complex structure coming from J) of $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$. The relations between them can be computed. The result is

$$\begin{array}{ccc}
[W,R^+] = 2JR^+ & [W,R^-] = -2JR^- & [R^+,R^-] = -4JW \\
[W',R'^+] = -2JR^+ & [W',R'^-] = 2JR^- & [R'^+,R'^-] = 4JW \\
[W,R'^+] = 2JR'^+ & [W,R'^-] = -2JR'^- & [W,W'] = 0 \\
[R^+,R'^+] = 0 & [R^+,R'^-] = -4JW' & [R^+,W'] = -2JR'^+ \\
[R^-,R'^+] = 4JW' & [R^-,R'^-] = 0 & [R^-,W'] = +2JR'^-
\end{array}$$

The first row of this table corresponds to the known relations between W, E^+, E^-. The second row is a trivial consequence of the first row. The rest can be verified directly.

We want to work out the action of W', R'_+, R'_- on $\mathcal{H}$. Recall that

$$\mathcal{H} = \bigoplus_{\substack{l\in S\\ -l\le m\le l}} H(l,m)$$

where the occurring $H(l,m) = \mathbb{C}e_{lm}$ are one dimensional. We know already the action of W, R^+, R^-. The basis elements e_{lm} can be taken such that

$$\begin{aligned}
&W : H(l,m) \xrightarrow{\sim} H(l,m), \quad We_{l,m} = 2\mathrm{i}me_{lm},\\
&R^+ : H(l,m) \xrightarrow{\sim} H(l,m+1), \quad R^+e_{lm} = e_{l,m+1} \qquad (-l \le m < l),\\
&R^- : H(l,m+1) \xrightarrow{\sim} H(l,m), \quad R^-e_{l,m+1} = c_{lm}e_{lm} \qquad (-l \le m < l)
\end{aligned}$$

with known constants c_{lm}. We notice that we can change for each l the basis element e_{lm} by a constant depending on l.

Our next goal is to determine the action of R'^+ on $H(l,l)$. We know that this space is in the kernel of R^+. (The kernel of R^+ is the sum of all $H(l,l)$ that occur in $\mathcal{H}$.) Since R^+ and R'^+ commute, the element $R^+(v)$ for every $v \in H(l,l)$ is contained in the kernel of R^+. Hence it is contained in the sum $\sum H(\nu,\nu)$. Now we make use of the commutation rule $[W, R'^+] = 2JR'^+$. It implies

$$W(R'^+v) = 2\mathrm{i}(l+1)R'^+v.$$

A similar argument works for R'^-. So we get the following result.

10.2 Lemma. *Let $l \in S$. We have*

$$\begin{aligned}
R'^+ &: H(l,l) \longrightarrow H(l+1,l+1),\\
R'^- &: H(l,-l) \longrightarrow H(l+1,-l-1).
\end{aligned}$$

The right hand sides have to be set to zero of there are outside the range.

Next we want to compute the action of W' on $H(l,l)$. This is more involved. We have to bring the two Casimir operators $\square_\pm$ into the game. A straight forward computation shows

$$\begin{aligned}
\square_+ &= 2(\varphi(W)^2 - \varphi(W')^2) + 2\varphi(R'^-)\varphi(R'^+) + 2\varphi(R^+)\varphi(R^-),\\
\square_- &= 4\varphi(W)\varphi(W') - 8\mathrm{i}\varphi(W') + 2\mathrm{i}\varphi(R^-)\varphi(R'^+) + 2\mathrm{i}\varphi(R'^-)\varphi(R^+).
\end{aligned}$$

We will use also the simplified notation

$$\begin{aligned}
\square_+ &= 2(W^2 - W'^2) + 2R'^-R'^+ + 2R^+R^-,\\
\square_- &= 4WW' - 8\mathrm{i}W' + 2\mathrm{i}R^-R'^+ + 2\mathrm{i}R'^-R^+.
\end{aligned}$$

We know that these operators act by scalars on $\mathcal{H}$,

$$\square_+a = \mu^+a, \quad \square_-a = \mu^-a.$$

We apply the two equations to $a = e_{ll}$. Making use of

$$R^+e_{ll} = 0, \quad WW' = W'W, \quad We_{ll} = 2\mathrm{i}le_{ll}, \quad [R^+, R^-] = -4JW$$

the expressions for $\square_\pm$ give

$$\begin{aligned}
2W'^2e_{ll} &= -(8l^2 + 16l - \mu^+)e_{ll} - R'^-R'^+e_{ll}\\
8\mathrm{i}(l+1)W'e_{ll} &= -\mu^-e_{ll} + 2R^-R'^+e_{ll}
\end{aligned}$$

The last relation shows the next lemma.

10.3 Lemma. *We have*

$$W' : H(l,l) \longrightarrow H(l,l) \oplus H(l+1,l),$$

more precisely

$$W'e_{ll} = \beta_{ll}e_{ll} + \gamma_{l+1,l}e_{l+1,l}$$

where

$$\beta_{ll} = \frac{-\mu^-}{8\mathrm{i}(l+1)}, \quad \gamma_{l+1,l}e_{l+1,l} = \frac{2R^-R'^+e_{ll}}{8\mathrm{i}(l+1)}.$$

The image of W' *is contained in* $H(l,l)$ *if and only if* R'^+ *is zero on* $H(l,l)$.

Next we determine the action of W' on all spaces $H(l,m)$.

10.4 Lemma. *Let* $l \in S$, $-l \le m \le l$. *We have*

$$W' : H(l,m) \longrightarrow H(l-1,m) \oplus H(l,m) \oplus H(l+1,m).$$

We will use the notation

$$W'e_{lm} = \alpha_{l-1,m}e_{l-1,m} + \beta_{lm}e_{lm} + \gamma_{l+1,m}e_{l+1,m}$$

(Again terms have to be set to 0 if they are outside the range.)

Proof. The case $m = l$ has been settled. We continue with $m = l-1$. We will make use of the relation

$$\boxed{[R^+,[R^-,W']] = 8W'}$$

which follows from the table of relations above. It means

$$R^+R^-W' - R^+W'R^- - R^-W'R^+ + W'R^-R^+ = 8W'$$

We apply both sides of this relation to e_{ll}. Since $R^+e_{ll} = 0$ we obtain

$$R^+W' : H(l,l-1) \longrightarrow H(l,l) + H(l+1,l).$$

We combine this with the relation $[W,W'] = 0$. It implies

$$W(W'e_{lm}) = 2\mathrm{i}mW'e_{lm}.$$

This means

$$W'e_{lm} = \sum_{|m|\le\nu} C_{\nu m}e_{\nu m}.$$

We set $m = l - 1$ and apply R^+. Since $R^+ e_{l-1,l-1} = 0$ but $R^+ e_{l-1,m} = e_{l-1,m}$ for the other m, we get

$$W' e_{l,l-1} = \alpha_{l-1,l-1} e_{l-1,l-1} + \beta_{l,l-1} e_{l,l-1} + \gamma_{l+1,l-1} e_{l+1,l-1}.$$

This is the first step of an induction to prove Lemma 10.6 for $m = l, l-1, \ldots$. The induction is performed with the help of the relation

$$R^+ R^- W' - R^+ W' R^- - R^- W' R^+ + W' R^- R^+ = 8W'.$$

This proves the lemma. □

Now we make use of the fact that our representation is unitarizible. Hence $\mathrm{SL}(2, \mathbb{C})$ acts by skew symmetric operators and more generally

$$\varphi(A + JB) = -\varphi(A) + \mathrm{i}\varphi(B).$$

We take the scalar product of the two sides of the first of the above two equations with e_{ll} to obtain

$$-2\|W' e_{ll}\|^2 = -(8l^2 - 16l + \mu^+) + \|R'^+ e_{ll}\|^2$$

Now we make use of the second equation. We want to take the square of the norm on both sides. We claim that the two terms on the right hand side are orthogonal, $\langle R^- R'^+ e_{ll}, e_{ll} \rangle = 0$. This follows from the fact that the adjoint of R^- is $-R^+$ and that R^+ annihilates $e_{ll} \in H(l, l)$. So we get

$$64(l+1)^2 \|W' e_{ll}\|^2 = (\mu^-)^2 + 4\langle R^- R'^+ e_{ll}, R^- R'^+ e_{ll} \rangle.$$

To simplify this, we use Lemma 10.2.

$$\langle R^- R'^+ e_{ll}, R^- R'^+ e_{ll} \rangle = -\langle R'^+ e_{ll}, R^+ R^- R'^+ e_{ll} \rangle = 8(l+1)\|R'^+ e_{ll}\|^2.$$

This gives the following result.

$$64(l+1)^2 \|W' e_{ll}\|^2 = (\mu^-)^2 + 32(l+1)\|R'^+ e_{ll}\|^2$$

Now we can use the two equations to eliminate $\|W' e_{ll}\|$.

$$(\mu^-)^2 + 32(l+1)\|R'^+ e_{ll}\|^2 = 32(l+1)^2 \left[(8l^2 - 16l + \mu^+) - \|R'^+ e_{ll}\|^2\right]$$

or

$$(32(l+1)+1)\|R'^+ e_{ll}\|^2 = 32(l+1)^2(8l^2 - 16l + \mu^+) - (\mu^-)^2.$$

The right hand side has to be non-negative. This is nearly to get the inequality in Theorem 10.1. To get it completely we need $R'^+ e_{ll} \neq 0$.

10.5 Lemma. *Let $H(l)$ be non zero. Then R'^+ is non zero on $H(l,l)$.*

Corollary. *Let l_0 be the smallest l_0 such that $H(l_0,l_0)$ is non zero. Then $H(l,l)$ is non zero for all $l \geq l_0$, $l \equiv l_0 \bmod 2$. This means*

$$S = \{l_0, l_0+1, l_0+2, \dots\}$$

Proof. Assume $H(l,l) \neq 0$ and $R'^+(H(l,l)) = 0$. We will show that then

$$\sum_{\lambda > l} H(\lambda)$$

is invariant under $\mathfrak{g}$. This contradicts the irreducibility. From Lemma 10.3 we get that $\langle W' e_{ll}, e_{l+1,l}\rangle = 0$. From the skew-symmetry we get $\langle W' e_{l+1,l}, e_{ll}\rangle = 0$. □

We still can normalize the bases $e_{l,-l}, \dots e_{ll}$ for each $l \geq l_0$, $l - l_0 \in \mathbb{Z}$ (i.e. multiply them by a constant). We do it in such a way that

$$R'^+ e_{ll} = e_{l+1,l+1}, \quad l \geq l_0,\ l - l_0 \in \mathbb{Z}.$$

We want to show that the representation is determined by $\mu^\pm$ and l_0. This implies a lot of formulas which we collect in a table. Some of them are known already and some of them will be proved later.

Recall that the space $\mathcal{H}$ has the basis

$$e_{l,-l}, e_{l,-l+1}, \dots, e_{l,l-1}, e_{ll}$$

where l runs through certain integral or half integral numbers.

$$\begin{aligned}
We_{lm} &= 2\mathrm{i}me_{lm}\\
R^+ e_{l,m} &= e_{l,m+1}\\
R^- e_{l,m+1} &= c_{lm} e_{lm}\\
R'^+ e_{ll} &= e_{l+1,l+1}\\
W' e_{ll} &= \beta_{ll} e_{ll} + \gamma_{l+1,l} e_{l+1,l}\\
W' e_{lm} &= \alpha_{l-1,m} e_{l-1,m} + \beta_{lm} e_{lm} + \gamma_{l+1,m} e_{l+1,m}\\
R'^+ e_{lm} &= \alpha^+_{l-1,m+1} e_{l-1,m+1} + \beta^+_{l,m+1} e_{l,m+1} + \gamma^+_{l+1,m+1} e_{l+1,m+1}\\
R'^- e_{lm} &= \alpha^-_{l-1,m-1} e_{l-1,m-1} + \beta^-_{l,m-1} e_{l,m-1} + \gamma^-_{l+1,m-1} e_{l+1,m-1}\\
[R'^+, R'^-] e_{ll} &= 8l e_{ll}\\
8W' &= R^+R^-W' - R^+W'R^- - R^-W'R^+ + W'R^-R^+\\
\beta_{ll} &= -\mu^-/(8\mathrm{i}(l+1))\\
\gamma_{l+1,l} e_{l+1,l} &= 2R^-R'^+ e_{ll}/(8\mathrm{i}(l+1))
\end{aligned}$$

In the following we have to show that W' , R'^- and the rest of R'^+ can bee computed from l_0, μ^+, μ^- only.

10.6 Lemma. *We have*

$$\begin{aligned}R'^{+} &: H(l,m) \longrightarrow H(l-1,m+1)\oplus H(l,m+1)\oplus H(l+1,m+1),\\ R'^{-} &: H(l,m) \longrightarrow H(l-1,m-1)\oplus H(l,m-1)\oplus H(l+1,m-1).\end{aligned}$$

We will use the notation

$$\begin{aligned}R'^{+}e_{lm} &= \alpha^{+}_{l-1,m+1}e_{l-1,m+1}+\beta^{+}_{l,m+1}e_{l,m+1}+\gamma^{+}_{l+1,m+1}e_{l+1,m+1},\\ R'^{-}e_{lm} &= \alpha^{-}_{l-1,m-1}e_{l-1,m-1}+\beta^{-}_{l,m-1}e_{l,m-1}+\gamma^{-}_{l+1,m-1}e_{l+1,m-1},\end{aligned}$$

Proof. The lemma follows from the formulas

$$R'^{+}=-\mathrm{i}[R^{+},W'],\quad R'^{-}=\mathrm{i}[R^{-},W']$$

and from Lemma 10.3. □

We will use the following notation. A function $f:[a,\infty)\to\mathbb{R}$ is called *rational* if it can by written as quotient $f=P/Q$ where P,Q are two polynomial functions such that Q has no zeros in (a,∞).

10.7 Lemma. *The functions α_{lm}, β_{lm}, γ_{lm} and $\alpha^{\pm}_{lm}$, $\beta^{\pm}_{lm}$, $\gamma^{\pm}_{lm}$ are rational on $[l_0,\infty)$ for fixed m. They depend only on $l_0,\mu^{\pm}$.*

Proof. We have determined the operators $W',R'^{\pm}$. The operators $W,R^{\pm}$ are also known, they leave each single $H(l)$ invariant. The formulas show that $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$ acts on the sum of all $\mathcal{H}(l)$, $l\equiv l_0 \bmod 2$. Since we assume that $\mathcal{H}$ is irreducible we see that $\mathcal{H}(l)$ is different from 0 if and only if $l\equiv l_0$, $l\ge 0$ and we see that the module is determined up to isomorphism by $l_0,\mu^{\pm}$. This is a good deal of Theorem 10.1. We still have to prove the constraint $(\mu^-)^2=32l_0^2(\mu^++8l_0^2-8)$ in the theorem. For this we make use of the relations

$$\begin{aligned}2W'^2e_{ll} &= -(8l^2+16l-\mu^+)e_{ll}-R'^{-}R'^{+}e_{ll},\\ 8\mathrm{i}(l+1)W'e_{ll} &= -\mu^-e_{ll}+2R^-R'^{+}e_{ll}\end{aligned}$$

We insert

$$\begin{aligned}R'^{-}R'^{+}e_{ll} &= \alpha^-_{ll}e_{ll}+\beta^-_{l+1,l}e_{l+1,l}+\gamma^-_{l+2,l}e_{l+2,l}\\ R^-R'^{+}e_{ll} &= c_{l+1,l}e_{l+1,l}.\end{aligned}$$

From the second relation we get

$$8\mathrm{i}(l+1)W'e_{ll}=-\mu^-e_{ll}+2c_{l+1,l}e_{l+1,l}$$

and then

$$\begin{aligned}8\mathrm{i}(l+1)W'^2e_{ll} =& -\mu^-(\beta_{ll}e_{ll}+\gamma_{l+1,l}e_{l+1,l})\\ &+c_{l+1,l}\alpha_{ll}e_{ll}+\beta_{l+1,l}e_{l+1,l}+\gamma_{l+2,l}e_{l+2,l},\\ 2W'^2e_{ll} =& -(8l^2+16l-\mu^+)e_{ll}-(\alpha^-_{ll}e_{ll}+\beta^-_{l+1,l}e_{l+1,l}+\gamma^-_{l+2,l}e_{l+2,l}).\end{aligned}$$

Comparing the e_{ll}-terms gives

$$-4\mathrm{i}(l+1)(8l^2+16l-\mu^+) = c_{l+1,l}\alpha_{ll} + 4\mathrm{i}(l+1)\alpha_{ll}^- - \mu^-\beta_{ll}.$$

The point is now that in the cases which are described in the theorem, all parameters under the described constraints can be actually realized by an irreducible unitary representation of $\mathfrak{g}$. Similar to the $\mathrm{SL}(2,\mathbb{R})$-case, it is better to introduce a new parameter s be the definition

$$s^2 = (\mu_1 + 8l^2 - 8)/8.$$

This means

$$\mu_1 = 8(s^2+1-l^2), \quad \mu_2^2 = (16ls)^2.$$

In the case $l \neq 0$ we can fix s such that

$$\mu_2 = 16ls \qquad (l \neq 0).$$

In the case $s = 0$ this makes no sense. So in this case we have to be satisfied with the fact that s is only determined up to sign.

Now we have to check for which s the inequality in Theorem 10.1 is satisfied. Obviously it is satisfied if s is a real number. We call the triples l, μ^+, μ^- which came from real s the principal series. But that ist all. In the case $l = 0$ one This parameter is defined up to the sign. With this parameter we can also take $s = \mathrm{i}t$ where $t \in (-1, 1)$. This is called the complementary series.

10.8 Theorem. ***Principal Series.***
*For every $l \in \{0, 1/2, 1, \dots\}$ and for any **real** s there exists an irreducible unitary representation $\pi_{l,s}$ which produces the parameters (l, μ^+, μ^-) where*

$$\mu^+ = 8(s^2+1-l^2), \quad \mu^- = 16ls.$$

The parameter s is uniquely determined if $l \neq 0$ and up to the sign if $l = 0$.

Complementary Series.
The same statement is true for $l = 0$ and $s = \mathrm{i}t$, $t \in (-1, 1)$. Here s is also determined up to the sign.

Every unitary irreducible representation of $\mathrm{GL}(2,\mathbb{C})$ is unitary isomorphic to a representation of these two lists.

In the following we will describe the realization of the principal series.

Chapter IV. The Poincaré group

1. The Lorentz group

The Minkowski space of dimension $n+1$ is the vector space $\mathbb{R}^{n+1}$ that has been equipped with the symmetric bilinear form

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots x_{n+1}y_{n+1}.$$

A vector is called time-like if $\langle x, x\rangle = 0$. The set of all time like vectors consists of two connected cones. One of them is define by $x_1 > 0$. We call this the future cone.

The Lorentz group is the subgroup of $\mathrm{GL}(\mathbb{R}^{n+1})$ that preserves this form, $\langle gx, gy\rangle = \langle x, y\rangle$. If one identifies $\mathrm{GL}(\mathbb{R}^{n+1})$ with $\mathrm{GL}(n+1, \mathbb{R})$ in the usual manner, then this means

$$A'JA = J \quad \text{where} \quad J = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

We denote the Lorentz group by $\mathrm{O}(n, 1)$. We always assume $n > 0$. There are two important subgroups. The first is the special orthogonal group $\mathrm{SO}(n, 1)$ which consists of all elements with determinant one. The second is the subgroup $\mathrm{O}^+(n, 1)$ that preserves the future cone. Since time like vectors are mapped to time like vectors, it is sufficient to know that the vector $(1, 0, \dots, 0)$ is mapped to a vector a with $a_1 \geq 0$. For the matrix A this means that $a_{11} > 0$. Hence we have seen that the set of all matrices in the Lorentz group with this property build a group. The elements of this group are called orthochronous. The matrix J is in the Lorentz group and has determinant -1. This shows

$$\mathrm{O}(n, 1) = \mathrm{SO}(n, 1) \cup \mathrm{SO}(n, 1)J.$$

The negative of the unit matrix E is not orthochronous. Hence we see

$$\mathrm{O}(n, 1) = \mathrm{O}^+(n, 1) \cup \mathrm{O}^+(n, 1)(-E).$$

We use the notation

$$\mathrm{SO}^+(n, 1) = \mathrm{O}^+(n, 1) \cap \mathrm{SO}(n, 1).$$

We see

$$\mathrm{O}(n,1) = \mathrm{SO}^+(n,1) \,\cup\, \mathrm{SO}^+(n,1)J \,\cup\, \mathrm{SO}^+(n,1)(-E) \,\cup\, \mathrm{SO}^+(n,1)(-J).$$

It can be shown that $\mathrm{SO}^+(n,1)$ is open in $\mathrm{O}(n,1)$ and connected. Hence $\mathrm{O}(n,1)$ has 4 connected components.

For small n one can find different descriptions. We start with $\mathrm{O}(2,1)$. For this we consider the vector space $\mathcal{X}$ of all skew symmetric real 2×2-matrices

$$X = \begin{pmatrix} x_2 & x_1 \\ -x_1 & x_3 \end{pmatrix}.$$

Their determinant is $-x_1^2 + x_2^2 + x_3^2$. We identify $\mathcal{X}$ with $\mathbb{R}^3$ in the obvious way. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{X}$ through $(A,X) \longmapsto AXA'$. For given A this transformation can be considered as element of $\mathrm{GL}(3,\mathbb{R})$. The above formula for the determinant shows that it is in $\mathrm{O}(3,\mathbb{R})$. From the Iwasawa decomposition one can see that $\mathrm{SL}(2,\mathbb{R})$ is connected. Hence we constructed a homomorphism $\mathrm{SO}^+(2,1) \longrightarrow \mathrm{SL}(2,\mathbb{R})$.. This is surjective and each element of the target has two pre-images. Hence we write

$$\mathrm{SL}(2,\mathbb{R}) = \mathrm{Spin}(2,1).$$

1.1 Proposition. *The homomorphism*

$$\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SO}^+(2,1)$$

is continuous and surjective. Each element of $\mathrm{SO}^+(2,1)$ *has precisely two inverse images which differ only by the sign.*

We skip the proof of the surjectivity. □

Proposition 1.1 is only a special case of a general result. For each n there exists connected locally compact group G and a continuous surjective homomorphism $G \to \mathrm{SO}^+(n,1)$ such that each element of the image has precisely two pre-images. This group is (in an obvious sense) essentially unique and called the spin covering. The usual notation is $\mathrm{Spin}(n,1)$ for this group. We don't give this (not quite trivial construction) in the general case and treat besides $n = 2$ only the case $n = 3$ which is fundamental for physics.

Similar constructions hold for the compact groups $\mathrm{O}(n)$, $n > 1$. It is known that $\mathrm{SO}(n)$ is already connected (there are no plus groups in this case). The spin group in this case is a connected group $\mathrm{Spin}(n)$ together with a continuous homomorphism $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ such that each element of the target has precisely two pre images.

So far we constructed $\mathrm{Spin}(2,1) = \mathrm{SL}(2,\mathbb{R})$. For the construction of $\mathrm{Spin}(3,1)$ we consider the space of all Hermitian 2×2-matrices

$$H = \begin{pmatrix} h_0 & h_1 \\ \bar{h}_1 & h_2 \end{pmatrix}.$$

We identify $\mathcal{H}$ with $\mathbb{R}^4$ through

$$H \longmapsto \Big(\frac{h_0+h_2}{2}, \frac{h_0-h_2}{2}, \operatorname{Re} h_1, \operatorname{Im} h_1\Big).$$

Then we have

$$\det H = x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

The group $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathcal{H}$ through $(A,H) \longmapsto AH\bar{A}'$ It preserves the determinant. Hence we obtain a Lorentz transformation. It can be shown that $\mathrm{SL}(2,\mathbb{C})$ is connected two. Hence we get a homomorphism

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{SO}^+(3,1).$$

1.2 Proposition. *The homomorphism*

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{SO}^+(3,1)$$

is continuous and surjective. Each element of $\mathrm{SO}^+(3,1)$ *has precisely two inverse images which differ only by the sign.*

This allows us to write

$$\mathrm{Spin}(3,1) = \mathrm{SL}(2,\mathbb{C}).$$

The existence of spin coverings is not tied to signature $(n,1)$. For example we can consider the Euclidian orthogonal group $\mathrm{O}(3,\mathbb{R})$. Recall that $\mathrm{O}(n,\mathbb{R})$ consists of all $A \in \mathrm{GL}(n,\mathbb{R})$ with the property $A'A = E$. This is a closed subgroup. The rows and columns have Euclidean length 1. Hence $\mathrm{O}(n,\mathbb{R})$ is a compact group (in contrast to the Lorentz group!). The subgroup $\mathrm{SO}(n,\mathbb{R})$ of elements of determinant one is called the special orthogonal group. One can show that it is connected. The group $\mathrm{O}(n,\mathbb{R})$ can be embedded into the Lorentz group $O(n,1)$ by means of

$$A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

We consider this in the case $n = 3$. We can consider the inverse image in $\mathrm{SL}(2,\mathbb{C})$. One can check that this inverse image is the special unitary group $\mathrm{SU}(2)$. Recall that The unitary group $\mathrm{U}(n)$ is the subgroup of all $A \in \mathrm{GL}(n,\mathbb{C})$ with the property $\bar{A}'A = E$. This is a compact group. The special unitary group is the subgroup of all A with $\det A = 1$. One can show that it is connected.

1.3 Proposition. *The homomorphism*

$$\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3,\mathbb{R})$$

is continuous and surjective. Each element of $\mathrm{SO}(3,\mathbb{R})$ *has precisely two inverse images which differ only by the sign.*

Hence we can write

$$\mathrm{Spin}(3) = \mathrm{SU}(2, \mathbb{C}).$$

What could be Spin(2). It should be a two fold covering of SO(2). This group isomorphic to S^1. Here we have the natural map

$$S^1 \longrightarrow S^1, \quad \zeta \longmapsto \zeta^2.$$

Hence it looks natural to define

$$\mathrm{Spin}(2) = S^1$$

together with the map

$$\mathrm{Spin}(2) \longrightarrow \mathrm{SO}(2), \quad e^{\mathrm{i}\theta} \longmapsto \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix}.$$

2. The Poincaré group

In the following we call $\mathrm{O}(n, 1)$ the *homogeneous Lorentz group.* The *inhomogeneous Lorentz group* is the set of all transformations of $\mathbb{R}^{n+1}$ of the form

$$v \longmapsto A(v) + b$$

where A is a Lorentz transformation and $b \in \mathbb{R}^{n+1}$. This group can be identified with the set $\mathrm{O}(n, 1) \times \mathbb{R}^{n+1}$. The group law then is

$$(g, a)(h, b) = (gh, a + gb).$$

We write for the inhomogeneous Lorentz group simply

$$\mathrm{O}(n, 1)\mathbb{R}^{n+1}.$$

We want to define also a "spin covering". For this we have to consider the action of $\mathrm{Spin}(n, 1)$ on $\mathbb{R}^{n+1}$ which is defined by means of the natural homomorphism $\mathrm{Spin}(n, 1) \to \mathrm{O}(n, 1)$ and the natural action of $\mathrm{O}(n, 1)$ on $\mathbb{R}^{n+1}$. We write this action simply in the form $(g, v) \mapsto gv$. The *Poincaré group* $P(n)$ is the set

$$P(n) = \mathrm{Spin}(n, 1) \times \mathbb{R}^{n+1}$$

together with the group law

$$(g, a)(h, b) = (gh, a + gb).$$

It is clear that this is a group and that the natural map

$$P(n) \longrightarrow \mathrm{O}(n,1)\mathbb{R}^{n+1}$$

(spin covering on the first factor and identity on the second factor) is a homomorphism. This image is $\mathrm{SO}^+(n,1)\mathbb{R}^{n+1}$ and each element has two inverse images.

There is a Euclidian pendent of the inhomogenous Lorentz group. The *Euclidian group* is the set of all transformations of $\mathbb{R}^n$ of the form

$$v \longmapsto A(v) + b$$

where $A \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$. This group can be identified with the set $\mathrm{O}(n) \times \mathbb{R}^n$. The group law then is

$$(g,a)(h,b) = (gh, a + gb).$$

We write for the inhomogeneous Lorentz group simply

$$E(n) = \mathrm{O}(n)\mathbb{R}^n$$

and

$$E_0(n) = \mathrm{SO}(n)\mathbb{R}^n.$$

Orbits

The Lorentz group $\mathrm{O}(3,1)$ acts on $\mathbb{R}^4$ in a natural way. Two elements a, b are in the same orbit if and only if $\langle a,a\rangle = \langle b,b\rangle$. Here $\langle\cdot,\cdot\rangle$ means the Lorentz scalar product. It is easy to derive a system of representatives of the orbits and the corresponding stabilizers.

Representatives of orbits and their stabilizers for the Lorentz group
(natural action of $\mathrm{O}(3,1)$ on $\mathbb{R}^4$)

1)	$(0,0,0,0)$	$\mathrm{O}(3,1)$
2)	$(0,m,0,0),\ m>0$	$\mathrm{O}(2,1)$
3)	$(m,0,0,0),\ m>0$	$\mathrm{O}(3)$
4)	$(1,1,0,0)$	$E(2)$

The subgroup $\mathrm{SO}(3,1)$ has the same orbits since the representatives are fixed by substitution that has determinant -1. Simply change the sign of the last coordinate. But this is not true if on takes $\mathrm{SO}^+(3,1)$. Here the representatives are as follows.

Representatives of orbits and their stabilizers for the restricted Lorentz group
(natural action of $\mathrm{SO}^+(3,1)$ on $\mathbb{R}^4$)

1)	$(0,0,0,0)$		$\mathrm{SO}^+(3,1)$
2)	$(0,m,0,0)$,	$m>0$	$\mathrm{SO}^+(2,1)$
3)	$(0,m,0,0)$,	$m<0$	$\mathrm{SO}^+(2,1)$
4)	$(m,0,0,0)$,	$m>0$	$\mathrm{SO}(3)$
5)	$(1,1,0,0)$		$E_0(2)$
6)	$-(1,1,0,0)$		$E_0(2)$

Finally we treat the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{R}^4$. The orbits are the same as that of $\mathrm{SO}^+(3,1)$. The stabilizers are the inverse images of the stabilizers in $\mathrm{SO}^+(3,1)$.

Representatives of orbits and their stabilizers for the Spin group
(natural action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{R}^4$)

1)	$(0,0,0,0)$		$\mathrm{SL}(2,\mathbb{C})$
2)	$(0,m,0,0)$,	$m>0$	$\mathrm{SL}(2,\mathbb{R})$
3)	$(0,m,0,0)$,	$m<0$	$\mathrm{SL}(2,\mathbb{R})$
4)	$(m,0,0,0)$,	$m>0$	$\mathrm{SU}(2)$
5)	$(1,1,0,0)$		$\mathrm{Iso}(2)$
6)	$-(1,1,0,0)$		$\mathrm{Iso}(2)$

Here $\mathrm{Iso}(2)$ means the inverse image of $E_0(2)$ in $\mathrm{SL}(2,\mathbb{C})$. Hence we see that the irreducible unitary representations of the Poincaré group come from the irreducible unitary representations of the little groups

$$\mathrm{SL}(2,\mathbb{C}),\quad \mathrm{SL}(2,\mathbb{R}) \text{ two possibilities},\quad \mathrm{SU}(2),\quad \mathrm{Iso}(2) \text{ two possibilities.}$$

They have been all determined in the book. Not all irreducible representations of $P(3)$ are of physical significance.

There is the notion of "positive energy" for an irreducible unitary representation of $P(3)$. The definition needs rests on the Hamilton operator which we will introduce a little later (Definition 3.3). Here we just mention that the cases 4) and 5) lead to representations of positive energy. These are the representations of physical interest. In both cases $\langle a,a\rangle$ is non-positive and we can define the mass of such a representation by

$$m := \sqrt{-\langle a,a\rangle}.$$

So we have

$a=(0,m,0,0)$	representation of positive energy and mass $m>0$,
$a=(1,1,0,0)$	representation of positive energy and mass $m=0$.

These representations come from the little groups $\mathrm{SU}(2)$ and $\mathrm{Iso}(2)$.

3. Physical relevance

We keep short.

Special relativity

Minkowski space. The space of all space time events. It is a four dimensional real vector space.

*Observer.*An isomorphism $M \cong \mathbb{R}^4$. So any event corresponds to a point (x_0, x_1, x_2, x_3), x_0 is the time coordinate and the other are the space coordinates.

*Change of an observer.*An isomorphism $\mathbb{R}^4 \to \mathbb{R}^4$. It preserves the form $-x_0^2 + x_1^2 + x_2^2 + x_3^2$. The set of all possible changes of observers is the Lorentz group $\mathrm{O}(3,1)$.

Quantum Mechanics

The physical Hilbert space $\mathcal{H}$.An observer sees states, these are elements of the associated projective space $\hat{\mathcal{H}} = (\mathcal{H} - \{0\})/\mathbb{C}^*$. Change of an observer induces a bijection $\hat{\mathcal{H}} \to \hat{\mathcal{H}}$. The transition probabilities

$$\frac{(\phi, \psi)}{\|\phi\|^2 \|\psi\|^2}$$

are preserved. The set of all these maps is denoted by $\mathrm{Aut}(\hat{\mathcal{H}})$.

Change of observers. This given by a homomorphism

$$G \cong \mathrm{O}^\circ(1,3) \times M \longrightarrow \mathrm{Aut}(\hat{\mathcal{H}}).$$

Special elements of $\mathrm{Aut}(\hat{\mathcal{H}})$ come from unitary operators $U : \mathcal{H} \to \mathcal{H}$. But also antiunitary operators induce elements of $\mathrm{Aut}(\mathcal{H})$. Wigner has shown that each element of $\mathrm{Aut}(\hat{\mathcal{H}})$ comes from a unitary or antiunitary operator. Hence The image of

$$\mathrm{U}(\mathcal{H}) \longrightarrow \mathrm{Aut}(\hat{\mathcal{H}})$$

is a subgroup of index two One can show that the image of $\mathrm{SO}^+(3,1)$ is contained in this subgroup. Such a homomorphism usually can not be lifted to a homomorphism into $\mathrm{U}(\mathcal{H})$ But now the Poincaré group comes into the game. A not quite trivial theorem says that there exists a homomorphism of the Poincaré group $P \to \mathrm{U}(\mathcal{H})$ such that the diagram

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{U}(\mathcal{H}) \\ \downarrow & & \downarrow \\ G & \longrightarrow & \mathrm{Aut}(\hat{\mathcal{H}}) \end{array}$$

commutes. Hence a unitary representation of the Poincarè group is basic for the (special) relativistic Quantum Mechanics.

The Poincaré algebra

We start with some remarks about Lie groups and Lie algebras. There is no need to prove them here, since in all cases we need them, one can verify them directly.

Let G, H be two Lie groups (one of the groups we consider) and let $G \to H$ be a continuous homomorphism. One can show that there exists a unique homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{h}$ such that the diagram

$$\begin{array}{ccc} G & \longrightarrow & H \\ \uparrow & & \uparrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{h} \end{array}$$

commutes. In the case that the fibres of $G \to H$ are discrete (for example finite) this map is injective and even more an isomorphism if their dimensions agree. Examples are our spin coverings, for example

$$\mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{SO}(3, 1).$$

The induced homomorphism of Lie algebras $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{s}(3, 1)$ must be an isomorphism. It is a good exercise to work it out. Similarly $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, and so on. Another observation is that a Lie group G has the same Lie algebra as its connected component. Roughly speaking: the Lie algebra cannot see discrete stuff.

Next we consider the extended Lorentz group. We want to consider it as a matrix group. For this we consider

$$\mathrm{O}(3, 1)\mathbb{R}^4 \longrightarrow \mathrm{GL}(5, \mathbb{R}), \quad (g, a) \longmapsto \begin{pmatrix} g & a \\ 0 & 1 \end{pmatrix}.$$

This defines an isomorphism of the extended Lorentz group with a closed subgroup of $\mathrm{GL}(5, \mathbb{R})$. Hence the Lie algebra is defined. Following our general remarks, the groups $\mathrm{SO}^+(3, 1)\mathbb{R}^4$ and the Poincaré group $P(3)$ should have the same Lie algebra. We skip the direct construction of the Lie algebra $\mathfrak{p}$ of $P(3)$ and just define

$$\mathfrak{p} := \quad \text{Lie algebra of the extended Lorentz group.}$$

Besides $\mathfrak{p}$ we also consider its complexification $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p} + \mathrm{i}\mathfrak{p}$. From definition, $\mathfrak{p}$ consists of al real 5×5-matrices $\tilde{A}$ such that $\exp(t\tilde{B})$ is in the group. It is easy to check that B is of the form

$$\tilde{A} = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}.$$

One computes the Lie bracket as

$$[\tilde{A}, \tilde{B}] = \begin{pmatrix} [A, B] & Ab - Ba \\ 0 & 0 \end{pmatrix}.$$

This shows that we can identify $\mathfrak{p}$ with pairs (A, a). We take this description now as final definition

3.1 Definition. *The Poincaré algebra* $\mathfrak{p}$ *is as vector space*

$$\mathfrak{p} := \mathfrak{so}(3,1) \times \mathbb{R}^4$$

with the Lie bracket

$$([(A,a),(B,b)] = ([A,B], Ab - Ba).$$

In this formula a, b *is understood as column vector.*

The natural embedding

$$\mathfrak{so}(3,1) \longrightarrow \mathfrak{p}, \quad A \longmapsto (A,0)$$

is a Lie homomorphism. Hence $\mathfrak{so}(3,1)$ can be considered as Lie sub-algebra of $\mathfrak{p}$. Similarly $\mathbb{R}^4$ can be considered as subspace of $\mathfrak{p}$, via $a \mapsto (0,a)$. We can consider $\mathbb{R}^4$ as Lie sub-algebra if we equip it with the trivial structure $[A,B]$. (On calls then $\mathbb{R}^4$ an abelian sub-algebra.)

The dimension of $\mathfrak{p}$ is obviously 10. It is easy to write down a basis. In the following we denote the coordinates of $\mathbb{R}^4$ by $(x_0, \dots, x_3)$ and we denote the standard basis of $\mathbb{R}^4$ by $e_0, \dots, e_4$. Similarly the lables of matrices run form 0 to 3.

First we consider the standard basis $e_0, \dots, e_3$ of $\mathbb{R}^4$ and consider them as elements of $\mathfrak{p}$. Then we recall that the elements of $\mathfrak{so}$ are of the form

$$E_{3,1}A, \quad A' = -A, \qquad E_{3,1} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

We denote by $\omega^{(ij)}$ $0 \le i < j \le 3$ the skew symmetric matrix that has entry 1 at the position (i,j) and -1 at the position an der Stelle (j,i) den Eintrag -1 und zeros else.

The 10 elements $e_i \in M$ und $E_{3,1}\omega^{(ij)} \in \mathfrak{p}$ form a (real) basis of $\mathfrak{p}$. Of course they form also a $\mathbb{C}$-basis of $\mathfrak{p}_{\mathbb{C}}$.

Physicists us a slight modification. They use

$$P^i = -\mathrm{i}e_i, \quad J^{ij} = \frac{\mathrm{i}}{2}\omega^{ij}.$$

3.2 Remark. *The* 10 *elements* P^i, J^{ij} *form a complex basis of* $\mathfrak{p}_{\mathbb{C}}$ *The commutation rules*

$$\begin{aligned} \mathrm{i}[J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\sigma\nu}J^{\rho\mu}, \\ \mathrm{i}[P^{\mu}, J^{\rho\sigma}] &= \eta^{\mu\rho}P^{\sigma} - \eta^{\mu\sigma}P^{\rho}, \\ [P^{\mu}, P^{\rho}] &= 0. \end{aligned}$$

determine the structure of the Poincaré algebra.

If we have a unitary representation of the Poincaré group on the Hilbert space $\mathcal{H}$, we can define as earlier the dense subspace of differentiable vectors. The elements of $\mathfrak{p}$ and then, by $\mathbb{C}$-linear extension, then induce operators $H^\infty \to \mathcal{H}^\infty$. This operators often are denoted by the same letter, and, even more, the elements of $\mathfrak{p}$ are called "operators". Their physical names are:

Hamilton operator	$H = P^0$
Momentum operators	$\mathbf{P} := (P^1, P^2, P^3)$
Angular momentum operators	$\mathbf{J} := (J^{23}, J^{31}, J^{12})$
Boost operators	$\mathbf{K} := (J^{01}, J^{02}, J^{03})$

The Hamilton operator commutes with momentum and angular momentum operators, but not with the boost operators.

The Hamilton operator is of great importance for the study of unitary representations $P(3) \to \mathrm{U}(\mathcal{H})$. The corresponding operator $H : \mathcal{H}^\infty \to \mathcal{H}^\infty$ is symmetric,

$$\langle Ha, b\rangle = \langle a, Hb\rangle \quad \text{for } a, b \in \mathcal{H}^\infty.$$

Hence $\langle Ha, a\rangle$ is real for all $a \in \mathcal{H}^\infty$.

3.3 Definition. *The unitary representation $P \to \mathrm{U}(\mathcal{H})$ is of positive energy if the exists $\varepsilon > 0$ such that*

$$\langle Ha, a\rangle \geq \varepsilon\langle a, a\rangle.$$

The eigen values of H for a representation of positive energy are all positive.

Chapter V. Mackey's theory of the induced representation

1. Induced representations, simple case

The basic idea of induced representations is easy to explain. Let $P \subset G$ be a subgroup of a group and let $\sigma : P \to \mathrm{GL}(H)$ be a representation of the subgroup. We consider the space $\mathrm{Ind}(\sigma)$ of all functions $f : G \to H$ with the property

$$f(px) = \sigma(p)f(x) \quad \text{for} \quad p \in P,\ x \in G.$$

Then G acts by right translation on $\mathrm{Ind}(\sigma)$. Assume that G is a locally compact group and that P is a closed subgroup. We want to modify this construction in such a way that we get – for certain σ – a *unitary* induced representation. An example was already given by the construction of the principal series in Chap. I, Sect. 7. Here $G = \mathrm{SL}(2,\mathbb{R})$ and P is the subgroup of all upper triangular matrices with positive diagonal, σ was the one dimensional unitary representation given by the character $\sigma(p) = a^s$ where $\mathrm{Re}\, s = 0$.

Already in this case we had to deal with the problem is that the condition $\Delta_G|P = \Delta_P$ may be false so there is no G-invariant measure on $P\backslash G$.

Before we go to the general case we make a very restrictive assumption which was satisfied in the example of the principal series. We assume that there exists a closed subgroup $K \subset G$ be a closed subgroup of the locally compact group G such that the multiplication map $P \times K \to G$ is a topological map. We assume that G and K are unimodular but we do not assume that P is unimodular. Let Δ be the modular function of P.

1.1 Lemma. *Let $y \in G$. We consider the (continuous) maps $\alpha : K \to K$ and $\beta : K \to P$ which are defined by $ky = \beta(k)\alpha(k)$. Then for each $f \in \mathcal{C}_c(K)$ the formula*

$$\int_K f(\alpha(k))\Delta(\beta(k))dk = \int_k f(k)dk$$

holds.

Proof. This is a generalization of Lemma II.3.3. The same proof works.

□

Now we can give a straight forward generalization of the principal series. Let $\sigma : P \to \mathrm{GL}(H)$ be a *unitary* representation of P. We consider functions $f : G \to V$ with the transformation property

$$f(py) = \Delta(p)^{1/2}\sigma(p)f(y), \qquad p \in P,\ y \in G.$$

(It is essential that we do not induce σ directly but modify it with the factor $\Delta(p)^{1/2}$.) Such a function is determined by its restriction to K and every function on K can be extended to a function with this transformation property on G. The group G acts by translation from the right on the space of functions with this transformation property. We can this consider as an action of G on the space of all functions $f : K \to V$.

1.2 Proposition. *We assume that $G = PK$ and that G and K are both unimodular. Let $\sigma : P \to \mathrm{GL}(V)$ be a unitary representation. The group G acts on functions $f : G \to V$ with the transformation property*

$$f(py) = \Delta(p)^{1/2}\sigma(p)f(y), \qquad p \in P,\ y \in G$$

by translation from the right. These functions can be identified with functions $f : K \to V$. Zero functions on K are transformed into zero functions and square integrable functions into square integrable ones. This induces a ***unitary*** *representation π of G on $L^2(K, V, dk)$.*

The proof is the same as that of Proposition II.3.1.

2. Induced representations, the general case

Unfortunately this construction which we gave in Sect. 1 is not good enough. We want to give up the existence of a decomposition $G = KL$. We simply assume that $L \subset G$ is a closed subgroup of a locally compact group.

The following procedure to overcome this difficulty is due to Mackey.

2.1 Proposition. *Let G be a locally compact group and $L \subset G$ be a closed subgroup. There exists a continuous function $\rho : G \to \mathbb{R}_{>0}$ and with the property*

$$\rho(x\ell) = \frac{\Delta_L(\ell)}{\Delta_G(\ell)}\rho(x).$$

A proof of this Proposition can be found in Follands book [Fo], Proposition 2.54. We do not need this proposition in full generality and can be satisfied with the weaker version that we will explain now.

We start with an important special case where the existence of a function ρ is trivial.

2.2 Remark. *Let $L, K \subset G$ we closed subgroups of a locally compact group such that the map*

$$K \times L \xrightarrow{\sim} G, \quad (k,\ell) \longmapsto k\ell,$$

is topological. Then the function

$$\rho(k\ell) = \frac{\Delta_L(\ell)}{\Delta_G(\ell)}$$

satisfies the condition in Proposition 2.1.

The proof is trivial. □

But this case not enough for our purpose. Let us assume the following:

There exists an open non-empty subset $U \subset G/L$ and a continuous section $s : U \to G$.

(Section means that $s(a)$ is a representative of the coset $aL \in G/L$.) Let $\tilde{U}$ be the inverse image of U in G and $\pi : \tilde{U} \to U$ the natural projection. We can consider the continuous function

$$\rho_U : \tilde{U} \to \mathbb{R}_{>0}, \quad \rho_U(x) = \frac{\Delta_L}{\Delta_G}(xs(\pi(x))^{-1}).$$

Then ρ_U has the desired transformation on $\tilde{U}$. Taking translates we can cover $L\backslash G$ with sets U. Since we have countable basis of the topology we can write

$$L\backslash G = U_1 \cup U_2 \cup \cdots$$

such that in each inverse image $\tilde{U}_i$ a function $\rho_i = \rho_{U_i}$) with such a property exist. We want to glue the ρ_i and do this in the most simple way. We consider the *disjoint* decomposition

$$L\backslash G = B_1 \dot{\cup} B_2 \dot{\cup} \dots \quad \text{where} \quad B_n = U_n - (U_1 \cup \ldots \cup U_{n-1}).$$

We now define ρ such that its restriction to B_i is ρ_i. This is a measurable function for any Radon measure, since the sets B_i are measurable sets. (They are Borel sets).

The assumption that a local continuous section s exists is weak. It is always satisfied in the context of Lie groups. The reason is that for a Lie group G there is a vector space $\mathfrak{g}$ (the Lie algebra) and a surjective map $\mathfrak{g} \to G$ which is a local homeomorphism close to the origin. Even more, there exists a subspace $\mathfrak{p} \subset \mathfrak{g}$ which plays the same role for L. Consider a decomposition of vector spaces $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{a}$. Then $\mathfrak{a} \to G/L$ is a local homeomorphism close to the origin. Now the existence of a local section is clear, it just corresponds to the natural imbedding of $\mathfrak{a}$ into $\mathfrak{g}$.

This argument applies in all situations which we need. Just to keep the presentation as simple as possible, we assume that Proposition 2.1 has been proved.

We need a generalization of the construction of quotient measures (Proposition I.5.5).

2.3 Proposition. *Assume that $L \subset G$ is a closed subgroup of a locally compact group and that ρ is a function as in Assumption 2.1. Let $d\ell$, dx be left invariant Haar measures on L, G. Then there exists a unique Radon measure $d\bar{x}$ on G/L (depending on ρ) such that the formula*

$$\int_G f(x)\rho(x)dx = \int_{G/L} \left[\int_L f(x\ell)d\ell \right] d\bar{x}$$

holds for $f \in \mathcal{C}_c(G)$.

Proof. The proof is the same as that of the existence of the quotient measure, Proposition I.5.5. First we mention the essential fact that the inner integral is right invariant as function of x, since $d\ell$ has been taken to be left invariant.

The essential point of the proof is

$$\int_L f(x\ell)d\ell = 0 \Longrightarrow \int_G f(x)\varrho(x)dx = 0.$$

The left hand side of this equation implies

$$\int_G g(x)\varrho(x) \int_L f(x\ell)d\ell dx = 0 \quad \text{for } g \in \mathcal{C}_c(G).$$

Interchanging the integrations we get

$$\int_L \int_G g(x)\varrho(x)f(x\ell)dxd\ell = 0.$$

In the inner integral we transform $x \mapsto x\ell^{-1}$. Since dx is left invariant, a factor $\Delta_G(\ell)$ comes up.

$$\int_L \int_G g(x\ell^{-1})\varrho(x\ell^{-1})f(x)\Delta_G(\ell)^{-1} = 0.$$

We interchange again the integrations and transform then $\ell \mapsto \ell^{-1}$. A factor $\Delta_L(\ell)$ enters. We also insert $\varrho(x\ell) = \varrho(x)\Delta_L(\ell)/\Delta_G(\ell)$. Then all Δ-factors cancel and we get

$$\int_G f(x)\varrho(x)\int_L g(x\ell)d\ell dx = 0.$$

Since this is true for all g we obtain that $\int_G f(x)\varrho(x)dx = 0$. □

The measure $d\bar{x}$ on G/L is not invariant under the action of G. But it has still the weaker problem that the space of zero functions is invariant under (left-) translation by elements of G.

One can use this measure to define the induced representation of a unitary representation $\sigma : L \to \mathrm{GL}(H)$.

2.4 Definition and Remark. *Assume that $L \subset G$ is a closed subgroup of a locally compact group and that ρ is a function as in Proposition 2.1. Let dx be the corresponding measure on G/L. Let $\sigma : L \to \mathrm{GL}(H)$ be a unitary representation. Consider the space of all measurable functions $f : G \to H$ with the property $f(x\ell) = \sigma(\ell)f(x)$ and such that $\|f(x)\|_\sigma^2$ is integrable considered as function on G/L. The quotient of this space by the subspace of all functions, such that $\|f(x)\|_\sigma^2$ is a zero function (considered on G/L), is a Hilbert space $H(\sigma)$ with the Hermitian inner product*

$$\langle f, g\rangle = \int_{G/L} \langle f(x), g(x)\rangle_\sigma dx.$$

The group G acts on it by means of the modified translation from the right: for $g \in G$ the operator L_g is defined by

$$(L_g f)(x) = f(g^{-1}x)\sqrt{\frac{\rho(g^{-1}x)}{\rho(x)}}.$$

This is a unitary representation, called the (unitary) induced representation of σ to G. It is independent of the choice of ρ up to unitary isomorphism.

Proof. First one checks that the transformation formula $f(x\ell) = \sigma(\ell)f(x)$ is preserved by the action L_g Then one checks $L_{gh} = L_g \circ L_h$ which is very easy. Now we prove that L_g is unitary. This means $\langle L_g f, L_g f_2\rangle = \langle f_1, f_2\rangle$ where $f_i \in \mathcal{C}_c(G/P)$. We can assume that $f_i(x) = \int_L (xl)F_i(x)$ where $F_i \in \mathcal{C}_c(G)$. If we set $\varphi(x) = \langle F_1(x), F_2(x)\rangle$ the claimed formula reads as

$$\int_G \varphi(x)\varrho(x)dx = \int_G \varphi(g^{-1}x)\frac{\varrho(g^{-1}x)}{\varrho(x)}\varrho(x)dx.$$

It is just an application of the left invariance of dx.

It remains to show that the obtained representations for two ϱ_1, ϱ_2 are isomorphic. This is also easy, the intertwining operator is $f \mapsto f\rho_2/\rho_1$.

□

3. Pseudomeasures

A *pseudomeasure* on a locally compact space by definition is just a continuous $\mathbb{C}$-linear map

$$\mu : \mathcal{C}_c(X) \longrightarrow \mathbb{C}$$

Here continuous means the following. Let $f_n : X \to \mathbb{C}$ be a sequence of continuous functions which vanish outside a joint compact set and which converges uniformly to a function f (that is automatically continuous and with compact support), then $\mu(f_n) \to \mu(f)$.

A more formal way is to introduce for compact $K \subset X$ the space $\mathcal{C}_K(X)$ of all a continuous functions with support in K. Then $\mathcal{C}_K(X)$ is a subspace of $\mathcal{C}_c(X)$. The space $\mathcal{C}_K(X)$ is a normed space (maximum norm) and hence a topological space. We can consider the weakest topology on $\mathcal{C}_c(X)$ such that the inclusions are all continuous. Then the convergence explained above is the same as convergence with respect to this topology. It is easy to see that each point $f \in \mathcal{C}_c(X)$ has a countable basis of the system of neighbourhoods. This means that the topology is determined by the convergent sequences. A set is A closed if the limit f_n of an arbitrary convergent sequence in A is contained in A.

Please notice that Radon measures are continuous. More examples of pseudomeasures are obtained as follows. Let (X, dx) be a Radon measure and let $h : X \to \mathbb{C}$ be a continuous function. Then

$$\mu(f) = \int_X f(x)h(x)dx$$

is a pseudomeasure.

Let $Y \subset X$ a closed subset and let μ be a pseudomeasure on Y. Then one can define a pseudomeasure ν on X by

$$\nu(f) = \mu(f|K)$$

We call ν the *injection* from μ to X.

Another simple construction is as follows. Let μ be a pseudomeasure and h be a continuous function on X. The one can define the pseudomeasure $h\mu$ through

$$(h\mu)(f) = h\mu(f).$$

Occasionally we will use the notation

$$\int_X f(x)d\mu(x) := \mu(f)$$

In this notation the injection writes as

$$\int_X f(x)d\nu(x) := \int_Y f(y)d\mu(y).$$

In the following definition we have to consider $\mathcal{C}_c(X)$ for a locally compact space. It can carry different interesting structures as $*$-algebra. For example one can take the usual (pointwise) multiplication of functions and $f^*(x) = \overline{f(x)}$. In the case that G is an locally compact group one can consider the convolution product. The convolution algebra $\mathcal{C}_c(G)$ is also a $*$-Algebra. One can mix the two examples as follows.

3.1 Lemma. *Let G be a locally compact group and let X be a locally compact space together with a continuous map*

$$G \times X \longrightarrow X, \quad (g, x) \longmapsto gx,$$

which defines an action from G on X from the left. Then $\mathcal{C}_c(X \times G)$ can be equipped with a structure as $*$-algebra as follows:

$$(f_1 * f_2)(x, g) = \int_G f_1(x, y) f_2(y^{-1}x, y^{-1}g)dy,$$
$$f^*(x, g) = \overline{f(g^{-1}x, g^{-1})}\Delta_G(g^{-1}).$$

If take $G = \{e\}$ or for X one point, we recover the two examples above.

3.2 Definition. *Let X be a locally compact space and let $\mathcal{C}_c(X)$ equipped with a structure as $*$-algebra. A pseudomeasure μ on $(X, *)$ is called of* ***positive type*** *if*

$$\mu(f * f^*) \text{ is real and non-negative for all } f \in \mathcal{C}_c(X)$$

For the rest of this section we consider a locally compact group G and equip $\mathcal{C}_c(G)$ with the structure as $*$-algebra coming from the convolution. We want to associate to a pseudomeasure μ of positive type on G a unitary representation of G. For this we start with the convolution algebra $(\mathcal{C}_c(G), *)$. We define a pairing

$$\langle f, g \rangle = \mu(g^* * f).$$

This is a Hermitian pairing ($\mathbb{C}$ linear in the first variable and with property $\langle g, f \rangle = \overline{\langle f, g \rangle}$). It is semidefinite, $\langle f, f \rangle \geq 0$. The null space of all f, $\langle f, f \rangle = 0$, is a sub vector space. We denote the factor space by

$$H(\mu) = \mathcal{C}_c(G)/\text{nullspace}.$$

The pairing factors through this factor space and defines a positive Hermitian form on $H(\mu)$ which we denote also by $\langle\cdot,\cdot\rangle$. The completion $\overline{H(\mu)}$ is a Hilbert space. The group G acts on $\mathcal{C}_c(G)$ through translation from the left

$$L_g f(x) = f(g^{-1}x)$$

This action preserves the above null space and hence defines an action of G on $H(\mu)$. This extends to a unitary representation of G on $\overline{H(\mu)}$. The continuity of this representation is clear since right- and left- translation are continuous maps

$$G \times \mathcal{C}_c(G) \longrightarrow \mathcal{C}_c(G).$$

So we obtain the following result.

3.3 Remark. *Let G be a locally compact group and let $\mathcal{C}_c(G)$ be the convolution algebra (which is a $*$-algebra). Let μ be a pseudomeasure of positve type on $(G,*)$. Then the pairing*

$$\langle f, g\rangle = \mu(g^* * f)$$

factors through $H(\mu) = \mathcal{C}_c(G)/\mathrm{nullspace}$ and induces a unitary representation of G on its completion.

This means that pseudomeasures of positive type on a locally compact group produce unitary representations. It is not trivial to construct such pseudomeasures. Here is an example. Consider the pseudomeasure

$$\mu : \mathcal{C}_c(G) \longrightarrow \mathbb{C}, \quad \mu(f) = f(e).$$

Actually this is a Radon measure. It is easy to check that it is of positive type. The associated unitary representation is the regular representation.

The question arises which representations come from a pseudomeasure of positive type. Here is a result in this direction. Assume that $L \subset G$ is a closed subgroup. Let σ be a unitary representation of L that comes from a pseudomeasure of positive type. Then the induced representation $\mathrm{Ind}_L^G(\sigma)$ also comes from a pseudomeasure of positive type.

3.4 Theorem. *Let $L \subset G$ be a closed subgroup and μ be a pseudomeasure of positive type on L and let σ be the associated representation. Consider the injection T of the pseudomeasure $\sqrt{\Delta_G/\Delta_L}\mu$ into G. Then T is of positive type and the associated unitary representation is unitarily equivalent to $\mathrm{Ind}_L^G(\sigma)$.*

We will nod make use of this theorem an omit a proof. $\square$

4. Imprimitivity

Induced representations from a proper closed subgroup $L \subset G$ have an important property. They are *imprimitive.* To define this we have first to introduce the notion of a *system of imprimitivity.*

We will have to consider several $*$-representations

$$S : \mathcal{A} \longrightarrow B(H)$$

of a $*$-algebra $\mathcal{A}$. This is a $*$-homomorphism into the algebra of bounded operators of a Hilbert space. The latter is a $*$-algebra where A^* is the adjoint of A. Such a homomorphism is called *nondegenerate* if a vector $h \in H$ with the property $S(A)h = 0$ for all $A \in \mathcal{A}$ is zero. This is the case if $\mathcal{A}$ contains a unit element and if this is mapped to the identity.

4.1 Definition. *Let Y be a locally compact space and let $\mathcal{C}_c(Y)$ equipped with a structure as $*$-algebra. A homomorphism*

$$\mathcal{C}_c(Y) \longrightarrow B(H), \ \ H \text{ some Hilbert space}$$

is called regular if it is a continuous $*$-homomorphism and if it is non-degenerate.

Continuity refers of course to the SOT-topology of $B(H)$ (and the topology we introduced on $\mathcal{C}_c(Y)$).

4.2 Definition. *A system of imprimivity is a triple (π, Y, S) consisting of*

1) *a unitary representation $\pi : G \to \mathrm{U}(H)$ of a locally compact group,*
2) *a continuous action from the left*

$$G \times Y \longrightarrow Y, \quad (g, y) \longmapsto gy,$$

of G on a locally compact space Y,

3) *a regular homomorphism $S : \mathcal{C}_c(Y) \longrightarrow B(H)$ satisfying*

$$\pi(g)S(f)\pi(g)^{-1} = S(R_g f)$$

Here the star algebra structure of $\mathcal{C}_c(Y)$ is given by usual (pointwise) multiplication and by $f^*(y) = \overline{f(y)}$.

There is an obvious notion of isomorphy of systems of imprimitivity.

A system of imprimitivity is called non-trivial if S consists of more then one point. Every unitary representation can be extended into a trivial system of primitivity in an essentially unique way. Hence systems of imprimitivity can be considered as generalizations of ordinary unitary representations.

4.3 Definition. *A unitary representation of a locally compact group is called **imprimitive** if it belongs to a system of imprimitivity which is non-trivial in the sense that S consists of more than one point.*

4.4 Proposition. *Every unitary representation that is induced from a closed subgroup $L \subset G$ is imprimitive.*

To prove this we consider a locally compact group G and a closed subgroup $L \subset G$ and a unitary representation $\sigma : L \to \mathrm{U}(H)$. We want to associate a concrete system of imprimitivity. We will call this the *canonical system of imprimitivity.* Let $H(\sigma)$ be the representation space of the induced representation π (see Definition and Remark 2.4). An element $f \in \mathcal{C}_c(G/L)$ act on $H(\sigma)$ by multiplication. This gives a map

$$S : \mathcal{C}_c(G/L) \longrightarrow B(H(\sigma)), \quad f \longmapsto m_f \ (m_f(g) = fg).$$

It is easy to verify that $(\pi, G/L, S)$ is a system of imprimitivity. □

Let G be a locally compact group and $\pi : G \to \mathrm{U}(H)$ a unitary representation. As we have learnt this extends to a $*$-homomorphism

$$\mathcal{C}_c(G) \longrightarrow B(H).$$

where the $*$-structure on $\mathcal{C}_c(G)$ comes from the convolution. It is clear that this homomorphism is non-degenerate. Hence it is regular in the sense of 4.1.

One knows that a closed subspace $A \subset H$ is invariant under G if and only if it is invariant under $\mathcal{C}_c(G)$.

This construction can be extended to systems of imprimitivity in a natural way as follows. Let (π, Y, S), $\pi : G \to \mathrm{U}(H)$ be a system of imprimitivity. We consider the algebra $\mathcal{C}_c(Y \times G)$ as star algebra as described in Lemma 3.1. Then one can construct a map

$$T : \mathcal{C}_c(Y \times G) \longrightarrow B(H)$$

as follows. Consider an element $f(y, x)$ from $\mathcal{C}_c(Y \times G)$ for fixed $x \in G$ as function of $y \in Y$. Apply to this function S. The result is a map

$$F : G \longrightarrow B(H), \quad F(x) = S(f(\cdot, x))$$

Now we define an operator on H by

$$h \longmapsto \int_G F(x)\pi(x)h dx.$$

It is easy to show that this operator is bounded. So we obtain a map

$$T : \mathcal{C}_c(Y \times G) \longrightarrow B(H)$$

which we call the natural one.

4.5 Lemma. *Let (π, Y, S), $\pi : G \to \mathrm{U}(H)$ be a system of imprimitivity. We consider the algebra $\mathcal{C}_c(Y \times G)$ as star algebra as described in Lemma 3.1. The natural map*

$$T : \mathcal{C}_c(Y \times G) \longrightarrow B(H)$$

is regular. A closed subspace of H is invariant under G and $\mathcal{C}_c(Y)$ if and only if it is invariant under $\mathcal{C}_c(Y \times G)$.

The proof is given by straight forward calculation which we omit. □

In Remark 3.3 we learnt how to construct unitary representations from pseudomeasures of positive type. There is a straightforward generalization to systems of imprimitivity.

Assume that the locally compact group G acts from the left on a locally compact space Y. Recall that we equipped $\mathcal{C}_c(Y \times G)$ (Lemma 3.1) with a structure as $*$-algebra. Let μ be a pseudomeasure of positive type. Then we can construct a system of imprimitivity (π, Y, S), $\pi : G \to B(H)$, as follows. Consider on $\mathcal{C}_c(Y \times G)$ the pairing

$$\langle f, g \rangle = \mu(g^* * f).$$

This is Hermitian and semipositive. We quotient out the null space to obtain the Hilbert space

$$H = \mathcal{C}_c(Y \times G)/\text{nullspace}$$

The group G acts on $\mathcal{C}_c(Y \times G)$ through translation from the left on both factors

$$(\pi(g)f)(y, h) = f(g^{-1}y, g^{-1}h)$$

One checks that this factors through H and gives a unitary representation. It remains to define the operator $S(f)$ for $f \in \mathcal{C}_c(Y)$. It is induced from the multiplication operator on $\mathcal{C}_c(Y \times G)$

$$S(f)h(y, x) = f(y)h(y, x) \qquad (h \in \mathcal{C}_c(Y \times G)).$$

A straight forward calculation shows that we constructed a system of imprimivity.

4.6 Lemma. *Let G be a locally compact group that acts on a locally compact space Y from the left and let $\mathcal{C}_c(Y \times G)$ the associated $*$-algebra. To every pseudomeasure of positive type on this algebra there is associated a system of imprimitivity (π, Y, S), $\pi : G \to \mathrm{U}(H)$, in a natural way*

There is also a converse way. One can associate to a system of imprimitivity (π, Y, S), $\pi : G \to \mathrm{U}(H)$, several pseudomeasures of positive type on $\mathcal{C}_c(Y \times G)$

4.7 Remark. *Let (π, Y, S), $\pi : G \to \mathrm{U}(H)$ be a system of imprimitivity and let $h \in H$. Here $T : \mathcal{C}_c(Y \times G) \to B(H)$ is the associated regular homomorphism. Then*

$$\mu(f) = \langle T(f)h, h\rangle$$

is a pseudomeasure of positive type on $\mathcal{C}_c(Y \times G)$.

The proof is easy and can be omitted. $\square$

4.8 Definition. *A system (π, Y, S), $\pi : G \to \mathrm{U}(H)$, of imprimitivity is called*
1) ***transitive*** *if G acts transitively on Y.*
2) ***cyclic*** *if there exists a vector $h \in H$, such that every closed subspace of H that contains h and is invariant under $\mathcal{C}_c(Y \times G)$, equals H.*

The vector h then is called a cyclic vector. Transitivity means that for each $a, b \in Y$ there exists g such that $b = ga$. If $a \in Y$ is an arbitrary point, and if $L = G_a$ is the stabilizer of a, then the natural map

$$G/L \longrightarrow Y$$

is bijective and continuous. Even more, one can show that it is topological, [He], Chap. II, Sect. 3, Theorem 3.2.

We associated to a system of imprimitivity (π, Y, S), $\pi : G \to \mathrm{U}(H)$, with a cyclic vector H a map $T : \mathcal{C}_c(Y \times G) \to B(H)$ (Lemma 4.5). If h is a vector of H wa can associate a pseudomeasure μ on $Y \times G$. To μ we can associate a system of transitivity (π', Y', S') as explained in Remark 4.7. It is natural to ask whether the two systems are the same. We recall that the representation space $H(\mu)$ of the representation associated to μ comes from $\mathcal{C}_c(Y \times G)$ (taking a quotient by a nullspace and completing). There is a natural map

$$\mathcal{C}_c(Y \times G) \longrightarrow H, \quad f \longmapsto T(f)h.$$

It is easy to check that this map factors through $H(\mu)$,

$$H(\mu) \longrightarrow H.$$

This is a linear map that preserves the scalar product. Hence it is an isomorphism onto the image which hence is closed. On the other side this image is dense since h is cyclic. Therefore it is an isomorphism of Hilbert spaces that intertwines the two representations of G.

4.9 Lemma. *Let (π, Y, S), $\pi : G \to \mathrm{U}(H)$, be a cyclic system of imprimitivity. Let $h \in H$ a cyclic vector. We consider the associated pseudo measure μ (s. Remark 4.7) on $Y \times G$. To this pseudo measure we can associate a system of imprimitivity again (Lemma 4.6). The two systems of imprimitivity are isomorphic.*

4.10 Imprimitivity theorem. *Let (π, Y, S), $\pi : G \to \mathrm{U}(H)$, be a transitive and cyclic system of imprimitivity. Choose a point $a \in Y$ and denote by $L = G_a$ its stabilizer. Then there exists a unitary representation σ of L such that π is unitary isomorphic to the induced representation $\mathrm{Ind}_L^G(\sigma)$. The representation σ is unique up to unitary isomorphism.*

Proof. We choose a cyclic vector h and denote by μ the corresponding pseudomeasure of positive type on $Y \times G$. We want to pull it back to a pseudomeasure on $G \times G$ and need for this a map

$$\mathcal{C}_c(G \times G) \longrightarrow \mathcal{C}_c(Y \times G).$$

To get it we define first a map

$$\mathcal{C}_c(G \times G) \longrightarrow \mathcal{C}_c(G \times G), \quad \varphi \longmapsto \Phi$$

through

$$\Phi(y, x) = \int_L \varphi(x^{-1}y\ell, y\ell)\Delta_G(y\ell)^{-1}d\ell.$$

The function $\Phi(y, x)$ is invariant under $y \mapsto y\ell$ for $\ell \in L$. Hence it can considered as function on $\mathcal{C}_c(Y \times G)$. So we obtain an obviously continuous map

$$\mathcal{C}_c(G \times G) \longrightarrow \mathcal{C}_c(Y \times G)$$

and we can pull back the pseudomeasure μ to a pseudomeasure λ on $G \times G$. Let $f, g \in \mathcal{C}_c(G)$ we then we can consider the function $f(x)\overline{g(y)}$ on $G \times G$. We can apply λ to this function to define

$$\langle f, g\rangle_\lambda = \lambda(f(x)\overline{g(y)}) = \int_L f(x)\overline{g(y)}d\lambda(x, y).$$

This is a Hermitian form on $\mathcal{C}_c(G)$. We will prove a little later that it is semipositive. First we want to define an action σ of L on $\mathcal{C}_c(G)$ from the left. It is a modified translation from the right, namely

$$(\sigma(\ell)f)(x) = \sqrt{\frac{\Delta_L(\ell)}{\Delta_G(\ell)}}f(x\ell) \quad (f \in \mathcal{C}_c(G),\ x \in G,\ \ell \in L).$$

So $\sigma(\ell)f$ is in $\mathcal{C}_c(G)$ (as f). A straightforward computation shows that σ preserves $\langle f, g\rangle_\lambda$. So, in the case that the form is semipositive, we can use σ to define a unitary representation of L on the completion $H(\lambda)$ of a quotient of $\mathcal{C}_c(G)$ by a nullspace.

Proof of the semipositivity

We want to compare the two spaces $\mathcal{C}_c(Y \times G)$ and the space $\mathcal{C}(G, \mathcal{C}_c(G))$ of all continuous maps from G into $\mathcal{C}_c(G)$ (which has been equipped with a topology). Actually we construct an operator

$$U : \mathcal{C}_c(Y \times G) \longrightarrow \mathcal{C}(G, \mathcal{C}_c(G)).$$

Recall $Y = G/L$. For this we must associate to a function $f \in \mathcal{C}_c(Y \times G)$ a map $Uf : G \to \mathcal{C}_c(G)$. So $Uf(x)$, $x \in G$, should be a function on G. We define it by

$$(Uf(x))(y) = f(xL, xy^{-1}) \quad (y \in G).$$

Claim. *For each $x \in G$ the map*

$$\mathcal{C}_c(Y, G) \longrightarrow \mathcal{C}_c(G), \quad f \longmapsto Uf(x)$$

is surjective.

Hence, for the proof of the semipositivity, it is sufficient to prove

$$\langle Uf(x), Uf(x) \rangle_\lambda \geq 0$$

for $x \in G$. Since this is a continuous function on G, it is sufficient to prove

$$\int_G \phi(x) \langle Uf(x), Uf(x) \rangle dx \geq 0$$

for all nonnegative $\phi \in \mathcal{C}_c(G)$. To prove this we introduce $\phi'(x) = \int_L \phi(x\ell) d\ell$ and consider it as function on Y. Then we define

$$g(y, x) = \sqrt{\phi'(y)} f(y, x).$$

Now a straightforward computation shows

$$\int_G \phi(x) \langle Uf(x), Uf(x) \rangle dx = \mu(g^* * g).$$

This finishes the proof of the semipositivity.

In $\mathcal{C}_c(G)$ we can consider the nullspace that consists of all f with $\langle f, f \rangle_\lambda = 0$. Since this is nonnegative the nullspace is a sub-vector space. The scalar product $\langle f, g \rangle_\lambda$ induces a positive definite Hermitian form on

$$\mathcal{C}_c(G)/\text{nullspace}(\lambda).$$

The completion $H(\lambda)$ is a Hilbert space with an unitary representation of L. We denote this representation by σ.

We want to show that the given π is isomorphic to the induced representation $\operatorname{Ind}_L^G(\sigma)$. From Lemma 4.9 we know that π is isomorphic to the representation of G on $H(\mu)$ which is associated to the pseudomeasure μ on $Y \times G$. So we have to construct an (unitary) isomorphism

$$H(\mu) \to \operatorname{Ind}_L^G(\sigma).$$

Recall that $H(\mu)$ is constructed from $\mathcal{C}_c(Y \times G)$ (by taking a quotient and then completing). The induced representation is built from certain functions $G \to H(\lambda)$ and $H(\lambda)$ is the completion of a quotient of $\mathcal{C}_c(G)$. This suggests that $\operatorname{Ind}_L^G(\sigma)$ is related to $\mathcal{C}(G, \mathcal{C}_c(G))$ and that the desired isomorphism comes from the map U. We describe the steps that need to be done.

1) We denote by $\mathcal{C}(G, \mathcal{C}_c(G))_0 \subset \mathcal{C}(G, \mathcal{C}_c(G))$ the subspace that consists of all continuous functions $f : G \to \mathcal{C}_c(G)$ such that the composition with $\mathcal{C}_c(G) \to H(\lambda)$ is contained in $\operatorname{Ind}_L^G(\sigma)$. Then there is a natural map

$$\mathcal{C}(G, \mathcal{C}_c(G))_0 \longrightarrow \operatorname{Ind}_L^G(\sigma).$$

2) Let $f \in \mathcal{C}_c(Y \times G)$ and $g = Uf : G \to \mathcal{C}_c(G)$. One can check

$$g(x\ell) = \sqrt{\frac{\Delta_H(\ell)}{\Delta_G(\ell)}}\sigma(\ell^{-1})g(x)$$

Hence the image of the map $U : \mathcal{C}_c(Y \times G) \longrightarrow \mathcal{C}(G, \mathcal{C}_c(G))$ is contained in $\mathcal{C}(G, \mathcal{C}_c(G))_0$.

3) The diagram

$$\begin{array}{ccc} \mathcal{C}_c(Y \times G) & \longrightarrow & \mathcal{C}(G, \mathcal{C}_c(G))_0 \\ \downarrow & & \downarrow \\ H(\mu) & \longrightarrow & \operatorname{Ind}_L^G(\sigma) \end{array}$$

is commutative.

4) The map $H(\mu) \longrightarrow \operatorname{Ind}_L^G(\sigma)$ preserves the scalar products. Hence it is injective and the image is complete and hence closed.

5) The image is dense, since the vector h is cyclic. Hence $H(\mu) \longrightarrow \operatorname{Ind}_L^G(\sigma)$ is an isomorphism of Hilbert spaces.

5. Stone's theorem

We study unitary representations of the additive group $\mathbb{R}^n$ which are not necessarily irreducible. We give an example. Let (X, dx) be a Radon measure and $f : X \to \mathbb{C}$ be a measurable and bounded function. Then we can define the *multiplication operator*

$$m_f : L^2(X, dx) \longrightarrow L^2(X, dx), \quad g \longmapsto fg.$$

This is a bonded linear operator. A bound is given by $\sup_{x\in X}|f(x)|$. The adjoint of m_f is $m_{\bar f}$. Hence m_f is self adjoint for real f and unitary if $|f(x)|=1$ for all x. If f is the characteristic function of a measurable set, we have $m_f^2 = m_f$. This means that $P = m_f$ is an orthogonal projection. This implies that there exists an orthogonal decomposition $H = H_1 \oplus H_2$ such that $P(h_1+h_2) = h_2$. Just take for H_1 the kernel of P and for H_2 its orthogonal complement. This is the image of P.

If f_n is a sequence of uniformly bounded functions that converges pointwise to f then m_{f_n} converges pointwise to m_f.

We also have to consider the group $\widehat{\mathbb{R}^n}$ of unitary characters of $\mathbb{R}^n$. This group is isomorphic to $\mathbb{R}^n$ and hence a locally compact group as well. An isomorphism can be obtained after the choice of a non degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$. Then on can associate to $a\in\mathbb{R}^n$ the character $\chi(x) = e^{\mathrm{i}\langle a,x\rangle}$. In any case $\widehat{\mathbb{R}^n}$ carries a structure as real finite dimensional vector space. In particular, it carries a structure as topological space

Now we assume that $f : X \to \widehat{\mathbb{R}^n}$ is a measurable (not necessarily bounded) function. Then we can consider for each $a\in\mathbb{R}^n$ the bounded and measurable function

$$x \longmapsto f(x)(a).$$

We denote by $U(a)$ the multiplication operator or for this function. Obviously this is a unitary operator and moreover $a \longmapsto U(a)$ is an unitary representation. We call it the multiplication representation related to f.

5.1 Stone's theorem. *Let $U : \mathbb{R}^n \to \mathrm{GL}(H)$ be a unitary representation. Then there exists a Radon measure (X, dx) and a continuous function $f : X \to \widehat{\mathbb{R}^n}$ and a Hilbert space isomorphism $\sigma : H \to L^2(X, dx)$ such that the transport of U to $L^2(X, dx)$ equals the multiplication representation related to f.*

The space (X, dx) is not uniquely determined.

In the following we use the notations of Stone's theorem. We consider a bounded function $\varphi : \widehat{\mathbb{R}^n} \to \mathbb{C}$. We always assume that $\varphi\circ f$ is measurable with respect to dx. This is for example the case when φ is continuous. Another case which we will use is that φ is the characteristic function of a Borel set $\mathcal{B}\subset\mathbb{R}^n$, since then $\varphi\circ f$ is the characteristic function of $f^{-1}(\mathcal{B})$ which is also a Borel set. Both type of functions are Borel functions in the following sense.

5.2 Definition. *A map $f : X \to Y$ between topological spaces is called a Borel map if the inverse images of Borel sets are Borel sets.*

Then we can consider the multiplication operator $m_{\varphi\circ f}$. We use the isomorphism in Theorem 5.1 to transport it to a bounded linear operator which we denote by same letter. We also use the notation

$$S(\varphi) = m_{\varphi\circ f} : H \longrightarrow H.$$

5.3 Remark. *Denote by $\mathcal{B}_b(\widehat{\mathbb{R}^n})$ the space of bounded Borel functions, The operator valued map*

$$S : \mathcal{B}_b(\widehat{\mathbb{R}^n}) \longrightarrow B(H)$$

depends only on the representation $U(a)$ (and not on the choice of (X, dx) and the isomorphism σ).

In particular we obtain a map $S : \mathcal{C}_c(\widehat{\mathbb{R}^n}) \to B(H)$.

5.4 Remark. *Let $U : \mathbb{R}^n \to \mathrm{GL}(H)$ be a unitary representation. Consider $\mathcal{C}_c(\widehat{\mathbb{R}^n})$ as $*$-Algebra where the $*$-Multiplikation is the usual product and $f^*(x) = \overline{f(x)}$. The associated map*

$$S : \mathcal{C}_c(\widehat{\mathbb{R}^n}) \to B(H)$$

is a regular homomorphism.

For any Borel set $\mathcal{B} \subset \widehat{\mathbb{R}^n}$ we can evaluate S at the characteristic function $\chi_{\mathcal{B}}$. We denote this by $P(\mathcal{B})$. This is a operator valued measure in the following sense.

5.5 Definition. *A operator valued measure on $\widehat{\mathbb{R}^n}$ is map P from the set of all Borel sets $E \subset \widehat{\mathbb{R}^n}$ into the set $B(H)$ of bounded operators of a Hilbert space with the properties*

1) *$P(\mathcal{B})$ is a projector.*
2) *$P(\emptyset) = 0, \quad P(\mathbb{R}^n) = \mathrm{id}$.*
3) *$P(E \cap F) = P(E)P(F)$.*
4) *If $E_1, E_2, \dots$ are pairwise disjoint then*

$$P(\cup E_i) = \sum P(E_i) \quad \textit{(pointwise).}$$

Let $Y \subset \widehat{\mathbb{R}^n}$ be a Borel subset such that $P(Y) = \mathrm{id}$. Then we say that P is supported at Y. Is Y' any Borel subset disjoint to Y then $P(Y') = 0$.

5.6 Lemma. *Let $\mathbb{R}^n \to \mathrm{U}(H)$ be a unitary representation and let $Y \subset \widehat{\mathbb{R}^n}$ be a locally closed subset such that the associated projection valued measure is supported at Y. Then there exists a unique map*

$$\mathcal{C}_c(Y) \to \mathrm{U}(H)$$

such that the diagram

$$\begin{array}{ccc} \mathcal{C}_c(\widehat{\mathbb{R}^n}) & \to & \mathrm{U}(H) \\ \downarrow & \nearrow & \\ \mathcal{C}_c(Y) & & \end{array}$$

commutes.

6. Mackey's theorem

Let M be a locally compact group and let

$$M \longrightarrow \mathrm{GL}(n, \mathbb{R})$$

be a continuous homomorphism. We denote by

$$M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad (m, a) \longmapsto ma,$$

the associated action of M on $\mathbb{R}^n$. Similar to the Poincaré group we can define on $M \times \mathbb{R}^n$ a group structure

$$(m, a)(n, b) = (mn, a + nb).$$

We call this group by

$$G = M\mathbb{R}^n.$$

The group M acts also on $\widehat{\mathbb{R}^n}$,

$$m(\chi)(x) = \chi(mx).$$

Let $\chi \in \widehat{\mathbb{R}^n}$. We denote its stabilizer by

$$L = M_\chi = \{m \in M;\ \chi(mx) = \chi(x) \text{ for } x \in \mathbb{R}^n\}.$$

This is called a little subgroup of M (with respect to the given action on $\mathbb{R}^n$. It depends up to conjugation only on the orbit of χ. Hence in the following χ runs through a fixed system of representatives.

This concept differs slightly from our construction of the Poincaré group. But it is easy to bring both concepts together is we assume that there exists non degenerate symmetric bilinear form on $\mathbb{R}^n$. (as in the case of the Poincaré group). In this case the induced isomorphism $\mathbb{R}^n \to \widehat{\mathbb{R}^n}$ has the property that the orbits map to orbits and the stabilizers are preserved.

Besides L we will have to consider the group

$$L\widehat{\mathbb{R}^n}.$$

Now we consider an irreducible unitary representation

$$\sigma : L_\alpha \longrightarrow \mathrm{GL}(H).$$

We can extend this to a representation

$$\sigma \cdot \chi : L_\alpha\widehat{\mathbb{R}^n} \longrightarrow B(H), \quad (x, a) \longmapsto \chi(a)\sigma(x).$$

It is easy to check that this is representation. (One has to use that L fixes χ. We can induce this representation to an unitary representation of G. We say that a unitary representation of G comes from a pair (L, σ) if it is isomorphic to the representation constructed in this way.

Mackey's theorem states that – under a certain assumption – this is an irreducible unitary representation of G and that each irreducible unitary representation is isomorphic to such one.

Now we can formulate the assumptions for Mackey's theorem.

6.1 Assumption. 1) *There exists a closed subset in $\widehat{\mathbb{R}^n}$ which intersects with each M-orbit in exactly one point.*

2) *The orbits are locally closed.*

A set $Y \subset X$ of a topological space is called "locally closed"if it is open in its closure. Assumption 1) means that we can choose from each orbit a representative in some regular way. Assumption 1) can be weakened and assumption 2) is not really necessary. But the proof gets a little easier under the sharpened assumptions. In our examples, in particular in the case of the Poincaé group the sharpened assumptions are met.

Now we can formulate Mackey's theorem.

6.2 Mackey's theorem. *Assume that $G = M\mathbb{R}^n$ satisfies the assumption. Then each unitary representation of G that comes from an irreducible unitary representations of a little group is unitary and irreducible. Each irreducible unitary representation of G is isomorphic to one of this type.*

One can ask when two irreducible representations of G are isomorphic.

6.3 Theorem, Mackey. *Two irreducible unitary representations that come from pairs (L,σ), (L,τ) are (unitary) isomorphic if and only if there exist $g \in L$, $\beta = g(\alpha)$ and a commutative diagram*

$$\begin{array}{ccc} L_\alpha & \longrightarrow & \mathrm{U}(H_\alpha) \\ \downarrow & & \downarrow \\ L_\beta & \longrightarrow & \mathrm{U}(H_\beta). \end{array}$$

The right vertical arrow has to come from a Hilbert space isomorphism $H_\alpha \to H_\beta$.

Hence we can choose a system S of representatives of the orbits and then write

$$\hat{G} \cong \dot{\bigcup_{\alpha\in S}} \hat{L}_\alpha.$$

This means that we have to determine a system of representatives of the orbits and the irreducible unitary representations of the corresponding little groups.

An example

We consider the group

$$\mathrm{Iso}(2) := \left\{ \begin{pmatrix} \zeta & z \\ 0 & \zeta^{-1} \end{pmatrix}; \quad \zeta \in S^1,\ z \in \mathbb{C} \right\}.$$

(Its name will be explained later.) The subgroup L is defined through $z = 0$ and isomorphic to S^1. The subgroup A is defined through $\zeta = 0$ and is isomorphic to $\mathbb{C}$. The action of S^1 on $\mathbb{C}$ is given by

$$(\zeta, z) \longmapsto \zeta^2 z.$$

As representatives of the orbits we can take $z = r$ real, $r \geq 0$. The corresponding little group is S^1 in the case $r = 0$ and the trivial group $\{1, -1\}$ else. The case $r = 0$ leads to the one dimensional representations that factor through $\mathrm{Iso}(2) \to S^1$. For each $r > 0$ we get one infinite dimensional irreducible representation of $\mathrm{Iso}(2)$.

6.4 Theorem. *The group* $\mathrm{Iso}(2)$ *has two series of irreducible unitary representations. The first series is parameterized through* $\mathbb{Z}$ *and corresponds to the one-dimensional characters that factor through* $\mathrm{Iso}(2) \to S^1$*. The second series is parameterized through* $\mathbb{R}_{>0} \times \{1, -1\}$*. They all come from the little group* $\{1, -1\}$*.*

We write the representation coming from (r, ε), $r > 0$, explicitly. Here ε as a character of $\{1, -1\}$. It is either the trivial representation or it corresponds to the non-trivial character $\mathbb{Z}/2\mathbb{Z} \to S^1$. Then we have to extend this to the character (one-dimensional representation)

$$\{1, -1\} \times \mathbb{C} \longrightarrow S^1; \quad (\alpha, z) \longmapsto \varepsilon(\alpha) e^{\mathrm{i}(r,z)}.$$

7. Proof of Mackey's theorem

Let $\pi : G \to \mathrm{U}(H)$ be a unitary representation where $G = M\mathbb{R}^n$. The idea is to apply the imprimitivity theorem. Restricting π we get a unitary representation $\mathbb{R}^n \to \mathrm{U}(H)$, $x \mapsto \pi(e, x)$. Stone's theorem gives us a map

$$T : \mathcal{C}_B(\mathbb{R}^n) \longrightarrow \mathrm{U}(H)$$

Recall that $\mathcal{C}_B(\mathbb{R}^n)$ denotes the space of bounded Borel functions. In particular we get a map

$$T : \mathcal{C}_c(\mathbb{R}^n) \longrightarrow B(H).$$

This is a regular homomorphism and we get the projection valued measure $P(\mathcal{B})$. It is ergodic in the following sense.

7.1 Lemma. *Let* $\mathcal{B} \subset \mathbb{R}^n$ *be a Borel set which is invariant under* M*. Then* $P(\mathcal{B}) = 0$ *or* $\mathcal{B} = \mathrm{id}$*,*

This is a very strong property.

7.2 Proposition. *Assume Assumption 6.1. There exists an orbit $\mathcal{B} \subset \mathbb{R}^n$ such that $P(\mathcal{B}) = \mathrm{id}$.*

Proof. We choose a closed subset $A \subset \widehat{\mathbb{R}^n}$ which is a system of representatives of the orbits. Let $U_1, U_2, \ldots$ be a basis of the topology of A. Consider for each i the set

$$\mathcal{U}_i = MU_i = \{mu;\ m \in M,\ u \in U_i\}$$

This are invariant sets, hence $P(\mathcal{U}_i) = 0$ or id. Now we consider the set

$$\mathcal{U} = \bigcap_{P(\mathcal{U}_i)=\mathrm{id}} \mathcal{U}_i.$$

Obviously $P(\mathcal{U}) = \mathrm{id}$. We claim now that $\mathcal{U}$ is an orbit. Since orbits are closed we can apply Lemma 5.6which gives us a map

$$S : \mathcal{C}_c(\mathcal{A}) \longrightarrow B(H).$$

This is a regular homomorphism which gives a system of imprimitivity

$$(\pi : M \to \mathrm{U}(H), \mathcal{A}, S)$$

Since $\mathcal{A}$ is an orbit this system is *transitive*.

We can apply the imprimitivity theorem to obtain a unitary representation $\sigma : L \to \mathrm{U}(H)$ such that $\pi|M$ is the induced representation.

Chapter VI. Appendices

1. The spectral theorem

We have to consider the space $\mathcal{C}_c^\infty(\mathbb{R})$ of infinitely many differentiable complex valued functions on the real line. By a "Radon measure on $\mathcal{C}_c^\infty(\mathbb{R})$" we understand a $\mathbb{C}$-linear map $I : \mathcal{C}_c^\infty(\mathbb{R}) \to \mathbb{C}$ with the properties $I(\bar{f}) = \overline{I(f)}$ and $I(f) \geq 0$ if $f \geq 0$. It is easy to show that such an I extends uniquely to a Radon measure (on $\mathcal{C}_c(\mathbb{R})$). This follows from the fact that each $f \in \mathcal{C}_c(\mathbb{R})$ is the uniform limit of a sequence $f_n \in \mathcal{C}_c^\infty(\mathbb{R})$ whose supports are contained in a joint compact set.

A function $f : \mathbb{R} \to \mathbb{C}$ is called rapidly decreasing if Pf is bounded for all polynomials P. If f is measurable and rapidly decreasing, the (usual Lebesgue) integral

$$\int_{-\infty}^{\infty} f(t)dt$$

exists. A function is called tempered (or a Schwartz function) if it is infinitely often differentiable and if all derivatives of arbitrary order are rapidly decreasing. The space of all tempered functions is called by $S(\mathbb{R})$. For tempered functions f the Fourier transform

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}tx} f(t)dt$$

exists. It is easy to show (using partial integration) that it is tempered again. In particular, the Fourier transform of a C^∞-function with compact support is tempered.

1.1 Fourier inversion theorem. *The map $S(\mathbb{R}) \to S(\mathbb{R})$, $f \mapsto \hat{f}$, is an isomorphism of vector spaces. It extends to an isomorphism of Hilbert spaces*

$$\mathcal{F} : L^2(\mathbb{R}, dt) \longrightarrow L^2(\mathbb{R}, dt)$$

where dt means the standard Lebesgue measure on the line. Moreover, one has

$$\mathcal{F}\mathcal{F}(f)(x) = f(-x).$$

Let (X, dx) be Radon measure and let $f : X \to \mathbb{C}$ be a bounded measurable function. Then we can define a bounded and linear operator

$$L^2(X, dx) \longrightarrow L^2(X, dx), \quad g \longmapsto fg.$$

In the case $\bar{f}f = 1$ this operator is unitary.

1.2 Spectral theorem. *Let $U : \mathbb{R} \to \mathrm{U}(H)$ be a unitary representation of the additive group $\mathbb{R}$ on a Hilbert space. Then there exists a Radon measure (X, dx) and a real continuous function $f : X \to \mathbb{R}$ such that the representation U is equivalent to the representation*

$$\tilde{U} : \mathbb{R} \longrightarrow \mathrm{U}(L^2(X, dx)), \quad \tilde{U}(t)(g) = e^{itf}g.$$

Equivalence means of course that there exists a Hilbert space isomorphism $W : L^2(\mathbb{R}, dx) \xrightarrow{\sim} H$ with the property $\tilde{U}(t) = W^{-1}U(t)W$.

We first treat a reduction of the spectral theorem to a special case. Let $\pi : G \to \mathrm{GL}(E)$ be a continuous representation. A vector a is called cyclic if the subspace generated by all $\pi(g)a$ is dense in E. This means that E is the only closed invariant subspace that contains a. The representation is (topologically) irreducible if and only if each non zero vector is cyclic. The existence of a cyclic vector is a much weaker condition.

When a cyclic vector exists, then the spectral theorem can be sharpened slightly as follows.

1.3 Proposition. *Let $U : \mathbb{R} \to \mathrm{U}(H)$ a unitary representation of the additive group $\mathbb{R}$ on a Hilbert space. Assume that a cyclic vector exists. Then in the spectral theorem we can take $X = \mathbb{R}$ (and dx some Radon measure) and $f(t) = t$.*

We first show that the general spectral theorem follows from Proposition 1.3 and hence after we prove the proposition.

Proposition 1.3 implies the spectral theorem 1.2. We claim the following.

Every unitary representation $\pi : G \to \mathrm{U}(H)$ has the following property. H can be written as direct Hilbert sum of a finite or countable set of sub Hilbert spaces H_i which are invariant and such that each of them admits a cyclic vector.

This can be proved by a standard argument using Zorn's lemma. We leave the details to the reader. Such a decomposition is not at all unique. Hence one should not overemphasize its meaning.

The Radon measure that is used for the spectral theorem of (π, H) is the direct sum of the Radon measures for the single H_i. We explain briefly the notion of the direct sum. Let (X_i, dx_i) be a finite or countable collection of Radon measures. Then one defines their direct sum as follows. One takes the disjoint union X of the X_i. This is the set of all pairs (x, i), $x \in X_i$. There is a natural inclusion $X_i \to X$, $x \mapsto (x, i)$, and X is the disjoint union of the images. We equip X with the direct sum topology. This means that the (images of the) X_i are open subsets and that the induced topology is the original one. Then one defines in an obvious way a Radon measure on X such that the restriction to the X_i are the given dx_i. □

Proof of Proposition 1.3. To any bounded continuous function $h : \mathbb{R} \to \mathbb{C}$ we associate the functional

$$I_h : \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}, \quad I_h(g) = \int_{-\infty}^{\infty} h(t)\hat{g}(t)dt.$$

The integral exists, since $\hat{g}$ and hence hg are rapidly decreasing. We apply this to the function $h(t) = \langle U(t)a, a\rangle$ where a is a cyclic vector. This function has the property

$$h(-t) = \langle U(-t)a, a\rangle = \langle a, U(t)a\rangle = \overline{h(t)}.$$

Using this it is easy to check that I_h is real, i.e. real valued for real g. One just has to use the simpel rule

$$\overline{\hat{g}(t)} = \hat{g}(-t).$$

We will see a little that I_h is actually a Radon measure on $\mathcal{C}_c^\infty(\mathbb{R})$. For this reason, we later use already now the notation

$$\int_{\mathbb{R}} g(x)d\mu = \int_{-\infty}^{\infty} \langle U(t)a, a\rangle \hat{g}(t)dt \qquad (g \in \mathcal{C}_c^\infty(\mathbb{R})).$$

Next we define a linear map

$$W : \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow H, \quad g \longmapsto \int_{-\infty}^{\infty} \hat{g}(t)U(t)adt.$$

This is a Bochner integral with values in the Hilbert space H. The integrand is continuous, hence measurable and it is bounded by the integrable function $|\hat{g}|$. Hence the Bochner integral exists.

For $g_1, g_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ we compute

$$\int_{\mathbb{R}} g_1(t)\overline{g_2(t)}d\mu$$

as follows. We make use of the fact that the Fourier transformation of the product g_1g_2 of two functions equals the convolution of the two Fourier transforms

$$\widehat{g_1g_2} = \hat{g}_1 * \hat{g}_2.$$

Recall that the convolution of two functions on the line is

$$(g_1 * g_2)(x) = \int_{-\infty}^{\infty} g_1(x-t)g_2(t)dt.$$

So we get

$$\begin{aligned}\int_{\mathbb{R}} g_1(t)\overline{g_2(t)}d\mu &= \int_{-\infty}^{\infty} \langle U(t)a, a\rangle \widehat{g_1\bar{g}_2}(t)dt \\ &= \int_{-\infty}^{\infty} \langle U(t)a, a\rangle \int_{-\infty}^{\infty} \hat{g}_1(t-s)\hat{\bar{g}}_2(s)dsdt.\end{aligned}$$

We compare this with the inner product of $W(g_1)$ and $W(g_2)$ in the Hilbert space H. It is

$$\langle W(g_1), W(g_2)\rangle = \left\langle \int_{-\infty}^{\infty} \hat{g}_1(t)U(t)adt, \int_{-\infty}^{\infty} \hat{g}_2(s)U(s)ads \right\rangle.$$

The integrals are standard integrals along continuous functions with compact support. They can be considered as Riemann integrals and hence approximated by finite sums. In this way we see

$$\langle W(g_1), W(g_2)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle U(t)a, U(s)a\rangle \hat{g}_1(t)\hat{\bar{g}}_2(s)dtds.$$

By means of the integral transformation $(s,t) \mapsto (s, t-s)$ we obtain

$$\boxed{\int_{\mathbb{R}} g_1(t)\overline{g_2(t)}d\mu = \langle W(g_1), W(g_2)\rangle}$$

Now, let $g \in \mathcal{C}_c^\infty(\mathbb{R})$ be a real nonnegative function. In the case that $\sqrt{g}$ is differentiable, we set $g_1 = g_2 = \sqrt{g}$ to show that $\int g d\mu$ is non negative. But $\sqrt{g}$ needs not to be differentiable (notice that the square root of x^2 is $|x|$ which is not differentiable at the origin). But it is always possible to approximate g by functions g_1^2 where g_1 is differentiable So we see that $d\mu$ is a Radon measure as we have claimed. The map $W : \mathcal{C}_c^\infty(\mathbb{R}) \to H$ is unitary. Hence it is injective and it extends to a unitary map

$$L^2(\mathbb{R}, d\mu) \longrightarrow H.$$

(Here one uses that a bounded linear map $E \to F$ of normed spaces extends to the completions.) In particular, W extends to $S(\mathbb{R})$. The image is a complete and hence a closed subspace of H. Next we have to compare $U(t)$ with the representation

$$\tilde{U} : \mathbb{R} \longrightarrow \mathrm{U}(L^2(\mathbb{R}, d\mu)), \quad (\tilde{U}(t)(g))(x) = e^{itx}g(x).$$

We prove $W\tilde{U}(s) = U(s)W$. This follows simply from the fact that the Fourier transform of the function $x \mapsto e^{\mathrm{i}sx}g(x)$ is the function $x \mapsto \hat{g}(x-s)$.

It remains to show that W is surjective. Here we have to use that a is a cyclic vector. It is sufficient that a is in the image, or even that there exists a sequence g_n of tempered functions such that $W(g_n) \to a$. For this purpose we choose a differentiable Dirac sequence h_n. Then the integrals $\int h_n(t)U(t)a$ converge to $U(0)a = a$. We can write $h_n = \hat{g}_n$ where g_n is tempered. $\square$

2. Variants of the spectral theorem

Let (X, dx) be a Radon measure. A function $f : X \to \mathbb{C}$ is called *essentially bounded* if there exists $C \geq 0$ such that $|f(x)| \leq C$ outside a zero set. We denote by $\|f\|_\infty$ the infimum of all C. This is a semi norm on the space $\mathcal{L}^\infty(X)$ of all measurable essentially bounded f. Zero functions are essentially bounded and their infinity-norm is 0. The quotient $L^\infty(X)$ of $\mathcal{L}^\infty(X)$ by the subspace of zero functions is a normed space. It can be shown that it is a Banach space. (Sometimes it is considered as a "limit space" of all L^p-spaces).

2.1 Remark. *Let f be an essentially bounded measurable function on X. Then gf is square integrable if g is and multiplication by f defines a bounded linear operator*

$$m_f : L^2(X, dx) \longrightarrow L^2(X, dx).$$

The norm of m_f equals $\|f\|_\infty$.

Proof. Let $g \in \mathcal{L}^2(X, dx)$. Then $\|fg\|_2 \leq \|f\|_\infty \|g\|_2$. This shows that m_f is bounded and $\|m_f\| \leq \|f\|_\infty$. We have to prove the inverse inequality. For this we assume for a moment that f is square integrable. Then we have $\|f\|_2^2 = \|f\bar{f}\|_2 \leq \|m_f\| \|f\|_2$. This shows $\|f\|_\infty \leq \|m_f\|$. in the case that f is not square integrable we replace f by $f\chi_K$ where χ_K is the characteristic function of a compact subset K. Take the supremum along all K in the inequality $\|f\chi_K\|_\infty \leq \|m_{f\chi_K}\|$ we obtain the claim. □

These are the most general multiplication operators due to the following Lemma.

2.2 Lemma. *Let f be an essentially bounded measurable function on X such that fg is square integrable if g is square integrable. Then f is essentially bounded.*

Proof. First we show that multiplication by f is a bounded operator m_f : $L^2(X, dx) \to L^2(X, dx)$. Here we use the closed graph theorem. It is enough to show that the graph $\{g, fg)$; $g \in L^2(X)$ is closed. Consider a sequence (g_n, fg_n) that converges in the qraph. This means that

$$g_n \longrightarrow g, \quad fg_n \longrightarrow h \quad (\text{both in } L^2(X, dx)).$$

Convergence in $L^2(X, dx)$ implies pointwise convergence of a suitable subsequence outside a zero set. Hence we can assume that g_n and fg_n converge pointwise outside the zero set. This shows $\psi = fg$ in $L^2(X, dx)$.

The boundedness of m_f implies the existence of a constant such that

$$\|fg\|_2 \leq C\|g\|_2.$$

Now we choose a constant $A > 0$ such that $A^2 > C$. We consider the characteristic function χ of the set

$$\{x \in X; \quad |f(x)| \geq A\}.$$

For a moment we assume that this set has a finite volume. From $A^2\chi(x) \leq |f(x)|\chi(x)$ and from $\chi = \chi^2$ we obtain

$$A^2 \int_X \chi(x)dx \leq \int_X |f(x)|\,\chi(x)^2 dx \leq C \int_X \chi(x)dx = C \int_X \chi(x)dx$$

we obtain that χ is a zero function. This means that $|f|$ is bounded by A outside a zero set.

If χ is not integrable, then we make a similar trick as in the proof of Remark 2.1. We multiply χ with the characteristic function of an arbitrary compact set K. □

We want to work out when m_f is an isomorphism. For this we introduce a notation. Let $f : X \to \mathbb{C}$ be a measurable function. We say that $1/f$ exists if the set of zeros is a zero set. In this case we define

$$(1/f)(x) = \begin{cases} 1/f(x) & \text{if } f(x) \neq 0, \\ 0 & \text{else.} \end{cases}$$

2.3 Lemma. *Let f be an essentially bounded measurable function on X. The multiplication operator $m_f : L^2(X, dx) \to L^2(X, dx)$ is an isomorphism if and only if $1/f$ exists and is essentially bounded.*

Proof. Assume that m_f is an isomorphism. Then every $h \in L^2(X, dx)$ is of the form fg, $g \in L^2(X, dx)$. In particular $(1/f)h = g$ is in $L^2(X, dx)$. Now we can apply Lemma 2.2. □

A bounded linear operator $A : H \to H$ is called self adjoint if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$. We derive a spectral theorem for such operators. For this we introduce the exponential

$$E^A = \sum_{n=1}^{\infty} \frac{A^n}{n!}.$$

This series converges in the Banach space $\mathcal{B}(H)$, since $\|A^n\| = \|A\|^n$ and the norm of e^A is bounded by $e^{\|A\|}$. As for complex numbers one can prove

$$e^{A+B} = e^A e^B \quad \text{if} \quad AB = BA.$$

Using this one can show that

$$U(t) = e^{\mathrm{i}tA}$$

is a unitary representation of $\mathbb{R}$ on H. We can apply the spectral theorem 1.2. It says that there exists a Radon measure (X, dx), an isomorphism $H \to L^2(X, dx)$ and a continuous function f such that $U(t)$ corresponds to multiplication with $e^{\mathrm{i}tf}$ on $L^2(X, dx)$. We want to follow that

$$Ag = fg \quad \text{on} \quad L^2(X, dx)$$

For this we consider a sequence $t_n \to 0$, $t_n \neq 0$. Then

$$\lim_{n\to\infty} \frac{e^{\mathrm{i}t_nA}g - g}{t_n} = A_n g.$$

This holds in $L^2(X, dx)$ but then, after replacing t_n by a sub-sequence, pointwise outside a zero set. Since we have also

$$\lim_{n\to\infty} \frac{e^{\mathrm{i}t_nf(x)}g(x) - g(x)}{t_n} = f(x)g(x)$$

we get

$$Ag = fg \quad \text{on} \quad L^2(X, dx)$$

as stated This implies that f is essentially bounded. So we obtain the following variant of the spectral theorem.

2.4 Theorem. *Assume that A is a self adjoint (bounded) operator on a Hilbert space. Then there exists a Radon measure (X, dx), a real continuous and essentially bounded function f on X, and a Hilbert space isomorphism $H \cong L^2(X)$ such that A corresponds to multiplication by f.*

The spectral theorem of compact self adjoint operators (Theorem I.8.3) is a special case of this. To prove it we have to study when a multiplication operator $m_f : L^2(X) \to L^2(X)$ for a real bounded continuous function is compact. A necessary condition for this is that X carries the discrete topology. In particular X must be a countable set. The measure is known if one knows the masses (=volumes) $m(a)$ of the single points. The set of all points with mass zero is a zero set. Hence we can replace X by their complement. without changing $L^2(X)$. This means that we can assume $m(a) > 0$ for all a. We then consider the functions

$$f_a(x) = \begin{cases} 1/m(a) & \text{for } x = a, \\ 0 & \text{else} \end{cases}.$$

This is an orthonormal basis. The functions f_a are eigen functions of m_f with eigen value $f(a)$. So we have proved.

Let $A : H \to H$ be a compact self adjoint bounded operator on a Hilbert space. Then there exists an orthonormal basis $e_1, e_2, \dots$ of eigen vectors, $Ae_i = \lambda e_i$.

We notice that the eigenvalues are bounded. This follows easily from $\|\lambda a\| \leq \|Aa\| \leq \|A\|\|a\|$ which shows that the eigenvalues are bounded by $\|A\|$. The compactness of A implies that the multiplicities of the eigenvalues are finite. It remains to show that the set of eigenvalues has no accumulation point different from zero. So assume that $\lambda = \lim \lambda_n$ is different from zero and the limit of a sequence of eigenvalues. We choose eigen vectors a, a_n of norm 1.

Functional calculus for self adjoint operators

The spectrum $\sigma(A)$ of a bounded operator on a Banach space consists of all $\lambda \in \mathbb{C}$ such that $A - \lambda E$ is not a bounded isomorphism of Banach spaces. From the spectral theorem we derive the following three propositions (which in other approaches are proved directly and then the spectral theorem is a consequence of them).

2.5 Proposition. *Let $A : H \to H$ be a self adjoint operator, then the spectrum $\sigma(A)$ is real and compact.*

Proof. An argument with the geometric series shows that $\mathcal{B}^*(H)$ is open in $\mathcal{B}(H)$. This implies that the spectrum is closed.

For the rest of the proof we can assume that A is a multiplication operator $m_f : L^2(X, dx) \to L^2(X, dx)$ where f is a real, locally bounded function. Now, let λ be a non-real number. Then $1/(f - \lambda)$ exists and is locally bounded. The same is true if λ is a real number with $|\lambda| > \|f\|_\infty$. Hence in both cases λ is not in the spectrum. □

Let $P \in \mathbb{C}[X]$ be a polynomial and let $\mathcal{A}$ be an associative algebra with unit. Then $P(a) \in \mathcal{A}$ can be defined for arbitrary $a \in \mathcal{A}$ in an obvious way. In particular, one can define $P(A)$ for an endomorphism A of a vector space. If A is a a self adjoint operator on a Hilbert space and if P is real then $P(A)$ is self adjoint too.

2.6 Proposition. *Let A be a self adjoint operator and let $P \in \mathbb{C}[X]$ be a polynomial whose restriction to $\sigma(A)$ vanishes, then $P(A) = 0$.*

There is an obvious conclusion.

2.7 Lemma. *Let A be a self adjoint operator such that its spectrum consists of one point $a \in \mathbb{R}$. Then A is a multiple of the identity.*

Proof. Consider $P(x) = x - a$. Then P vanishes on the spectrum. This implies $P - a\,\mathrm{id} = 0$. □

2.8 Proposition. *Let A be a self adjoint operator and let $P \in \mathbb{C}[X]$ be a polynomial. Then we have*

a) $$\sigma(P(A)) = P(\sigma(A)),$$

b) $$\|P(A)\| = \|P\|_{\sigma(A)}.$$

Here $\|P(A)\|$ means the operator norm and $\|P\|_{\sigma(A)}$ the maximum of $|P|$ on $\sigma(A)$.

The three propositions immediately imply what is called "functional calculus".

2.9 Theorem. *Let A be a self adjoint operator. For each real continuous function P on $\sigma(A)$ the bounded operator $P(A)$ can be defined in a unique way such that the following properties holds. The map*

$$\mathcal{C}(\sigma(A)) \longrightarrow \mathcal{B}(H)$$

is a norm preserving Banach algebra homomorphism.

Additional Remark. *Let B be a bounded linear operator that commutes with A. Then B commutes with all $P(A)$, $P \in \mathcal{C}(\sigma(A))$.*

Let A be a bounded linear operator on a Hilbert space. The commutator of A consists of all bounded linear operators that commute with A. The bi-commutant consists of all bounded linear operators that commute with all operators of the commutator of A. Clearly $A \in \mathcal{G}(A)$. The Additional Remark in Theorem 2.9 shows that all $P(A)$ are in the bi-commutant of A.

2.10 Lemma. *Let A be a self adjoint operator on a Hilbert space H, Assume that the bi-commutant contains only operators whose kernel is H or 0. Then A is a multiple of the identity.*

Proof. We assume that A is not a multiple of the identity. Then the spectrum consists of more than one point. Hence we can find two continuous real functions f_1, f_2 on the spectrum which are not zero but their product is zero. Let $A_i = f_i(A)$. These are two non-zero operators in the bi-commutant of A with the property $A_1 \circ A_2 =$. This shows that $A_2(H)$ is in the kernel of A_1. So the kernel is neither 0 nor H. This proves the Lemma. □

3. The von-Neumann bi-commutant theorem

Let H be a Hilbert space and $\mathcal{B}(H)$ the algebra of bounded linear spaces. This is a Banach space with the operator norm and hence a topological space. But there are several other topologies. One of them is the strong operator topology (SOT) which is defined with the help of a family of seminorms. For each $a \in H$ we consider

$$p_a(A) = \|Aa\|.$$

The SOT-topology on $\mathcal{B}(H)$ is the weakest topology such that these seminorms are continuous. It can be described concretely as follows. For $A \in \mathcal{B}(H)$ and $a \in H$ and for $\varepsilon > 0$ we denote by

$$B_a(A, \varepsilon) = \{B \in \mathcal{B}; \quad p_a(B - A) < \varepsilon\}$$

In the SOT-topology these sets are open and each open subset is the union of finite intersections

$$B_{a_1}(A, \varepsilon_1) \cap \ldots \cap B_{a_n}(A, \varepsilon_n).$$

We have to consider sub-algebras $\mathcal{A} \subset \mathcal{B}(H)$. The commutant

$$\mathcal{A}' = \{B \in \mathcal{B}(H); \quad AB = BA \text{ for all } A \in \mathcal{A}\}$$

is a subalgebra too. The bi-commutant $\mathcal{A}''$ contains $\mathcal{A}$. We are mainly interested in star-subalgebras of $\mathcal{B}$. This means that with A also the adjoint operator A^* is contained in $\mathcal{A}$. The von-Neumann density theorem states.

3.1 Theorem (von Neumann bi-commutant theorem). *Let $\mathcal{A}$ be a star-subalgebra of $\mathcal{B}(H)$ which contains the identity. Then $\mathcal{A}$ is* SOT*-dense in $\mathcal{A}''$.*

Proof. The proof rests on the following simple lemma.

3.2 Lemma. *Let $\mathcal{A} \subset \mathcal{B}(H)$ be a $*$-subalgebra and let $P \in \mathcal{B}(H)$ be a projector (i.e. $P^2 = P$). The space $P(H)$ is invariant under $\mathcal{A}$ if and only if $P \in \mathcal{A}'$.*

This theorem has an important consequence for unitary representations.

3.3 Theorem. *Let $\pi : G \to \mathrm{U}(H)$ be an irreducible unitary representation of a locally compact group. Then the image of $\mathcal{C}_c(G)$ in $\mathcal{B}(H)$ is* SOT*-dense in $\mathcal{B}(H)$.*

Proof. Consider the SOT-closure $\mathcal{A}$ of the image of $\mathcal{C}_c(G)$ in $\mathcal{B}(H)$. This is a star-algebra in $\mathcal{B}(H)$. It contains then unity. By Schur's lemma, the commutator $\mathcal{A}'$ of $\mathcal{A}$ consists of multiples of the identity only. Hence $\mathcal{A}'' = \mathcal{B}(H)$ and we can apply the density theorem. □

4. The Peter-Weyl theorem

Let K be a compact group and let $\sigma : K \to \mathrm{U}(H)$ be a finite dimensional unitary representation. Recall that we defined the character

$$\chi(x) = \chi_\sigma(x) = \mathrm{tr}(\sigma(x))$$

and a modified version

$$e_\sigma = \dim(H)\overline{\chi(x)}.$$

This a continuous function on K. Unitary equivalent representations have the same character.

Other important functions on K are the matrix coefficients of an unitary representation σ (Here K needs not to be compact.) They are defined for two $a, b \in H$ through

$$\langle \sigma(k)a, b\rangle.$$

The span a space of continuous functions that we denote by

$$\mathcal{E}_\sigma \subset \mathcal{C}(K).$$

In the case that the representation is finite dimensional one can choose an orthonormal basis e_i of H. Then $\mathcal{E}_\sigma$ is generated by the entries of the matrix

$$\langle \sigma(k)e_i, e_j\rangle.$$

They satisfy the famous orthogonality relations.

4.1 Theorem. *Let σ, τ be two irreducible unitary representation of a compact group K. Then the corresponding spaces $\mathcal{E}_\sigma$, $\mathcal{E}_\tau$ are orthogonal.*

Proof. Let σ, τ be the two irreducible unitary (hence finite dimensional) representations. Let $B : H_\sigma \to H_\tau$ be an linear map. Then we can build the operator

$$A = \int_K \pi_\tau(k)B\pi_\sigma(k^{-1})dk.$$

Then the invariance of the Haar measure shows

$$A\sigma(k) = \tau(k)A.$$

The operator A can not be injective, since otherwise it would be an isomorphism and σ and τ would be equivalent. So the kernel of A is not trivial. But the formula above shows that the kernel of A is invariant under σ. Hence it must be the whole space. So A is zero, whatever B might be. We will apply this for a well chosen B. First we choose $a \in H_\sigma$, $b \in H_\tau$. Then we define

$$B(x) = \langle x, a\rangle b.$$

Then we choose two other vectors $c \in H_\sigma$, $d \in H_\tau$. Then

$$\begin{aligned}
0 = \langle Ac, d\rangle &= \int_K \langle \tau(k^{-1})B\sigma(k^{-1})c, d\rangle dk \\
&= \int_K \big\langle \langle \sigma(k^{-1})c, a\rangle b, \tau(k^{-1})d\big\rangle dk \\
&= \int_K \langle \sigma(k^{-1})c, a\rangle \langle b, \tau(k^{-1})d\rangle dk \\
&= \int_K \langle \sigma(k^{-1})c, a\rangle \overline{\langle \tau(k^{-1})d, b\rangle} dk
\end{aligned}$$

This proves the theorem. □

The space $\mathcal{E}$ for a irreducible unitary representation is non-zero. It follows that there there are only finitely or countably many isomorphy classes of irreducible unitary representations of a compact group K (recall that we assume that K has countable topology and that the Hilbert spaces are assumed to be separable). Recall that we defined the unitary dual $\hat{G}$ to be the set of isomorphy classes of unitary irreducible representations.

We have introduced the convolution algebra $\mathcal{C}(K)$. It depends on the choice of a Haar measure which we normalize such that the volume of K is one. Recall that an unitary representation of K can be extended to a representation $\pi : \mathcal{C}(K) \to \mathcal{B}(H)$. Now, let $\sigma : K \to \mathrm{GL}(H_\sigma)$ be an irreducible unitary representation. Then we can consider the operator $\pi(e_\sigma)$.

4.2 Theorem. *Let $\pi : \mathcal{C}(K) \to \mathcal{B}(H)$ be a unitary representation of the compact group K and let $\sigma \in \hat{K}$. Then $\pi(e_\sigma)$ is the orthogonal projection of H onto the isotypic component $H(\sigma)$.*

4.3 Theorem. *The functions e_σ, $\sigma \in \hat{K}$ satisfy the following relations.*

$$e_\sigma * e_\sigma = e_\sigma, \quad e_\sigma * e_\tau = 0 \text{ for different } \sigma, \tau \in \hat{K}.$$

4.4 Theorem. *Let K be a compact group and let $f \in L^2(K)$. One has*

$$\langle f, f \rangle = \sum_{\sigma \in \hat{K}} \dim(\sigma) \operatorname{tr}(\sigma(f)\sigma(\bar{f})).$$

This means of course that the sum is absolute convergent.

4.5 Theorem. *Every irreducible unitary representation of a compact group K occurs in the regular representation $L^2(K)$ and its multiplicity equals its dimension.*

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