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# Complex Analysis of Several Variables II

Stein spaces

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# Contents

## Preface

We assume that the reader is acquainted with the local theory of complex spaces as we treated in the first volume. In particular, we will make freely use of elementary sheaf theory without cohomology. In this volume we present an introduction into the cohomology theory of sheaves in an appendix (Chapt. III).

We also need some knowledge in functional analysis, in particular in the theory of Fréchet spaces. We collected them in another appendix (Chapt. IV) without proves.

The main subject is the theory of Stein spaces. These are complex spaces which are opposite to compact spaces in the sense that they admit many global holomorphic functions. The central result about Stein spaces is Cartan's theorem B that asserts that the higher cohomology groups of a coherent sheaf $\mathcal{M}$ vanishes, $H^n(X, \mathcal{M}) = 0$ for $n > 1$.

We treat also some applications as the construction of meromorphic functions on a Stein space with prescribed poles.

The proof of Theorem B needs several different techniques. One of them rests on a certain approximation procedure. For this we have to equip the space $\mathcal{M}(X)$ of global sections of a coherent sheaf on a complex space with a structure as a Frèchet space. This is not trivial and will take a while.

We have written the notes in the language of complex spaces in the sense of Grothendieck. But it turns out that such a space $X$ is Stein if and only if the associated complex space in the sense of Serre, $X_{\text{red}}$, is so. If the reader feels more comfortable, he can restrict to complex spaces in the sense of Serre. But he will not get a big profit from this.

In a following third volume we will prove Grauert's finiteness theorem (projection theorem). The theory of Stein spaces is an essential tool for this.

# Chapter I. Local theory of complex spaces

## 1. Complex spaces

This sections collects basic results from vol. I, Sect. 1. In the following by a *ringed space* $(X, \mathcal{O}_X)$ we always understand a topological space that has been equipped with a sheaf of $\mathbb{C}$-algebras. For a brief introduction into sheaf theory including the notion of coherent sheaf we refer to vol. I, Chapt. IV. Instead of ringed space we should better say "algebred space", but this sounds ugly. By definition, a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, \varphi)$, consisting of a continuous map $f : X \to Y$ and a homomorphism

$$\varphi : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X)$$

of sheaves of $\mathbb{C}$-algebras. In practice this means that we have homomorphisms of algebras

$$\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V))$$

which are compatible with restrictions. It is clear how to define the composition of two morphisms and there is the identity morphism. (This means that we defined a category). In particular, we have the notion of an isomorphism of ringed spaces. We write the morphism as $(f, \varphi)$ simply by $f$ it is clear which $\varphi$ is considered. But one should have in mind that $\varphi$ is usually not determined by $f$.

We equip $\mathbb{C}^n$ with the sheaf of all holomorphic functions (on open subsets). We denote this sheaf by $\mathcal{O}_{\mathbb{C}^n}$. The restricted sheaf to an open subset we denote by $\mathcal{O}_U = \mathcal{O}_{\mathbb{C}^n}|U$. Let $f_1, \dots, f_m$ be a finite system of holomorphic functions on $U$. Then we can consider the ideal sheaf $\mathcal{J}$ generated by the $f_i$ . The factor $\mathcal{O}_U/\mathcal{J}$ is a sheaf of $\mathbb{C}$-algebras. The support of this sheaf is

$$\begin{aligned} Y &= \{\, a \in X; \quad \mathcal{J}_a \neq \mathcal{O}_{X,a} \,\} \\ &= \{\, z \in \mathbb{C}^n; \quad f_1(z) = \cdots = f_m(z) = 0 \,\} \end{aligned}$$

which is a closed subset. Then we can consider the ringed space

$$(Y, \mathcal{O}_Y) \quad \text{where} \quad \mathcal{O}_Y = (\mathcal{O}_X/\mathcal{I})|Y.$$

Of course $\mathcal{O}_Y$ depends on the choice of $f_1, \dots, f_m$. Such a ringed space is called a *model space*. Open subsets of $\mathbb{C}^n$ equipped with the sheaf of all holomorphic functions are special model spaces. (Take $f_i = 0$).

For the definition of $|Y$ for closed subspaces we refer to vol. I, Chapt. IV, Sect. 5, in particular to Remark IV.5.1 there. We recall the essentials briefly. We have to use here a very special case of the pull back of a sheaf with respect to a continuous map $f : Y \to X$. This associates to a sheaf $F$ on $X$ a sheaf $f^{-1}F$ on $Y$. In the special case that $Y$ is a subspace of $X$ (equipped with the induced topology) and that $f$ is the canonical injection, one writes $f^{-1}F = F|Y$. In the case that $Y$ is open in $X$ this agrees with the naiv restriction of a sheaf to an open subset which we used already. We don't need the pull back construction in general and do not presume its knowledge. We only need another special case which we describe briefly now.

Let $Y \subset X$ be a closed subspace and let $i : Y \to X$ be the natural injection. We denote by $\mathcal{A}$ the category of sheaves of abelian groups on $Y$ and we denote by $\mathcal{B}$ the full subcategory of the category of abelian sheaves on $X$ whose objects are sheaves $F$ of abelian groups on $X$ with the property $F|(X - Y) = 0$. The functor "direct image" (see vol. I, Chapt. II, Sect.6) defines an equivalence of categories

$$i_* : \mathcal{A} \longrightarrow \mathcal{B}.$$

There exists an inverse functor which we write as $F \mapsto F|Y$. It has the following property. Assume that $U \subset X$ is open and that $V = U \cap Y$. Then there is canonical isomorphism $F(U) \to (F|Y)(V)$. This property lays close the following definition of $F|Y$.

$$(F|Y)(V) := \varinjlim F(U).$$

Here $U$ runs through all open subsets $U \subset X$ such that $U \cap Y = V$. The basic results of this construction are:

There are canonical isomorphisms (i.e. isomorphisms in the sense of functors)

$$i_*(F|Y) \cong F, \quad (i_*G)|Y \cong G.$$

One has canonical isomorphisms for $a \in Y$

$$(i_*G)_a \cong G_a, \quad (F|Y)_a \cong F_a.$$

There are variants of this construction. One can take for take $\mathcal{A}$ the category of sheaves of rings on $Y$ and by $\mathcal{B}$ the category of sheaves of rings on $X$ with the property $F|(X-Y) = 0$. Or one can fix a sheaf of rings $\mathcal{O}_Y$ and consider $\mathcal{O}_X = i_*\mathcal{O}_Y$. Then one can take for $\mathcal{A}$ the category of $\mathcal{O}_Y$-modules and for $\mathcal{B}$ the category of $\mathcal{O}_X$-modules. Notice that every $\mathcal{O}_X$-module $\mathcal{M}$ has the property $\mathcal{M}|(X - Y) = 0$ (since every module over the zero ring is zero).

**1.1 Definition.** *A complex space (in the sense of Grothendieck) is a ringed space that is locally isomorphic to a model space. A holomorphic map between two complex spaces* $(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ *is a morphism in the sense of ringed spaces.*

We can talk about the category of complex spaces in the sense of Grothendieck. Notice that a holomorphic map consists of two parts, a continuous map $f:X\to Y$ and a homomorphism of sheaves of algebras $\varphi:\mathcal{O}_Y\to f_*\mathcal{O}_X$. From vol. I, Theorem I.11.3 we recall the following basic result.

**1.2 Oka's coherence theorem.** *The structure sheaf of a complex space is coherent.*

We consider the stalk $\mathcal{O}_{X,a}$ of a complex space. In the case $\mathbb{C}^n$ (equipped with the sheaf of holomorphic functions) this algebra is isomorphic as $\mathbb{C}$-algebra to the ring of convergent power series.

$$\mathcal{O}_n:=\mathcal{O}_{\mathbb{C}^n,0}=\mathbb{C}\{z_1,\dots,z_n\}.$$

An analytic algebra $A$ is a $\mathbb{C}$-algebra $A$ that is different from 0 and is isomorphic to a factor algebra of $\mathcal{O}_n$, $n$ suitable,

$$A\cong\mathcal{O}_n/\mathfrak{a},\quad \mathfrak{a}\neq A.$$

Analytic algebras are local algebras. If $\mathcal{O}_n\to A$ is a surjective algebra homomorphism, then the image of the maximal ideal of $\mathcal{O}_n$ is the maximal ideal of $A$. In particular we get natural homomorphisms

$$\mathbb{C}\longrightarrow A\longrightarrow A/\mathfrak{m}$$

The composition $\mathbb{C}\to A/\mathfrak{m}$ is an isomorphism. We will use it to identify $A/\mathfrak{m}=\mathbb{C}$. We finally mention that homomorphisms of analytic algebras $A\to B$ are automatically local, i.e. the image of the maximal ideal of $A$ is contained in the maximal ideal of $B$. So we get a natural homomorphism

$$A/\mathfrak{m}(A)\longrightarrow B/\mathfrak{m}(B)$$

which is the identity if we identify both sides with $\mathbb{C}$.

Let $f\in\mathcal{O}_X(X)$ be a global section of the structure sheaf of a complex space and let $x\in X$ be a point. We can consider the germ $f_x$ and take its coset mod $\mathfrak{m}(\mathcal{O}_{X,x})$. This is a number which we denote by $f(x)$. In this way we get a function

$$\tilde{f}:X\longrightarrow\mathbb{C},\quad \tilde{f}(x):=f(x).$$

A look at the definition of model spaces shows that $\tilde{f}$ is continuous. Hence we have constructed an algebra homomorphism

$$\mathcal{O}_X(X)\longrightarrow\mathcal{C}_X(X).$$

The same can be done for open subsets. We can read this as map of sheaves of $\mathbb{C}$-algebras

$$\mathcal{O}_X \longrightarrow \mathcal{C}_X.$$

We denote the kernel of this map by $\mathcal{N}_X$. Clearly $\mathcal{N}_X(U)$ contains all nilpotent elements of $\mathcal{O}_X(U)$.

Let $(X, \mathcal{O}_X)$ be an arbitrary ringed space. The nilradical $\mathcal{N}$ is the subsheaf of $\mathcal{O}_X$ that is defined through

$$U \longmapsto \{g \in \mathcal{O}_X(U), \quad \text{locally nilpotent}\}.$$

It also can be defined through

$$\mathcal{N}_X(U) = \{f \in \mathcal{O}_X(U); \quad f_a \text{ nilpotent in } \ \mathcal{O}_{X,a} \text{ for all } a \in U\}.$$

Basic results of local complex analysis show.

**1.3 Hilbert-Rückert nullstellensatz.** *Let $(X, \mathcal{O}_X)$ be a complex space. Then the kernel of the natural map $\mathcal{O}_X \to \mathcal{C}_X$ is the nilradical $\mathcal{N}_X$.*

**1.4 Cartan's coherence theorem.** *Let $(X, \mathcal{O}_X)$ be a complex space. The nilradical is coherent.*

(This is equivalent to the fact that the nilradical is locally finitely generated.)

### Holomorphic functions on complex spaces

By a holomorphic function on a complex space $(X, \mathcal{O}_X)$ we understand a holomorphic map

$$(f, \varphi) : (X, \mathcal{O}_X) \longrightarrow (\mathbb{C}, \mathcal{O}_{\mathbb{C}}).$$

So $\varphi : \mathcal{O}_{\mathbb{C}} \to f_* \mathcal{O}_X$. Such a morphism is determined by the image of the global section $1 \in \mathcal{O}_{\mathbb{C}}$. This is an element of $\mathcal{O}_X(X)$. This gives the following result.

**1.5 Remark.** *The holomorphic mappings $(X, \mathcal{O}_X) \to (\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ are in one-to-one correspondence to the global sections in $\mathcal{O}_X(X)$.*

### Open subspaces

Let $(X, \mathcal{O}_X)$ be a complex space and let $U \subset X$ be an open subset. Then $(U, \mathcal{O}_X|U)$ is a complex space too, The natural inclusion $i : U \to X$ together with the natural map $\varphi : \mathcal{O}_U \to i_* \mathcal{O}_X)$ gives a holomorphic map $(U, \mathcal{O}_X|U) \to (X, \mathcal{O}_X)$. The following universal property is satisfied. Let $(g, \psi) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a holomorphic map of a third complex space $Z$ into $X$ such that $f(X) \subset U$, then $(p, \psi)$ factors through a unique holomorphic map $(g_0, \psi_0) : (Z, \mathcal{O}_Z) \to (U, \mathcal{O}_X|U)$.

A holomorphic map $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called an *open embedding* if there is an open subset $U \subset Y$ such that $(f, \varphi)$ factors through an isomorphism $(X, \mathcal{O}_X) \xrightarrow{\sim} (U, \mathcal{O}_X|U)$. The composition of two open embeddings is an open embedding.

**Closed subspaces**

Let $(X, \mathcal{O}_X)$ be a complex space and let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent ideal sheaf. (It is enough to know that $\mathcal{J}$ is locally finitely generated.) We then can consider the sheaf $\mathcal{O}_X/J$. The support of this sheaf is a closed subset $Y \subset X$. We then can consider the restriction

$$\mathcal{O}_Y := (\mathcal{O}_X/\mathcal{J})|Y.$$

Then $(Y, \mathcal{O}_Y)$ is a complex space. We call this the closed complex subspace of $(X, \mathcal{O}_X)$ related to the ideal sheaf $\mathcal{J}$. There is a natural holomorphic map $i : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$. A holomorphic map $j : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is called a closed embedding (of complex spaces) if there exists a coherent ideal sheaf $J \subset \mathcal{O}_X$) such that $j$ factors through a biholomorphic map

$$(Z, \mathcal{O}_Z) \xrightarrow{\sim} (Y, \mathcal{O}_Y) \quad \text{where} \quad Y = \operatorname{supp}(\mathcal{O}_X/J),\ \mathcal{O}_Y = (\mathcal{O}_X/J)|Y.$$

It is easy to show that the composition of two closed embeddings is a closed embedding.

Finally we call a holomorphic map $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ a locally closed embedding if it is the composition of a closed embedding $f : (Y, \mathcal{O}_Y) \to (U, \mathcal{O}_U)$ and an open embedding $(U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$.

A subset $Y \subset X$ is called a closed analytic subset of the complex space $(X, \mathcal{O}_X)$ if there exists a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$ such that $Y = \operatorname{supp}(\mathcal{O}_X/J)$. The union and intersection of finitely many closed analytic subsets are closed analytic subsets.

**Complex spaces in the sense of Serre**

A complex space is called a complex space in the sense of Serre if the natural map $\mathcal{O}_X \to \mathcal{C}_X$ is injective. Due to the nullstellensatz this is equivalent to the fact that the rings $\mathcal{O}_{X,a}$ are nilpotent-free. Then we can consider the elements of $\mathcal{O}_X(U)$ as usual functions on $U$. The category of complex spaces in the sense of Serre is the full subcategory of the category of complex spaces in the sense of Grothendieck. If $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a holomorphic map between complex spaces in the sense of Serre, then $\varphi$ is determined by $f$. It is just the usual pull-back of functions. There is a natural functor $X \mapsto X_{\text{red}}$ of the category of complex spaces of Grothendieck to that of Serre. Just associate to $(X, \mathcal{O}_X)$ the ringed space $(X, \mathcal{O}_X/\mathcal{N})$ where $\mathcal{N}$ is the nil-radical. Due to Cartan's coherence theorem this is a complex space. Notice also that there is a natural morphism

$$(X, \mathcal{O}_X/\mathcal{N}) \longrightarrow (X, \mathcal{X}).$$

This is a closed embedding. So $X_{\text{red}}$ is a closed complex subspace of $X$ with the same underlying spaces. (But usually it is not an open subspace).

**Example of a non-reduced complex space**

Consider the complex plane $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$. Consider the ideal sheaf that is generated by $z^2$. This is coherent. Its support is one point (the origin) and the restricted sheaf is given by $\mathbb{C}\{z\}/z^2 \cong \mathbb{C}^2$. This ring is not nilpotent free. So the ringed space $(\mathrm{pt}, \mathbb{C}\{z\}/z^2 \cong \mathbb{C}^2)$ is a non-reduced complex space.

**The singular locus**

A point $a \in X$ of a complex space is called *regular* if there exists an open neighborhood $U$ such that $(U, \mathcal{O}_X)$ is isomorphic to $(V, \mathcal{O}_V)$ for an open subset $V \subset \mathbb{C}^n$ (and $\mathcal{O}_V$ is the standard sheaf of holomorphic functions). A complex space is called a complex manifold if all points are regular. A Riemann surface is a complex manifold of (pure) dimension one. Let $S$ be the singular locus of $(X, \mathcal{O}_X)$. The main theorem of local complex analyis is the following result.

*Let $(X, \mathcal{O}_X)$ be a complex space in the sense of Serre. The singular locus $S$ is a closed analytic subset of $X$. It is thin in $X$.*

**Topological assumptions**

In this lecture we will always assume that complex spaces are Hausdorff. In particular they are locally compact. We also assume that their exists a countable basis of the topology. This is a countable system of open sets such that each open set can be written as union from sets of the system.

## 2. Finite maps

We recall the basic results of vol. I, Chapt. II, Sect. 4 and 5. A holomorphic map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_X)$ is called *finite* if the underlying map between topological spaces is finite. This means that it is proper and the fibres are finite sets. A holomorphic map $f : X \to Y$ is locally finite at a point $a \in X$ if there exist open sets $a \in U \subset X$ and $f(a) \in V \subset Y$ such that $f(U) \subset V$ and that $f : U \to V$ is finite.

**2.1 Theorem.** *A holomorphic map $f : X \to Y$ is locally finite at $a$ if an only if the corresponding map of analytic algebras $\mathcal{O}_{Y,f(a)} \to \mathcal{O}_{X,a}$ is finite.*

Here “finite” is understood in the sense that $\mathcal{O}_{X,y}$ is a finitely generated $\mathcal{O}_{Y,f(a)}$-module. An important result of Grauert states.

**2.2 Theorem.** *Let $X \to Y$ be a finite holomorphic map between complex spaces. Let $\mathcal{M}$ be a coherent sheaf on $X$. Then the direct image $f_*\mathcal{M}$ is coherent too. The functor $\mathcal{M} \mapsto f_*\mathcal{M}$, starting from the category of coherent sheaves on $X$, is exact.*

A direct consequence of the theorem of Grauert is the following theorem of Remmert.

**2.3 Theorem.** *Let $X \to Y$ be a finite holomorphic map between complex spaces. Then the image is a closed analytic subset of $Y$.*

# Chapter II. Stein spaces

## 1. The notion of a Stein space

Probably the reader knows that on a connected compact complex manifold any holomorphic function is constant. Assume that the dimension is $> 1$. If one removes from this manifold a single point the situation does not remedy, since in more than one variable there do not exist isolated singularities. Hence there exist also non-compact manifolds that admit no non-constant analytic function. Stein spaces are opposite to this situation. They are spaces that admit many holomorphic functions. We are going to explain in which sense this has to be understood.

Let $K$ be a non-empty compact subset of a topological space $X$. We use the notation

$$||f||_K := \max\{|f(x)|;\ x \in K\}$$

for a continuous function $f$ on $X$.

**1.1 Definition.** *Let $K$ be a non-empty compact subset of a complex space. The* ***holomorphic convex hull*** *$\hat{K}$ of $K$ is the set of all $x \in X$ such that $|f(x)| \le ||f||_K$ for all $f \in \mathcal{O}_X(X)$.*

**1.2 Definition.** *A complex space is called* ***holomorphically convex*** *if the holomorphic convex hull of any compact subset is compact.*

Assume that $X$ is a complex space with the following property: for every infinite closed discrete subset $S \subset X$ there exists a holomorphic function $f \in \mathcal{O}_X(X)$ that is unbounded on $S$. Then $X$ is holomorphically convex. This can be seen by an indirect argument. Let $K$ be a compact subset such that $\hat{K}$ is not compact. Then their exists a sequence in $\hat{K}$ with no convergent subsequence. This gives an infinite subset $S \subset \hat{K}$ that is closed in $X$ and discrete. Then there exists a global holomorphic function which is unbounded on $\hat{K}$. This is not possible.

From this observation we can deduce that open subsets $U$ of the plane $\mathbb{C}$ are holomorphically convex. To show this we consider an infinity closed discrete subset $S$. If $S$ is unbounded, then we take $f(z) = z$. In the case that $S$ is bounded there must be an accumulation point $a$ of $S$ which lies on the boundary of $U$. Then take $f(z) = 1/(z - a)$.

In more then one variable the situation is completely different. Let

$$U = U_r(0) = \{z \in \mathbb{C}^n;\ |z_i| < r_i\}$$

be a polydisk around zero. We claim that $U - \{0\}$ is not holomorphically convex. For this we consider the subset $K$ consisting of all $z$ with $|z_i| = r_i/2$. We know that every holomorphic function $f$ on $U - \{0\}$ extends holomorphically to $U$. From the maximum principle one deduces $\hat{K} = \{z \in U;\ |z_i| \le r_i/2\}$. This set is not compact.

**1.3 Definition.** *A complex space $X$ is called a* ***Stein space*** *if the following conditions are satisfied:*

1) *It is holomorphically convex.*
2) *(Point separation) For two different points $x, y \in X$ there exists a global $f \in \mathcal{O}_X(X)$ with $f(x) = 0,\ f(y) = 1$.*
3) *(Infinitesimal point separation) For any point $a \in X$ there exist global $f_1, \dots, f_m \in \mathcal{O}_X(X)$ whose germs generate the maximal ideal of $\mathcal{O}_{X,x}$.*

It is clear that open subsets of the complex plane are Stein spaces. More generally it is clear that a cartesian product $D = D_1 \times \cdots \times D_n$ of open subsets $D_i \subset \mathbb{C}$ is Stein. It is already a deep result that all non-compact connected Riemann surfaces are Stein spaces. We will not proof this result here. A proof can be found in [Fo]. As we have seen it is false that open subsets of $\mathbb{C}^n$ are always Stein in the case $n > 1$.

**1.4 Remark.** *Let $X$ be a Stein space. Then every closed analytic subspace is a Stein space too. The cartesian product of two Stein spaces is Stein.*

**1.5 Definition.** *An* ***Oka domain*** *in a complex space $(X, \mathcal{O}_X)$ is an open subset $U \subset X$ such that there exists a holomorphic map $f : (X, \mathcal{O}_X) \to (\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ with the following property. The restriction of $f$ to $U$ defines a closed embedding*

$$f_0 : (U, \mathcal{O}_X|U) \longrightarrow (P, \mathcal{O}_P)$$

*into some polydisk in $\mathbb{C}^n$.*

The basic exhaustion theorem states:

**1.6 Theorem.** *Let $X$ by a Stein space. Any compact subset $K$ is contained in an Oka domain $U$.*

**Additional remark.** *In the case that $K = \hat{K}$ and that $W$ is some open subset containing $K$ one can get $U \subset W$.*

Before we start with the proof, we formulate a technical lemma. For this it is convenient to introduce the notion of a "local embedding". A holomorphic map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of complex spaces is called a local embedding of complex spaces if for every $a \in X$ there exist open neighbourhoods $a \in U \subset X$, $f(a) \in V \subset Y$ such that $f$ factors through a closed embedding $U \to V$.

**1.7 Lemma.** *Let $f : X \to Y$ be a holomorphic map of complex spaces. We make two assumptions:*

a) *$f$ is a closed embedding of topological spaces. This means that $f(X)$ is closed and $X \to f(X)$ is topological. (This is more than injective).*
b) *$f$ is a local embedding.*

*Then $f$ is a closed embedding of complex spaces.*

*Proof of the lemma.* Let $a \in X$ be a point and $U$ a neighborhood with property b). Then we know from a) that $f(U)$ is an open subset from $f(X)$. By assumption b) $f(U)$ is analytic. This means that every point of $f(X)$ admits an open neighborhood that is analytic. But then $f(X)$ is analytic. The inverse map $f(X) \to X$ is analytic since this is locally the case. □

Property a) has been used essentially in the proof. So one should have in mind that bijective continuous maps between topological spaces need not to be topological. There is an exceptional case where the situation is better.

Recall that a continuous map $f : X \to Y$ between locally compact Hausdorff spaces is called *proper*, if the inverse image of any compact set $K \subset Y$ is compact. Proper maps have the basic property that they are closed. This means that the images of closed subsets of $X$ are closed in $Y$. This immediately gives:

*Let $f : X \to Y$ be a bijective continuous and proper map between complex Hausdorff spaces. Then $f$ is topological.*

This is clear: The inverses under $f^{-1}$ are the images under $f$. Hence the assumption says that the inverse images of closed sets under $f^{-1}$ are closed. This means that $f^{-1}$ is open.

*Proof of 1.6 continued.* We can assume that $K = \hat{K}$. We will prove the sharpened form where we have to consider an open neighborhood $W$ of $K$. For each $a \in K$ we can choose finitely many global functions that map an open neighborhood $U(a)$ of $a$ biholomorphically onto an analytic subset of some $\mathbb{C}^n$. The compact subset $K$ can be covered by finitely many of these neighborhoods, $K \subset U(a_1) \cup \cdots \cup U(a_m)$. We collect the functions for each $a$ and obtain a holomorphic map such is locally biholomorphic on $U(a_1) \cup \cdots \cup U(a_m)$. We choose an open neighborhood $U$ of $W$ whose closure is compact and contained in $U(a_1) \cup \cdots \cup U(a_m)$. We would like to manage that $f$ is injective on $U$. For this we consider the set $A$ of all $(a, b) \in \bar{U} \times \bar{U}$ such that $f(a) = f(b)$. The diagonal $\Delta$ of $\bar{U} \times \bar{U}$ is contained in $A$. Actually $\Delta$ is an open subset of $A$. To show this we consider some diagonal point $(a, a)$. Then $a \in U(a_i)$ for some $i$. Then all points. Then $U(i) \times U(i)$ is an open neighborhood of $(a, a)$ in $X$. Its intersection with $A$ is contained in $\Delta$ since $f$ is injective on $U(a_i)$. Since $\Delta$ is open in $A$ we get that the complement $A - \Delta$ is compact. For each pair $(a, b) \in A - \Delta$ we can choose a global holomorphic function $h$ with $h(a) \neq h(b)$. Then $h(x) \neq h(y)$ for all $(x, y)$ in a full open neighborhood of $(a, b)$ in $A - \Delta$. We can cover $A - \Delta$ by finitely many such open sets. We add the finitely man functions $h$ as new components to the map $f$. In this way we produce a globally defined map that is injective. Without loss of generality we can assume that $f$ is injective on $\bar{U}$ (and locally biholomorphic on $U$).

It remains to manage that $f$ defines a proper map of $U$ onto an analytic set of some polydisk. The polydisk we want to take is just the product of unit discs ($|z_j| < 1$). For this we can assume without loss of generality $|f_i(z)| \le 1$ for $z$ in $K$. One just has to multiply $f$ with a suitable constant. Now we will make use of the holomorphic convexity: For each boundary point $a \in \partial U$ we can choose a global holomorphic function $g$ such that $||g||_K < g(a)$. Multiplying with a suitable constant we can get $||g||_K < 1 < |g(a)|$. This inequality remains true in a full open neighborhood of $a$. We can cover $\partial U$ with finitely many of these neighborhoods. We add the corresponding functions $g$ as additional components to $f$. Now we modify $U$. We replace $U$ by the set of all $x \in U$ such that $|f_i(x)| < 1$. We still have that $f$ is injective and locally biholomorphic on this new $U \supset K$. But now we have the advantage that $f$ defines a proper map of $U$ into the polydisk. For this one has just to show that the inverse image of the compact set $|z_i| \le \varrho < 1$ is compact in $U$. This is clear since this set is away from the boundary of $U$. □

## 2. Approximation theorems for cuboids

In the theory of Stein spaces it turned out to be of some advantage to work with rectangles of the form

$$Q = \{z \in \mathbb{C}; \quad a_1 < x_1 < a_2;\ b_1 < y_1 < b_2\}.$$

Here $a_1 < a_2$ and $b_1 < b_2$ real numbers. In the following we understand by an open *cuboid* a set $Q = Q_1 \times \cdots \times Q_n$, where the $Q_i$ are rectangles in the above sense. Cuboids are Stein spaces and every closed analytic subset of a cuboid is Stein.

A very special case of the so-called Runge approximation theorem states:

**2.1 Runge's approximation theorem (special case).** *Every holomorphic function on a cuboid is the locally uniform limit of a sequence of polynomials.*

(For polydisks instead of cuboids this result follows from the Taylor expansion of a holomorphic function. But we need it for cuboids). We just give a hint to the proof in the one-dimensional case. Let $f : Q \to \mathbb{C}$ be a holomorphic map on a rectangle $Q$. We have to show that for each shrunken rectangle $Q_0 \subset Q$ and for each $\varepsilon > 0$ there exists a polynomial $P$ with $|P(z) - f(z)| < \varepsilon$ for all $z \in Q_0$. Cauchy's integral formula gives for $z \in Q_0$

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial Q_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Here we have chosen some cuboid $Q_1$ between $Q_0$ and $Q$. Using an approximation by step functions we can approximate $f$ by functions of the type $C/(z-a)$ where $a$ is on the boundary of $Q_1$. Hence it is sufficient to assume $f(z) = C/(z-a)$. Since $a$ is outside of the closure of $Q_0$, we find a disk that contains the closure of $Q_0$ but not $a$. In this disc we can expand $C/(z-a)$ into a power series and then approximate it by its Taylor polynomials. □

We need a certain matrix valued version of this approximation theorem. For this it is convenient to use the matrix norm for a square matrix $A$.

$$|A| = \max\{|Az|; \quad |z| = 1\}.$$

Here $|z|$ denotes the Euclidean norm of a vector $z$. This matrix norm has the properties:

a) $|a_{ik}| \le |A|$.
b) $|AB| \le |A||B|$.
c) $|A+B| \le |A| + |B|$.

From these inequalities immediately follows that the series

$$e^A := \sum_{\nu=0}^{\infty} \frac{A^\nu}{\nu!},$$
$$\log(E-A) := -\sum_{\nu=1}^{\infty} \frac{A^\nu}{\nu} \quad \text{for} \quad |A| < 1$$

converge. The rule

$$A^{\log(E-A)} = E - A$$

holds. It follows from the known case $n = 1$ since it can be expressed as a formal relation in factorials. We have to give some warning. The rule $e^{A+B} = e^A e^B$ is usually false. It holds if the matrices $A, B$ commute.

We will have to consider matrix valued function $F : D \to \mathbb{C}^{(m,m)}$ on open subsets $D \subset \mathbb{C}^n$. Of course holomorphy means that each component of is holomorphic. An immediate application of the above consideration is:

**2.2 Lemma.** *Let $F : D \to \mathbb{C}^{(m,m)}$ be some matrix valued holomorphic function on an open subset $D \subset \mathbb{C}^n$. Assume that $|F(z) - E| < 1$ for all $z \in D$, Then there exists a holomorphic function $G : D \subset \mathbb{C}^{(m,m)}$ with the property $F = e^G$.*

In contrast to the case $m = 1$ it is very difficult to get holomorphic logarithms without an estimate as in 2.2. This will cause some difficulties. To come around them we prove:

**2.3 Lemma.** *Let $F : D \to \mathrm{GL}(m, \mathbb{C})$ be an invertible holomorphic matrix valued function on an open convex subset $D \subset \mathbb{C}^n$. Let $K \subset D$ be a compact subset and $\varepsilon > 0$. Then $F = F_1 \cdots F_k$ can be written as finite product of holomorphic functions*

$$F_i : D \to \mathrm{GL}(m, \mathbb{C}), \quad |F_i(z) - E| \le \varepsilon \quad \textit{for} \quad z \in K.$$

*Proof.* We will use a simple fact about topological groups. Let $G$ be the set of all holomorphic maps $F : D \to \mathrm{GL}(m, \mathbb{C})$. This is a group under multiplication. For any holomorphic $F : D \to \mathbb{C}^{(m,m)}$ and a compact subset $K \subset D$ we define

$$||F||_K = \max\{|F(z)|,\ z \in K\}.$$

Eventually replacing $K$ by a bigger compact set (with non-empty interior) we can assume that $||\cdot||$ is definite. Then $||F - G||_K$ defines a metric on $G$ and $G$ gets a topological space. It is clear that multiplication $G \times G \to G$ and inversion $G \to G$ are continuous. This means that $G$ is a topological group. We claim that this topological space is arcwise connected. To show this we can assume that $0 \in D$. For any $F \in G$ we can consider $F_t(z) = F(tz)$, $0 \le t \le 1$. Notice that $F_t \in G$ and that $t \to F_t$ is continuous. Hence it defines a curve in $G$ that combines $F$ with the constant function $F_0$. Now the connectedness of $G$ follows from the known fact that $\mathrm{GL}(m, \mathbb{C})$ is connected. For sake of completeness we recall the argument. Any invertible matrix can be written as finite product of diagonal matrices and strict triangular matrices. Each of them, hence also an finite product of them can be combined with the unit matrix. This follows just from the connectedness of $\mathbb{C}^{\bullet}$ and $\mathbb{C}$.

*Proof of 2.3 continued.* We denote by $U \subset G$ the set of all $F \in G$ with $||F||_K < \varepsilon$ and $||F^{-1}||_K < \varepsilon$. This is an open subset. Then we denote by $G_0$ the subgroup of $G$ generated by $U$. It consists of all finite products of elements of $U$. Since $G_0$ is the union of translates of $G$ it is an open subgroup of $G$. But an open subgroup is automatically closed. This follows from the decomposition of $G$ into (say right-) cosets $G_0 g$. The complement of $G_0$ is the union of all cosets different from $G_0$ and hence open. From the fact that $G$ is arcwise connected we get $G = G_0$. This finishes the proof of 2.3. □

Now we are able to prove a multiplicative analogue of Runge's approximation theorem.

**2.4 Multiplicative version of Runge's approximation theorem.** *Let $F : Q \to \mathrm{GL}(m, \mathbb{Z})$ be an invertible holomorphic matrix valued function on a cuboid $Q \subset \mathbb{C}^n$. There exists a sequence $F_\nu : \mathbb{C}^n \to \mathrm{GL}(n, \mathbb{C})$ of invertible holomorphic matrix valued functions on the whole $\mathbb{C}^n$ that converges on $Q$ locally uniformly to $F$.*

*Proof.* Let $K \subset Q$ be a compact subset and $\varepsilon > 0$. We have to construct a holomorphic $G : \mathbb{C}^n \to \mathrm{GL}(m, \mathbb{C})$ such that $||F - G||_K < \varepsilon$. We choose a cuboid $K \subset Q_0$ whose compact closure is contained in $Q$. Because of 2.3 we can restrict to the case $|F(z) - E| < 1$ for $z \in Q_0$. Then there exists a holomorphic logarithm $e^H = F$ on $Q_0$. By Runge's approximation theorem we can approximate $H$ by a polynomial function $P$. Hence we can manage $||F - e^P||_K < \varepsilon$. □

The usual theory of infinite products can be generalized to matrix valued functions. Recall that an infinite product $(1+a_1)(1+a_2)\cdots$ is called absolutely convergent if the series $|a_1| + |a_2| + \cdots$ converges. It is known that then the limit

$$\lim_{\nu\to\infty}(1 + a_1)\cdots(1 + a_\nu)$$

exists and that it is zero if and only of one of the factors $1 + a_i$ is zero. Here is a matrix valued variant.

**2.5 Lemma.** *Let $G_\nu$ be a sequence of holomorphic matrix valued functions on some open domain in $\mathbb{C}^n$ such that there exists a convergent series $a_1+a_2+\cdots$ of numbers with the property $|G_\nu(z)| \le a_\nu$ for all $z$. Then the limit*

$$F(z) = \lim_{m\to\infty} F_1\cdots F_m, \qquad F_\nu := E + G_\nu,$$

*exists and is a holomorphic function. It is invertible if all $F_\nu$ are.*

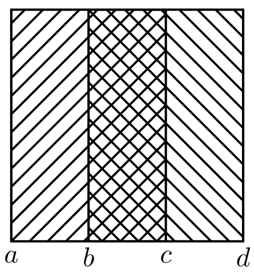
*Proof.* The usual theory of infinite products shows that $(1 + a_1)\cdots(1 + a_\nu)$ converges, say to $a$. $P_\nu = F_1\cdots F_\nu$ are bounded ba $a$ in the sense $|P_\nu(z)| \le a$ for all $z$. This follows from $|E + G_i(z)| \le 1 + a_i$. Now we get

$$|P_{\nu+1}(z) - P_\nu(z)| = |P_\nu(z)G_{\nu+1}(z)| \le a \cdot a_\nu$$

From this follows that $P_\nu$ is a uniform Cauchy sequence. Hence its limit $F$ exists and is a holomorphic function. We have still to show that it is invertible if all $F_\nu$ are. For this it is sufficient to show that the product of the $\det F_\mu$ converges absolutely in the sense of infinite products. This means the the series $\sum(1 - \det F_\nu)$ converges absolutely. Since $1 - \det F_\nu$ is polynomial without constant coefficient in the entries of $F_\nu$ it can be bounded for all $\nu$ with $a_\nu < 1$ by a bound $C|a_\nu|$. This shows the convergence. □

## 3. Cartan's gluing lemma

We consider two rectangles $R', R'' \subset \mathbb{C}$ in a very special position. We identify $\mathbb{C}$ with $\mathbb{R}^2$. In fact we assume that there are real numbers $a < b < c < d$ such that the rectangles are of the form $R' = (a, c) \times I$ and $R'' = (b, d) \times I$, where $I \subset \mathbb{R}$ is a bounded open interval.




For a cuboid $D \subset \mathbb{C}^{n-1}$ we can consider $Q' = R' \times D$ and $Q'' = R'' \times D$.

**3.1 Cousin's additive gluing lemma.** *Let $Q', Q''$ be two cuboids in $\mathbb{C}^n$ in the special position $Q' := R' \times D$, $Q'' := R'' \times D$, where $R', R'' \subset \mathbb{C}$ are rectangles of the form*

$$R' = (a, c) \times I,\ R'' = (b, d) \times I \quad (a < b < c < d).$$

*Furthermore let $f$ be an analytic function on $Q' \cap Q''$ Then one has:*
*There exist analytic functions*

$$f' : Q' \longrightarrow \mathbb{C}, \quad f'' : Q'' \longrightarrow \mathbb{C}$$

*with the property*

$$f(z) = f'(z) + f''(z) \text{ for } z \in Q' \cap Q''.$$

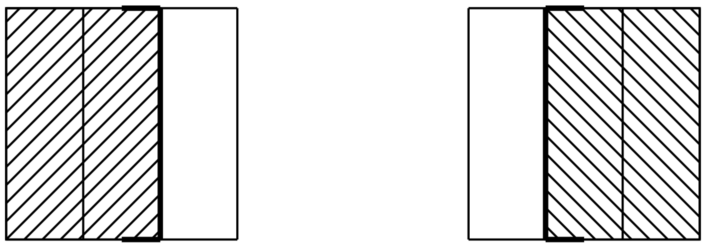
*Proof.* We know that the cohomology of $\mathcal{O}$ on a cuboid vanishes. By Leray's Lemma the cohomology $H^1(Q, \mathcal{O})$ can be computed by means of the Čech cohomology with respect to the covering $Q = Q' \cup Q''$. Its vanishing is just the statement of Lemma 3.1. □

We give a second proof of Lemma 3.1 under a slightly stronger assumption. We assume that $f$ can be extended to a an analytic function on an open set $U$ which contains $\overline{Q' \cap Q''}$.

This proof uses the Cauchy integral formula applied to $f$ as function of $z_1$. During the proof, $z_2, \dots, z_n$ will kept fixed. The integrals in consideration will depend analytically on $z_2, \dots, z_n$ by Leibniz's criterion. Hence it is sufficient to restrict to the case $n = 1$. The Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial(R' \cap R'')} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad \text{for } z \in Q' \cap Q''.$$

It is clear that the boundary $\partial(R' \cap R'')$ is the composition of two paths $W'$ and $W''$, where $W'$ is contained in the boundary of $R'$ and $W''$ in the boundary of $R''$.



Then one has

$$f(z) = f'(z) + f''(z) \text{ for } z \in Q' \cap Q''$$

with

$$f'(z) := \oint_{W'} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

and similarly $f''$. The functions $f', f''$ are analytic in the complements of $W', W''$, hence in the whole $Q', Q''$ (actually in a much bigger domain!)

This second proof of the gluing lemma has the advantage to admit estimates for the functions $f'$, $f''$. For this improvement we assume that the set $U$ is bounded and also that the function $f$ is bounded on $U$. Recall that the construction of the gluing functions is given by a Cauchy integral along $f(z)/(\zeta_1 - z_1)$. This integral can be estimated by the standard estimate of line integrals. This estimate involves the length of the curve. This is bounded by the bounds of the domain $U$. We obtain.

**3.2 Lemma.** *Assume that a bounded open set $U \subset \mathbb{C}^n$ which contains the closure of $Q' \cap Q''$ is given. There exists a constant $M$ depending only on $U$ such that for each bounded holomorphic function $f$ on $U$ the solution $f' : Q' \to \mathbb{C}$, $f'' : Q'' \to \mathbb{C}$ of the additive gluing lemma can be obtained with the estimate*

$$|f'(z)| \le \frac{M||f||}{\delta'(z)} \qquad (z \in Q').$$

*Here $||f||$ denotes the supremum of $|f(z)|$ on $U$ and $\delta'(z)$ denotes the minimal distance of $z$ to a boundary point of $Q'$ (similarly for $f''$).*

**Supplement.** *For $M$ one can take 3 times the diameter of $U$. (The diameter is the supremum of the Euclidean lengths of line segments contained in $U$.)*

There is a multiplicative version of the gluing lemma that produces a decomposition of the typ $f(z) = f'(z)f''(z)$. The proof is easy for scalar valued functions. One takes a holomorphic logarithm of $f$ and applies the additive lemma to the logarithm and exponentiates then. The result follows then from the rule $e^{a+b} = e^a e^b$. Due to Cartan the multiplicative lemma is also valid for matrix valued $f$. But the proof is more involved. One reason is that the rule $e^{a+b} = e^a e^b$ is false for matrices $a, b$.

**3.3 Lemma (Cartan's multiplicative gluing lemma).** *We take the same assumptions as in 3.1. Furthermore let $F : U \to \mathrm{GL}(m, \mathbb{C})$ be a holomorphic function on an open set $U$ which contains the closure of $Q' \cap Q''$. Then there exist holomorphic functions $F' : Q' \to \mathrm{GL}(m, \mathbb{C})$, $F'' : Q'' \to \mathrm{GL}(m, \mathbb{C})$ such that*

$$F(z) = F'(z) \cdot F''(z) \text{ for } z \in Q' \cap Q''.$$

*Proof.* In a first step we mention that for the proof of the gluing lemma we can assume that $F(z)$ is close to the identity matrix (in the sense $|F(z)| < \varepsilon$ for a given $\varepsilon > 0$). The reason is that be the multiplicative Runge approximation we can choose for an arbitrary $F$ a $G : \mathbb{C}^n \to \mathrm{GL}(m, \mathbb{C})$ such that $FG^{-1}$ is small in the sense we need. So we get a decomposition $FG^{-1} = F'F''$ and then a decomposition $F = F' \cdots (F''G)$. □

In the next step we will explain the strategy of the proof (which only will work if $F$ is close enough to the unit matrix). We write $F(Z) = E + G(Z)$ where $E$ is the unit matrix. Then we apply the additive lemma to the components of $G$ to produce a decomposition $G(Z) = G'(Z) + G''(Z)$, where $G', G''$ are holomorphic on $Q', Q''$. Then as a first trial we set $F' = E + G'$, $F'' = E + G''$. Then

$$F'F'' = (E + G')(E + G'') = E + G'G'' + G'G'' = F + G'G''.$$

The term $G'G''$ is a failure term. We want to get rid of it through an approximation method. What we described is only the first step of an approximation. Hence we set

$$G_0 = G, \quad G'_0 = G', \quad G''_0 = G''.$$

By induction we will define a sequence $G_\nu, G'_\nu, G''_\nu$. Here $G_\nu$ should be an invertible matrix valued function on some open neighborhood of $\overline{Q' \cap Q''}$ and $G_\nu = G'_\nu + G''_\nu$ a decomposition in sense of the additive lemma. The basic formula for the procedure is

$$(E + G'_\nu)(E + G_{\nu+1})(E + G''_\nu) = (E + G_\nu).$$

Assume that we have constructed this sequence. Then we can define

$$F'_\nu = E + G'_\nu, \quad F''_\nu = E + G''_\nu.$$

Then we have

$$F = [F'_1 F'_2 \cdots F'_\nu] F_{\nu+1} [F''_1 F''_2 \cdots F''_\nu] \quad \text{on} \quad Q' \cap Q''$$

and the solution of the multiplicative decomposition should be obtained by

$$F' := \lim_{\nu\to\infty}[F_1' \cdots F_\nu']$$

and similarly $F''$. Of course the hope is that $G_\nu$ tends to zero for $\nu \to \infty$ and that the infinite products converge.

Before we start with the proof of the convergence, we have to overcome a small technical difficulty. Of course we can define $G_{\nu+1}$ through the equation $(E + G_\nu')(E + G_{\nu+1})(E + G_\nu'') = E + G_\nu$ if $E + G_\nu$ is invertible and we get a function that is holomorphic on $Q' \cap Q''$. But to apply the additive gluing lemma we should have a holomorphic function on some open neighborhood of $\overline{Q' \times Q''}$. We will overcome this difficult through a small modification. We enlarge the cuboid a little bit: We write $Q$ as the intersection of a decreasing sequence of cuboids (all contained in $U$) $Q_1 \supset Q_2 \supset \cdots$ such that each $Q_\nu$ contains the (compact) closure of $Q_{\nu+1}$. We define the decomposition $Q_\nu = Q_\nu' \cup Q_\nu''$ into two sub-cuboids in the obvious way such that $Q_\nu' \cap Q = Q'$ and $Q_\nu'' = Q''$.

Now we can define the functions $G_\nu$ inductively as holomorphic functions on $Q_\nu' \cap Q_\nu''$ and then apply the additive gluing lemma to define $G_\nu'$, $G_\nu''$ on $Q_{\nu+1}'$, $Q_{\nu+1}''$. So lets recall:

The functions $G_\nu$ are holomorphic on $Q_\nu' \cap Q_\nu''$. One has the decomposition $G_\nu = G_\nu' + G_\nu''$ on $Q_{\nu+1}' \cap Q_{\nu+1}''$. Moreover one has (by definition of $G_{\nu+1}$)

$$(E + G_\nu')(E + G_{\nu+1})(E + G_\nu'') = (E + G_\nu) \quad \text{on} \quad Q_{\nu+1}' \cap Q_{\nu+1}''.$$

Of course the start is $G_0 = E - F$.

Now we come to the problem of convergence of $F_1' \cdots F_\nu'$ (where $F_\nu' = E + G_\nu'$). We want to use a standard criterion for convergence of infinite products.

*Proof of 3.3 continued.* The strategy to enforce convergence is to construct the $G_\nu'$ with an estimate. What we finally want to have is an estimate of the forms

$$|G_\nu(z)| \le \varrho \cdot 4^{-\nu} \text{ for } z \in Q_\nu' \cap Q_\nu'',$$
$$|G_\nu'(z)| \le C \cdot 2^{-\nu} \text{ for } z \in Q_{\nu+1}'$$

with certain constants $C < 1/2$, $\varrho$. The condition on $C$ will ensure that $E + G_\nu' + G_\nu''$ is invertible. If we succeed to get such an estimate we are obviously through.

### Estimates for the gluing functions

We will obtain the estimates for $G_{\nu+1}$ from estimates of the $G_\nu', G_\nu''$ inductively. But this demands also an estimate for the $G_\nu$. Recall that $G_{\nu+1}$ is defined by

$$(E + G_\nu')(E + G_{\nu+1})(E + G_\nu'') = (E + G_\nu' + G_\nu'') \quad \text{on} \quad Q_{\nu+1}' \cap Q_{\nu+1}''.$$

**3.4 Lemma.** *Let $A, B$ be $m \times m$-matrices such that $|A| \le 1/2$ and $|B| \le 1/2$ and let be $C$ a matric such that*

$$(E + A)(E + C)(E + B) = E + A + B.$$

*There exists a constant $P$ depending only on $m$ such that*

$$|C| \le P|A||B|.$$

*Proof.* The set of all $A$ with $|A| \le 1/2$ is compact. The matrix $E + A$ is invertible for these $A$. This can be shown by means of the geometric series. The function $|(E+A)^{-1}|$ takes a maximum on $|A| \le 1/2$. Let $P$ be the square of this maximum. An easy computation gives

$$C = (E + A)^{-1}(-AB)(E + B)^{-1}.$$

This shows $|C| \le P|A||B|$. □

*Proof of 3.3 continued.* It is our goal to get an estimate for $G'_\nu, G''_\nu$. To apply this lemma to our situation we make an assumption about our system of enlarged cuboids. We assume that the minimal distance of any point of $Q_{\nu+1}$ to a boundary point of $Q_\nu$ is $\ge \delta 2^{-\nu}$ with some positive constant $\delta$. It is clear such a constant $\delta$ exists (depending on the shape of $\overline{Q \cap Q'} \subset U$).

We will proceed by induction to produce

$$\begin{aligned} &|G_\nu(z)| \le \varrho \cdot 4^{-\nu} \text{ for } z \in Q'_\nu \cap Q''_\nu, \\ &|G'_\nu(z)| \le C 2^{-\nu} \text{ for } z \in Q'_\nu \\ &|G''_\nu(z)| \le C 2^{-\nu} \text{ for } z \in Q''_\nu \end{aligned}$$

The constants $C, \varrho$ will be determined during the proof. Whatever the constants will be, we can get the beginning of the induction $G_0, G'_0, G''_0$ since, as we mentioned at the beginning of the proof, $G$ can be assumed as small as we want. Assume that $G_\nu$ and $G'_\nu$, $G''_\nu$ have been constructed. Then we construct $G_{\nu+1}$ and then the decomposition $G_{\nu+1} = G''_{\nu+1} + G''_{\nu+1}$. For $G_{\nu+1}$ we get the estimate (Lemma 3.4)

$$|G_{\nu+1}(z)| \le PC^2 4^{-\nu}.$$

So, if we make the choice

$$\varrho := 4PC^2,$$

we get the desired inequality $|G_{\nu+1}(z)| \le \varrho \cdot 4^{-(\nu+1)}$. For $G'_{\nu+1}$ (similarly $G''_{\nu+1}$) we get from Lemma 3.2 the estimate

$$|G'_{\nu+1}(z)| \le \frac{2^\nu M}{\delta} \cdot \varrho \cdot 4^{-(\nu+1)} = \frac{2MPC^2}{\delta} 2^{-(\nu+1)}.$$

So alle we need is the estimate

$$2MPC^2 \le \delta C.$$

This is true if $C$ is small enough.

## 4. The syzygy theorem

We need a sheaf theoretic version of a famous result, namely Hilbert's syzygy theorem. Hilbert expressed this theorem for the polynomial ring but the proof works literally also for the ring of power series. It states:

**4.1 Hilbert's syzygy theorem.** *Let $M$ be a finitely generated module over the ring $R = \mathbb{C}\{z_1, \dots, z_n\}$ of convergent power series in $n$ variables. Let*

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

*be a an exact sequence where the modules $F_i$ are finitely generated free modules. Then the kernel of $F_n \to F_{n-1}$ is free.*

**Corollary.** *For any finitely generated module $M$ there exists an exact sequence*

$$0 \longrightarrow F_{n+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

*with free modules $F_i$.*

There is an immediate sheaf theoretic consequence.

**4.2 Remark.** *Let $\mathcal{M}$ be a coherent sheaf on some open subset $U \subset \mathbb{C}^n$ and $a \in U$ a point. There exists an open neighborhood $a \in V \subset U$ and an exact sequence*

$$0 \longrightarrow \mathcal{F}_{n+1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{M}|V \longrightarrow 0$$

*where $\mathcal{F}_i \cong \mathcal{O}_V^{n_i}$ are free sheaves on $V$.*

*Proof.* We choose a resolution of the module $\mathcal{M}_a$

$$0 \longrightarrow F_{n+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow M_a \longrightarrow 0$$

by free $\mathcal{O}_{U,a}$-modules. We can extend this sequence using ??? and ???. □

There is a much better result:

**4.3 Proposition.** *Let $\mathcal{M}$ be a coherent sheaf on a cuboid $Q$ and $Q_0 \subset Q$ a shrunken cuboid. Then there exists an exact sequence*

$$0 \longrightarrow \mathcal{F}_{n+1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{M}|Q_0 \longrightarrow 0$$

*where $\mathcal{F}_i \cong \mathcal{O}_{Q_0}^{n_i}$ are free sheaves on $Q_0$.*

The proof of this proposition rests on the *Cartan gluing lemma* 3.3. During the proof we use the following short notation. Let $\mathcal{M}$ be a coherent sheaf on some open subset $U \subset \mathbb{C}^n$. The sheaf $\mathcal{M}$ admits a free resolution over a compact subset $K \subset U$ if there exists an open set $K \subset V \subset U$ and an exact sequence

$$0 \longrightarrow \mathcal{F}_{n+1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{M}|V \longrightarrow 0$$

with free $\mathcal{O}_V$-modules $\mathcal{F}_i$.

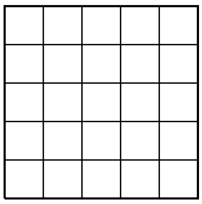
**4.4 Lemma.** *As in 3.1 we consider two rectangles in the position*

$$R' = (a,c) \times I,\ \ R'' = (b,d) \times I \quad (a < b < c < d)$$

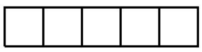
*and then the cuboids* $Q' = R' \times D$, $Q'' = R'' \times D$ *with a cuboid* $D \subset \mathbb{C}^n$. *Let* $\mathcal{M}$ *be a coherent sheaf over some open neighborhood of* $\overline{Q' \cup Q''}$.

*Assume that* $\mathcal{M}$ *admits free resolutions over* $\bar{Q}'$ *and* $\bar{Q}''$. *Then* $\mathcal{M}$ *admits a free resolution over* $\overline{Q' \cup Q''}$.

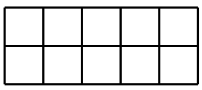
Before we prove this lemma we show that 4.3 follows from it. For this we decompose the cuboid $Q_0$ into $N^{2n}$ closed small sub-cuboids, by dividing each edge into $N$ equidistant sub-cuboids as indicated in the figure.



By means of 4.2 and a simple compactness argument this can be done in such a way that $\mathcal{M}$ admits a free resolution over the closure of each small sub-cuboid. Application of the gluing lemma 4.4 several times leads to a free resolution over $\bar{Q}_0$. We describe this in more detail in the case $n = 1$: In the first step one produces a resolution over the first row of squares in the above figure



Then we do the same with the second row and then glue the first with the second row. This gives a free resolution over



It should be clear that this argument works in arbitrary dimension. So we are reduced to the

*Proof of 4.4.* The resolutions over $\bar{Q}'$ and $\bar{Q}''$ give two different resolutions over the intersection. So we need a method to compare two different resolutions. The principle can be understood already in the local case. So let us assume that we have a finitely generated module $M$ over a ring $R$ and that we have two different free resolutions

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F_{n+1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ 0 & \longrightarrow & G_{n+1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Two such resolutions are called isomorphic if there is a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & F_{n+1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & G_{n+1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the vertical arrows are isomorphisms. It is not true that two resolutions are isomorphic. The reason that there exist trivial resolutions of 0. By a trivial resolution of 0 we understand a resolution of the form

$$0 \cdots \longrightarrow F \xrightarrow{\text{id}} F \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

with a free module $F$. One can define the direct sum of a resolution with such a trivial resolutions. (The direct sum two resolutions

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F_{n+1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ 0 & \longrightarrow & G_{n+1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

is

$$0 \longrightarrow F_{n+1} \oplus G_{n+1} \longrightarrow \cdots \longrightarrow F_1 \oplus G_1 \longrightarrow M \oplus N \longrightarrow 0$$

with obvious arrows. In the case $N = 0$ we can identify $M \oplus 0$ and $M$.)

By an *elementary modification* of a free resolution we understand a new free resolution which one obtains if one takes the direct sum with a trivial resolution of 0 as described above.

**4.5 Lemma.** *Two free resolutions of an $R$-module $M$ get isomorphic after performing a finitely many elementary modifications (to both of them).*

*Proof.* The proof is given by some induction. The first step is to modify $F_1, G_1$ if necessary. We take free generators of $F_1$ and consider their images in $M$. Taking inverse images of them in $G_1$ we construct an $R$-linear map $\sigma : F_1 \to G_1$ and similarly $\tau : G_1 \to F_1$ such that the diagrams

$$\begin{array}{ccc} F_1 & \text{———} & M \\ \sigma \Big| & & \Big| \| \\ G_1 & \text{———} & M \end{array} , \qquad \begin{array}{ccc} F_1 & \text{———} & M \\ \tau \Big| & & \Big| \| \\ G_1 & \text{———} & M \end{array}$$

commute. It may be that $\sigma$ and $\tau^{-1}$ are isomorphisms. Then we do nothing. Otherwise we add to the $F$-resolution the trivial resolution $0 \to G_1 \to G_1 \to 0$ and to the $G$-resolution the trivial resolution $0 \to F_1 \to F_1 \to 0$. We get new resolutions

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & F_3 & \longrightarrow & F_2 \oplus G_1 & \longrightarrow & F_1 \oplus G_1 & \longrightarrow & M & \longrightarrow 0 \\ & & & & & & \downarrow & & \| & \\ \cdots & \longrightarrow & G_3 & \longrightarrow & G_2 \oplus F_1 & \longrightarrow & G_1 \oplus F_1 & \longrightarrow & M & \longrightarrow 0 \end{array}$$

where the vertical arrows have to be explained. The map $F_1 \oplus G_1 \to G_1 \oplus F_1$ is defined by means of the matrix

$$\begin{pmatrix} \sigma & 1-\sigma\tau \\ 1 & -\tau \end{pmatrix}.$$

This has to be understood as follows. The action on a pair $(f, g)$ is given by

$$\begin{pmatrix} \sigma & 1-\sigma\tau \\ 1 & -\tau \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \sigma(f) + g - \sigma\tau(g) \\ f - \tau(g) \end{pmatrix}.$$

The essential point is that this map is an isomorphism. The inverse map is given though the matrix

$$\begin{pmatrix} -\tau & -1+\sigma\tau \\ -1 & \sigma \end{pmatrix}.$$

One checks that the above diagram is commutative. This shows that we can reduce to the situation

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F_{n+1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & G_{n+1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the vertical arrow is an isomorphism. This was the first step of the induction. We explain, how to continue. It might happen that $F_\nu = G_\nu = 0$ for $\nu \geq 0$. Then $F_2, G_2$ can be considered as submodules of $F_1, G_1$. The map $F_1 \to G_1$ maps $F_2$ into $G_2$ and conversely. Hence we have isomorphic resolutions $0 \to F_2 \to F_1 \to M \to 0$ and $0 \to G_2 \to G_1 \to M$ and we are done. Otherwise we construct now a linear map $\sigma : F_2 \to G_2$ such the diagram

$$\begin{array}{ccccccc} F_2 & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ \downarrow & & \downarrow & & \| & & \\ G_2 & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

commutes. This can easily done by means of the free generators. Similarly we construct $\tau : G_2 \to F_2$. We modify now with the complexes $\cdots 0 \to G_2 \to G_2 \to 0 \to 0$ and $\cdots 0 \to F_2 \to F_2 \to 0 \to 0$ and reduce to a situation

$$\begin{array}{cccccccc} \cdots & F_2 & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ & \downarrow & & \downarrow & & \| & & \\ \cdots & G_2 & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where both vertical arrows are isomorphism. I should be clear now how the induction runs and terminates.

*Proof of 4.4 continued.* We come back to the resolutions of $\mathcal{M}$ over $\bar{Q}'$ and $\bar{Q}''$. This means that there are two cuboids $\bar{Q}' \subset \tilde{Q}'$ and similarly $\tilde{Q}''$ that are located similarly as described in 4.4 and such that the resolutions of $\mathcal{M}$

are defined over $\tilde{Q}', \tilde{Q}''$. After finitely many modifications they are isomorphic over the intersection. This means that the resolutions are of the form

$$0 \longrightarrow \mathcal{O}_{\tilde{Q}'}^{m_{n+1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\tilde{Q}'}^{m_1} \longrightarrow \mathcal{M}|\tilde{Q}' \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{\tilde{Q}''}^{m_{n+1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\tilde{Q}''}^{m_1} \longrightarrow \mathcal{M}|\tilde{Q}'' \longrightarrow 0$$

and over $\tilde{Q}' \cap \tilde{Q}''$ there are isomorphisms $\sigma$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\tilde{Q}'\cap\tilde{Q}''}^{m_{n+1}} & \longrightarrow \cdots \longrightarrow & \mathcal{O}_{\tilde{Q}'\cap\tilde{Q}''}^{m_1} & \longrightarrow & \mathcal{M}|(\tilde{Q}' \cap \tilde{Q}'') & \longrightarrow & 0 \\ & & \Big| \sigma & & \Big| \sigma & & & & \\ 0 & \longrightarrow & \mathcal{O}_{'\cap\tilde{Q}''}^{m_{n+1}} & \longrightarrow \cdots \longrightarrow & \mathcal{O}_{\tilde{Q}'\cap\tilde{Q}''}^{m_1} & \longrightarrow & \mathcal{M}|(\tilde{Q}' \cap \tilde{Q}'') & \longrightarrow & 0 \end{array}$$

gets commutative.

The isomorphisms $\sigma$ are given by invertible holomorphic functions $\tilde{Q}'\cap\tilde{Q}'' \to \mathrm{GL}(m_i, \mathcal{O}(\tilde{Q}' \cap \tilde{Q}''))$. Now can Cartan's gluing lemma to write $\sigma$ as product $\sigma = \sigma'\sigma''$, where $\sigma'$ is a holomorphic map from $Q'$ to $\mathrm{GL}(m_i, \mathcal{O}(Q'))$ and similarly $\sigma''$. To be precise we first have to shrink $\tilde{Q}'$ and $\tilde{Q}''$ a little. We use the isomorphisms $\sigma', \sigma''$ to modify the resolution of $\mathcal{M}|\tilde{Q}', \mathcal{M}|\tilde{Q}''$ in such a way that now the two resolutions over $\tilde{Q}' \cap \tilde{Q}''$ are identical. If this is the case they glue to single resolution of $\mathcal{M}$ over $\tilde{Q}' \cup \tilde{Q}''$. This finishes the proof of 4.4 and then of 4.3. □

## 5. Theorem B for cuboids

We know from the lemma of Dolbeault III.6.9 that the cohomology groups $H^q(Q, \mathcal{O}_Q)$, $q > 0$, vanish for a poly disk $Q$. Since every rectangle is is biholomorphic equivalent to the unit disk this is also true for cuboids. This section is devoted to the proof of

**5.1 Theorem B for cuboids.** *Let $\mathcal{M}$ be a coherent sheaf on a cuboid $Q$. Then*

$$H^q(Q, \mathcal{M}) = 0 \quad \textit{for} \quad q > 0.$$

**Corollary.** *Theorem B is true for polydisks.*

The corollary follows since each rectangle in the complex plane is biholomorphic equivalent to a disk. Technically it has advantages to work with cuboids instead of polydisks.

It will be necessary to shrink $Q$ a little. This means that we have to consider a cube $Q_0$ whose (compact) closure is contained in $Q$. We write $Q_0 \subset\subset Q$ to indicated this. There are two different steps. In the first basic step we will prove

**5.2 Theorem.** *let $\mathcal{M}$ be a coherent sheaf on a cuboid $Q$ and $Q_0$ a shrunken cuboid. Then*

$$H^q(Q_0, \mathcal{M}|Q_0) = 0 \quad \textit{for} \quad q > 0.$$

*Proof.* We just have to show: Let

$$0 \to F_{n+1} \longrightarrow \cdots F_1 \longrightarrow F \longrightarrow 0$$

be an exact sequence of sheaves such that all $F_i$ are acyclic. (This means that the higher cohomology groups vanish). Then $F$ acyclic. For the proof one considers the sequence

$$0 \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow\longrightarrow 0.$$

(So $K$ is a co-kernel). From the long exact cohomology sequence follows that $K$ is acyclic. There is an obvious exact sequence

$$0 \longrightarrow K \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow K.$$

Now we can argue by induction on $n$. □

*Proof of 5.1.* The proof for arbitrary cuboids uses an exhaustion argument. This argument also will work in the general case of arbitrary Steil spaces. But in the case of a cuboid is it is technically easier. Hence we give the details already in the case of the cuboid.

If $X$ is an complex manifold we know that $\mathcal{O}_X(X)$ gets the structure as a Frèchet space if one equips it with the topology of uniform convergence on compact subsets. Slightly more generally $\mathcal{O}_X(X)^n$ gets a Frèchet space if we equip it with the product topology. Our starting point for constructing topologies is:

**5.3 Lemma.** *Let $X$ be a complex manifold and $\mathcal{M} \subset \mathcal{O}_U^m$ be a coherent subsheaf of a free sheaf. Then $\mathcal{M}(X)$ is a closed subspace of $\mathcal{O}_X^m(X)$.*

*Proof.* Let $s_k$ be sequence in $\mathcal{M}(X)$ that converges (uniformly on compact subsets) to $s \in \mathcal{O}_X(X)^n$. We have to show that $s \in \mathcal{M}(X)$. This means that for any point $a \in X$ we have $s_a \in \mathcal{M}_a$. We use the notation $F = \mathcal{O}_{X,a}^n$ and $M = \mathcal{M}_a$. We consider the maximal ideal $\mathfrak{m}$ in the local ring $\mathcal{O}_{X,a}$. The vector space $F/\mathfrak{m}^m F$ is finite dimensional for any $m$. Hence it carries a natural topology. Now we consider the images $\bar{s}_n, \bar{s}$ of $s_n, s$ in $F/\mathfrak{m}^m F$. The essential point is that $\bar{s}_n$ converges to $\bar{s}$ in this finite dimensional vector space. We have to explain the reason for this. Taking coordinates we can identify $\mathcal{O}_{X,a}$ with the ring of power series. Then $\mathcal{O}_{X,a}/\mathfrak{m}^m$ can be identified with a $\mathbb{C}^N$ where the map $\mathcal{O}_{X,a}/\mathfrak{m}^m \to \mathbb{C}^N$ associates to a power series the vector of coefficients $a_\nu$ of degree $\le m$. Now we have to use from complex analysis that the locally uniform convergence $s_k \to s$ implies the locally uniform convergence of all partial derivatives and hence also of the Taylor coefficients. This proves that $\bar{s}_n \to \bar{s}$ in $F/\mathfrak{m}^m F$. Now we use that every sub-vector space of a finite dimensional vector space is closed. This gives us that $\bar{s}$ is in the image of $M$ since this is a sub-vector space. This can be expressed as

$$s \in M + \mathfrak{m}^m F.$$

From this follows $s \in M$ by a pure algebraic argument: One has to use Krull's intersection theorem. Application of the polydisks intersection theorem completes the proof of 5.3. □

Let $\mathcal{M}$ be a coherent sheaf on a cuboid $Q$. We shrink $Q$ to a cuboid $Q_0$. We want to construct a topology on $\mathcal{M}(Q_0)$ For this purpose we slightly enlarge the shrink. That is we choose a cuboid $Q_1$ such that $Q_0$ is a shrink of $Q_1$ and $Q_1$ is a shrink of $Q$. We know that $\mathcal{M}|Q_1$ is finitely generated. This means that there exists a surjective map $\mathcal{O}_{Q_1}^n \to \mathcal{M}|Q_1$. From the weak form of Theorem B (5.2) we we get the surjectivity

$$\mathcal{O}_Q(Q_0)^n \longrightarrow \mathcal{M}(Q_0).$$

From 5.3 follows that the kernel is closed. In this way we get a structure as Frèchet space on $\mathcal{M}(Q_0)$. It is rather clear that this structure is independent of the presentation $\mathcal{O}_{Q_1}^n \to \mathcal{M}|Q_1$. Hence we obtain:

**5.4 Lemma.** *Let $Q$ be a cuboid and $\mathcal{M}$ a finitely generated coherent sheaf on $Q$. Let $Q_0$ be a shrunken cuboid. Then $\mathcal{M}(Q_0)$ carries a unique structure as Frèchet space with the following property. For each cuboid $Q_0 \subset\subset Q_1 \subset\subset Q$ and all surjective maps $\mathcal{O}_{Q_1}^n \to \mathcal{M}|Q_1$ the induced map $\mathcal{O}_Q^n(Q_0) \to \mathcal{M}(Q_0)$ is continuous.*

A direct consequence of Runge's approximation theorem 2.1 is:

**5.5 Runge approximation theorem for coherent sheaves (weak form).** *Let $Q_0 \subset\subset Q_1 \subset\subset Q$ be cuboids and let $\mathcal{M}$ be a coherent sheaf on $Q$. The image of the restriction map $\mathcal{M}(Q_1) \to \mathcal{M}(Q_0)$ is dense.*

Now we collected all tools for:

*Proof of Theorem B for cubes 5.1.* We choose a sequence of cuboids

$$Q_1 \subset\subset Q_2 \subset\subset Q_3 \subset\subset \cdots \subset\subset Q$$

whose union is $Q$. This is an open covering $\mathfrak{U}$ of $Q$. We know $H^q(Q_\nu, \mathcal{M}|Q_\nu) = 0$ for $q > 0$. We want to show that $H^q(Q, \mathcal{M}) = 0$ for $q > 0$. From Leray's theorem follows that this cohomology group can be computed by means of Čech cohomology

$$H^q(\mathfrak{U}, \mathcal{M}) = H^q(Q, \mathcal{M}).$$

Similarly we get

$$H^q(\mathfrak{U}_m, \mathcal{M}|Q_m) = H^q(Q_m, \mathcal{M}|Q_m) \qquad (= 0),$$

where $\mathfrak{U}_m$ denotes the (finite) covering of $U_m$ by $U_1, \dots, U_m$. We recall that the Čech complex has been denoted by $C^q(\mathfrak{U}, \mathcal{M})$. For sake of simplicity we use the notation $C^q(\mathfrak{U}_m, \mathcal{M}) := C^q(\mathfrak{U}_m, \mathcal{M}|Q_m)$. There are natural restriction maps

$$C^q(\mathfrak{U}, \mathcal{M}) \longrightarrow C^q(\mathfrak{U}_m, \mathcal{M}) \longrightarrow C^q(\mathfrak{U}_k, \mathcal{M}) \quad \text{for} \quad m > k$$

and there is a natural (injective) extension map

$$C^q(\mathfrak{U}_k, \mathcal{M}) \longrightarrow C^q(\mathfrak{U}_m, \mathcal{M}) \quad \text{for} \quad m > k,$$

a cochain $s$ is extended by the definition $s(i_0, \dots, i_q) = 0$ if one of the indices is out of the range (greater than $k$).

We consider now a cochain $s \in C^q(\mathfrak{U}, \mathcal{M})$, $q > 0$, with the property $ds = 0$ We have to show that there is cochain $t \in C^{q-1}(\mathfrak{U}, \mathcal{M})$ with $dt = s$. We denote by $s^{(m)} \in C^q(\mathfrak{U}_m, \mathcal{M})$ for some $m > 0$ the restriction of $s$. Since $ds = 0$ implies $ds^{(m)} = 0$ we get $s^{(m)} = dt^{(m)}$ with $t^{(m)} \in C^{q-1}(\mathfrak{U}_m, \mathcal{M})$. We can restrict $t^{(m)}$ to $C^{q-1}(\mathfrak{U}_{m-1}, \mathcal{M})$. From the restriction we can subtract $t^{(m-1)}$. We denote the result simply by $t^{(m)} - t^{(m-1)}$. We know $d(t^{(m)} - t^{(m-1)}) = 0$. There are two different cases. The case $q > 1$ is very easy, the difficult part will be the case $q = 1$.

*First case,* $q > 1$. In this case we have still $H^{q-1}(\mathfrak{U}_{m-1}, \mathcal{M}) = 0$. Hence there exists

$$\alpha_{m-1} \in C^{q-2}(\mathfrak{U}_{m-1}, \mathcal{M}) \quad \text{such that} \quad t^{(m)} - t^{(m-1)} = d\alpha_{m-1}.$$

We denote the natural extension of $\alpha_{m-1}$ to $C^{q-2}(\mathfrak{U}_k, \mathcal{M})$, $k > m-1$, by the same letter. Then we can define

$$T^{(m)} := t^{(m)} - d\left(\sum_{k=1}^{m-1} \alpha^{(k)}\right) \qquad \bigl(\in C^{q-1}(\mathfrak{U}_m, \mathcal{M})\bigr).$$

The $T^{(m)}$ are modifications of the $t^{(m)}$ in the sense that the satisfy $s^{(m)} = dT^{(m)}$. The advantage of the modification is that we now have that the system $T^{(m)}$ is compatible. We omit the simple calculation for it. This means that the restriction of $T^{(m)}$ to $\mathfrak{U}_{m-1}$ is $T^{(m-1)}$. This implies that they glue to a cochain

$t \in C^{q-1}(\mathfrak{U}, \mathcal{M})$. But with this cochain we clearly have $dt = s$. This is want we wanted to prove.

*Second case,* $q = 1$. We consider the sequence of Frèchet spaces

$$\mathcal{M}(Q_1) \longleftarrow \mathcal{M}(Q_2) \longleftarrow \mathcal{M}(Q_2) \longleftarrow \cdots$$

The image of each arrow is dense. Recall that we have chosen $t^{(m)}$ with $dt^{(m)} = s^{(m)}$. Now $q = 1$. The elements of $C^0(\mathcal{U}_m, \mathcal{M})$ attach to each index $k \le m$ a section from $\mathcal{M}(Q_k)$. If the element is closed, then these sections glue to a section from $\mathcal{M}(Q_m)$. Hence $\mathcal{M}(Q_m)$ can be identified with the closed elements from $C^0(\mathcal{U}_m, \mathcal{M})$. In this way $t^{(m)} - t^{(m-1)}$ can be considered as element of $\mathcal{M}(Q_{m-1})$. As in the first case we will have to replace $t^{(m)}$ by some other $T^{(m)} = t^{(m)} + \alpha^{(m)}$. Here $\alpha^{(m)}$ should be a zero cochain with the property $d\alpha^{(m)} = 0$. As we explained this can be considered as element of $\mathcal{M}(Q_m)$. The construction of $\alpha^{(m)}$ will use Runge approximation. The aim of the construction is that the sequence $T^{(m)}$ converges. Since the entries of this sequence are in different spaces, we have to explain what convergence means: It means that there exist an $T \in \mathcal{M}(Q)$ such that for each $k$ the sequence $(t^{(m)})_{\ge k}$, more precisely its image in $\mathcal{M}(Q_k)$ converges to $T|\mathcal{M}(Q_k)$. To prove the convergence, we will use the Cauchy criterion: For each $k$ we will have to show:

For each neighborhood $0 \in U \subset \mathcal{M}(Q_k)$ there exists an $N$ such that the image of $T^{(\mu)} - T^{(\nu)}$ in $\mathcal{M}(Q_k)$ is contained in $U$ for $\mu > \nu \ge 0$.

We will use also that each space $\mathcal{M}(Q_m)$ has a countable fundamental system of neighborhoods of the origin (Frèchet spaces are metrisable).

For each $m$ we choose a fundamental system of neighborhoods of the origin as indicated in the figure

$$\begin{array}{ccccccc}
\mathcal{M}(Q_1) & \longleftarrow & \mathcal{M}(Q_2) & \longleftarrow & \mathcal{M}(Q_3) & \longleftarrow & \cdots \\
\cup & & \cup & & \cup & & \\
U_{11} & \longleftarrow & U_{21} & \longleftarrow & U_{31} & & \cdots \\
\cup & & \cup & & \cup & & \\
U_{12} & \longleftarrow & U_{22} & \longleftarrow & U_{32} & & \cdots \\
\cup & & \cup & & \cup & & \\
U_{13} & \longleftarrow & U_{23} & \longleftarrow & U_{33} & \longleftarrow & \cdots \\
\cup & & \cup & & \cup & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

The horizontal arrows indicate that $U_{km}$ is mapped to $U_{k,m-1}$ under the restriction map $\mathcal{M}(Q_m) \to \mathcal{M}(Q_{m-1})$. We also want to have that the neighborhoods shrink rapidly in the sense $U_{m,k+1} + U_{m,k+1} \subset U_{m,k}$. It clear that such a system of neighborhoods can be constructed. Then induction shows

$$\overbrace{U_{m,k+\nu} + \cdots + U_{m,k+\nu}}^{\nu+1 \text{ copies}} \subset U_{m,k}.$$

After this preparation we come the construction of $T^{(m)} = t^{(m)} + \alpha^{(m)}$. What we want to have is $T^{(m+1)} - T^{(m)} \in U_{m,m}$. It is now problem to construct this by induction. One starts with $T^{(1)} = t^{(1)}$. Assume that $T^{(1)}, \dots T^{(m)}$ have been constructed. We construct $T^{(m+1)}$. For this we consider $T^{(m+1)} - t^{(n)} \in \mathcal{M}(Q_m)$. By the approximation theorem there exists an element $\alpha^{(m+1)} \in \mathcal{M}(Q_{m+1})$ such that $T^{(m)} - t^{(m+1)} - \alpha^{(m+1)} \in U_{m,m}$. Now $T^{(m+1} = t^{(m+1)} + \alpha^{(m+1)}$ has the desired property.

We have to check that $T^{(m)}$ is a Cauchy sequence in the described sense. For this wa have to fix an $k$ and to consider a neighborhood of the origin in $\mathcal{M}(Q_k)$. We can take this neighborhood in the form $U = U_{k,l}$ with some $l$. We have to construct an $N \geq k$ such that

$$T^{(\mu)} - T^{(\nu)} \longrightarrow U \quad \text{for} \quad \mu \geq \nu \geq N.$$

(The arrow just indicates that —after restriction to $\mathcal{M}(Q_k)$— the element should be contained in $U$.) We claim that a possible choice is $N = \max(k, l{+}2)$. For this we decompose

$$T^{(\mu)} - T^{(\nu)} = (T^{(\mu)} - T^{(\mu-1)}) + \cdots + (T^{(\nu+1)} - T^{(\nu)}).$$

We can consider this element in $\mathcal{M}(Q_\nu)$. There it lies in

$$U_{\nu,\mu-1} + U_{\nu,\mu-2} + \cdots + U_{\nu,\nu} \subset \overbrace{U_{\nu,\nu} + \cdots + U_{\nu,\nu}}^{\mu-\nu-2 \text{ copies}} \subset U_{\nu,\nu-(\mu-\nu-2)} = U_{\nu,2\nu-\mu-2}.$$

Hence the image of $T^{(\mu)} - T^{(\nu)}$ in $\mathcal{M}(Q_k)$ is in $U_{k,2\nu-\mu-2}$. Since $2\nu - \mu - 2 \geq \nu - 2 \geq N - 2 \geq l$ we obtain $T^{(\mu)} - T^{(\nu)} \in U_{k,l}$ as desired. So the global section $T$ "$= \lim T^{(m)}$" has been constructed.

Finally we claim $dT = s$ (globally). Since $dT^{(k)} = s^{(k)}$ we only have to show that $T|U_k - T^{(k)}$ is closed. From construction is a limit of the close elements. Now $d$ is clearly a continuous operator. This finishes the proof of 5.1. □

## 6. Theorem A and B for Stein spaces

The basic theorems about Stein spaces are

**6.1 Theorem A for Stein spaces.** *Let $X$ be a Stein space and $\mathcal{M}$ a coherent sheaf. For each $a \in X$ the stalk $\mathcal{M}_a$ can be generated by (the germs of) finitely many global sections.*

**6.2 Theorem B for Stein spaces.** *Let $X$ be a Stein space and $\mathcal{M}$ a coherent sheaf. Then*

$$H^q(X,\mathcal{M})=0 \quad \textit{for} \quad q>0.$$

The formulation seems to indicate that we have two independent theorems. Actually theorem A is an easy consequence of theorem B. To prove this we consider the vanishing ideal sheaf $\mathcal{J}\subset\mathcal{O}_X$ of the point $a$ and then for an arbitrary natural number Then we use the exact sequence

$$0\longrightarrow\mathcal{J}\mathcal{M}\longrightarrow\mathcal{M}\longrightarrow\mathcal{M}/\mathcal{J}\longrightarrow 0.$$

From theorem B we get that $\mathcal{M}(X)\to(\mathcal{M}/\mathcal{J})(X)$ is surjective. Notice that $(\mathcal{M}/\mathcal{J})(X)=\mathcal{M}_a/\mathfrak{m}\mathcal{M}_a$. Here $\mathfrak{m}$ means the maximal ideal of $\mathcal{O}_{X,a}$. We denote by $M$ the submodule of $\mathcal{M}_a$ that is generated by the image of $\mathcal{M}(X)$ and by $N=\mathcal{M}_a/M$ the factor module. The above argument shows $\mathcal{M}_a=M+\mathfrak{m}\mathcal{M}_a$ or $\mathfrak{m}N=N$. The proof now follows from the lemma of Nakayama.

For the proof of theorem B we will use an exhaustion by Oka domains. So the procedure is similar as in the proof of Theorem B for cuboids. But there are some technical difficulties that arise. One of them is to define a structure as Frèchet space on $\mathcal{M}(X)$ for singular $X$. Actually this is possible for each coherent sheaf on an arbitrary complex space in a natural way. But the construction is difficult. This is already visible for the structure sheaf. Actually on can try to equip $\mathcal{O}_X(X)$ with the topology of uniform convergence on compact sets. To make this work correctly one needs that the limit of a sequence of analytic functions that converges uniformly on each compact subset is analytic too. Actually this is true but unfortunately rather deep and not at reach at the moment. Hence we restrict to topologize $\mathcal{M}(X)$ only in special cases.

We will use 5.3 to construct a Frèchet topology on $\mathcal{O}_X(X)$ for special non-smooth complex spaces. Let $P\subset\mathbb{C}^n$ be a polydisk and $X\subset P$ a closed analytic subset. We have a natural map $\mathcal{O}_P(P)\to\mathcal{O}_X(X)$. This map is surjective by Theorem B for polydisks. To see this just consider the ideal sheaf $\mathcal{J}\subset\mathcal{O}_P$ corresponding to $X$. Then we have a short exact sequence

$$0\longrightarrow\mathcal{J}\longrightarrow\mathcal{O}_P\longrightarrow\mathcal{O}_P/\mathcal{J}\longrightarrow 0.$$

From theorem $B$ we get $H^1(P,\mathcal{J})=0$ and form this the surjectivity of $\mathcal{O}_P(P)\to\mathcal{O}_P(P)/\mathcal{J}(P)$. There is a natural isomorphism $\mathcal{O}_X(X)\cong\mathcal{O}_P/\mathcal{J}(P)$. This gives the claimed surjectivity $\mathcal{O}_P(P)\to\mathcal{O}_X(X)$. We know from 5.3 that the kernel is closed. Hence the factor space of $\mathcal{X}_P(P)$ by this kernel carries a natural structure as Frèchet space. We transport this structure to $\mathcal{O}_X(X)$ to get a structure as Frèchet space there.

**6.3 Proposition.** *Let $X$ be a complex space such there exists polydisk $P$ and a closed holomorphic embedding $\alpha : X \to \mathcal{P}$. There exists a unique structure as Frèchet space on $\mathcal{O}_X(X)$ such that the induced map $\mathcal{O}_P(P) \to \mathcal{O}_X(X)$ is continuous. This structure is independent of the choice of the embedding $\alpha$.*

The open mapping theorem for Frèchet spaces shows that $\mathcal{O}_P(P)$ must carry the quotient topology of $\mathcal{O}_P(P)$. Hence we only have to show the independence of the choice of the embedding $\alpha$. Let $\beta : X \to P'$ be another closed embedding. We connect both embeddings to an embedding

$$(\alpha, \beta) : X \to P \times P'.$$

We consider the natural maps

$$\mathcal{O}_P(P) \longrightarrow \mathcal{O}_{P\times P'}(P \times P') \longrightarrow \mathcal{O}_X(X).$$

the first one is associated to the projection $P \times P' \to P$. Now we first equip $\mathcal{O}_X(X)$ with the quotient topology of $\mathcal{O}_{P\times P'}(P \times P')$. Since $\mathcal{O}_P(P) \to \mathcal{O}_{P\times P'}(P \times P')$ is continuous by trivial reasons we get with this topology that $\mathcal{O}_P(P) \to \mathcal{O}_X(X)$ is continuous. By the open mapping theorem $\mathcal{O}_X(X)$ must carry the quotient topology of $\mathcal{O}_P(P)$. So we see that $\mathcal{O}_P(P)$ and $\mathcal{O}_{P\times P'}(P \times P')$ induce the same topology. Since the roles of $P$ and $P'$ can be interchanged, we see that $\mathcal{O}_P(P)$ and $\mathcal{O}_{P'}(P')$ induce the same topology.

□

We get a first version of a variant of Runge's approximation theorem.

**6.4 Approximation theorem, first version.** *Let $X'$ be a complex space that can be embedded into some polydisk $\beta : X' \hookrightarrow Q' \subset \mathbb{C}^n$ as closed complex subspace. Let $X$ be an open subset of $X'$ that also can be embedded into some polydisk $\alpha : X \hookrightarrow Q \subset \mathbb{C}^m$ as closed complex subspace. We assume that the function $\alpha$ extends to a holomorphic map $X' \to \mathbb{C}^m$. Then the following holds:*

1) *The natural map $\mathcal{O}_{X'}(X') \to \mathcal{O}_X(X)$ is continuous.*
2) *The image of this map is dense.*

*Proof.* We denote the extension of $\alpha$ also by $\alpha : X' \to \mathbb{C}^m$. The two polydisks can be very different and not be compared directly. We improve this by modifying them. Instead of $\alpha : X \to Q$ we consider

$$X \longrightarrow Q \times Q', \quad x \longmapsto (\alpha(x), \beta(x)).$$

This is also an closed embedding. Similarly we consider

$$X' \longrightarrow \mathbb{C}^m \times Q', \quad x \longmapsto (\alpha(x), \beta(x)).$$

which is also a closed embedding. Since the topologies don't depend on the choice of the embeddings, we can assume from advance.

*The polydisks $Q, Q'$ are in the same $\mathbb{C}^n$ and we have $Q \subset Q'$. The diagram*

$$\begin{array}{ccc} X & \hookrightarrow & Q \\ \cap & & \cap \\ X' & \hookrightarrow & Q' \end{array}$$

*commutes.*

From this diagram we get a map $\mathcal{O}_{Q'}(Q') \to \mathcal{O}_Q(Q) \to \mathcal{O}_X(X)$ that clearly is continuous. From the universal property of the quotient topology we get that $\mathcal{O}_{X'}(X') \to \mathcal{O}_X(X)$ is continuous. The claimed density now follows from the density of the image of $\mathcal{O}_{Q'}(Q') \to \mathcal{O}_Q(Q)$. This is a consequence of the possibility power series expansions in polydisk.

We need an extension of 6.4 to finitely generated coherent sheaves. For this we need a generalization of 5.3.

**6.5 Lemma.** *Let $X$ be a complex space that can be embedded as closed analytic subset into a polydisk. Let $\mathcal{M} \subset \mathcal{O}_X^n$ be a coherent subsheaf of a free sheaf. Then $\mathcal{M}(X)$ is closed in $\mathcal{O}_X(X)^n$.*

*Proof.* Let $X \to P$ be the closed embedding into a polydisk. It is sufficient to show that inverse image of $\mathcal{M}(X)$ in $\mathcal{O}_P(P)^m$ is closed. But $\mathcal{M}$ is the module of global sections of a coherent sub-sheaf of $\mathcal{O}_P^n$. Hence we can apply 5.3.

□

**6.6 Lemma.** *Let $X$ be a complex space that is embeddable as as closed analytic subset into a polydisk. Let $\mathcal{M}$ be a finitely generated coherent sheaf on $X$. Then there exists a unique structure as Frèchet space on $\mathcal{M}(X)$ such that for each presentation $\mathcal{O}_X^n \to \mathcal{M}$ the map $\mathcal{O}_X(X)^n \to \mathcal{M}(X)$ is continuous.*

The approximation theorem 6.4 now has an obvious generalization.

**6.7 Runge's approximation theorem, second version.** *Let $X'$ be a complex space that can be embedded into some polydisk $\beta : X' \hookrightarrow Q' \subset \mathbb{C}^n$ as closed complex subspace. Let $X$ be an open subset of $X$ that also can be embedded into some polydisk $\alpha : X \hookrightarrow Q \subset \mathbb{C}^m$ as closed complex subspace. We assume that the function $\alpha$ extends to a holomorphic map $X' \to \mathbb{C}^m$. Assume that $\mathcal{M}$ is a finitely generated coherent sheaf on $X'$. Then the following holds:*

1) *The natural map $\mathcal{M}(X') \to \mathcal{M}(X)$ is continuous.*
2) *The image of this map is dense.*

With the so far developed tools the proof of theorem B is literally the same as for a cube 5.1. So can keep short. Using 1.6 we can construct an exhaustion

$$U_1 \subset\subset U_2 \subset\subset U_3 \subset\subset \cdots \quad \subset\subset X$$

by Oka domains. We know that the cohomology of $\mathcal{M}$ vanishes on each $U_m$. This follows from Theorem $B$ for polydisks. We also have a Frèchet space structure on $\mathcal{M}(U_m)$ such the image in $\mathcal{M}(U_m)$ is dense. (Notice that 5.5 can be applied since Oka domains are embedded into polydisks by global functions.) So we have produced the analogue situation as we had in the case of a cuboid. The proof that we started behind 5.5 now works literally. □

## 7. Meromorphic functions

An element $a$ of a ring $R$ is called a non-zero divisor if $ax = 0 \Rightarrow x = 0$. Let $S$ be a set of all non-zero divisors. Assume that $1 \in S$ and that $s, t \in S$ implies $st \in S$. Then we call $S$ a multiplicative subset. There exists a ring $R_S$ that contains $R$ as subring such that the elements of $S$ are invertible in $R_S$ and such that each element of $R_S$ can be written in the form $a/s$, $a \in R$, $s \in S$. Such a ring is uniquely determined up to canonical isomorphism. It is called the total quotient ring of $R$. In the case that $R$ is an integral domain, one can take for $S$ the set of all non-zero elements and $R_S$ then is the quotient field of $R$. Let $f : R_1 \to R_2$ be a ring homomorphism and let $S_1 \subset R_1$, $S_2 \subset R_2$ be multiplicative subsets such that $f(S_1) \subset S_2$ then the homomorphism $f$ extends in a natural way to a homomorphism $R_{S_1} \to R_{S_2}$.

Let $\mathcal{O}$ be sheaf of rings. For an open subset $U$ we consider the set $S(U)$ of all $f \in \mathcal{O}(U)$ such that $f|V$ is a non-zero divisor in $\mathcal{O}(V)$ for each open $V \subset U$. In particular, the elements of $S(U)$ are non-zero divisors in $\mathcal{O}(U)$. Hence one can consider

$$\mathcal{O}(U)_{S(U)} = \{\ f/g; \quad f \in \mathcal{O}(U),\ g \in S(U)\ \}.$$

There are obvious restriction maps, such that this assignment gives a presheaf. We denote the generated sheaf of rings by $\mathcal{M}$. The natural map $\mathcal{O} \to \mathcal{M}$ is injective since the functor „generated sheaf“ is exact. Hence we can consider $\mathcal{O}$ as a subsheaf of $\mathcal{M}$.

There is a natural map $\mathcal{O} \to \mathcal{M}$ of sheaves of rings and this map is injective, since the functor “generated sheaf” is exact.

We call $\mathcal{M}$ the sheaf of meromorphic sections of $\mathcal{O}$. The construction of $\mathcal{M}$ is compatible with restriction to open subsets $U$. This means that $\mathcal{M}|U$ can be identified with the sheaf of meromorphic sections of $\mathcal{O}|U$. Let $f \in \mathcal{M}(X)$ be a section of $\mathcal{M}$. Consider the set of all open subsets $U \subset X$ such that $f|U \in \mathcal{O}(U)$. The union of all these $U$ is an open subset $U_f$ of $X$. Clearly $f|U_f \in \mathcal{O}(U_f)$. We call $U_f$ the domain of holomorphy of $f$.

Let $a$ be a point. There is a natural map from $\mathcal{M}_a$ into the total quotient ring of $\mathcal{O}_a$. Clearly this is injective.

**7.1 Lemma.** *Let $X$ be a topological space and let $\mathcal{O}$ be a coherent sheaf of rings. The natural homomorphism of $\mathcal{M}_a$ into the total quotient ring of $\mathcal{O}_{X,a}$ is an isomorphism.*

*Proof.* Let $f \in \mathcal{O}_X(U)$ be an element such that $f_a$ is a non-zero divisor in $\mathcal{O}_{X,b}$. We know from the coherence theorems that then $f_b$ is a non-zero divisor in a full neighborhood. This implies Lemma 7.1. □

Let $f \in \mathcal{M}(X)$ be a section of $\mathcal{M}$. Consider the set of all open subsets $U \subset X$ such that $f|U \in \mathcal{O}(U)$. The union of all these $U$ is an open subset $U_f$ of $X$. Clearly $f|U_f \in \mathcal{O}(U_f)$. We call $U_f$ the domain of holomorphy of $f$.

So far $\mathcal{M}$ is a rather abstract object, even if $\mathcal{O} \subset \mathcal{C}_X$ is a sheaf of continuous functions, for example if $(X, \mathcal{O}_X)$ is a complex space in the sense of Serre. To remedy this situation, we make the following assumption.

**7.2 Assumption.** *Assume that $\mathcal{O}$ is a subsheaf of rings of $\mathcal{C}_X$. Assume that for each open subset $U \in \mathcal{O}$ and that each $f \in \mathcal{O}(U)$ with the property $f(x) \neq 0$ is invertible in $\mathcal{O}(U)$.*

This property is fulfilled of course for complex spaces in the sense of Serre.

**7.3 Proposition.** *Assume that the assumption above is fulfilled. Let $f$ a global section of $\mathcal{M}_X$. The domain of holomorphy $U_f$ is open and dense in $X$. Assume that there exists an open and dense subset $U$ that is contained in $U_f$ and such that $f(x) = 0$ for all $x \in U$. Then $f = 0$.*

*Proof.* Let $a \in X$ be a point in the complement of $U$. There exists a non-zero divisor $h_a \in \mathcal{O}_{X,a}$ such that $h_a f_a \in \mathcal{O}_{X,a}$. This implies $hf =$ in a full neighborhood $W$ of $a$ where $h \in \mathcal{O}_X(W)$ is a representative of $h_a$. The function $fh$ is zero in $U \cap W$. By continuity it is zero in $W$. We can assume that $h_b$ is a non-zero divisor of all $b \in W$. We obtain $f|W = 0$. This shows $f = 0$. □

So we see that global sections of $\mathcal{M}_X$ can be considered as holomorphic functions on dense open subsets with additional properties. For this reason we call sections of $\mathcal{M}_X$ simply "meromorphic functions".

## 8. Cousin problems

An additive Cousin datum on a complex $X$ space is an open covering $\mathfrak{U} = (U_i)_{i \in I}$ on $X$ and a collection of meromorphic functions $f_i \in \mathcal{M}_X(U_i)$ for all indices $i$ such that for each two indices the difference $f_i - f_j$ is holomorphic on $U_i \cap U_j$. One then can ask whether there exists a global meromorphic function $f \in \mathcal{M}_X(X)$ such that $f - f_i$ is holomorphic on $U_i$ for all $i$.

**8.1 The first Cousin problem.** *Let $X = (X, \mathcal{O}_X)$ be a complex space. Does any additive Cousin datum admit a solution?*

In standard courses about complex analysis one proves the Mittag-Leffler theorem which in a constructive way gives an positive answer in the case $X = \mathbb{C}$. Again we consider an open covering $\mathfrak{U} = (U_i)_{i \in I}$ of a complex space.

A multiplicative Cousin datum is a collection of holomorphic functions $f_i$ with the following property:

a) The set of zeros of $f_i$ is thin in $U_i$.
b) There exists a holomorphic function $f_{ij}$ on $U_i \cap U_j$ without zeros such that $f_i = f_{ij} f_j$ on $U_i \cap U_j$.

This means that the zeros of $f_i$ and $f_j$ in $U_i \cap U_j$ are the same. Hence a multiplicative Cousin datum should be considered as prescription of zeros. On can ask whether there exists a global holomorphic function $f : X \to \mathbb{C}$ such that $f = \varphi_i f_i$ on $U_i$ with a holomorphic function $\varphi : U_i \to \mathbb{C}$ without zeros.

**8.2 Second Cousin problem.** *Let $X$ be a complex space. Does every multiplicative Cousin datum admit a solution?*

We are able to prove now:

**8.3 Theorem.** *Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open covering of a Stein space $X$ $f_i \in \mathcal{M}_X(U_i)$ collection of meromorphic functions for all indices $i$ such that for each two indices the difference $f_i - f_j$ is holomorphic on $U_i \cap U_j$. Then there exists a global meromorphic function $f \in \mathcal{M}_X(X)$ such that $f - f_i$ is holomorphic on $U_i$ for all $i$.*

*Proof.* We consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{M}_X \longrightarrow \mathcal{M}_X/\mathcal{O}_X \longrightarrow 0.$$

We consider the images $s_i$ of $f_i$ in $(\mathcal{M}_X/\mathcal{O}_X)(U_i)$. By assumption they agree in the intersections and hence define a global section $(\mathcal{M}_X/\mathcal{O}_X)(X)$. Now from the long exact cohomology sequence follows that $\mathcal{M}(X) \longrightarrow (\mathcal{M}_X/\mathcal{O}_X)(X)$ is surjective. Choose $f \in \mathcal{M}(X)$ with image $s$. Then clearly $f_a - (f_i)_a$ is contained in $\mathcal{O}_{X,a}$ for all $a \in U_i$. This shows that $f - f_i$ is holomorphic on $U_i$. □

**The second Cousin problem**

Again we consider an open covering $\mathfrak{U} = (U_i)_{i\in I}$ of a complex space. A multiplicative Cousin datum is a collection of holomorphic functions $f_i$ with the following property:

a) The set of zeros of $f_i$ is thin in $U_i$.
b) There exists a holomorphic function $f_{ij}$ on $U_i \cap U_j$ without zeros such that $f_i = f_{ij} f_j$ on $U_i \cap U_j$.

This means that the zeros of $f_i$ and $f_j$ in $U_i \cap U_j$ are the same. Hence a multiplicative Cousin datum should be considered as prescription of zeros. On can ask whether there exists a global holomorphic function $f : X \to \mathbb{C}$ such that $f = \varphi_i f_i$ on $U_i$ with a holomorphic function $\varphi : U_i \to \mathbb{C}$ without zeros.

For a solution of this Cousin problem we need the sheaf $\mathbb{Z}_X$ of locally constant functions with values in $\mathbb{Z}$.

**8.4 Theorem.** *Let $X$ be a Stein space with the property $H^2(X, \mathbb{Z}_X) = 0$. Then any multiplicative Cousin problem has a solution.*

*Proof.*. For any open subset $U \subset X$ we consider the set $\mathcal{O}_X^*(U)$ of holomorphic functions without zeros on $U$. This is a group under multiplication. (This statement easily can be reduced to the case $\mathbb{C}^n$ where it is known.) With usual restriction maps we get a sheaf $\mathcal{O}_X^*$ of abelian groups. Let $f \in \mathcal{O}_X(U)$. Then $e^{2\pi i f}$ is holomorphic too. (Again this follows from the case $X = \mathbb{C}^n$.) We claim that the sequence

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

is exact. The only problem is the surjectivity. For this one has to show: Let $a \in X$ be a point and $f$ a holomorphic function without zeros on some open neighborhood of $a$. Then there is a holomorphic function $g$ in a maybe smaller open neighborhood of $a$ with the property $e^g = f$. This also can be reduced easily to the case $X = \mathbb{C}^n$. For the construction of $g$ one may assume that $|f(a) - 1| < 1$. Then one can make use of the logarithm series.

Now the proof of 8.4 is easy. From the assumptions and the long exact cohomology sequence we get $H^1(X, \mathcal{O}_X^*) = 0$. A Cousin distribution is nothing but a Čech cocycle. The solution of the Cousin problem means that this cocycle is trivial. Hence we have to show $\check{H}^1((\mathfrak{U}, \mathcal{O}_X^*) = 0$. But we know that the first Čech cohomology groups are embedded into the true cohomology. □

One should investigate now what it means that $H^2(X, \mathbb{Z}_X)$ is zero. Clearly this depends only on the topological nature of $X$. Hence it is more a problem of topology than of complex analysis. Hence we only mention

1) $H^2(\mathbb{C}^n, \mathbb{Z}_{\mathbb{C}^n}) = 0$. (This has been proved in III.6.8.)
2) $H^2(U, \mathbb{Z}_U) = 0$ for an open subset $U \subset \mathbb{C}$. (We will not prove it here.)

We recall that in standard courses on complex functions the solution of the multiplicative Cousin problem for $X = \mathbb{C}$ is given in a constructive way by means of Weierstrass products. So we obtained a very remarkable generalization using cohomological methods.

# Chapter III. Cohomology of sheaves

## 1. Some homological algebra

A complex $A^{\bullet}$ is a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \longrightarrow \cdots$$

such that the composition of two consecutive is 0, $d_n \circ d_{n-1} = 0$. Usually one omits indices at the $d$-s and writes simply $d = d_n$ and hence $d \circ d = 0$, which sometimes is written as $d^2 = 0$. The cohomology groups of $A^{\bullet}$ are defined as

$$H^n(A^{\bullet}) := \frac{\mathrm{Kernel}(A^n \to A^{n+1})}{\mathrm{Image}(A^{n-1} \to A^n)} \qquad (n \in \mathbb{Z}).$$

They vanish if and only if the complex is exact. Hence the cohomology groups measure the absence of exactness of a complex.

A homomorphism $f^{\bullet} : A^{\bullet} \to B^{\bullet}$ of complexes is a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & B^{n-1} & \longrightarrow & B^n & \longrightarrow & B^{n+1} & \longrightarrow & \cdots \end{array}$$

It is clear how to compose two complex homomorphisms $f^{\bullet}; A^{\bullet} \to B^{\bullet}$, $g^{\bullet}; B^{\bullet} \to C^{\bullet}$ to a complex homomorphism $g^{\bullet} \circ f^{\bullet} : A^{\bullet} \to C^{\bullet}$. A sequence of complex homomorphisms

$$\cdots \longrightarrow A^{\bullet} \longrightarrow B^{\bullet} \longrightarrow C^{\bullet} \longrightarrow \cdots$$

is called exact, if all the induced sequences

$$\cdots \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow \cdots$$

are exact. There is also the notion of a short exact sequence of complexes

$$0 \longrightarrow A^{\bullet} \longrightarrow B^{\bullet} \longrightarrow C^{\bullet} \longrightarrow 0$$

Here 0 stands for the zero-complex ($0^n = 0$, $d^n = 0$ for all $n$).

A homomorphism of complexes $A^\bullet \to B^\bullet$ induces naturally homomorphisms

$$H^n(A^\bullet) \longrightarrow H^n(B^\bullet)$$

of the cohomology groups (use ???). These homomorphisms are compatible with the composition of complex-homomorphisms. A less obvious construction is as follows: Let

$$0 \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow 0$$

be a short exact sequence of complexes. We construct a homomorphism

$$\delta : H^n(C^\bullet) \longrightarrow H^{n+1}(A^\bullet).$$

Let $[c] \in H^n(C^\bullet)$ be represented by an element $c \in C^n$. Take a pre-image $b \in B^n$ and consider $\beta = db \in B^{n+1}$. Since $\beta$ goes to $d(c) = 0$ in $C^{n+1}$ there exists a pre-image $a \in A^{n+1}$. This goes to 0 in $A^{n+2}$ (because $A^{n+2}$ is imbedded in $B^{n+2}$ and $b$ goes to $d^2(b) = 0$ there). Hence $a$ defines a cohomology class $[a] \in H^{n+1}(A^\bullet)$. It is easy to check that this class doesn't depend on the above choices.

**1.1 Fundamental lemma of homological algebra.** *Let*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

*be a short exact sequence of complexes. Then the long sequence*

$$\cdots \to H^{n-1}(C^\bullet) \xrightarrow{\delta} H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(C^\bullet) \to \cdots$$

*is exact.*

We leave the details to the reader. □

There is a second lemma of homological algebra which we will need.

**1.2 Lemma.** *Let*

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & | & & | & & | & & | & & \\
0 & — & A^{00} & — & A^{01} & — & A^{02} & — & A^{03} & — & \cdots \\
 & & | & & | & & | & & | & & \\
0 & — & A^{10} & — & A^{11} & — & A^{12} & — & A^{13} & — & \cdots \\
 & & | & & | & & | & & | & & \\
0 & — & A^{20} & — & A^{21} & — & A^{22} & — & A^{23} & — & \cdots \\
 & & | & & | & & | & & | & & \\
0 & — & A^{30} & — & A^{31} & — & A^{32} & — & A^{33} & — & \cdots \\
 & & | & & | & & | & & | & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

*be a commutative diagram where all lines and columns are exact besides the first column and the first row (those containing $A^{00}$). Then there is a natural isomorphism between the cohomology groups of the first row and the first column,*

$$H^n(A^{\bullet,0}) \cong H^n(A^{0,\bullet})$$

*For $n = 0$ this is understood as*

$$\text{Kernel}(A^{00} \longrightarrow A^{01}) = \text{Kernel}(A^{00} \longrightarrow A^{10}).$$

The proof is given by "diagram chasing". We only give a hint how it works. Assume $n = 1$. Let $[a] \in H^1(A^{0,\bullet})$ be a cohomology class represented by an element $a \in A^{0,1}$. This element goes to 0 in $A^{0,2}$. As a consequence the image of $a$ in $A^{1,1}$ goes to 0 in $A^{1,2}$. Hence this image comes from an element $\alpha \in A^{1,0}$. Clearly this element goes to zero in $A^{2,0}$ (since it goes to 0 in $A^{2,1}$.) Now $\alpha$ defines a cohomology class $[\alpha] \in H^1(A^{\bullet,0})$. There is some extra work to show that this map is well-defined. □

## 2. The canonical flabby resolution

A sheaf $F$ is called *flabby,* if $F(X) \to F(U)$ is surjective of all $U$. Then $F(U) \to F(V)$ is surjective for all $V \subset U$. An example for a flabby sheaf is the Godement sheaf $F^{(0)}$. Recall that we have the exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)}.$$

We want to extend this sequence. For this we consider the sheaf $F^{(0)}/F$ and embed it into its Godement sheaf,

$$F^{(1)} := \left(F^{(0)}/F\right)^{(0)}.$$

In this way we get a long exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)} \longrightarrow F^{(1)} \longrightarrow F^{(2)} \longrightarrow \cdots$$

If $F^{(n)}$ has been already constructed then we define

$$F^{(n+1)} := \left(F^{(n)}/F^{(n-1)}\right)^{(0)}.$$

The sheaves $F^{(n)}$ are all flabby. We call this sequence the *canonical flabby resolution* or the *Godement resolution.* Sometimes it is useful to write the resolution in the form

$$\begin{array}{ccccccccccccc}
\cdots & — & 0 & — & F & — & 0 & — & 0 & — & 0 & — & \cdots \\
 & & | & & | & & | & & | & & | & & \\
\cdots & — & 0 & — & F^{(0)} & — & F^{(1)} & — & F^{(2)} & — & F^{(3)} & — & \cdots
\end{array}$$

Both lines are complexes. The vertical arrows can be considered as a complex homomorphism. The induced homomorphism of the cohomology groups are isomorphisms. Notice that only the 0-cohomology group of both complexes is different from 0. This zero cohomology group is naturally isomorphic $F$.

Now we apply the global section functor $\Gamma$ to the resolution. This is

$$\Gamma F := F(X).$$

We obtain a long sequence

$$0 \longrightarrow \Gamma F \longrightarrow \Gamma F^{(0)} \longrightarrow \Gamma F^{(1)} \longrightarrow \Gamma F^{(2)} \longrightarrow \cdots$$

The essential point is that this sequence is no longer exact. we only can say that it is a complex. We prefer to write in the form

$$\begin{array}{ccccccccccccc} \cdots & \text{———} & 0 & \text{———} & \Gamma F & \text{———} & 0 & \text{———} & 0 & \text{———} & 0 & \text{———} & \cdots \\ & & \Big| & & \Big| & & \Big| & & \Big| & & \Big| & & \\ \cdots & \text{———} & 0 & \text{———} & \Gamma F^{(0)} & \text{———} & \Gamma F^{(1)} & \text{———} & \Gamma F^{(2)} & \text{———} & \Gamma F^{(3)} & \text{———} & \cdots \end{array}$$

The second line is

$$\begin{array}{c} \cdots \longrightarrow 0 \longrightarrow \Gamma F^{(0)} \longrightarrow \Gamma F^{(1)} \longrightarrow \Gamma F^{(2)} \longrightarrow \cdots \\ \uparrow \\ \text{zero position} \end{array}$$

Now we define the cohomology groups $H^{\bullet}(X, F)$ to be the cohomology groups of this complex:

$$H^n(X, F) := \frac{\text{Kernel}(\Gamma F^{(n)} \longrightarrow \Gamma F^{(n+1)})}{\text{Kernel}(\Gamma F^{(n-1)} \longrightarrow \Gamma F^{(n)})}$$

(We define $\Gamma F^{(n)} = 0$ for $n < 0$.) Clearly

$$H^n(X, F) = 0 \qquad \text{for} \quad n < 0.$$

Next we treat the special case $n = 0$,

$$H^0(X, F) = \text{Kernel}(\Gamma F^{(0)} \longrightarrow \Gamma F^{(1)}).$$

Since the kernel can be taken in the presheaf sense, we can write

$$H^0(X, F) = \Gamma\text{Kernel}(F^{(0)} \longrightarrow F^{(1)}).$$

Recall that $F^{(1)}$ is a sheaf, which contains $F^{(0)}/F$ as subsheaf. We obtain

$$H^0(X, F) = \Gamma\text{Kernel}(F^{(0)} \longrightarrow F(0)/F)$$

This is the image of $F$ in $F^{(0)}$ an hence a sheaf which is canonically isomorphic to $F$.

**2.1 Remark.** *There is a natural isomorphism*

$$H^0(X,F) \cong \Gamma F = F(X).$$

If $F \to G$ is a homomorphism of sheaves, then the homomorphism $F_a \to G_a$ induce a homomorphism $F^{(0)} \to G^{(0)}$. If $F \to G \to H$ is an exact sequence. Then $F^{(0)} \to G^{(0)} \to H^{(0)}$ is also exact (already as sequence of presheaves). More generally

**2.2 Lemma.** *Let $0 \to F \to G \to H \longrightarrow 0$ be an exact sequence of sheaves. Then the induced sequence $0 \to F^{(n)} \to G^{(n)} \to H^{(n)} \to 0$ is exact for every $n$.*

The proof is by induction. One needs the following lemma about abelian groups:

*Let*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & | & & | & & | & & \\
0 & — & A^{00} & — & A^{01} & — & A^{02} & — & 0 \\
 & & | & & | & & | & & \\
0 & — & A^{10} & — & A^{11} & — & A^{12} & — & 0 \\
 & & | & & | & & | & & \\
0 & — & A^{20} & — & A^{21} & — & A^{22} & — & 0 \\
 & & | & & | & & | & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

*be a commutative diagram such that the three columns and the first to lines are exact. Then the third line is also exact.*

This follows from 1.2. □

Before we continue we need a basic lemma:

**2.3 Lemma.** *Let $0 \to F \to G \to H \to 0$ be a short exact sequence of sheaves. Assume that $F$ is flabby. Then*

$$0 \to \Gamma F \to \Gamma G \to \Gamma H \to 0$$

*is exact.*

*Proof.* Let $h \in H(X)$. We have to show that $h$ is the image of an $g \in G(X)$. For the proof one considers the set of all pairs $(U, g)$, where $U$ is an open subset and $g \in G(U)$ and such that $g$ maps to $h|U$. This set is ordered by

$$(U, g) \geq (U', g') \Longleftrightarrow U' \subset U \text{ and } g|U' = g'.$$

From the sheaf axioms follows that every inductive subset has an upper bound. By Zorns's lemma there exists a maximal $(U, g)$. We have to show $U = X$. If this is not the case, we can find a pair $(U', g')$ in the above set such that $U'$ is not contained in $U$. The difference $g - g'$ defines a section in $F(U \cap U')$. Since $F$ is flabby, this extends to a global section. This allows us to modify $g'$ such that it glues with $g$ to a section on $U \cup U'$. □

An immediate corollary of 2.3 states:

**2.4 Lemma.** *Let $0 \to F \to G \to H \to 0$ an exact sequence of sheaves. If $F$ and $G$ are flabby then $H$ is flabby too.*

Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheafs. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n-1)} & \longrightarrow & G^{(n-1)} & \longrightarrow & H^{(n-1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n)} & \longrightarrow & G^{(n)} & \longrightarrow & H^{(n)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n+1)} & \longrightarrow & G^{(n+1)} & \longrightarrow & H^{(n+1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

From 2.2 we know that all lines of this diagram are exact From 2.3 follows that they remain exact after applying $\Gamma$. Hence the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n-1)} & \longrightarrow & \Gamma G^{(n-1)} & \longrightarrow & \Gamma H^{(n-1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n)} & \longrightarrow & \Gamma G^{(n)} & \longrightarrow & \Gamma H^{(n)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n+1)} & \longrightarrow & \Gamma G^{(n+1)} & \longrightarrow & \Gamma H^{(n+1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

can be considered as a short exact sequence of complexes. We can apply 1.1 to obtain the long exact cohomology sequence:

**2.5 Theorem.** *Every short exact sequence* $0 \to F \to G \to H \to 0$ *induces a natural long exact cohomology sequence*

$$0 \to \Gamma F \longrightarrow \Gamma G \longrightarrow \Gamma H \stackrel{\delta}{\longrightarrow} H^1(X,F) \longrightarrow H^1(X,G) \longrightarrow H^1(X,H)$$
$$\stackrel{\delta}{\longrightarrow} H^2(X,F) \longrightarrow \cdots$$

The next Lemma shows that the cohomology of flabby sheaves is trivial.

**2.6 Lemma.** *Let*

$$0 \to F \longrightarrow F_0 \to F_1 \longrightarrow \cdots$$

*be an exact sequence of flabby sheaves (finite or infinite). Then*

$$0 \to \Gamma F \longrightarrow \Gamma F_0 \to \Gamma F_1 \longrightarrow \cdots$$

*is exact.*

**Corollary.** *For flabby* $F$ *one has:*

$$H^i(X,F) = 0 \quad \textit{for} \quad i > 0.$$

*Proof.* We use the so-called splitting principle. The long exact sequence can be splitted into short exact sequences

$$0 \longrightarrow F \longrightarrow F_0 \longrightarrow F_0/F \longrightarrow 0, \quad 0 \longrightarrow F_0/F \longrightarrow F_1 \longrightarrow F_1/F_0 \longrightarrow 0, \ldots.$$

From 2.4 we get that the $F_0/F, F_1/F_0, \ldots$ are flabby. The claim now follows from 2.3. □

A sheaf $F$ is called *acyclic* if $H^n(X,F) =$ for $n > 0$. Hence flabby sheaves are acyclic. By an *acyclic* resolution of a sheaf we understand an exact sequence

$$0 \longrightarrow F \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

with acyclic $F_i$.

**2.7 Proposition.** *Let* $0 \to F \to F_0 \to F_1 \to \cdots$ *be an acyclic resolution of* $F$. *Then there is a natural isomorphism between the* $n$*-the cohomology group* $H^n(X,F)$ *and the* $n$*-th cohomology group of the complex*

$$\cdots \longrightarrow 0 \longrightarrow \underset{\substack{\uparrow \\ \textit{zero position}}}{\Gamma F_0} \longrightarrow \Gamma F_1 \longrightarrow \Gamma F_2 \longrightarrow \cdots$$

*Proof*. Taking the canonical flabby resolutions of $F$ and of all $F_n$ on gets a diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & | & & | & & | & & | & & \\
0 & — & F & — & F_0 & — & F_2 & — & F_3 & — & \cdots \\
 & & | & & | & & | & & | & & \\
0 & — & F^{(0)} & — & F_0^{(0)} & — & F_1^{(0)} & — & F_2^{(0)} & — & \cdots \\
 & & | & & | & & | & & | & & \\
0 & — & F^{(1)} & — & F_0^{(1)} & — & F_1^{(1)} & — & F_2^{(1)} & — & \cdots \\
 & & | & & | & & | & & | & & \\
0 & — & F^{(2)} & — & F_0^{(2)} & — & F_1^{(2)} & — & F_2^{(2)} & — & \cdots \\
 & & | & & | & & | & & | & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

All lines and columns are exact. We apply $\Gamma$ to this complex. Then all lines and columns besides the first ones remain exact. We can apply 1.2. □

One may ask what “natural” means in 2.7 means. It means that certain diagrams in which this isomorphism appears are commutative. Since it is the best to check this when it is used we give just one example: Consider the above commutative diagram in the following new meaning: All occurring sheaves besides $F$ are acyclic. Then 1.2 gives an isomorphism between the $n$-th cohomology groups of the complexes $0 \to \Gamma F_0 \to \Gamma F_1 \to \cdots$ and $0 \to \Gamma F^{(0)} \to \Gamma F^{(1)} \to \cdots$. Both are isomorphic to $H^n(X, F)$. This gives a commutative triangle.

## 3. Paracompactness

We consider a very special case. We take for $\mathcal{O}$ the sheaf $\mathcal{C}$ of continuous functions. There are two possibilities: $\mathcal{C}_{\mathbb{R}}$ is the sheaf of continuous real-valued and $\mathcal{C}_{\mathbb{C}}$ the sheaf of continuous complex-valued functions. If we write $\mathcal{C}$ we mean one of both. The sheaf $\mathcal{C}$ or more generally a module over this sheaf have over paracompact spaces a property which can be considered as a weakened form of flabbyness.

**3.1 Remark.** *Let $X$ be paracompact space and $\mathcal{M}$ a $\mathcal{C}$-module on $X$. Assume that $U$ is an open subset and $V \subset\subset U$ an open subset which is relatively compact* *in* $U$. *Assume that* $s \in \mathcal{M}(U)$ *is a section over* $U$. *Then there is a global section* $S \in \mathcal{M}(X)$ *such that* $S|V = s|V$.

*Proof.* We choose a continuous real valued function $\varphi$ on $X$, which is one on $V$ and whose support is compact and contained in $U$. Then we consider the open covering $X = U \cup U'$, where $U'$ denotes the complement of the support of $\varphi$. On $U$ we consider the section $\varphi s$ and on $U'$ the zero section. Since both are zero on $U \cap U'$ they glue to a section $S$ on $X$.

**3.2 Lemma.** *Let* $X$ *be a paracompact space and* $\mathcal{M} \to \mathcal{N}$ *a surjective* $\mathcal{C}$*-linear map of* $\mathcal{C}$*-modules. Then* $\mathcal{M}(X) \to \mathcal{N}(X)$ *is surjective.*

*Proof.* Let $s \in \mathcal{N}(X)$. There exists an open covering $(U_i)_{i\in I}$ of $X$ such that $s|U_i$ is the image of an section $t_i \in \mathcal{M}(U_i)$. We can assume that the covering is locally finite. We take relatively compact open subsets $V_i \subset U_i$ such that $(V_i)$ is still a covering. Then we choose a partition of unity $(\varphi_i)$ with respect to $(V_i)$. By 3.2 there exists global sections $T_i \in \mathcal{M}(X)$ with $T_i|V_i = t_i|V_i$. We now consider

$$T := \sum_{i\in I} \varphi_i T_i.$$

Since $I$ can be infinite we have to explain what this means. Let $a \in X$ a point. There exists an open neighborhood $U(a)$ such $V_i \cap U(a) \neq \emptyset$ only for a finite subset $J \subset I$. We can define the section

$$T(a) := \sum_{i\in J} \varphi T_i|U(a).$$

The sets $U(a)$ cover $X$ and the sections $T(a)$ glue to a section $T$. Clearly $T$ maps to $s$. □

**3.3 Lemma.** *Let* $X$ *be a paracompact space and* $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ *an exact sequence of* $\mathcal{C}$*-modules. Then* $\mathcal{M}(X) \to \mathcal{N}(X) \to \mathcal{P}(X)$ *is exact too.*

*Proof.* The exactness of the sequence implies the exactness of

$$0 \longrightarrow \mathrm{Image}(\mathcal{M} \to \mathcal{N}) \longrightarrow \mathcal{N} \longrightarrow \mathrm{Kernel}(\mathcal{N} \to \mathcal{P}) \longrightarrow 0.$$

From 3.2 we get

$$0 \longrightarrow \mathrm{Image}(\mathcal{M} \to \mathcal{N})(X) \longrightarrow \mathcal{N}(X) \longrightarrow \mathrm{Kernel}(\mathcal{N} \to \mathcal{P})(X) \longrightarrow 0.$$

Applying 3.2 to $\mathcal{M} \to \mathrm{Image}(\mathcal{M} \to \mathcal{N})$ we obtain

$$\mathrm{Image}(\mathcal{M} \to \mathcal{N})(X) = \mathrm{Image}(\mathcal{M}(X) \to \mathcal{N}(X)).$$

Since also

$$\mathrm{Kernel}(\mathcal{N} \to \mathcal{P})(X) = \mathrm{Kernel}(\mathcal{N}(X) \to \mathcal{P}(X))$$

we get the exactness of

$$0 \longrightarrow \text{Image}(\mathcal{M}(X) \to \mathcal{N}(X)) \longrightarrow \mathcal{N}(X) \longrightarrow \text{Kernel}(\mathcal{N}(X) \to \mathcal{P}(X)) \longrightarrow 0.$$

This proves 3.3. □

Let $\mathcal{M}$ b an $\mathcal{C}$-module over a paracompact space. Then the canonical flabby resolution is also a sequence of $\mathcal{C}$-modules. From 3.3 follows that the resolution remains exact after the application of $\Gamma$. We obtain.

**3.4 Proposition.** *Let $X$ be paracompact. Every $\mathcal{C}$-module is acyclic, i.e. $H^n(X, \mathcal{M}) = 0$ for $n > 0$.*

The essential tool of the proofs has been the existence of a partition of unity. Partitions of unity exist also in the differentiable world. Hence there is the following variant of 3.3.

**3.5 Proposition.** *Let $X$ be a paracompact differentiable manifold, then every $\mathcal{C}^\infty$-modul is acyclic.*

## 4. Čech Cohomology

We have to work with open coverings $\mathfrak{U} = (U_i)_{i \in I}$ of the given topological space $X$. For indices $i_0, \dots, i_p$ we use the notation

$$U_{i_0,\dots,i_p} = U_{i_0} \cap \ldots \cap U_{i_p}.$$

Let $F$ be sheaf on $X$. A $p$-cochain of $F$ with respect to the covering $\mathfrak{U}$ is family of sections is an element of

$$\prod_{(i_0,\dots,i_p) \in I^{p+1}} F(U_{i_0,\dots,i_p}).$$

This means that to any $(p+1)$-tuple of indices $i_0, \dots, i_p$ there is associated a section $s(i_0, \dots, i_p) \in F(U_{i_0,\dots,i_p})$. We denote the group of all cochains by $C^p(\mathfrak{U}, F)$. The derivative $ds$ of a $p$-cochain the $(p+1)$-cochain defined by

$$ds(s_0, \dots s_{p+1}) = \sum_{j=0}^{p+1} (-1)^j s(i_0, \dots, \hat{i}_j, \dots, i_{p+1}) | U_{i_0,\dots,i_{p+1}}.$$

The rule $d^2 = 0$ is obvious, hence we obtain a complex

$$\cdots \longrightarrow C^{p-1}(\mathfrak{U}, F) \longrightarrow C^p(\mathfrak{U}, F) \longrightarrow C^{p+1}(\mathfrak{U}, F) \longrightarrow \cdots$$

Here for negative $p$ we set $C^p(\mathfrak{U}, F) = 0$. The cohomology groups of this complex are the Čech cohomology groups $\check{\mathrm{H}}^p(\mathfrak{U}, F)$.

**4.1 Lemma.** *There is a natural isomorphism*

$$\check{H}^0(\mathfrak{U}, F) = H^0(X, F) \qquad (= F(X)).$$

*Proof.* A zero-cochain $s$ is just a family $s_i \in F(U_i)$. The condition $ds = 0$ means $s_i|U_i \cap U_j = s_j|U_i \cap U_j$. By the sheaf axioms they glue to a global section. □

**4.2 Remark.** *Let $F$ be a flabby sheaf. Then for every open covering*

$$\check{H}^p(\mathfrak{U}, F) = 0 \quad \textit{for} \quad p > 0.$$

*Proof.* Just to save notation we restrict to the case $p = 1$. The general case works in the same way. We start with a little remark. Assume that the whole space $X = U_{i_0}$ is a member of the covering. Then the Čech cohomology vanishes (for every sheaf): if $(s_{ij})$ is a cocycle one defines $s_i = s_{i,i_0}$. Then $d((s_i)) = (s_{ij})$.

For the general proof of 4.2 (in the case $p = 1$) we now consider the sequence

$$\begin{array}{ccccccc} 0 \longrightarrow F(X) & \longrightarrow & \prod_i F(U_i) & \longrightarrow & \prod_{ij} F(U_i \cap U_j) & \longrightarrow & \prod_{ijk} F(U_i \cap U_j \cap U_k) \\ s & \longmapsto & (s|U_i) & & & & \\ & & (s_i) & \longmapsto & (s_i - s_j) & & \\ & & & & (s_{ij}) & \longmapsto & (s_{ij} + s_{jk} - s_{ik}) \end{array}$$

We will proof that this sequence is exact. (Then 4.2 follows.) The idea is to sheafify this sequence: For an open subset $U \subset X$ one considers $F|U$ and also the restricted covering $U \cap U_i$. Repeating the above construction for $U$ instead of $X$ on obtains a sequence of sheaves

$$0 \longrightarrow F \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}.$$

Since $F$ is flabby, also $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are flabby. The remark at the beginning of the proof shows that $0 \longrightarrow F(U) \longrightarrow \mathcal{A}(U) \longrightarrow \mathcal{B}(U) \longrightarrow \mathcal{C}(U)$ is exact, when $U$ is contained in some $U_i$. Hence the sequence of sheaves is exact. From 2.6 follows that the exactness is also true for $U = X$. □

**4.3 Theorem of Leray.** *Let $F$ be a sheaf on $X$ and $\mathfrak{U} = (U_i)$ an open covering of $X$. Assume that $H^p(U, F|U) = 0$ for all $p > 0$ and for arbitrary intersection of finitely many $U_i$. Then there is a natural isomorphism*

$$H^p(X, F) \cong \check{H}^p(\mathfrak{U}, F)$$

*for all $p$.*

*Proof*. We consider a flabby resolution $0 \to F \to F_0 \to F_1 \to \cdots$. There is a natural diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F(X) & \to & F_0(X) & \to & F_1(X) & \to & F_2(X) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^0(\mathfrak{U}, F) & \to & C^0(\mathfrak{U}, F_0) & \to & C^0(\mathfrak{U}, F_1) & \to & C^0(\mathfrak{U}, F_2) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^1(\mathfrak{U}, F) & \to & C^1(\mathfrak{U}, F_0) & \to & C^1(\mathfrak{U}, F_1) & \to & C^1(\mathfrak{U}, F_2) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^2(\mathfrak{U}, F) & \to & C^2(\mathfrak{U}, F_0) & \to & C^2(\mathfrak{U}, F_1) & \to & C^2(\mathfrak{U}, F_2) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

All rows but the first one are exact. Similarly all columns but first one are exact. Now a homological lemma 1.2 gives the desired result. □

## 5. The first cohomology group

The first Čech cohomology group has some special properties: We will keep very short, since later we will use it only in applications. Let $f : G \to H$ be a surjective homomorphism of sheaves and $\mathfrak{U} = (U_i)$ an open covering of $X$. We denote by $H_{\mathfrak{U},f}(X)$ the set of all global sections of $H$ with the following property:

For every index $i$ there is a section $t_i \in G(U_i)$ with $f(t_i) = s|U_i$. By definition of (sheaf-)surjectivity for every global section $s \in H(X)$ there exists an open covering $\mathfrak{U}$ with $s \in H_{\mathfrak{U},f}(X)$. It follows

$$H(X) = \bigcup_{\mathfrak{U}} H_{\mathfrak{U},f}(X).$$

Let $0 \to F \to G \xrightarrow{f} H \to 0$ be an exact sequence and $\mathfrak{U}$ an open covering. There exists a natural homomorphism

$$\delta : H_{\mathfrak{U},f}(X) \longrightarrow \check{H}^1(\mathfrak{U}, F),$$

which is constructed as follows: Let be $s \in H_{\mathfrak{U},f}(X)$. We choose elements $t_i \in G(U_i)$ which are mapped to $s|U_i$. The differences $t_i - t_j$ come from sections $t_{ij} \in F(U_i \cap U_j)$. They define a 1-cocycle $\delta(s)$. It is easy to check that this corresponding element of $\check{H}^1(\mathfrak{U}, F)$ doesn't depend on the choice of the $t_i$.

**5.1 Lemma.** *Let* $0 \to F \xrightarrow{f} G \to H \to 0$ *be an exact sequence of sheaves and* $\mathfrak{U}$ *an open covering. The sequence*

$$0 \to F(X) \longrightarrow G(X) \longrightarrow H_{\mathcal{U},f}(X) \xrightarrow{\delta} \check{H}^1(\mathcal{U}, F) \longrightarrow \check{H}^1(\mathcal{U}, G) \longrightarrow \check{H}^1(\mathcal{U}, H)$$

*is exact.*

The simple proof is left to the reader. □

Let now $F$ be an arbitrary sheaf, $F^{(0)}$ the associated flabby sheaf. We get an exact sequence $0 \to F \to F^{(0)} \to H \to 0$. let $\mathfrak{U}$ be an open covering. We know that $\check{H}^1(\mathfrak{U}, F^{(0)})$ vanishes, 4.2. From 4.2 we obtain an isomorphy

$$\check{H}^1(\mathfrak{U}, F) \cong H_{\mathfrak{U},f}(X)/G(X).$$

From the long exact cohomology sequence we get for the usual cohomology

$$H^1(X, F) \cong H(X)/G(X).$$

This gives an *injective* homomorphism

$$\check{H}^1(\mathfrak{U}, F) \longrightarrow H^1(X, F).$$

In the following we consider $\check{H}^1(\mathfrak{U}, F)$ as subset of $H^1(X, F)$. Now it is easy to check:

**5.2 Proposition.** *Let* $F$ *be a sheaf. Then*

$$H^1(X, F) = \bigcup_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, F).$$

The following commutative diagram that the Čech combining $\delta$ from 5.1 and that of general sheaf theory 2.5 coincide:

**5.3 Remark.** *For a short exact sequence* $0 \to F \to G \xrightarrow{f} H \to 0$ *the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H_{\mathfrak{U},f}(X) & \xrightarrow{\delta} & \check{H}^1(\mathfrak{U}, F) \\ & & \| & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H(X) & \xrightarrow{\delta} & H^1(X, F) \end{array}$$

*is commutative.*

The proof is left to the reader. □

Let $\mathfrak{V} = (V_j)_{j \in J}$ be a refinement of $\mathfrak{U} = (U_i)_{i \in I}$ and $\varphi : J \longrightarrow I$ a refinement map ($V_\varphi \subset U_i$). Using this refinement map one obtains a natural map

$$\check{H}^1(\mathfrak{U}, F) \longrightarrow \check{H}^1(\mathfrak{V}, F).$$

This shows:

**5.4 Remark.** *Let $\mathfrak{V}$ be an refinement of $\mathfrak{U}$ and $\varphi : J \to I$ a refinement map. The diagram*

$$\begin{array}{ccc} \check{H}^1(\mathfrak{U}, F) & \longrightarrow & \check{H}^1(\mathfrak{V}, F) \\ & H^1(X, F) & \end{array}$$

*commutes. Especially it doesn't depend on the choice of the refinement map.*

We also mention a refinement of Leray's lemma 4.3 in case of the first cohomology group.

**5.5 Theorem (refinement of Leray's theorem in case of the first cohomology group).** *Let $F$ be a sheaf on $X$ and $\mathfrak{U} = (U_i)$ an open covering of $X$. Assume that $H^1(U_i, F|U_i) = 0$ for all $\subset\in I$. Then there is a natural isomorphism*

$$H^1(X, F) \cong \check{H}^1(\mathfrak{U}, F).$$

*Hint for the proof.* One has to show that for any refinement $\mathfrak{V}$ the map $H^1(\mathfrak{U}, F) \to H^1(\mathfrak{V}, F)$ is surjective. The proof is easy and left to the reader. Details can be found in Forster's book "Riemann surfaces", Proposition II.12.8. □

## 6. Some vanishing results

Let $X$ be a topological space and $A$ an abelian group. We denote by $A_X$ the sheaf of locally constant functions with values in $A$. This sheaf can be identified with the sheaf which is generated by the presheaf of constant functions. We will write

$$H^n(X, A) := H^n(X, A_X).$$

**6.1 Proposition.** *Let $U$ be an open and convex subset of $\mathbb{R}^n$. Then for every abelian group $A$*

$$H^1(U, A) = 0.$$

Actually this is true for all $H^n$, $n > 0$. The best way to prove this is to use the comparison theorem with singular cohomology as defined in algebraic topology. We restrict to $H^1$.

*Proof of 6.1.* Every convex open subset of $\mathbb{R}^n$ is topologically equivalent to $\mathbb{R}^n$. Hence it is sufficient to restrict to $U = \mathbb{R}^n$. Just for simplicity we assume $n = 1$. (The general case should then be clear.) We use Čhech cohomology and show that every open covering admits a refinement $\mathfrak{U}$ such that $H^1(\mathfrak{U}, A_X) = 0$. To show this we take a refinement of a very simple nature. It is easy to show that there exists a refinement of the following form. The index set is $\mathbb{Z}$. There exists a sequence of real numbers $(a_n)$ with the following properties:

a) $a_n \le a_{n+1}$
b) $a_n \to +\infty$ for $n \to \infty$ and $a_n \to -\infty$ for $n \to -\infty$
c) $U_n = (a_n, a_{n+2})$.

Assume that $s_{n,m}$ is a cocycle with respect to this covering. Notice that $U_n$ has non empty intersection only with $U_{n-1}$ and $U_{n+1}$. Hence only $s_{n-1,n}$ is of relevance. This a locally constant function on $U_{n-1} \cap U_n = (a_n, a_{n+1})$. Since this is connected, the function $s_{n-1,n}$ is constant. We want to show that it is coboundary, i.e. we want to construct constant functions $s_n$ on $U_n$ such that $s_{n-1,n} = s_n - s_{n-1}$ on $(a_n, a_{n+1})$. This is easy. One starts with $s_0 = 0$ and then constructs inductively $s_1, s_2, \dots$ and in the same way for negative $n$. □

Consider on the real line $\mathbb{R}$ the sheaf of real valued differentiable functions $\mathcal{C}^\infty$. Taking derivatives one gets a sheaf homomorphism $\mathcal{C}^\infty \to \mathcal{C}^\infty$, $f \mapsto f'$. The kernel is the sheaf of all locally constant functions, which we denote simply by $\mathbb{R}$. Hence we get an sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty \longrightarrow 0.$$

This sequence is exact since every differentiable function has an integral. Hence this sequence can be considered as acyclic resolution of $\mathbb{R}$. We obtain $H^n(\mathbb{R}, \mathbb{R}) = 0$ for all $n > 0$. For $n = 1$ this follows already from 6.1. There is a generalization to higher dimensions. For example a standard result of vector analysis states in the case $n = 2$.

**6.2 Lemma.** *Let $E \subset \mathbb{R}^n$ be an open and convex subset, $f, g \in \mathcal{C}^\infty$ a pair of differentiable functions with the property*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

*Then there is a differentiable function $h$ with the property*

$$f = \frac{\partial h}{\partial x}, \quad g = \frac{\partial h}{\partial y}.$$

In the sequence of exact sequences this means:

The sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{C}^\infty(E) & \longrightarrow & \mathcal{C}^\infty(E) \times \mathcal{C}^\infty(E) & \longrightarrow & \mathcal{C}^\infty(E) & \longrightarrow 0 \\ f & \longmapsto & \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) & & & \\ & & (f,g) & \longmapsto & \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} & \end{array}$$

is exact. When $E$ is not convex, this sequence needs not to be exact. But since every point in $\mathbb{R}^2$ has an open convex neighborhood, the sequence of sheaves

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \mathcal{C}^\infty_X \longrightarrow \mathcal{C}^\infty_X \times \mathcal{C}^\infty_X \longrightarrow \mathcal{C}^\infty_X \longrightarrow 0$$

is exact. This is an acyclic resolution and we obtain:

**6.3 Proposition.** *For convex open $E \subset \mathbb{R}^2$ we have*

$$H^i(E, \mathbb{R}) = 0 \quad \textit{for} \quad i > 0.$$

The sequence is a special case of the de-Rham complex

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0_X \longrightarrow A^1_X \longrightarrow \cdots \longrightarrow A^n_X \longrightarrow 0$$

Here $X$ is a differentiable manifold of dimension $n$ and $A^i_X$ denotes the sheaf of alternating differential forms of degree $i$.

**6.4 Lemma of Poincaré.** *Let $U \subset \mathbb{R}^n$ be an open convex subset. Then*

$$H^p(U, \mathbb{R}) = 0 \quad \textit{for} \quad p > 0.$$

*Proof.* Let $\omega$ be a closed form. We decompose it as

$$\omega = \alpha + \beta \wedge dx_n,$$

where $\alpha$ doesn't contain any term with $dx_n$. We write

$$\beta = \sum f_a dx_a$$

where $a$ are subsets of $\{1, \dots, n-1\}$ that do nor contain $n$. (We use the notation $dx_a = dx_{a_1} \wedge \dots \wedge dx_{a_p}$, where $a_1 < \dots < a_p$ are the elements of $a$ in their natural order.) Integrating with respect to the last variable we find differentiable functions $F_a$ such that $\partial_n F_a = f_a$. Now the difference $\omega - d\sum_a F_a dx_a$ doesn't contain any term in which $dx_n$ occurs. Hence we can assume that in $\omega$ no term with $dx_n$ occurs. We write

$$\omega = \sum_a g_a dx_a,$$

where all $a$ are subsets of $\{1, \dots, n-1\}$. Now we use $d\omega = 0$. We obtain $\partial_n g_a = 0$. Hence $g_a$ do not depend on $x_n$. But now $\omega$ can be considered as differential form in one dimension less (on the image of $U$ with respect to the projection map that cancels the last variable) and an induction argument completes the proof. □

We obtain

**6.5 Theorem of de Rham.** *For a differentiable manifold $X$ on has*

$$\dim H^i(X,\mathbb{R}) \cong \frac{\mathrm{Kernel}(A^i(X) \longrightarrow A^{i+1}(X))}{\mathrm{Image}((A^{i-1}(X) \longrightarrow A^i(X))}.$$

Applying the Lemma of Poincarè again we obtain:

**6.6 Proposition.** *For convex open $E \subset \mathbb{R}^n$ on has*

$$H^i(E,\mathbb{R}) = 0 \quad \textit{fur} \quad i > 0.$$

Differential forms can also be considered complex valued. The Lemma of Poincarè remains true by trivial reasons. Hence we see also:

**6.7 Proposition.** *For convex open $E \subset \mathbb{R}^n$ on has*

$$H^i(E,\mathbb{C}_X) = 0 \quad \textit{fur} \quad i > 0.$$

As an application we prove

**6.8 Proposition.** *For convex open $E \subset \mathbb{R}^n$ on has*

$$H^2(E,\mathbb{Z}) = 0.$$

*Proof.* We consider the homomorphism

$$\mathbb{C} \longrightarrow \mathbb{C}^{\bullet}, \qquad z \longmapsto e^{2\pi \mathrm{i} z}.$$

The kernel is $\mathbb{Z}$. This can be considered as a exact sequence of sheaves for example on an open convex $E \subset \mathbb{R}^n$. A small part of the long exact cohomology sequence is

$$H^1(E,\mathbb{C}^*) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow H^2(E,\mathbb{C}).$$

Since the first and the third member of this sequence vanish (6.1 and 6.3) we get the proof of 6.6. □

Next we treat an example of complex analysis. For this wee need the Dolbeault complex

$$0 \longrightarrow \Omega^p(U) \xrightarrow{\bar\partial} A^{p,0}(U) \xrightarrow{\bar\partial} A^{p,1}(U) \xrightarrow{\bar\partial} \cdots$$

for an open subset $U \subset \mathbb{C}^n$.

**6.9 Lemma of Dolbeault.** *Let $U \subset \mathbb{C}^n$ be a polydisk. The sequence*

$$0 \longrightarrow \Omega^p(U) \xrightarrow{\bar\partial} A^{p,0}(U) \xrightarrow{\bar\partial} A^{p,1}(U) \xrightarrow{\bar\partial} \cdots$$

*is exact.*

**Corollary.** *One has*

$$H^q(U, \mathcal{O}_U) = 0 \quad \text{for} \quad q > 0.$$

**6.10 Basic Lemma.** *Let $f : E \to \mathbb{C}$ be a $\mathcal{C}^\infty$-function on the unit disk $E$. Then there exists a $\mathcal{C}^\infty$-function $g : E \to \mathbb{C}$ with the property*

$$\frac{\partial g}{\partial \bar z} = f(z).$$

**Additional Remark.** *If $f$ depends differentiably on more variables, one can get that the seme is true for $g$.*

*Proof of the basic lemma.* In a first step we assume that $f$ is defined on some open neighborhood of $\bar E$. The proof uses Stokes's theorem. The idea is to define $g$ as an surface integral:

$$g(a) = \frac{1}{2\pi \mathrm{i}} \int_E f(z) \frac{dz \wedge d\bar z}{z - a}.$$

Since there is a singular point in the integrand, the integral needs an interpretation. For this we use polar coordinates $z = a + re^{\mathrm{i}\varphi}$ in a small disk around $a$. We get

$$dz \wedge d\bar z = 2\mathrm{i} dx \wedge dy = 2\mathrm{i} r dr d\varphi.$$

The new integrand is $2\mathrm{i} f(z) e^{-\mathrm{i}\varphi}$. The singularity disappeared!. This considerations shows that as precise definition of the integral one can take

$$g(a) = \frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon \to 0} \int_{E(\varepsilon)} f(z) \frac{dz \wedge d\bar z}{z - a}$$

where $E(\varepsilon)$ denotes the complement of the disk $|z - a| \le \varepsilon$. Here $\varepsilon$ should be taken small enough such that this closed disk is contained in $E$. We will apply the theorem of Stokes to $E(\varepsilon)$ and the differential form

$$\omega := f(z) \log |z - a|^2 d\bar z.$$

Since

$$d\omega = \partial\omega = \frac{\partial f}{z} \log |z - a|^2 + \frac{f(z)}{z - a},$$

we get from Stoke's theorem

$$\oint_{|z|=1} f(z)\log|z-a|^2 d\bar z - \oint_{|z-a|=\varepsilon} f(z)\log|z-a|^2 d\bar z$$
$$= \int_{E(\varepsilon)} \frac{\partial f}{\partial z}\log|z-a|^2 dz\wedge d\bar z + \int_{E(\varepsilon)} f(z)\frac{dz\wedge d\bar z}{z-a}.$$

Now we take the limit $\varepsilon$ to 0 the integral $\oint_{|z-a|=\varepsilon} f(z)\log|z-a|^2 d\bar z$ tends to 0. This follows from the standard estimate of line curve integrals and the fact $\lim_{\varepsilon\to 0}\varepsilon\log\varepsilon = 0$. Taking the limit now we get

$$2\pi \mathrm{i} g(a) = \oint_{|z|=1} f(z)\log|z-a|^2 d\bar z - \int_E \frac{\partial f}{\partial z}\log|z-a|^2 dz\wedge d\bar z.$$

One should notice that the integrand of the surface integral still has a singularity at $a$. But this is only a logarithmic singularity and $\log|z-a|$ is Lebesgue integrable over $E$. It is easy to verify that the Lebesgue limit theorem applies. The same argument applies to show that $g$ is differentiable and that differentiation can be interchanged with integration:

$$2\pi\mathrm{i}\frac{\partial g(a)}{\partial \bar a} = \int_E \frac{\partial f(z)}{\partial z}\frac{dz\wedge d\bar z}{\bar z - a} - \int_{|z|=1} f(z)\frac{d\bar z}{\bar z - a}.$$

Now the proof follows from the *generalized Cauchy integral formula*: Let $f$ be a $\mathcal{C}^\infty$ function on an open neighborhood of $\bar E$. Then

$$2\pi\mathrm{i} f(a) = \int_{|z|=1}\frac{f(z)}{z-a}dz + \int_E \frac{\partial f}{\partial \bar z}\frac{dz\wedge d\bar z}{z-a}.$$

(For holomorphic $f$ this is the usual Cauchy integral formula.

Since this formula may not be standard, we mention that it is also an application of Stokes theorem. One uses the formula

$$d\Big(f(z)\frac{dz}{z-a}\Big) = \frac{\partial f}{\partial \bar z}\frac{dz\wedge d\bar z}{z-a}$$

and again applies Stoke's theorem to the domain $E(\varepsilon)$, introduces polar coordinates and takes the limit $\varepsilon \to 0$.

Now we assume that $F$ is given only on $E$ (and not on a neighborhood of $\bar E$. This needs a new technique. The idea is to use an approximation argument. We choose an exhaustion of $E$ by the sequence of disks $E_n = \{z;\ |z| < 1-1/n\}$. We know already that there exists $g_n \in \mathcal{C}^\infty(E_n)$ such that $\partial g_n/\partial\bar z = f$ on $E_n$. The functions $g_n$ are not uniquely determined. The idea is to prepare them such that they converge. More precisely we want to have that for each $i$ the sequence $g_n, g_{n+1}, \dots$ converges on $E_n$. The limit will be a function on $E_n$ and all these differential forms glue to a function $g$ on the whole $E$. This will be the solution of our problem. $(\partial g/\partial\bar z = f)$.

We have to explain in which sense convergence is understood. For this we use the maximum norm $||h||_{E_n}$ for a function that is continuous on some open neighborhood of $E_n$. The strategy is to construct the $g_n$ inductively such that

$$||g_{n+1} - g_n||_{E_{n-1}} < 2^{-n}.$$

One starts with arbitrary $g_1$. The induction step is very easy. Assume that $g_1, \dots, g_n$ have been constructed. Then choose any $h$ such that $\partial h/\partial\bar z = f$ on $E_{n+1}$. We can modify $h$ by adding function. Hence we try to define $g_{n+1} = h + P$ with an analytic function. Now we use that $h - g_n$ is holomorphic on $E_n$. We can approximate this function on $E_{n-1}$ by a polynomial $P$ (taking a partial sum of the Taylor expansion). This gives the construction of $g_{n+1}$.

Now it is easy to show that the limit of the $g_n$ exists. Just write in (on $P_n$) in the form

$$g = g_n + \sum_{i=n}^{\infty}(f_{i+1} - f_i).$$

The sum is a series of holomorphic functions that converges uniformly on $E_n$. Hence the limit exists and differentiation can be exchanged with the limit.

This finishes the proof of the basic lemma. □

*Proof of 6.9 continued.* Now we go to several variables and consider a polydisk $P$. We assume that $\omega$ is a differential form of type $(p, q)$ not only on $P$ but on an open neighborhood of $\bar P$. We assume $\partial\omega = 0$ and claim that there exists a $(p, q-1)$-form $\omega'$ on $P$ with $\partial\omega' = \omega$. The proof can be given by induction in the same way as in the proof of the lemma of Picareè. The beginning of the induction now is the basic lemma 6.10. We skip details. □

We give a nice application. Let $\bar{\mathbb{C}}$ be the Riemann sphere.

**6.11 Theorem.** *One has*

$$H^1(\bar{\mathbb{C}}, \mathcal{O}_{\bar{\mathbb{C}}}) = 0.$$

For the proof we use a covering by two disks of the Riemann sphere $U = \{z \in \mathbb{C}; |z| < 2\}$ and $V = \{z \in \bar{\mathbb{C}}; |z| > 1\}$ (including $\infty$). We can apply the refinement of Leray's theorem 5.5 to obtain $H^1(\mathfrak{U}, \mathcal{O}) = H^1(\bar{\mathbb{C}}, \mathcal{O}_{\bar{\mathbb{C}}})$. A Čech 1-cocycle simply is given by a holomorphic function on the circular ring. We have to show that it can be written as difference $f_1 - f_2$ where $f_i$ is holomorphic on the disc $E_i$. This is possible by the theory of the *Laurent decomposition*.

□

# Chapter IV. Topological tools

## 1. Paracompact spaces

A covering $\mathfrak{U} = (U_i)_{i\in I}$ of a topological space is called *locally finite* if for every point $a \in X$ there exists a neighborhood $W$, such that the set of indices $i \in I$ with $U_i \cap W \neq \emptyset$ is finite.

A covering $\mathfrak{V} = (V_j)_{j\in J}$ is called a *refinement* of the covering $\mathfrak{U}$ if for every index $j \in J$ there exists an index $i \in I$ with $V_j \subset U_i$. If one chooses for each $j$ such an $i$ one obtains a so-called *refinement map* $J \to I$, which needs not to be unique.

**1.1 Definition.** *A Hausdorff space is called* ***paracompact*** *if every open covering admits a locally finite (open) refinement.*

We collect some results about paracompact spaces without proofs. Firstly we give examples:

*Every metric space is paracompact.*

*Every locally compact space with countable basis of topology is paracompact.*

Next we formulate the basic result about paracompactness: Let $\mathfrak{U} = (U_i)$ be a locally finite covering. A partition of unity with respect to $\mathfrak{U}$ is family $\varphi_i$ of continuous real valued functions on $X$ with the following property:

a) The support of $\varphi_i$ is compact and contained in $U_i$.
b) $0 \le \varphi_i \le 1$,
c) $\sum_{i\in I} \varphi_i(x) = 1$ for all $x \in X$.

(This sum is finite.)

**1.2 Proposition.** *Let $X$ be a paracompact space. For every locally finite open covering there exists a partition of unity.*

We mention two related results:

**1.3 Proposition.** *Let $X$ be a paracompact space and $\mathfrak{U} = (U_i)$ a locally finite open covering. There exist open subsets $V_i \subset U_i$ whose closure $\bar{V}_i$ (taken in $X$) is contained in $U_i$ and such that $\mathfrak{V} = (V_i)$ is still a covering.*

Another related result states:

**1.4 Proposition.** *Let $X$ be a locally compact paracompact space, $U$ an open subset and $V \subset\subset U$ a relatively compact open subset in $U$. Then there exists a continuous function on $X$ which is one on $V$ and whose support is compact and contained in $U$.*

(The symbol $V \subset\subset U$ means that the closure $\bar{V}$, taken in $X$, is compact and contained in $U$.)

## 2. Frèchet spaces

A topological vector space is (complex) vector space $E$ together with a topology such the addition map $E \times E \longrightarrow E$ and the multiplication with scalars $\mathbb{C} \times E \longrightarrow E$ is continuous. It is easy to derive then that or each fixed $a \in E$ the map $E \to E$, $x \mapsto x + a$, is topological. Topological vector spaces very often are constructed by means of semi-norms.

A semi-norm $p$ on a complex vector space $E$ is a map $p : E \to \mathbb{R}$ with the properties

a) $p(a) \geq 0$ for all $a \in E$,
b) $p(ta) = |t|p(a)$ for all $t \in \mathbb{C}$, $a \in E$,
c) $p(a + b) \leq p(a) + p(b)$.

The ball of radius $r > 0$ is defined as

$$U_r(a, p) := \{\, x \in E;\ p(a - x) < r \,\}.$$

Let $\mathcal{M}$ be a set of semi-norms. A subset $B \subset E$ is called a semi-ball around $a$ with respect to $\mathcal{M}$ if there exists a finite subset $\mathcal{N} \subset \mathcal{M}$ and for each $p \in \mathcal{N}$ a number $r_p > 0$ such that

$$B = \bigcap_{p \in \mathcal{N}} U_{r_p}(a, p).$$

A subset $U$ of $E$ is called open (with respect to $\mathcal{M}$) if for every $a \in U$ there exists a semi-ball $B$ around $a$ with $B \subset U$.

It is clear that this defines a topology on $E$ such that all $p : E \to \mathbb{C}$ are continuous. (It is actually the weakest topology with this property.) It is also easy to to see that $E$ is a topological vector space. Moreover a sequence $(a_n)$ in $E$ converges to $a \in E$ if and only if $p(a_n - a) \to 0$ for all $p \in \mathcal{M}$. Obviously the elements $p \in \mathcal{M}$ are continuous. Let $\mathcal{M}_{\max}$ be the set of all continuous semi-norms. Two sets $\mathcal{N}$ and $\mathcal{M}$ define the same topology if and only if $\mathcal{M}_{\max} = \mathcal{N}_{\max}$. Especially $\mathcal{M}_{\max}$ and $\mathcal{M}$ define the same topology.

The set $\mathcal{M}$ is called definite, if

$$p(a) = 0 \quad \text{for all} \quad p \in \mathcal{M} \quad \Longrightarrow \quad a = 0.$$

It is easy to prove that $\mathcal{M}$ is definit if and only if $E$ is a Hausdorff space.

A sequence $(a_n)$ in $E$ is called a *Cauchy sequence* with respect to $\mathcal{M}$, if for every $\varepsilon > 0$ and every $p \in \mathcal{M}$ there exists an $N = N(p, \varepsilon)$ such that

$$p(a_n - a_m) < \varepsilon \quad \text{for} \quad n, m \geq N.$$

Remarkably this notion only depends on the topology. Obviously a sequence is a Cauchy sequence if and only if for every neighborhood $U$ od the origin one has $a_n - a_m \in U$ if noth $n, m$ are sufficiently large.

The set $\mathcal{M}$ is called of countable type, if there exists a countable subset $\mathcal{N} \subset \mathcal{M}$ defining the same topology and the same Cauchy sequences.

**2.1 Definition.** *A Frèchet space $E$ is a topological vector space whose topology can be defined by a set $\mathcal{M}$ of seminorms such the following properties are satisfied:*

a) $\mathcal{M}$ is definite.
b) $\mathcal{M}$ is of countable type.
c) Every Cauchy sequence converges.

Notice that a Banach space is a Frèchet space, where $\mathcal{M}$ consists of a single element.

**2.2 Lemma.** *Frèchet spaces are metrisable.*

*Proof.* We choose some ordering of $\mathcal{N} = \{p_1, p_2, \ldots\}$. Then one defines

$$d(a, b) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(a - b)}{1 + p(a_n) + p(b_n)}.$$

It is easy to show that this is a metric which defines the original topology.

□

An important result about Frèchet spaces is:

**2.3 Open mapping theorem.** *Any surjective linear continuous map $E \to F$ between Frèchet spaces is open. Especially the topology on $F$ agrees with the quotient topology of $E$.*

An obvious corollary states that a bijective linear continuous map between Frèchet spaces is topological.

**Permanence properties of Frèchet spaces**

A closed subspace $F \subset E$ of a Frèchet space, equipped with the induced topology, is a Frèchet space too. A defining system of seminorms is obtained if one restricts the seminorms $p$ of a defining system on $E$ to $F$.

Let $F \subset E$ a closed subspace of a Frèchet space. Then the quotient space $E/F$ equipped with the quotient topology, is a Frèchet space. A defining system of seminorms is obtained as follows. Denote the quotient map by $f : E \to E/F$. Let $p$ be a continuous seminorm on $E$ (from a defining system is enough). Then

$$\tilde{p}(y) = \inf_{f(x)=y} p(x) \qquad (x \in E),$$

is a seminorm on $E/F$.

Let $(E_s)_s \in I$ be a finite or countable family of Frèchet spaces. Then their direct product

$$E = \prod_{s \in S} E_s$$

equipped with the product topology, is a Frèchet space. In terms of seminorms this can be described as follows. Take a finite subset $T \subset S$ and for each $t \in T$ take a continuous seminorm $p_i$, $i \in J$ on $E_i$ (from a defining system is enough). Then one can define a seminorm on the product

$$p((x_i)) = \max_{j \in J} p_j(x_j).$$

**Basic example of Frèchet spaces**

Let $X$ be a complex manifold and $\mathcal{O}(X)$ the set of all analytic functions on $X$. This is a complex vector space. For an arbitrary compact subset $K \subset X$ we define

$$p(f) = p_K(f) := \max_{z \in K} |f(z)|.$$

This is s semi norm. A sequence $(f_n)$ converges with respect to $p_K$ if and only if $f_n$ converges uniformly on $K$.

**2.4 Remark.** *Let $X$ be a complex manifold. The vector space $\mathcal{O}(X)$ equipped with the set of all norms of the form $p_K$, $K \subset X$ compact, is a Frèchet space.*

The set of all $p_K$ is of countable type since $X$ is assumed to have countable basis of topology. This implies that there is a sequence $K_1 \subset K_2 \subset \cdots$ of compact subsets whose union is $X$ and such that $K_i$ is contained in the interior of $K_{i+1}$. Then every compact subset is contained in one of the $K_i$. The convergence of Cauchy sequences follows from the theorem of Weierstrass, which states that analyticity is stable under uniform convergence. $\square$

The basic result about this Frèchet space is:

**2.5 Theorem of Montel.** *Let $X$ be a complex manifild and $C > 0$ a positive constant. The set*

$$\mathcal{O}(X, C) := \{\, f \in \mathcal{O}(X); \quad |f(z)| \le C \text{ for } z \in X \,\}$$

*is compact in $\mathcal{O}(X)$.*

For the proof one has to use the fact that a metric space is compact if every sequence admits a convergent subsequence. Hence the statement follows from the usual theorem of Montel which states that every sequence in $\mathcal{O}(X, C)$ admits a locally convergent sub-sequence. We notice that the analogue for real differentiable functions is false. The proof uses heavily the Cauchy integral.

### Compact operators

A well-known fact is that in a Banach space of infinite dimension the closed ball $||a|| \le 1$ is not compact. This result is also true for Frèchet spaces in the following form:

*Assume that the Frèchet space admits a non-empty open subset with compact closure. Then it is of finite dimension.*

We need a generalization of this result: A continuous linear map $f : E \to F$ between Frèchet spaces is a *compact operator,* if there exists a non-empty open subset of $E$ such that the closure of its image is compact. It is clear that this is the case if $f(E)$ is of finite dimension.

A linear map $f : E \to F$ is called *nearly surjective* if $F/f(E)$ has finite dimension. This is automatically the case when $F$ is finite dimensional.

**2.6 Theorem of Schwartz.** *Let $f : E \to F$ be a surjective continuous linear map between Frèchet spaces and let $g : E \to F$ be a compact operator. Then $f + g$ is nearly surjective.*

If one applies Schwartz's theorem in the case $E = F$, $f = -\operatorname{id}$ and $g = \operatorname{id}$ on obtains:

**2.7 Corollary.** *When the identity operator* $\operatorname{id} : E \to E$ *of a Frèchet space is compact, then $E$ is finite dimensional*

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