Eberhard Freitag

Complex Analysis of Several Variables I

Local Complex Analysis of Several Variables

Self-Publishing, 2016

Eberhard Freitag
Universität Heidelberg
Mathematisches Institut
Im Neuenheimer Feld 288
69120 Heidelberg
freitag@mathi.uni-heidelberg.de

Contents

Preface

These notes arose from lectures on complex spaces which I gave occasionally in Heidelberg. In a first part I treated the local theory of complex spaces. I tried to proceed as elementary as possible. So the first chapter that covers the local theory up to the coherence theorems is presented without using the language of sheaves. Instead of this we used some basic commutative algebra which makes things much more understandable. In an appendix the reader finds the tools of commutative algebra without proofs. But in the spirit of the notes commutative algebra should be considered as easy compared to the complex analysis.

In the second chapter we change the point of view. We introduce complex spaces in the sense of Grothendieck. This rests on sheaf theory. We presented basic sheaf theory (without cohomology) in an extra appendix. In the second chapter we reformulated the local result, in particular the coherence theorems sheaftheoretically.

There will follow two other volumes, one about Stein spaces and finally on Grauert's coherence theorem.

Chapter I. Local complex analysis

1. The ring of power series

All rings are assumed to be commutative and with unit element. Homomorphisms of rings are assumed to map the unit element into the unit element. Modules M over a ring R are always assumed to be unitary, $1_R m = m$.

Recall that an algebra over a ring A by definition is a ring B together with a distinguished ring homomorphism $\varphi : A \to B$. This ring homomorphism can be used to define on B a structure as A-module, namely

$$ab := \varphi(a)b \quad (a \in A,\ b \in B).$$

Let B, B' be two algebras. A ring homomorphism $B \to B'$ is called an algebra homomorphism if it is A-linear. This is equivalent to the fact that

$$\begin{tikzcd} B \arrow[rr] & & B' \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes.

The notion of a formal power series can be defined over an arbitrary ring R. A formal power series in n variables is just an expression of the type

$$P = \sum_\nu a_\nu z^\nu, \quad a_\nu \in R,$$

where ν runs through all multi-indices (n-tuples of nonnegative integers). Here $z = (z_1, \dots, z_n)$ and $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ just have a symbolic meaning. Strictly logically, power series are just maps $\mathbb{N}_0^n \to R$. Power series can be added and multiplied formally, i.e.

$$\sum_\nu a_\nu z^\nu + \sum_\nu b_\nu z^\nu = \sum_\nu (a_\nu + b_\nu) z^\nu,$$

$$\sum_\nu a_\nu z^\nu \cdot \sum_\nu b_\nu z^\nu = \sum_\nu \Big(\sum_{\nu_1+\nu_2=\nu} a_{\nu_1} b_{\nu_2} \Big)\, z^\nu.$$

The inner sum is finite. In this way we get a ring $R[[z_1,\dots,z_n]]$. Polynomials are just power series such that all but finitely many coefficients are zero. In this way, we can consider the polynomial ring $R[z_1,\dots,z_n]$ as subring of the ring of formal power series. The elements of R can be identified with polynomials such that all coefficients a_ν with $\nu\neq 0$ vanish. We recall that for a non-zero polynomial $P\in R[z]$ in one variable the degree $\deg P$ is well-defined. It is the greatest n such the the nth coefficient is different from 0. Sometimes it is useful to define the degree of the zero polynomial to be $-\infty$. If R is an integral domain, the rule $\deg(PQ)=\deg P+\deg Q$ is valid.

There is a natural isomorphism

$$R[[z_1,\dots,z_{n-1}]][[z_n]] \xrightarrow{\sim} R[[z_1,\dots,z_n]]$$

whose precise definition is left to the reader. In particular, $R[[z_1,\dots,z_{n-1}]]$ can be considered as a subring of $R[[z_1,\dots,z_n]]$. One can use this to show that $R[[z_1,\dots,z_n]]$ is an integral domain if R is so.

Let now R be the field of complex numbers $\mathbb{C}$. A formal power series is called convergent if there exists a small neighborhood of the origin where it is absolutely convergent. It is easy to show that this means just that there exist constants a,b such that $|a_\nu|\le ab^{\nu_1+\dots+\nu_n}$. The set

$$\mathcal{O}_n=\mathbb{C}\{z_1,\dots,z_n\}$$

of all convergent power series is a subring of the ring of formal power series. There is a natural homomorphism

$$\mathcal{O}_n\longrightarrow\mathbb{C},\quad P\longmapsto P(0):=a_0,$$

that sends a power series to its constant coefficient. Its kernel $\mathfrak{m}_n$ is the set of all power series whose constant coefficient vanishes. The power $\mathfrak{m}_n^k$ is the ideal generated by $P_1\cdots P_k$ where $P_i\in\mathfrak{m}_n$. It is easy to see that a power series P belongs to $\mathfrak{m}_n^k$ if and only if

$$a_\nu\neq 0\Longrightarrow \nu_1+\dots+\nu_n\ge k.$$

As a consequence, we have

$$\bigcap \mathfrak{m}_n^k=0.$$

2. Holomorphic functions

We consider open subsets $U \subset \mathbb{C}^n$. A function $f : U \to \mathbb{C}$ is called holomorphic if for any $a \in U$ there exists a convergent power $P \in \mathcal{O}_n$ such that

$$f(z+a) = P(z) = \sum_\nu a_\nu z^\nu$$

for all z in a small neighborhood of 0. We also can write

$$f(z) = \sum_\nu a_\nu (z-a)^\nu$$

in a small neighborhood of a. We associate to each $a \in U$ an own ring of power series

$$\mathcal{O}_{U,a} = \mathbb{C}\{z_1 - a_1, \dots, z_n - a_n\}$$

and we write

$$[f]_a = \sum_\nu a_\nu (z-a)^\nu,$$
$$[f]_a \in \mathbb{C}\{z_1 - a_1, \dots, z_n - a_n\}.$$

We denote be $\mathcal{O}(U)$ the ring of all holomorphic functions on U. So we get for each $a \in U$ a homomorphism

$$\mathcal{O}(U) \longrightarrow \mathcal{O}_{U,a}, \quad f \longmapsto [f]_a.$$

Let $V \subset \mathbb{C}^m$ be a second open subset. A maps $f : U \to V$ is called holomorphic if all its components are holomorphic. It is easy to show that holomorphic maps are complex differentiable in the following sense. They are continuously differentiable in the sense of real matrices and the Jacobian maps $\mathbb{C}^n \longrightarrow \mathbb{C}^m$ are $\mathbb{C}$-linear.

2.1 Proposition. *A map $f : U \to V$, $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ is holomorphic if and only if it is complex differentiable.* Phcd

In the case $n = 1$ this is proved in standard courses about complex calculus. Since the proof can be straightly generalized to the case $n > 1$ we omit a prof here. A possible reference is [Fr]. □

On can use this proposition to prove the theorem of invertible functions by reducing it to the known real case.

2.2 Proposition. *Let $f : U \to V$, $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^n$ be a holomorphic map and let $a \in U$ be a point such that the Jacobian map $J(f,a) : \mathbb{C}^n \to \mathbb{C}^n$ is invertible. Then there exists an open neighborhood $a \subset W \subset U$ such that $h(W)$ is open and such that the map $W \to f(W)$ is biholomorphic.* Psif

There is a natural homomorphism, sometimes called "pull back",

$$f^* : \mathcal{O}(V) \longrightarrow \mathcal{O}(U), \quad g \longmapsto g \circ f,$$

and similarly for each $a \in U$

$$f_a^* : \mathcal{O}_{V,f(a)} \longrightarrow \mathcal{O}_{U,a}.$$

3. Homomorphisms between rings of power series

Complex analysis deals with holomorphic functions.

3.1 Lemma. *An element $P \in \mathbb{C}[[z_1, \dots, z_n]]$ is a unit (i.e. an invertible element) if an only if $P(0) \neq 0$. The same is true in the ring $\mathcal{O}_n$* IsInv

Proof. We can assume $P(0) = 1$. Then P can be inverted by means of the geometric series

$$\frac{1}{P} = \frac{1}{1 + (P - 1)} = \sum_{m=0}^{\infty} (P - 1)^n.$$

Since $P - 1$ has no constant coefficient this series defines a formal power series. An easy argument shows that this is convergent of P is convergent- □

As a consequence of Lemma 3.1, the rings $\mathbb{C}[[z_1, \dots, z_n]]$, $\mathcal{O}_n$ are local rings. Recall that a ring R is called local if the sum of two non-units is a non-unit. Then the set of all non-units is an ideal, obviously the only maximal ideal.

3.2 Remark. *The rings $\mathbb{C}[[z_1, \dots, z_n]]$, $\mathcal{O}_n$ is a local ring. The maximal ideal in both cases consists of all P with $P(0) = 0$.* RisL

We denote by $\mathfrak{m}_n$ the maximal ideal of $\mathcal{O}_n$. The rings $\mathbb{C}[[z_1, \dots, z_n]]$, $\mathcal{O}_n$ contains $\mathbb{C}$ as a subring (constant power series). Hence they are $\mathbb{C}$-algebra.

Our next task is to describe the algebra homomorphisms

$$f : \mathbb{C}[[z_1, \dots, z_m]] \longrightarrow \mathbb{C}[[z_1, \dots, z_n]], \quad f : \mathcal{O}_m \to \mathcal{O}_n.$$

First we claim that non-units are mapped to non-units. This means that f is a local homomorphism. Otherwise there would be non-unit $P \in \mathcal{O}_m$ such that $Q = f(P)$ is a unit. Then we would have $f(P - Q(0)) = Q - Q(0)$. The element $P - Q(0)$ is a unit but its image $Q - Q(0)$ is not. This is not possible.

There is a special kind of such a homomorphism which we call a "substitution homomorphism". Let $P(z_1, \dots, z_n)$ is an element of $\mathbb{C}[[z_1, \dots, z_n]]$, and let $P_1, \dots, P_n$ contained in the maximal ideal Then one can define

$$P(P_1, \dots, P_n) = \sum_{\nu} a_\nu P_1^{\nu_1} \cdots P_n^{\nu_n}.$$

The right hand side can be read as formal power series (since that $P_1, \dots, P_n$ have no constant coefficients). So we get a homomorphism

$$\mathbb{C}[[z_1, \dots, z_n]] \longrightarrow \mathbb{C}[[z_1, \dots, z_m]], \quad P \longmapsto P(P_1, \dots, P_n).$$

We call it a *substitution homomorphism.* A simple convergence proof shows that the restriction to $\mathcal{O}_m$ defines a homomorphism $\mathcal{O}_m \to \mathcal{O}_n$. We call it also a substitution homomorphism.

3.3 Lemma. *Each algebra homomorphism* SubHom

$$\mathbb{C}[[z_1, \dots, z_n]] \longrightarrow \mathbb{C}[[z_1, \dots, z]], \quad \mathcal{O}_n \longrightarrow \mathcal{O}_m$$

is a substitution homomorphism.

Proof. We treat the convergent version. Let $\varphi : \mathcal{O}_n \to \mathcal{O}_m$ an algebra homomorphism. Since it is local, the elements $P_i := \varphi(z_i)$ are contained in the maximal ideal. Hence one can consider the substitution homomorphism ψ defined by them. We claim $\varphi = \psi$. At the moment we only know that φ and ψ agree on $\mathbb{C}[z_1, \dots, z_n]$. Let $P = \sum_\nu a_\nu z^\nu \in \mathcal{O}_n$. We claim $\varphi(P) = \psi(Q)$. For this we decompose for a natural number k

$$P = P_k + Q_k, \quad P_k = \sum_{\nu_1+\cdots+\nu_n \le k} a_\nu z^\nu.$$

Then Q_k is contained in the k-the power $\mathfrak{m}^k$ of the maximal ideal. (Obviously $\mathfrak{m}^k$ is generated by all z^ν where $\nu_1 + \cdots + \nu_n \ge k$.) We get

$$\varphi(P) - \psi(P) = \varphi(Q_k) - \psi(P_k) \in \mathfrak{m}^k.$$

This is true for all k. But the intersection of all $\mathfrak{m}_n^k$ is zero. □

Finally we treat a version of the theorem of inverse functions. First we mention that for a formal power series the partial derivatives

$$\frac{\partial P}{\partial z_j}$$

can be defined in an obvious formal way. It is clear the the partial derivatives of a convergent power series are convergent. If $P = (P_1, \dots, P_m)$ is a tuple of power series in $\mathbb{C}[[z_1, \dots, z_n]]$, then we can define the Jacobi matrix at 0

$$J(P, 0) := \Big(\frac{\partial P_i}{\partial z_j}\Big)_{i \le j}.$$

3.4 Proposition. *An algebra homomorphism* Pahi

$$\mathbb{C}[[z_1, \dots, z_n]] \longrightarrow \mathbb{C}[[z_1, \dots, z_n]]$$

is an isomorphism if and only if the Jacobi matrix $J(P, 0)$ is invertible, where $P = (P_1, \dots P_n)$ are the images of $z_1, \dots, z_n$. The same is true in the convergent case.

In the following we need only the convergent case. Nevertheless we want to indicated how the formal proof runs. To get the idea, it is sufficient to treat the case $n = 1$. We have

$$P = \sum_{n=1}^{\infty} a_n z^n.$$

By assumption $a_1 \neq 0$. We can assume $a_1 = 1$. The inverse isomorphism, if it exists, is a substitution homomorphism given by power series

$$Q = \sum_{n=1}^{\infty} b_n z^n.$$

The relation $P(Q) = 1$ gives a recursion b_n:

$$\sum_{n=1}^{\infty} a_n \left(\sum_{m=1}^{\infty} b_m x^m \right)^n = x,$$

or

$$\sum_{n=1}^{m} a_n \sum_{\nu_1+\nu_2+\cdots+\nu_n=m} b_{\nu_1} \dots b_{\nu_n} = \begin{cases} 1 & \text{für } m = 1, \\ 0 & \text{für } m > 1. \end{cases}$$

We obtain

$$(*) \qquad \begin{aligned} b_1 &= 1, \\ b_m &= -\sum_{n=2}^{m} a_n \sum_{\nu_1+\nu_2+\cdots+\nu_n=m} b_{\nu_1} \dots b_{\nu_n}, \; m > 1. \end{aligned}$$

One the right hand side we only have b_ν such that $\nu < m$. Hence the coefficients b_n can be determined inductively.

We know treat the convergent case. It would be enough to prove that Q is convergent if this is so for P. This can be proved directly by a clever somewhat tedious estimate. We will not give it here and propose another argument. It rests on Sect. 2., in particular on the Propositions 2.1 and 2.2.

An n-tupel $P = (P_1, \dots, P_n) \in \mathcal{O}_n^n$ induces a complex differentiable map $f : U \to \mathbb{C}^n$, where $0 \in U \subset \mathbb{C}^n$ is an open neighborhood of the origin. By assumption its Jacobian at 0 is invertible. Hence we can apply the theorem of invertible functions. After possible shrinking of U we obtain that $V = f(U)$ is open and that the map $U \to V$ is biholomorphic. Hence the inverse map can be expanded into power series $Q = (Q_1, \dots, Q_n)$. □

4. The Preparation and the Division Theorem

There is an division algorithm in the ring of power series analogous to the Euclidean algorithm in a polynomial ring. We recall this Euclidean algorithm.

The Euclidean Algorithm for Polynomials

let R be an integral domain and let

a) $P \in R[X]$ *be an arbitrary polynomial,*
b) $Q \in R[X]$ *be a normalized polynomial, i.e. the highest coefficient is 1.*

Then there exists a unique decomposition

$$P = AQ + B.$$

where $A, B \in R[X]$ are polynomials and

$$\deg(B) < d.$$

This includes the case $B = 0$ if one defines $\deg(0) = -\infty$. The proof of this result is trivial (induction on the degree of P). □

4.1 Definition. *A power series $P \in \mathbb{C}[[z_1, \dots, z_n]]$ is called z_n-general if* Dzna

$$P(0, \dots, 0, z_n) \neq 0.$$

Then we can write

$$P(0, z_n) = C z_n^d + \text{higher terms}, \ \ C \neq 0.$$

We call d the order of P. Frequently we will normalize to $C = 1$.

4.2 Division theorem. *Let $Q \in \mathbb{C}[[z_1, \dots, z_n]]$ be a z_n-general power series of order d. Every power series $P \in \mathbb{C}[z_1, \dots z_n]]$ admits a unique decomposition* weidiv

$$\begin{aligned} &P = AQ + B \quad \text{where} \\ &A \in \mathbb{C}[[z_1, \dots, z_n]], \quad B \in \mathbb{C}[[z_1, \dots, z_{n-1}]][z_n], \quad \deg_{z_n}(B) < d. \end{aligned}$$

The analoge theorem holds in $\mathcal{O}_n$.

Here $\deg_{z_n}(B)$ means the degree of the polynomial B in the variable z_n (taking $-\infty$ if $B = 0$).

Proof. In this proof we denote the coefficients of a power series A by the same letter, i.e.

$$A = \sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu .$$

So the product of two power series is

$$AB = \sum_\nu \sum_{\mu \le \nu} A_\mu B_{\nu-\mu}.$$

Here $\mu \le \nu$ is understood componentwise. Even more, $\mu < \nu$ means $\mu \le \nu$ and $\mu \ne \nu$. In our context it is convenient to separate ν into two parts, a vector of length $n-1$ and a number. We write for this pair (ν, j) where now $\nu \in \mathbb{N}_0^{n-1}$ and $j \in \mathbb{N}_0$. Now the formula $P = AQ + B$ reads as

$$(*) \qquad P_{\nu,j} = \sum_{\mu \le \nu} \sum_{m=0}^{j} A_{\mu,m} Q_{\nu-mu,j-m} + B_{\nu,j}.$$

The condition that B is a polynomial in z_n of degree $< d$ reads as

$$B_{\nu,j} = 0 \quad \text{for all} \quad j \ge d.$$

The series P, Q are given and Q has the property

$$Q_{00}, \dots, Q_{0,d-1} = 0, \quad Q_{0d} \ne 0.$$

We can and will assume that

$$Q_{0d} = 1.$$

We want to extract $A_{\nu,j}$ from $(*)$. For this purpose we will apply this formula only in the case $j \ge d$. Then the B-s don't occur. If we write $j = d + k$, the formula reads

$$P_{\nu,d+k} = \sum_{\mu \le \nu} \sum_{m=0}^{j+k} A_{\mu,m} Q_{\nu-\mu,d+k-m}.$$

We extract the terms such that $\mu = \nu$. By means of the fact that $Q_{0,d+k-m} = 0$ if $d + k - m < d$ we obtain

$$P_{\nu,d+k} = \sum_{m=0}^{k} A_{\nu,m} Q_{0,d+k-m} + \sum_{\mu \le \nu} \sum_{m=0}^{j+k} A_{\mu,m} Q_{\nu-\mu,d+k-m}$$

$$A_{\nu,k} + \sum_{m=0}^{k-1} A_{\nu,m} Q_{0,d+k-m} + \sum_{\mu \le \nu} \sum_{m=0}^{j+k} A_{\mu,m} Q_{\nu-\mu,d+k-m}$$

.

We solve this formula for $A_{\nu,k}$ ($\nu \in \mathbb{N}_0^{n-1}$, $k \in \mathbb{N}_0$).

$$\boxed{A_{\nu,k} = P_{\nu,d+k} - \sum_{m=0}^{k-1} A_{\nu,m} Q_{0,d+k-m} - \sum_{\mu \le \nu} \sum_{m=0}^{j+k} A_{\mu,m} Q_{\nu-\mu,d+k-m}}$$

It is easy to solve this system inductively. The beginning of the induction is

$$A_{00} = P_{0d}.$$

Then we can determine $A_{01}, A_{02}, \dots$ inductively. After this has been settled we consider an arbitrary ν with $|\nu| = 1$. From the framed formula we first get $A_{\nu,0}$ and then, again inductively, all $A_{\nu,k}$. After the ν with $|\nu| = 1$ have been settled, we jump to the ν with $\nu = 2$ and so on. So all in all we have a double induction where the outer induction runs over $|\nu|$ and the inner induction over k. This gives us a formal series A and then also B.

In the next part of the proof we show that everything works in $\mathcal{O}_n$. So we have to show that A (and then also B) converges if P and Q converge. The convergence of P, Q means that there exist constants a, b such that

$$P_{\nu,j}, Q_{nu,j} \le ab^{\nu+j}.$$

We will show that there exist positive numbers α, β, γ with the property

$$A_{\nu,k} \le \alpha \beta^{|\nu|} \gamma^k.$$

This implies of course the convergence of A. The constants α, β, γ depend only on a, b. The proof follows the same induction procedure as the proof of the determination of the $A_{\nu,k}$. The beginning of the induction is

$$|A_{00}| = |P_{0d}| \le ab^d \le \alpha.$$

So α should be greater of equal than ab^d. We say that (μ, m) comes before (ν, k) if $\mu < \nu$ or if $\mu = \nu$ and $m < k$. To prove the estimate for $A_{\nu,k}$ we can assume that the estimate has been proved for all (μ, m) before (ν, k). So we get

$$|A_{\nu,k}| \le ab^{d+k} + \sum_{m=0}^{k-1} \alpha \beta^{|\nu|} \gamma^m ab^{d+k-m} + \sum_{\mu < \nu} \sum_{m=0}^{d+k} \alpha \beta^{|\mu|} \gamma^m ab^{|\nu-\mu|+d+k-m}.$$

We want this be be smaller than $\alpha \beta^{|\nu|} \gamma^k$. Sufficient for this are the three inequalities

$$\begin{aligned} \alpha \beta^{|\nu|} \gamma^k \ge & 3ab^{d+k}, \\ & 3 \sum_{m=0}^{k-1} \alpha \beta^{|\nu|} \gamma^m ab^{d+k-m}, \\ & 3 \sum_{\mu < \nu} \sum_{m=0}^{d+k} \alpha \beta^{|\mu|} \gamma^m ab^{|\nu-\mu|+d+k-m}. \end{aligned}$$

The first of the three inequalities is fulfilled if we demand

$$\alpha \geq 3ab^d, \quad \beta \geq 1, \quad \gamma \geq b.$$

The next inequality reads

$$\gamma^k \geq 3ab^{d+k} \sum_{m=0}^{k-1} \Big(\frac{\gamma}{b}\Big)^m.$$

A polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$, $a_n \neq 0$, can be estimated as follows

$$|p(x)| \leq C|x^n| \quad \text{for } |x| \geq 1.$$

Here C is a constant that depends on the coefficients of p. So the claimed inequality follows from

$$\gamma^k \geq 3Cab^{d+k}\Big(\frac{\gamma}{b}\Big)^{k-1}$$

or

$$\gamma \geq 3Cab^{d+1}.$$

The treatment of the third inequality is analogous. First we notice that $\mu < \nu$ implies $|\mu - \nu| = |\nu| - |\mu|$. Hence the right hand side of the third inequality contains the sum

$$\sum_{\mu<\nu} \Big(\frac{\beta}{b}\Big)^{|\mu|}$$

occurs. This is a polynomial in β/b of degree $|\nu|$-1. Hence we can estimate it by

$$\sum_{\mu<\nu} \Big(\frac{\beta}{b}\Big)^{|\mu|} \leq C\Big(\frac{\beta}{b}\Big)^{|\nu|-1}.$$

Now the same argument as in the second case gives the desired equation for big enough β. □

The Weierstrass preparation theorem is related to the division theorem which – not correctly – sometimes is also called Weierstrass preparation theorem. But this is historically not correct. Weierstrass first proved the preparation theorem and short time after that Stickelberger proved the division theorem. We will do the converse, we derive the preparation from the division theorem.

4.3 Definition. *A Weierstrass polynomial is a normed polynomial* DWp

$$Q \in \mathbb{C}[[z_1, \dots, z_{n-1}]][z_n]$$

of the form

$$Q = z_n^d + Q_{d-1}z_m^{d-1} + \cdots + Q_0, \quad Q_j(0) = 0.$$

So a Weierstrass polynomial has the property

$$Q(0, z_n) = z_n^d.$$

In particular Weierstrass polynomials are z_n-general. We mention that units are also z_n-general and that the product of two z_n-general elements is z_n-general.

4.4 Weierstrass preparation theorem. *Let P be a z_n-general power series in $\mathbb{C}[[z_1,\dots,z_n]]$. Then there exists a unique decomposition* WPT

$$P = UQ$$

where Q is a Weierstrass polynomial and U is a unit, $U(0) \neq 0$. The same is true in the ring of convergent instead of formal power series. So $P \in \mathcal{O}_n$ implies $Q, U \in \mathcal{O}_n$.

Proof. We apply the division theorem for $(P, Q) = (z_n^d, P)$. So we get

$$z_n^d = AP + B$$

where B is a polynomial of degree $< d$ (=order(P)). This means

$$AP = z_n^d - B.$$

The right hand side, we call it $Q = z_n^d - B$, is a normalized polynomial in z_n of degree d. We claim that it is a $Weierstrasspolynomial$. This means $B(0, z_n) = 0$. This follows from the fact that the z_n-order of the left hand side is $\geq d$. The same argument shows that $A(0) \neq 0$. So A is a unit and we get

$$P = UQ, \quad U = A^{-1}.$$

It remains to prove the uniqueness of the decomposition. This means the following. Let $Q, \tilde{Q}$ be two Weierstrass polynomials and let $\tilde{Q} = UQ$ where U is a unit. Then $\tilde{Q} = Q$. For the prove we perform the (trivial) polynomial division $\tilde{Q} = AQ + B$. From the uniqueness in the division theorem we get $A = U$ and $B = 0$. So U is a polynomial. Comparing degrees and the value at 0 we obtain $U = 1$. □

The technique of comparing the Weierstrass division with the polynomial division gives the following result.

4.5 Remark. *Let Q be a Weierstrass polynomial and let A a power series such that AQ is a non-zero polynomial in z_n, then A is a polynomial in z_n as well. (This holds in the ring of formal and in the ring of convergent power series.)* Ldfc

Proof. Just compare $P := AQ$ with the polynomial division $P = A_1 Q + B_2$. The uniqueness statement in the division theorem gives $A_1 = A$ (and $B_1 = 0$). □

Let A be a complex $m \times n$-matrix. We consider A as linear map

$$A : \mathbb{C}^n \longrightarrow \mathbb{C}^m \quad z \longmapsto w, \qquad w_\mu = \sum a_{\mu\nu} z_\nu.$$

For a power series $P \in \mathcal{O}_n$, we obtain by substitution the power series $P(Az)$ If $m = n$ and if A is an invertible matrix, then

$$\mathcal{O}_n \xrightarrow{\sim} \mathcal{O}_n, \qquad P(z) \longmapsto P(A^{-1}z),$$

is an ring automorphism. The inverse map is given by A^{-1}.

4.6 Remark. *For every finite set of convergent power series $P \in \mathcal{O}_n$, $P \neq 0$, there exists an invertible $n \times n$-matrix A, such that all P^A are z_n-general.* AlAl

Proof. There exists a point $a \neq 0$ in a joint convergence polydisk, such that $P(a) \neq 0$ for all P. After the choice of suitable coordinate transformation (choice of A), one can assume $A(0, \dots, 0, 1) = a$. Then all P^A are z_n-general. □

5. Algebraic properties of the ring of power series

We restrict now to the ring of convergent power series. The ring $\mathcal{O}_0$ just coincides with the field of complex numbers. The ring $\mathcal{O}_1$ is also very simple. Every element can be written in the form $z^n P$ where P is a unit and $n \geq 0$ an integer. It follows that each ideal of $\mathcal{O}_1$ is of the form $\mathcal{O}_1 z^n$. The rings $\mathcal{O}_n$, $n > 1$, are much more complicated.

Let $Q \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial. We can consider the natural homomorphism

$$\mathcal{O}_{n-1}[z_n]/Q\mathcal{O}_{n-1}[z_n] \longrightarrow \mathcal{O}_n/Q\mathcal{O}_n.$$

The division theorem implies that this is an isomorphism.

5.1 Theorem. *For a Weierstrass polynomial $Q \in \mathcal{O}_{n-1}[z_n]$ the natural homomorphism* WDalg

$$\mathcal{O}_{n-1}[z_n]/Q\mathcal{O}_{n-1}[z_n] \longrightarrow \mathcal{O}_n/Q\mathcal{O}_n$$

is an isomorphism.

Proof. The surjectivity is an immediate consequence of the existence statement in the division theorem. The injectivity follows from the uniqueness statement in this theorem as follows. Let $P \in \mathcal{O}_{n-1}[z_n]$ a polynomial that goes to 0, i.e. $P = SA$, $Q \in \mathcal{O}_n$. We have to show that S is a polynomial in z_n. We compare with the elementary polynomial division $P = AQ + B$. The uniqueness statement in the division theorem shows $A = S$ and $B = 0$. □

Recall that an element $a \in R$ is a prime element if and only if Ra is a non-zero prime ideal. (A prime ideal $\mathfrak{p} \subset R$ is an ideal such that $R/\mathfrak{p}$ is an integral domain.) By our convention the zero ring is no integral domain. Hence prime ideals are proper ideals and prime elements are non-units. From Theorem 5.1 we obtain the following result.

5.2 Lemma. *A Weierstrass polynomial $P \in \mathcal{O}_{n-1}[z_n]$ is a prime element in $\mathcal{O}_n$, if and only if it is a prime element in $\mathcal{O}_{n-1}[z_n]$.* WePr

We recall that an integral domain R is called a UFD-domain if every nonzero and non-unit element of R can be written as a finite product of prime elements. This product then is unique in an obvious sense. Every principal ideal domain is UFD. As a consequence every field is UFD. But also $\mathbb{Z}$ and $\mathcal{O}_1$ are principal ideal rings and hence UFD. A famous result of Gauss states that the polynomial ring over a UFD domain is UFD. A non-unit and non-zero element a of an integral domain is called *indecomposable* if it cannot be written as product of two non-units. Primes are indecomposable. The converse is true in UFD-domains. It is often easy to show that any element of an integral domain is the product of finitely many indecomposable elements. For example this is case in $\mathcal{O}_n$. On can prove this by induction on

$$o(P) := \sup\{k; \quad P \in \mathfrak{m}_n^k\}.$$

An integral domain is UFD if and only if every element is the product of finitely many indecomposable elements and if each indecomposable element is a prime.

5.3 Theorem. *The ring $\mathcal{O}_n$ is a UFD-domain.* UFD

Proof. We have to show that every indecomposable element $P \in \mathcal{O}_n$ is a prime. The proof is given by induction on n. By the preparation theorem one can assume that $P \in \mathcal{O}_{n-1}[z_m]$ is a Weierstrass polynomial.

We claim that P is indecomposable in $\mathcal{O}_{n-1}[z_n]$. We argue indirect and consider a non-trivial decomposition $P = AB$; $A, B \in \mathcal{O}_{n-1}[z_n]$. Then $z_n^d = P(0, z_n) = A(0, z_n)B(0, z_n)$. But then we can assume $A(0, z_n) = z_n^\alpha$, $B(0, z_n) = z_n^\beta$ where $\alpha + \beta = d$, $\alpha > 0$, $\beta > 0$. But then $A(0) = B(0) = 0$ and $P = AB$ is a non-trivial decomposition in $\mathcal{O}_n$.

By induction assumption $\mathcal{O}_{n-1}$ is UFD. The theorem of Gauss implies that $\mathcal{O}_{n-1}[z_n]$ is UFD. Hence P is a prime element in $\mathcal{O}_{n-1}[z_n]$. By Theorem 5.1 then P is prime in $\mathcal{O}_n$. □

Recall that a ring R is called noetherian if each ideal $\mathfrak{a}$ is finitely generated, $\mathfrak{a} = Ra_1 + \cdots + Ra_n$. Then any sub-module of a finitely generated module is finitely generated.

5.4 Theorem. *The ring $\mathcal{O}_n$ is noetherian.* NOET

Proof. Again we argue by induction on n. Let $\mathfrak{a} \subset \mathcal{O}_n$ be an ideal. We want to show that it is finitely generated. We can assume that $\mathfrak{a}$ is non-zero. Take any non-zero element $P \in \mathfrak{a}$. By the preparation theorem we can assume that P is a Weierstrass polynomial. It is sufficient to show that the image of $\mathfrak{a}$ in $\mathcal{O}_n/(P)$ is finitely generated. This is the case, since $\mathcal{O}_{n-1}$ is noetherian by induction hypothesis and then $\mathcal{O}_{n-1}[z_n]$ is noetherian by Hilbert's basis theorem. □

6. Hypersurfaces

Under a hypersurface we understand here the set of zeros of a non-zero analytic function on a domain $D \subset \mathbb{C}^n$. For their study we will make use of the theory of the discriminant. It can be used to characterize square free elements of a polynomial ring over factorial rings.

An element a of an integral domain is called square free if $a = bc^2$ implies that c is a unit. Primes are square free. Notice our convention: units are square free but they are no primes.

There is a close relation between the question of divisibility of power series and their zeros.

6.1 Proposition. *Let $P, Q \in \mathcal{O}_n$, $Q \neq 0$, be two power series. We assume* divI
that there exists a neighborhood of the origin in which both series converge and such that every zero of Q in this neighborhood is also a zero of P. Then there exist a natural number n such that P^n is divisible by Q,

$$P^n = AQ, \quad A \in \mathcal{O}_n.$$

If Q is square free, one can take $n = 1$, i.e. then P is divisible by Q.

Proof. Because of the existence of the prime decomposition, we can assume that Q is square free. By our standard procedure, we can assume that Q is a Weierstrass polynomial. From the division theorem we obtain

$$P = AQ + B, \quad B \in \mathcal{O}_{n-1}[z_n], \quad \deg_{z_n} B < d.$$

By assumption we know in a small neigborhood of the origin

$$Q(z) = 0 \Longrightarrow B(z) = 0.$$

Now we make use of the fact that Q is a square free element of $\mathcal{O}_n$. We know then that Q is square free in $\mathcal{O}_{n-1}[z_n]$. Hence the discriminant of Q is different from 0. Now we consider the polynomial

$$Q_a(z) = Q(a_1, \dots, a_{n-1})(z) \in \mathbb{C}[z]$$

for fixed sufficiently small $a = (a_1, \dots, a_{n-1})$. The discriminant d_{Q_a} can be obtained from d_Q by specializing $z_1 = a_1, \dots, z_{n-1} = a_{n-1}$. This follows for example from the existence of the universal polynomial Δ_n. Therefore there exists a dense subset M of a small neighborhood of 0 such that d_{Q_a} is different from 0 for $a \in M$. This means that Q_a is a square free element from $\mathbb{C}[z]$. Since $\mathbb{C}$ is algebraically closed, this means nothing else that Q_a has no multiple zeros. Hence Q_a has d pairwise distinct zeros (for $a \in M$). As we pointed out several times the d zeros are arbitrarily small if a is sufficiently small. We obtain that $z \mapsto B(a, z)$ has d pairwise distinct zeros if a lies in a dense subset of a sufficiently small neighborhood of the origin. It follows that B_a vanishes for these a. This shows $B = 0$. □

6.2 Definition. *A holomorphic function* reD

$$f : D \longrightarrow \mathbb{C} \quad (D \subset \mathbb{C}^n \text{ open})$$

is called ***reduced*** *at a point $a \in D$ if the power series of f at a is a square free element of $\mathbb{C}\{z_1 - a_1, \dots z_n - a_n\}$.*

(The notation $\mathbb{C}\{z_1 - a_1, \dots z_n - a_n\}$ has been introduced for the same time. This ring is just the usual ring of power series. The notation just indicates that the elements now are consider as functions around a. It is the same to consider $f(z - a)$ and then to take the power series expansion around 0.) If a is a non-zero element of an UFC-domain one can define its "square free part" b. This is a square free element which divides a and such that a divides a suitable power of a. The square free part is determined up to a unit of R. The definition of b is obvious fron the prime decomposition of a. For example the square free part of $z_1^2 z_2^3$ is $z_1 z_2$. If we want to investigate local properties of a hypersurface A around a given point $a \in A$ we can assume that the defining equation $f(z) = 0$ in a small neighborhood of a is given by a function f which is reduced at a.

6.3 Proposition. *Let f be a holomorphic function on an open set $U \subset \mathbb{C}^n$.* redO
The set of all points $a \in U$ in which f is reduced is an open set.

For the prove of 6.3 we need the following two remarks:

6.4 Remark. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be a normalized polynomial, which is square* squFr
free in the ring $\mathcal{O}_{n-1}[z_n]$. Then P is square free in the bigger ring $\mathcal{O}_n$.

We already used this result for Weierstrass polynomials where it is a consequence of 5.2. For the general case, we use the preparation theorem

$$P = UQ, \quad U \text{ unit in } \mathcal{O}_n, \quad Q \text{ Weierstrass polynomial.}$$

We know that U is a polynomial in z_n (???). This implies that Q is square free in the ring $\mathcal{O}_{n-1}[z_n]$ and therefore in $\mathcal{O}_n$. But U is a unit in $\mathcal{O}_n$. Therefore P is square free in $\mathcal{O}_n$. □

The same argument shows:

6.5 Remark. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be a normalized polynomial which is prime* UoE
in the ring $\mathcal{O}_{n-1}[z_n]$. Then P either is a unit in $\mathcal{O}_n$ or it is a prime in $\mathcal{O}_n$

Proof of 6.3. Let $a \in D$ be a point in which f is reduced. We can assume $a = 0$ and that the power series $P = f_0$ is a Weierstrass polynomial. We consider the power series of f in all points b in a small polydisk around 0.

$$f_b \in \mathbb{C}\{z_1 - b_1, \dots, z_n - b_n\}$$

This power series is still a normalized polynomial in $\mathbb{C}\{z_1 - b_1, \dots, z_{n-1} - b_{n-1}\}[z_n - b_n]$ but usually not a Weierstrass polynomial. By assumption P is square free (in $\mathcal{O}_n$ but then also in $\mathcal{O}_{n-1}[z_n]$ since it is a Weierstrass polynomial). Therefore the discriminant does not vanish. This (and the universal formula for the discriminant) shows that the discriminant of P_b does dot vanish if b is close to 0. This means that P_b is square free in the polynomial ring and square free in $\mathcal{O}_a$ by 6.4. □

7. Analytic Algebras

All rings are assumed to be commutative and with unit element. Homomorphisms of rings are assumed to map the unit element into the unit element.

Recall that an algebra over a ring A by definition is a ring B together with a distinguished ring homomorphism $\varphi : A \to B$. This ring homomorphism can be used to define on B a structure as A-module, namely

$$ab := \varphi(a)b \quad (a \in A,\ b \in B).$$

Let B, B' be two algebras. A ring homomorphism $B \to B'$ is called an algebra homomorphism if it is A-linear. We will consider $\mathbb{C}$-algebras A. If A is different form zero then the structure homomorphism $\mathbb{C} \to A$ is injective. Usually identify complex numbers with their image in A. So each non-zero $\mathbb{C}$-algebra contains the field of complex numbers as sub-field.

7.1 Definition. *An analytic algebra A is a $\mathbb{C}$-algebra which is different from the zero algebra and such there exist an n and a surjective algebra homomorphism $\mathcal{O}_n \to A$.* AnAlgo

A ring R is called a *local ring* if it is not the zero ring and if the set of non-units is an ideal. This ideal is then a maximal ideal and moreover, it is the only maximal ideal. We denote this ideal by $\mathfrak{m}(R)$. Hence $R - \mathfrak{m}(R)$ is the set of units of R. The algebra $\mathcal{O}_n$ is a local ring. The maximal ideal $\mathfrak{m}_n$ consists of all P with $P(0) = 0$.

Let A be a local ring and $\mathfrak{a} \subset \mathfrak{m}$ be a proper ideal. Then $A/\mathfrak{a}$ is a local ring too and the maximal ideal of $A/\mathfrak{a}$ is the image of $\mathfrak{m}$. This shows the following. If A is a local ring and $A \to B$ is a surjective homomorphism onto a non-zero ring, then B is also a local ring and the maximal ideal of A is mapped onto the maximal ideal of B. In general, a homomorphism $A \to B$ between local rings is called local if it maps the maximal ideal of A into the maximal ideal of B. The natural map $A/\mathfrak{m}(A) \to B/\mathfrak{m}(B)$ is an isomorphism.

In particular, analytic algebras are local rings and the homomorphism $\mathcal{O}_n \to A$ in Definition 7.1 is a local homomorphism. The natural maps

$$\mathbb{C} \longrightarrow \mathcal{O}_n/\mathfrak{m}_n \longrightarrow A/\mathfrak{m}(A)$$

are isomorphisms. For $a \in A$ we denote by $a(0)$ the its image in $A/\mathfrak{m}(A)$. Recall that we identify this with a complex number. The maximal ideal of A consists of all $a \in A$ such that $a(0) = 0$.

We notice that an arbitrary algebra homomorphism $f : A \to B$ between analytic analytic algebras is local. This follows from the commutative diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

Our next task is to describe the homomorphisms $\mathcal{O}_m \to \mathcal{O}_n$. They are defined by means of elements $P_1, \dots, P_n \in \mathcal{O}_m$ that are contained in the maximal ideal. If $P(z_1, \dots, z_n)$ is an element of $\mathcal{O}_n$, one can substitute the variables z_i by the power series P_i. This substitution gives a homomorphism

$$\mathcal{O}_n \longrightarrow \mathcal{O}_m, \quad P \longmapsto P(P_1, \dots, P_n).$$

7.2 Lemma. *Each algebra homomorphism $\mathcal{O}_n \to \mathcal{O}_m$ is a substitution homomorphism.* SubsHom

Proof. Let $\varphi : \mathcal{O}_n \to \mathcal{O}_m$ an algebra homomorphism. Since it is local, the elements $P_i := \varphi(z_i)$ are contained in the maximal ideal. Hence one can consider the substitution homomorphism ψ defined by them. We claim $\varphi = \psi$. At the moment we only know that φ and ψ agree on $\mathbb{C}[z_1, \dots, z_n]$. Let $P = \sum_\nu a_\nu z^\nu \in \mathcal{O}_n$. We claim $\varphi(P) = \psi(Q)$. For this we decompose for a natural number k

$$P = P_k + Q_k, \quad P_k = \sum_{\nu_1 + \cdots + \nu_n \le k} a_\nu z^\nu.$$

Then Q_k is contained in the k-the power $\mathfrak{m}^k$ of the maximal ideal. (Obviously $\mathfrak{m}^k$ is generated by all z^ν where $\nu_1 + \cdots + \nu_n \ge k$.) We get

$$\varphi(P) - \psi(P) = \varphi(Q_k) - \psi(P_k) \in \mathfrak{m}^k.$$

This is true for all k. But the intersection of all $\mathfrak{m}^k$ is zero. This proves 7.4. $\square$

We have to generalize 7.4 to homomorphisms $\varphi : A \to B$ of arbitrary analytic algebras A, B. There is one problem. Let $\mathfrak{m}(B)$ be the maximal ideal of B. It is not obvious that the intersection of all powers of $\mathfrak{m}(B)$ is zero. But it is true by general commutative algebra (Krull's intersection theorem).

7.3 Lemma. *Let $A \to B$ a homomorphism of analytic algebras. Assume* SubBel
that surjective algebra homomorphism $\mathcal{O}_n \to A$ and $\mathcal{O}_m \to B$ are given. There exists a (substitution) homomorphism $\mathcal{O}_n \to \mathcal{O}_m$ such the the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathcal{O}_n & \longrightarrow & \mathcal{O}_m \end{array}$$

commutes.

The proof should be clear. The variable $z_i \in \mathcal{O}_n$ is mapped to an element of A then of B. Consider in $\mathcal{O}_m$ an inverse image P_m. These elements define a substitution homomorphism $\mathcal{O}_n \to \mathcal{O}_m$. From Krull's intersection theorem follows that the diagram commutes. □

From 7.3 follows:

7.4 Lemma. *Let $f_1, \dots, f_m$ be elements of the maximal ideal of an analytic* SubsHom
algebra A. There is a unique homomorphism $\mathbb{C}\{z_1, \dots, z_n\} \to A$ such that $z_i \mapsto f_i$.

We denote the image by $\mathbb{C}\{f_1, \dots, f_n\}$ and call it the analytic algebra generated by $f_1, \dots, f_n$. We want to derive a criterion that $\mathbb{C}\{f_1, \dots, f_n\} = A$. A necessary condition is that $f_1, \dots, f_n$ generate the maximal ideal. Actually it is also sufficient:

7.5 Lemma. *Let $f_1, \dots, f_n$ be elements of the maximal ideal of an analytic* MaxErz
algebra A. Then the following conditions are equivalent:

a) *They generate the maximal ideal.*
b) $A = \mathbb{C}\{f_1, \dots, f_n\}$.

It is easy to reduce this to the ring $A = \mathbb{C}\{z_1, \dots, z_n\}$. Let $P_1, \dots, P_m$ be generators of the maximal ideal. We can write

$$z_i = \sum_{ij} A_{ij} P_j.$$

Taking derivatives and evaluating at 0 we get: The rank of the Jacobian matrix of $P = (P_1, \dots, P_m)$ is n. We can find an system consisting of n elements, say $P_1, \dots, P_n$, such that the Jacobian is invertible. Now one can apply the theorem of invertible functions. □

8. Noether Normalization

We consider ideals $\mathfrak{a} \in \mathcal{O}_n$ and their intersection $\mathfrak{b} := \mathfrak{a} \cap \mathcal{O}_{n-1}$ with $\mathcal{O}_{n-1}$.

8.1 Lemma. *Let $\mathfrak{a} \subset \mathcal{O}_n$ ba a z_n-general ideal. Then $\mathcal{O}_n/\mathfrak{a}$ is an $\mathcal{O}_{n-1}/\mathfrak{b}$-module of finite type with respect to the natural inclusion* natE

$$\mathcal{O}_{n-1}/\mathfrak{b} \hookrightarrow \mathcal{O}_n/\mathfrak{a} \quad (\mathfrak{b} = \mathcal{O}_{n-1} \cap \mathfrak{a}).$$

Additional remark. *If $\mathfrak{a}$ contains a Weierstrass polynomial of degree d, then $\mathcal{O}_n/\mathfrak{a}$ is generated as $\mathcal{O}_{n-1}/\mathfrak{b}$-module by the images of the powers*

$$1, z_n, \dots, z_n^{d-1}.$$

The proof is an immediate consequence of the division theorem. □

8.2 Noether normalization theorem. *Let A be an analytic algebra. There exists an injective homomorphism of analytic algebras* noetH

$$\mathbb{C}\{z_1, \dots, z_d\} \hookrightarrow A \quad (d \text{ suitable})$$

such that A is a module of finite type over $\mathbb{C}\{z_1, \dots, z_d\}$. The number d is unique (it is the Krull dimension).

Proof. The existence of such an embedding follows from 8.1 by repeated application. One makes use of the following simple fact. If $A \subset B$ and $B \subset C$ are finite then $A \subset C$ is finite too. The essential point is the uniqueness of d. It follows from the characterization as Krull dimension. □

The Noether normalization admits a refinement if the starting ideal $\mathfrak{a}$ is a prime ideal. Recall that an ideal $\mathfrak{p} \subset R$ in a ring R is called a prime ideal if the factor ring is an integral domain.

So let $\mathfrak{P} \subset \mathcal{O}_n$ be a prime ideal and $\mathfrak{p} = \mathcal{O}_{n-1} \cap \mathfrak{P}$. We have an injective homomorphism

$$\mathcal{O}_{n-1}/\mathfrak{p} \hookrightarrow \mathcal{O}_n/\mathfrak{P}$$

which shows that $\mathfrak{p}$ is also a prime ideal. Let K resp. L be the field of quotients of $\mathcal{O}_{n-1}/\mathfrak{p}$ resp. $\mathcal{O}_n/\mathfrak{p}$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{n-1}/\mathfrak{p} & \hookrightarrow & \mathcal{O}_n/\mathfrak{P} \\ \cap & & \cap \\ K & \hookrightarrow & L\,. \end{array}$$

We distinguish two cases which behave completely different:

First alternative. $\mathfrak{P}$ is a principal ideal (i.e. generated by one element).
Second alternative. This is not the case.

8.3 Theorem, the first alternative. *Let* $\mathfrak{P} \subset \mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$ *be a* erstA
z_n*-general prime ideal. Assume*

$$\mathfrak{P} \cap \mathcal{O}_{n-1} = \{0\}.$$

Then $\mathfrak{P}$ *is a principal ideal.*

Proof. Let $Q \in \mathfrak{P}$ be a z_n-general element. One of the prime divisors of Q must be contained in $\mathfrak{P}$. It is z_n-general too. Hence we can assume that Q is prime. We will show that Q generates $\mathfrak{P}$. By the preparation theorem we can assume that Q is a Weierstrass polynomial. Let $P \in \mathfrak{P}$ be an arbitrary element. From 8.1 applied to the ideal $\mathfrak{a} = (Q)$ we get an equation

$$P^k + A_{k-1}P^{k-1} + \ldots + A_0 \equiv \operatorname{mod}(Q), \quad A_i \in \mathcal{O}_{n-1}\ (0 \le i < k).$$

The equation shows that A_0 is contained in $\mathfrak{P}$, hence in $\mathfrak{P} \cap \mathcal{O}_{n-1}$. By assumption this ideal is 0 and we obtain $A_0 = 0$. We see

$$P \cdot (P^{k-1} + \ldots + A_1) \equiv 0 \operatorname{mod} Q.$$

But (Q) is a prime ideal and we get

$$\text{either} \qquad P \in (Q) \qquad \text{or} \qquad P^{k-1} + \ldots + A_1 \equiv \operatorname{mod}(Q).$$

Repeated application of this argument shows $P \in (Q)$ in any case. □

8.4 Theorem, the second alternative. *Let* gleiQ

$$\mathfrak{P} \subset \mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$$

be a prime ideal which is not a principal ideal. After a suitable linear transformation of the coordiantes we can obtain:

a) $\mathfrak{P}$ *is* z_n*-general.*
b) *The two integral domains*

$$\mathcal{O}_{n-1}/\mathfrak{p} \hookrightarrow \mathcal{O}_n/\mathfrak{P} \quad (\mathfrak{p} = \mathcal{O}_{n-1} \cap \mathfrak{P})$$

have the same field of fractions.

"After a suitable linear transformation of the coordinates" means that we allow to replace $\mathfrak{P}$ by its image under the automorphism

$$\mathcal{O}_n \to \mathcal{O}_n, \quad P(z) \mapsto P(A^{-1}z),$$

for suitable $A \in \mathrm{GL}(n, \mathbb{C})$.

Proof of Theorem 8.4. We may assume that $\mathfrak{P}$ is already z_n-general. From 8.3 we know that

$$\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_{n-1}$$

is different from 0. After a linear transformation of the variables $(z_1, \dots, z_{n-1})$ we can assume that $\mathfrak{p}$ is z_{n-1}-general. The ideal $\mathfrak{P}$ remains z_n-general. Now we consider

$$\mathfrak{q} = \mathfrak{p} \cap \mathcal{O}_{n-2} = \mathfrak{p} \cap \mathcal{O}_{n-2}.$$

The extension

$$\mathcal{O}_{n-2}/\mathfrak{q} \subset \mathcal{O}_n/\mathfrak{P}$$

is of finite type. We denote the fields of fractions by $K \subset L$. This is a finite algebraic extension and we have $L = K[\bar{z}_{n-1}, \bar{z}_n]$. The bar indicates that we have to take cosets mod P. From elementary algebra we will use

Theorem of primitive element. *Let $K \subset L$ be a finite algebraic extension of fields of characteristic zero, which is generated by two elements, $L = K[a, b]$. Then for all $x \in K$ but a finite number of exceptions one has*

$$L = K[a + xb].$$

As a consequence every finite algebraic extension of fields of characteristic zero is generated by one element. This is the usual formulation of this theorem. The above variant is contained in the standard proofs.

We obtain that

$$L = K[\bar{z}_{n-1} + a\bar{z}_n].$$

for almost all $a \in \mathbb{C}$. We consider now the following (invertible) linear transformation of variables,

$$w_{n-1} = z_{n-1} + az_n, \qquad w_j = z_j \text{ for } j \neq n-1.$$

We have to take care that $\mathfrak{P}$ remains general in the new coordinates, which now means w_n-general. This possible because we have infinitely many possibilities for a.

Thus we have proved that we can assume without loss of generality $L = K[\bar{z}_{n-1}]$. But then the quotient fields of $\mathcal{O}_{n-1}/\mathfrak{p}$ and $\mathcal{O}_n/\mathfrak{P}$ agree. $\square$

9. Geometric Realization of Analytic Ideals

We will have to consider systems of ideals $\mathfrak{a}_a \subset \mathcal{O}_{U,a}$.

9.1 Definition. *Let $U \subset \mathbb{C}^n$ be an open subset. A system $\mathfrak{a} = (\mathfrak{a}_a)_{a\in U}$ of ideals $\mathfrak{a}_a$ in $\mathcal{O}_{U,a}$ is called* ***finitely generated*** *if there exist finitely many holomorphic functions $f_1, \dots, f_m$ on U such that* Dfgs

$$\mathfrak{a}_a = ([f_1]_a, \dots, [f_m]_a).$$

We call the functions $f_1, \dots, f_m$ a system of generators. This system is not uniquely determined. For example, $g_1, \dots, g_k$ defines the same system of they generate the same ideal in $\mathcal{O}(U)$. We leave the following statement as exercise to the reader. (We will not make use of it.)

The tuples $f_1, \dots, f_m$ and $g_1, \dots, g_k$ define the same system if and only if there exists an open covering $U = \bigcup U_i$ such that the restriction to U_i generate the same ideal in $\mathcal{O}(U_i)$.

9.2 Definition. *Let $(U, \mathfrak{a})$ be a finitely generated system. We consider* Daas

$$X := \{a \in U; \quad \mathfrak{a}_a \neq \mathcal{O}_{U,a}\}$$

and call it the ***associated analytic set****.*

In the notation of Definition 9.1 we have

$$X = \{a \in U; \quad f_1(a) = \cdots f_m(a) = 0\}.$$

So our associated analytic sets are zero sets of finitely many holomorphic functions.

9.3 Definition. *A holomorphic map $f : (U, \mathfrak{a}) \to (V, \mathfrak{b})$ of finitely generated systems is a holomorphic map $f : U \to V$ with the following condition:* Dhmf

$$f_a^*(\mathfrak{b}_{f(a)}) \subset \mathfrak{a}_a.$$

The condition formulated in the Definition means that we have an induced homomorphism of analytic algebras

$$f_a^* : \mathcal{O}_{V,f(a)}/\mathfrak{b}_{f(a)} \longrightarrow \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

9.4 Remark. *Let $(U, \mathfrak{a})$, $(V, \mathfrak{b})$ be two finitely generated systems and X, Y the associated analytic sets and let $f : (U, \mathfrak{a}) \to (V, \mathfrak{b})$ be an analytic map. Then $f(X) \subset Y$.* RfXY

Hence an analytic map f of analytic systems induces a map

$$f : X \longrightarrow Y$$

of the associated analytic sets. These maps are clearly continuous.

9.5 Remark. *Let* $\mathfrak{a}_0 \subset \mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$ *be an ideal. Then there exists a finitely generated system* $(U, \mathfrak{a})$ *on some open neighborhood* U *of the origin such that extends* $\mathfrak{a}_0$*.* Raan

Proof. The ring $\mathcal{O}_n$ being noetherian we can choose a finite system of generators $\mathfrak{a} = (P_1, \dots, P_m)$. The generators converge in a common open neighborhood U around 0. They can be considered as holomorphic functions there and for each point in U we can consider the ideal generated by their power series expansions in this point. □

We call this system $(U, \mathfrak{a})$ a *geometric realization* of $\mathfrak{a}$. It is clear that two geometric realizations agree in a small neighborhood of the origin. This means that for all local questions around the origin the geometric realization behaves as if it were unique.

The technique of the last section was to consider the intersection $\mathfrak{b}_0 = \mathfrak{a}_0 \cap \mathcal{O}_{n-1}$. Let $(U, \mathfrak{a})$ resp. $(V, \mathfrak{b})$ be geometric realizations of $\mathfrak{a}$ resp. $\mathfrak{b}$. We can assume that

$$U = V \times W, \quad 0 \in W \subset \mathbb{C}.$$

We consider the projection (cancelation of the last variable)

$$f : U \to V, \quad (z_1, \dots, z_n) \longmapsto (z_1, \dots, z_{n-1}).$$

Generators of $\mathfrak{b}$ can be expressed by means of generators of $\mathfrak{a}$. If we replace V, W by suitable smaller open neighborhoods of the origin, we get that the projection defines a holomorphic map

$$f : (U, \mathfrak{a}) \longrightarrow (V, \mathfrak{b}).$$

We call this map a *geometric realization* of the pair $(\mathfrak{a}, \mathfrak{b} = \mathfrak{a} \cap \mathcal{O}_{n-1})$. Again this realization is uniquely determined in an obvious local sense around 0.

In particular, we get a continuous map

$$f : X \longrightarrow Y$$

between the associated analytic sets.

9.6 Definition. *A continuous map* $f : X \to Y$ *between locally compact spaces is called* ***finite*** *if it is proper and if its fibres* $f^{-1}(b)$, $b \in Y$ *are finite.* Dcmf

Recall that proper means that inverse images of compact sets are compact. Here is an example of a finite map.

9.7 Lemma. *Let* $V \subset \mathbb{C}^{n-1}$ *open and let* $P \in \mathcal{O}(V)[z_n]$ *be a normalized polynomial. Let* $X \subset V \times \mathbb{C}$ *be the zero set of* P*. The projection* Lpfm

$$X \longrightarrow V$$

is a finite map.

An ideal $\mathfrak{a} \subset \mathcal{O}_n$ is called z_n-general if it contains a z_n-general element. For the theory of ideals in $\mathcal{O}_n$ it is sufficient to restrict to z_n-general ideals, since every non-zero ideal can be transformed into a z_n-general one by means of linear change of coordinates.

9.8 Remark. *Let $\mathfrak{a}_0$ be a z_n-general ideal in $\mathcal{O}_n$ and $\mathfrak{b}_0 = \mathfrak{a} \cap \mathcal{O}_{n-1}$. There exists a geometric realization* einP

$$f : (U, \mathfrak{a}) \longrightarrow (V, \mathfrak{b}), \qquad , \quad U = V \times W,$$

of $(\mathfrak{a}, \mathfrak{b})$ such that the map

$$X \longrightarrow V$$

is finite. The sets U, V can be taken to be arbitrarily small in the following sense. If $0 \in V' \subset \mathbb{C}^{n-1}$, $0 \in W' \subset W$ are open neighborhoods, then one can get $V \subset V'$, $W \subset W'$. In addition one can reach that the point $0 \in V$ has only one inverse image in X (namely 0).

Proof. There exists a Weierstrass polynomial $P \in \mathfrak{a}$. Close to the origin the inverse image is contained in the set of zeros of $P(0, \dots, 0, z_n) = 0$. But $P(0, \dots, 0, z_n) = z_n^d$ implies that 0 is the only solution. The rest comes from the frequently used argument of "continuity of zeros" of a Weierstrass polynomial. $\square$

We want to apply this in the case that $\mathfrak{P} \subset \mathcal{O}_n$ is a prime ideal of the second alternative, i.e. it is z_n-general and $\mathcal{O}_n/\mathfrak{P}$ and $\mathcal{O}_{n-1}/\mathfrak{p}$ ($\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_{n-1}$) have the same field of fractions.

9.9 Proposition. *Let $\mathfrak{P} \subset \mathcal{O}_n$ be a z_n-general prime ideal and $\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_{n-1}$. We assume that the fields of fractions of $\mathcal{O}_n/\mathfrak{P}$ and $\mathcal{O}_{n-1}/\mathfrak{p}$ agree (second alternative). There exists a geometric realization* bimeR

$$f : (U, (\mathfrak{P}_a)_{a \in U}) \longrightarrow (V, (\mathfrak{p}_b)_{b \in V})$$

where

$$U = V \times W, \quad V \subset \mathbb{C}^{n-1}, \ W \subset \mathbb{C},$$

(and f is the natural projection) such the following holds:

Let $f : X \to Y$ be the corresponding map of the associated analytic sets. There exists a power series $A \in \mathcal{O}_{n-1}$ which is not contained in $\mathfrak{p}$ and which converges in V. Let be

$$S := \left\{ z \in Y; \quad A(z) = 0 \right\} \quad \text{and} \quad T := f^{-1}(S).$$

The restriction

$$f_0 : X - T \longrightarrow Y - S$$

of f is topological.

Proof. We make use of the fact that the two fields of fractions agree. Expressing the coset of z_n as a fraction we obtain:

There exist power series $A, B \in \mathcal{O}_{n-1}$ with the properties

$$A \notin \mathfrak{p} \qquad Az_n - B \in \mathfrak{P}.$$

We can assume that that A and B both converge in V. In particular the sets S and T are defined now. All points $z \in X$ satisfy

$$z_n A(z_1, \dots, z_{n-1}) = B(z_1, \dots, z_{n-1}).$$

This means

$$z_n = \frac{B(z_1, \dots, z_{n-1})}{A(z_1, \dots, z_{n-1})}$$

if z is not contained in T. So we have proved the injectivitiy of the map $f_0 : X - T \to Y - S$.

It remains to show that f_0 is surjective. We define

$$g(z_1, \dots, z_{n-1}) := (z_1, \dots, z_n), \quad z_n := \frac{B(z_1, \dots, z_{n-1})}{A(z_1, \dots, z_{n-1})}.$$

What we need is $g(z) \in X$ for $z \in Y - S$. In a first step we show:

9.10 Lemma. *Let $P \in \mathfrak{P} \cap \mathcal{O}_{n-1}[z_n]$. There exists a r', $0 < r' \leq r$, such* teL
that

$$P(g(z)) = 0 \quad \text{for all} \quad z \in Y - S, \ ||z|| < r'.$$

($|| \cdot ||$ denotes the maximum norm.)

Proof. We choose r' small enough such that the coefficients of P converge in the polydisk with multiradius $(r', \dots, r')$. Let d be the degree of P. Then $A^d P$ can be written as polynomial in Az_n with coefficients from $\mathcal{O}_{n-1}$. By means of $Az_n = (Az_n - B) + B$ we can rearrange P as polynomial in $Az_n - B$,

$$P = \sum_{j=0}^{d} (Az_n - B)^d P_j \quad (P_j \in \mathcal{O}_{n-1}).$$

We want to show $P(g(z)) = 0$ which is equivalent to $P_0(z) = 0$. But this is clear because $P_0 \in \mathfrak{P} \cap \mathcal{O}_{n-1} = \mathfrak{p}$. This completes the proof of the Lemma. □

We continue the proof of Proposition 9.9 and claim:

There exists r', $0 < r' \leq r$, such that

$$|z_n| < \varepsilon \quad \text{for} \quad ||(z_1, \dots, z_{n-1})|| < r'.$$

One applies the Lemma 9.10 to a Weierstrass polynomial Q contained in $\mathfrak{P}$ and uses the standard argument of "continuity of roots".

The set X can be defined by a finite number of equations $P_1(z) = \cdots = P_m(z)$, $P_j \in \mathfrak{P}$, which converge in the polydisk of multiradius $(r, \ldots, r, \varepsilon)$. By means of the division theorem ($P_j = A_j Q + B_j$) and the above lemma 9.10 we obtain $P_j(g(z)) = 0$ and hence $g(z) \in X$ for $||z|| < r'$ and suitable $r' \leq r$. If we replace Y resp. X by their intersections with the polydisks of multiradius $(r', \ldots, r')$ resp. $(r', \ldots, r', \varepsilon)$ we obtain that f_0 is surjective and then that f_0 is bijective. The above formula for z_n shows that the inverse of f_0^{-1} is continuous. □

Lemma 9.9 should be interpreted as a result which states that the realization $X \to Y$ in case of the second alternative is close to a biholomorphic map. One could say that f is *bimeromorphic*. But there is a big problem up to now. In principle it could be that S equals the whole Y. The *Hilbert-Rückert Nullstellensatz* will show that this is not the case. This nullstellensatz will be the goal of the next section.

Appendix to Sect. 9. Continuity of roots

We recall a basic fact about proper maps $f : X \to Y$ between locally compact Hausdorff spaces. Recall that proper means that inverse images of compact sets are compact. Proper maps have the following two basic properties. They can be found in standard text books on topology.

9.11 Lemma. *Let $f : X \to Y$ be proper.* Lpmp

1) *The images of closed subsets are closed.*
2) *Let b be a point of Y and Let $f^{-1}(b) \subset U \subset X$ be an open neighborhood of the fibre over b. Then there exists a neighborhood $b \subset V \subset V$ such that its inverse image is contained in U, i.e. $f^{-1}(V) \subset U$.*

Here is an example of a proper map.

9.12 Lemma. *The map* LefC

$$\mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad a \longmapsto (E_1(a), \ldots, E_n(a)),$$

given by the elementary functions, is continuous and proper.

Proof. We make use of the Vieta formula

$$(X - a_1) \cdots (X - a_n) = X^n + \sum_{\nu=1}^{n} E_\nu(a) X^{n-\nu}.$$

We have to show that a set of a-s is bounded if the elementary functions E_ν are bounded on this set. This follows from the estimate

$$|a_m| \leq nC \quad \text{where} \quad C = \max\{1, |E_1(a)|, \ldots, |E_n(a)|\}.$$

This follows from the following simple computation.

$$a_m^n + \sum_{\nu=1}^{n} E_\nu(a) a_m^{n-\nu} = 0.$$

In the case $|a_m| \le 1$ it implies

$$|a_m|^n \le nC \quad \text{or} \quad |a_m|^n \le nC.$$

In the case $|a_m| > 1$ we get

$$|a_m|^n \le nC|a_m|^{n-1} \quad \text{or} \quad |a_m| \le nC.$$

Tho both cases are settled. □

There is another way ro express the lemma. Call two points $z, w \in \mathbb{C}^n$ equivalent if they agree up to the ordering. This means that there is a permutation $\sigma \in S_n$ in the group of permutations S_n such that $w_i = z_{\sigma(i)}$. We denote the equivalence class of z by $[z]$. The set of all equivalence classes is denoted by $\mathbb{C}^n/S_n$. There is a natural projection map

$$\mathbb{C}^n \longrightarrow \mathbb{C}^n/S_n, \quad z \longmapsto [z].$$

We equip $\mathbb{C}^n/S_n$ with the quotient topology. So a subset of $\mathbb{C}^n/S_n$ is open if and only if its inverse image in $\mathbb{C}^n$ is open. Then the projection map is continuous and open (the image of open subsets are open). It is also easy to check that $\mathbb{C}^n/S_n$ is locally compact and that the projection map is proper

Let $[z] \in \mathbb{C}^n/S_n$. The point $(E_1(z), ..., E_n(z)$ does not depend on the choice of tje representative z. Hence we get a natural map

$$\mathbb{C}^n/S_n \longrightarrow \mathbb{C}^n, \quad [z] \longmapsto (E_1(z), ..., E_n(z)).$$

9.13 Lemma. *The natural map* LSit

$$\mathbb{C}^n/S_n \longrightarrow \mathbb{C}^n, \quad [z] \longmapsto (E_1(z), ..., E_n(z)),$$

is topological.

Proof. This map is obviously bijective, continuous and proper. Hence images of closed subsets are closed. This means the inverse map has the property that inverse maps of closed subsets are closed. This implies continuity of the inverse map. □

The inverse map $\mathbb{C}^n \longrightarrow \mathbb{C}^n/S_n$ describes the map that associates to a normed polynomial of degree n its roots. Hence the Lemmas 9.12 and its corollary 9.13 can be considered as a precise mathematical statement of the *principle of continuity of roots*.

10. The Nullstellensatz

We associate to an ideal $\mathfrak{a} \subset \mathcal{O}_n$ an ideal $\mathfrak{A} \subset \mathcal{O}_n$. A power series $P \in \mathcal{O}_n$ belongs to $\mathfrak{A}$ if there exists a geometric realization $(U, \mathfrak{a})$ of $\mathfrak{a}$ with associated analytic set X P converges in U and such that P vanishes on X. The ideal $\mathfrak{A}$ is called the *vanishing ideal* associated to $\mathfrak{a}$. It is a proper ideal, i.e. contained in the maximal ideal $\mathfrak{m}_n$. It is clear that the $\mathfrak{a} \subset \mathfrak{A}$. We call $\mathfrak{A}$ also the *saturation* of $\mathfrak{a}$.

The Radical of an Ideal

Let R be a ring. The radical $\operatorname{rad}\mathfrak{a}$ of an ideal $\mathfrak{a}$ is the set of all elements $a \in R$ such that a suitable power a^n, $n \geq 1$ is contained in $\mathfrak{a}$. It is easy to prove that $\operatorname{rad}\mathfrak{a}$ is an ideal which contains $\mathfrak{a}$. Furthermore $\operatorname{rad}\operatorname{rad}\mathfrak{a}=\operatorname{rad}\mathfrak{a}$. An ideal is called a *radical ideal* is it coincides wit its radical. This e aquivalent with the property that R/a is a reduced ring, i.e. a ring which contains no nilpotent elements different form 0. Let R be a UFD-domain. A principal ideal Ra, $a \neq 0$ is a radical ideal if and only if a is square free. We are able to state and prove a fundamental result of local complex analysis:

10.1 The Hilbert-Rückert nullstellensatz. *The saturation $\mathfrak{A}$ of an ideal $\mathfrak{a} \subset \mathcal{O}_n$ is the radical of $\mathfrak{a}$,* RNS

$$\mathfrak{A} = \operatorname{rad}\mathfrak{a}.$$

Proof. We want to reduce the nullstellensatz to prime ideals $\mathfrak{a}$. Prime ideals are of course radical ideals. The easiest way to do this reduction is to use a little commutative algebra, namely:

Every proper radical ideal in a noetherian ring is the intersection of finitely many prime ideals.

We use this and write the radical of our given ideal as intersection of prime ideals:

$$\operatorname{rad}\mathfrak{a} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m.$$

The saturation $\mathfrak{A}$ of $\mathfrak{a}$ is contained in the intersection of the of the saturations of the prime ideals. If we assume the nullstellensatz for prime ideals we obtain

$$\mathfrak{A} \subset \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m = \operatorname{rad}\mathfrak{a}$$

This implies $\mathfrak{A} = \operatorname{rad}\mathfrak{a}$ because the converse inclusion is trivial.

Now we can assume that $\mathfrak{P} := \mathfrak{a}$ is a prime ideal. We have to distinguish the two alternatives:

First alternative. The ideal $\mathfrak{P}$ is principal, $\mathfrak{P} = (P)$. The element P is a prime element in $\mathcal{O}_n$. In this case the nullstellensatz is a consequence of the theory of hypersurfaces (6.1).

Second alternative. $\mathfrak{P}$ is not a principal ideal. Then we can assume that $\mathfrak{P}$ is z_n-general, that the extension

$$\mathcal{O}_{n-1}/\mathfrak{p} \hookrightarrow \mathcal{O}_n/\mathfrak{P} \qquad (\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_{n-1}).$$

is module-finite and that the two rings have the same field of fractions. We make use of the geometric realization as in Proposition 9.9.

$$f : (U, (\mathfrak{P}_a)_{a\in U}) \longrightarrow (V, (\mathfrak{p}_b)_{b\in V})$$

where

$$U = V \times W, \quad V \subset \mathbb{C}^{n-1},\ W \subset \mathbb{C},$$

(and f is the natural projection). Recall that it induces a diagram

$$\begin{array}{cccc} f: & X & \longrightarrow & Y \\ & \cup & & \cup \\ f_0: & X - T & \xrightarrow{\sim} & Y - S\ . \end{array}$$

We indicated already in the last section that in principle S could be the whole Y before the nullstellensatz is known. But now we are in a better situation. We can prove the nullstellensatz by induction on n and therefore assume:

The nullstellensatz is true for $\mathfrak{p}$.

From this we derive:

Let $P_0 \in \mathcal{O}_{n-1}$ be a power series which converges in a small polydisk V around 0 and vanishes on $(Y - S) \cap V$. Then P_0 is contained in $\mathfrak{p}$.

This is quite clear, because AP_0 (A as in 9.9) vanishes on $Y \cap V$. The nullstellensatz for $\mathfrak{p}$ gives $AP_0 \in \mathfrak{p}$ and get $P_0 \in \mathfrak{p}$ because $\mathfrak{p}$ is a prime ideal and A is not contained in $\mathfrak{p}$.

So in some sense the set S is negligible. The proof of the nullstellensatz now runs as follows. We take an element P from the saturation of $\mathfrak{P}$. The claim is $P \in \mathfrak{P}$. The idea is to use an integral equation

$$P^m + P_{m-1}P^{m-1} + \ldots + P_0 \in \mathfrak{P}, \quad P_i \in \mathcal{O}_{n-1}\ (0 \le i < m).$$

We take a minimal degree m. We distinguish two cases:

First case. P_0 is contained in $\mathfrak{p}$: Then

$$P \cdot (P^{m-1} + P_{m-1}P^{m-2} + \ldots + P_1) \in \mathrm{P}.$$

Because of the minimality of M the expression in the bracket is not contained in $\mathfrak{P}$. But $\mathfrak{P}$ is a prime ideal and we obtain $P \in \mathfrak{P}$ what we wanted to show.

Second case. P_0 is not contained in $\mathfrak{p}$: We know that P vanishes on X in a neighborhood of 0. We can assume that P vanishes on the whole X (use Remark 9.8). Using the bijection $X - T \to Y - S$ we obtain that P_0 vanishes on $Y - S$. But as we have seen this implies $P_0 \in \mathfrak{p}$ which is a contradiction. This completes the proof of the nullstellensatz. □

We want to introduce the notion "thin at" which reflects that the set S is negligible in Y in a certain sense.

10.2 Definition. *Let $Y \subset X \subset \mathbb{C}^n$ be analytic sets and $a \in Y$ a distinguished* thinA
point. We call Y ***thin at*** *a if the following is true:*

> *If f is an analytic function on a neighborhood $a \in U \subset \mathbb{C}^n$ which vanishes on $(X - Y) \cap U$ then f vanishes on X in a (possibly smaller) neighborhood of a.*

So the essential part of the proof of the nullstellensatz was to show:

10.3 Remark. *Let $\mathfrak{P} \subset \mathcal{O}_n$ be a prime ideal with geometric realization X. Let* thinL
$P \in \mathcal{O}_n$ be a power series which is not contained in $\mathfrak{P}$. Assume that P converges in a polydisk around 0 which contains X. Then $Y := \{z \in X;\ P(z) = 0\}$ is thin at 0.

Again we get an obvious problem. On should expect that the property "thin at a" extends to a full neighborhood of a and that Y is thin in the usual topological sense in X (in this neighborhood). At the moment we are not able to prove this. This needs the principle of *coherence* which will be our next goal. Before we have developed this basic tool we must (and can) be content with the notion "thin at". But the reader should have in mind that "thin at" is in reality the same as thin in a neighborhood.

11. Oka's Coherence Theorem

We introduced already the ring

$$\mathbb{C}\{z_1 - a_1, \dots, z_n - a_n\}$$

of power series. Every holomorphic function f on an open neighborhood of a has a power series expansion in this ring. (Instead of this one could consider the function $f(z - a)$ and take its power series expansion around 0.) We have a natural injection

$$\mathbb{C}\{z_1 - a_1, \dots, z_{n-1} - a_{n-1}\} \longrightarrow \mathbb{C}\{z_1 - a_1, \dots, z_n - a_n\}$$

and can define the ring

$$\mathbb{C}\{z_1 - a_1, \dots, z_{n-1} - a_{n-1}\}[z_n - a_n] \subset \mathbb{C}\{z_1 - a_1, \dots, z_n - a_n\}$$

in an obvious way. An element P of this ring is called a Weierstrass polynomial, if it is normalized as polynomial in $z_n - a_n$ and if it has the property $P(a_1, \dots, a_{n-1}, z_n - a_n) = (z_n - a_n)^d$, where d is the degree of P in the variable $z_n - a_n$.

Let f be a holomorphic function on an open subset $D \subset \mathbb{C}^n$. Let its power series expansion at some point $a \in U$ be a Weierstrass polynomial. Then the power series expansion at a different point $b \in U$ usually will be not a Weierstrass polynomial. But it is still a normalized polynomial! We give a simple example, $U = \mathbb{C}$. Then z is a Weierstrass polynomialat the origin. But at any other point we have $[f_a] = w + a$, where $w = z - a$. But this is not a Weierstrass polynomial.

Let m be a natural number. We are interested in $\mathcal{O}_{U,a}$-submodules of the free module $\mathcal{O}_{U,a}^m$. In the case $m = 1$ such a submodule is nothing else but an ideal and ideals are the modules in which we are mainly interested. For technical reasons it is important to allow arbitrary m. Every submodule of $\mathcal{O}_{U,a}^m$ is finitely generated because the ring of power series is noetherian.

We are not only interested in individual modules but in *systems of modules.* This means that we assume that for every $a \in U$ a submodule

$$\mathcal{M}_a \subset \mathcal{O}_{U,a}^m$$

is given. We denote this system usually by a single letter,

$$\mathcal{M} = (\mathcal{M}_a)_{a \in U}.$$

If V is an open subset of U, one defines in an obvious way the restricted system $\mathcal{M}|V := (\mathcal{M}_a)_{a \in V}$. We need a straight forward generalization of Definition 9.1.

11.1 Definition. *A system* endEr

$$\mathcal{M} = (\mathcal{M}_a)_{a \in U}, \quad \mathcal{M}_a \subset \mathcal{O}_{U,a}^m,$$

is called ***finitely generated****, if there exist finitely many vectors of holomorphic functions*

$$f^{(j)} \in \mathcal{O}(U)^m \quad \textit{for} \quad 1 \le j \le k,$$

such that the $\mathcal{O}_a$-module $\mathcal{M}_a$ is generated by the germs

$$(f^{(1)})_a, \dots, (f^{(k)})_a.$$

The germs are taken of course componentwise.

11.2 Definition. *The system $\mathcal{M} = (\mathcal{M}_a)_{a \in U}$ is called* ***coherent****, if it is locally finitely generated, which means that every point $a \in U$ admits an open neighborhood $a \in V \subset U$ such that $\mathcal{M}|V$ is finitely generated.* coH

Let p, q be natural numbers and let

$$F = \begin{pmatrix} F_{11} & \dots & F_{1p} \\ \vdots & & \vdots \\ F_{q1} & \dots & F_{qp} \end{pmatrix}$$

by a matrix of holomorphic functions on U. We can consider the $\mathcal{O}(U)$-linear map

$$F : \mathcal{O}(U)^p \longrightarrow \mathcal{O}(U)^q$$

which is defined by

$$Ff := g; \quad g_i := \sum_{j=1}^{p} F_{ij} f_j \quad (1 \le i \le q).$$

As the notation indicates we identify the matrix and the linear map. For every point $a \in U$ we can consider $F_a = ((F_{ik})_a)$ and the corresponding map

$$F_a : \mathcal{O}_{U,a}^p \to \mathcal{O}_{U,a}^q.$$

11.3 Oka's coherence theorem. *Let* OkC

$$F : \mathcal{O}(U)^p \to \mathcal{O}(U)^q \quad (U \subset \mathbb{C}^n \text{ open})$$

be an $\mathcal{O}(U)$–linear map. The system

$$\mathcal{M} = (\mathcal{M}_a)_{a \in U} \quad \mathcal{M}_a := \operatorname{kernel}(F_a)$$

is coherent.

The proof will be given in three steps:
First step, reduction to the case $q = 1$. This will be done by induction on q. So let's assume $q > 1$ and that the theorem is proved for $q - 1$ instead of q. Let $a_0 \in U$ be a distinguished point. We want to prove that $\mathcal{M}$ is finitely generated in a neighborhood of a. For this purpose we can replace U by a smaller neighborhood of a_0. We consider the two projections

$$\mathcal{O}(U)^q = \mathcal{O}(U)^{q-1} \times \mathcal{O}(U) \quad \begin{array}{l} \xrightarrow{\alpha} \mathcal{O}(U) \\ \xrightarrow{\beta} \mathcal{O}(U)^{q-1}. \end{array}$$

By the induction hypothesis, applied to

$$\alpha \circ F : \mathcal{O}(U)^p \to \mathcal{O}(U)^{q-1}$$

we can assume that there exist a finite system

$$A^{(1)}, \dots, A^{(m)} \in \mathcal{O}(U)^p,$$

such that the germs $A_a^{(1)}, \dots, A_a^{(m)}$ generated the kernel of $(\alpha \circ F)_a$ for each point $a \in U$. Now we consider the linear map

$$G : \mathcal{O}(U)^m \longrightarrow \mathcal{O}(U)^p, \quad (f_1, \dots, f_m) \longmapsto f_1 A^{(1)} + \dots + f_m A^{(m)},$$

and compose it with the projection β,

$$\beta \circ A : \mathcal{O}(U)^m \to \mathcal{O}(U).$$

We assumed that the case $q = 1$ is proved and can therefore assume that there exists a finite system

$$B^{(1)}, \dots, B^{(l)} \in \mathcal{O}(U)^m,$$

whose germs in an arbitrary point $a \in U$ generate $(\beta \circ A)_a$. It is easy to see that the germs of

$$C^{(i)} = G(B^{(i)}) \in \mathcal{O}(U)^p \quad (1 \le i \le m).$$

generate the kernel of our original F_a. Thus we have show:

If Oka's theorem is true for $q = 1$ in a given dimension n then it is true for all q in this dimension.

Second step. The proof of Oka's theorem rests on *Oka's Lemma*, which is a lemma for an individual ring of power series (not a system). Before we can formulate it, we need a notation:

$$\mathcal{O}_{n-1}[z_n : m] = \left\{ P \in \mathcal{O}_{n-1}[z_n]; \quad \deg_{z_n} P < m \right\}.$$

This is a free module over $\mathcal{O}_{n-1}$ with basis $1, z_n, \dots, z_n^{m-1}$,

$$\mathcal{O}_{n-1}[z_n : m] \cong \mathcal{O}_{n-1}^m.$$

11.4 Oka's Lemma. *Let* okaL

$$F : \mathcal{O}_n^p \to \mathcal{O}_n$$

be a $\mathcal{O}_n$-linear map and let K be its kernel.

Assumption. *The components of the matrix F are normalized polynomials in $\mathcal{O}_{n-1}[z_n]$ of degree $< d$ (in the variable z_n).*

We consider the restriction of F

$$\mathcal{O}_{n-1}[z_n : m]^p \to \mathcal{O}_{n-1}[z_n : m + d]$$

and denote by K_m its kernel.

Claim. *The $\mathcal{O}_n$-module K is generated by K_m for $m \ge 3d$.*

Proof. In a first step we assume that the first component of the map $F = (F_1, \dots, F_p)$ is aWeierstrass polynomial (and not only a normalized polynomial). We will prove Oka's Lemma in this case with the better bound $2d$ instead of $3d$. Let $G = (G_1, \dots, G_p) \in K$ be an element of the kernel. The division theorem gives

$$G = F_1 A + B, \quad A \in \mathcal{O}_n^p, \quad B \in \mathcal{O}_{n-1}[z_n : d]^p.$$

We notice that the elements

$$H^{(j)} = (-F_j, 0, \ldots, 0, F_1, 0, \ldots, 0) \quad (1 < j \leq p)$$

are contained in the kernel. The trivial formula

$$F_1 A = \sum_{j=2}^{p} A_j H^{(j)} + (A_1 F_1 + \ldots + A_p F_p, 0, \ldots, 0)$$

shows that besides G also the element $H := B + (A_1 F_1 + \ldots + A_p F_p, 0, \ldots, 0)$ is contained in the kernel, i.e.

$$F_1(B_1 + A_1 F_1 + \ldots + A_p F_p) + F_2 B_2 + \ldots + F_p B_p = 0.$$

This equation shows

$$F_1(A_1 F_1 + \ldots + A_p F_p) \in \mathcal{O}_{n-1}[z_n : 2d].$$

Using again that F_1 is a Weierstrass polynomial we obtain

$$A_1 F_1 + \ldots + A_p F_p \in \mathcal{O}_{n-1}[z_n : 2d].$$

Now we see that the components of H are contained in K_{2d}. The trivial formula

$$G = \sum_{j=2}^{p} A_j H^{(j)} + H$$

finally shows that G is contained in the module which generated by the $H^{(j)}$ and H, which are elements of K_{2d}.

Now we treat the general case where F_1 is not necessarily a Weierstrass polynomial. We apply the preparation theorem

$$F_1 = Q \cdot U, \quad Q \text{ Weierstrass polynomial,} \quad U \text{ unit in } \mathcal{O}_n.$$

We are interested in the solutions of the equation $F_1 P_1 + F_2 P_2 + \ldots + F_p P_p = 0$ or equivalently

$$Q \tilde{P}_1 + F_2 P_2 + \ldots + F_p P_p = 0 \quad (\tilde{P}_1 = U P_1).$$

Since Q is a Weierstrass polynomial, this system is generated by solutions of z_n-degree $< 2d$. But UP_1 is of degree $< 3d$ if P_1 is of degree $< 2d$. This completes the proof of Oka's lemma.

Third step, the proof of Oka's theorem in the case $q = 1$.

The proof now is given by induction on n. As beginning of the induction can be taken the trivial case $n = 0$. We have to consider a $\mathcal{O}(U)$-linear map

$$F : \mathcal{O}(U)^p \longrightarrow \mathcal{O}(U),$$

which is given by a vector $(F_1, \dots, F_P)$. We want to show that the kernel system is finitely generated in a neighborhood of a given point and can assume that this point is the origin 0 and that U is a polydisk with center 0. After a suitable linear coordinate transformation we can assume that the power series expansions of $F_1, \dots, F_p$ in the origin are z_n-general. By the preparation theorem we can assume that the all are Weierstrass polynomials. If we consider the power series expansions in other points $a \in U$ we still have normalized polynomials

$$(F_i)_a \in \mathbb{C}\{z_1 - a_1, \dots, z_{n-1} - a_{n-1}\}[z_n - a_n].$$

(but usually not Weierstrass polynomials). The degree of all those polynomials is bounded by a suitable number d. We write U in the form

$$U = V \times (-r, r) \quad (V \subset \mathbb{C}^{n-1})$$

and denote by $\mathcal{O}(V)[z_n : m]$ the set of all holomorphic functions on U which are polynomial s in z_n of degree $< m$ with coefficients independent of z_n. This is a free $\mathcal{O}(V)$ module,

$$\mathcal{O}(V)[z_n : m] \cong \mathcal{O}(V)^m.$$

Our given map F induces an $\mathcal{O}(V)$-linear map

$$\begin{array}{ccc} \mathcal{O}(V)[z_n : m]^p & \longrightarrow & \mathcal{O}(V)[z_n : m + d] \\ \| & & \| \\ \mathcal{O}(V)^{mp} & \longrightarrow & \mathcal{O}(V)^{m+d}. \end{array}$$

From the induction hypothesis we can assume that the kernel of this map is finitely generated. From Oka's lemma we obtain that the kernel system of F is finitely generated. Oka's theorem is proved. $\square$

Some Important Properties of Coherent Systems

The following trivial property of coherent systems will be used frequently:

11.5 Remark. *Let $\mathcal{M}, \mathcal{N}$ be two coherent systems on an open set $U \subset \mathbb{C}^n$. Assume $\mathcal{M}_{a_0} \subset \mathcal{N}_{a_0}$ for a distinguished point a_0. Then $\mathcal{M}_a \subset \mathcal{N}_a$ in a complete neighborhood of a_0 holds.* umgC

Corollary. *$\mathcal{M}_{a_0} = \mathcal{N}_{a_0}$ implies $\mathcal{M}|V = \mathcal{N}|V$ for an open neighborhood V of a_0.*

Another trivial observation is

11.6 Remark. *Let* bilC

$$F : \mathcal{O}(U)^m \to \mathcal{O}(U)^l \quad (U \subset \mathbb{C}^n \ \textit{open})$$

be an $\mathcal{O}(U)$ linear map and let

$$\mathcal{M} = (\mathcal{M}_a)_{a\in U}, \quad \mathcal{M}_a \subset \mathcal{O}^m_{U,a},$$

be a coherent system. The the image system

$$\mathcal{N} = (\mathcal{N}_a)_{a\in U}, \quad \mathcal{N}_a := F_a(\mathcal{M}_a) \subset \mathcal{O}^l_{U,a}.$$

is coherent. (The same is true already for "finitely generated" instead for "coherent".)

The next result is not trivial, it uses Oka's theorem:

11.7 Proposition. *Let $\mathcal{M},\mathcal{N}$ be two coherent systems on the open set* durC
$U \in \mathbb{C}^n$,

$$\mathcal{M}_a, \mathcal{N}_a \subset \mathcal{O}^m_{U,a} \quad (a \in U).$$

The the intersection system $\mathcal{M}\cap\mathcal{N}$ which is defined by

$$(\mathcal{M}\cap\mathcal{N})_a := \mathcal{M}_a \cap \mathcal{N}_a \quad (a \in U)$$

is coherent too.

Proof. The idea is to write the intersection as a kernel. We explain the principle for individual modules $M, N \subset R^n$ of finite type over a ring R instead of a system: We can write M resp. N as image of a linear map $F : R^p \to R^m$ resp. $G : R^q \to R^m$. We denote by K the kernel of the linear map

$$R^{p+q} \longrightarrow R^m, \quad (m, n) \longmapsto F(m) - G(n).$$

The image of K under the map

$$R^{p+q} \longrightarrow R^m, \quad (m, n) \longmapsto F(m).$$

is precisely the intersection $M \cap N$. The proof of 11.7 is clear now. On "reads" M, N as coherent systems. By Oka's theorem K now stands for a coherent system and the image $M \cap N$ is is coherent by 11.6. □

11.8 Proposition. *Let* invC

$$F : \mathcal{O}(U)^m \to \mathcal{O}(U)^l \quad (U \subset \mathbb{C}^n \text{ open})$$

be an $\mathcal{O}(U)$-linear map and let

$$\mathcal{N} = (\mathcal{N}_a)_{a\in U}, \quad \mathcal{N}_a \subset \mathcal{O}^l_{U,a},$$

be a coherent system. The inverse image system

$$\mathcal{M} = (\mathcal{M}_a)_{a\in U}, \quad \mathcal{M}_a := F_a^{-1}(\mathcal{N}_a) \subset \mathcal{O}^m_{U,a},$$

is coherent.

In the special case $\mathcal{N} = 0$ this is Oka's theorem.
Proof. We explain again the algebra behind this result. Let $F : R^m \to R^l$ be a R-linear map and $N \subset R^l$ be a R-module of finite type. We assume that $F(R^m) \cap N$ is finitely generated. Then there exists a finitely generated submodule $P \subset R^m$ such that $F(P) = F(R^m) \cap N$. We also assume that the kernel K of F is finitely generated. It is easily proved that $F^{-1}(N) = P + K$ and we obtain that the inverse image is finitely generated. These argument works in an obvious way for coherent systems and gives a proof of 11.8.

□

12. Rings of Power Series are Henselian

The fact that power series are henselian rings can be considered as an abstract formulation of the Weierstrass theorems. We don't need the notion of a henselian ring to formulate this result, but for sake of completeness we give the definition of this property.

A local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$ is called a ***henselian ring*** *if the following is true:*

Let $P \in R[X]$ be a normalized polynomial. We denote by p its image in $k[X]$. Assume that $a, b \in k[X]$ are two coprime normalized polynomials with the property $p = ab$. Then there exist normalized polynomials $A, B \in R[X]$ with cosets a, b such that $P = AB$.

We recall that the polynomial ring in one variable over a field is a principal ideal ring. Therefore two polynomials a, b are coprime if and only if they generate the unit ideal $k[X]$.

We consider the special case where k is algebraically closed. Then every normalized polynomial $p \in k[X]$ is a product of linear factors, if $b_1, \dots b_m$ are the pairwise distinct zeros and $d_1, \dots, d_m$ their multiplicities then

$$p(X) = \prod_{j=1}^{m} (X - b_j)^{d_j}.$$

This is a decomposition of p into m pairwise coprime factors. So the henselian property means in this case:

Let R be a local ring with an algebraically closed residue field $k = R/\mathfrak{m}$. Let $P \in R[X]$ be a normalized polynomial and let p its image in $k[X]$,

$$p(X) = \prod p_j(X), \quad p_j(X) = (X - b_j)^{d_j}.$$

There exists a decomposition $P = P_1 \cdots P_m$ of P as product of normalized polynomials such that $p_j(X) = (X - b_j)^{d_j}$ where p_j is the image of P_j in $k[X]$.

We want to show that the ring of power series $\mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$ is henselian. The residue field $\mathcal{O}_n/\mathfrak{m}_n$ can be identified with $\mathbb{C}$ and the projection $\mathcal{O}_n \to \mathcal{O}_n/\mathfrak{m}_n$ corresponds to the map

$$\mathbb{C}\{z_1, \dots, z_n\} \longrightarrow \mathbb{C}, \quad P \longmapsto P(0).$$

We have to consider the polynomial ring over $\mathbb{C}\{z_1, \dots, z_n\}$. Therefore we need a letter for the variable. To stay close to previous notations we consider $\mathcal{O}_{n-1}$ instead of $\mathcal{O}_n$ and formulate the Hensel property for this ring. Then we have the letter z_n free for the variable of the polynomial ring. After this preparation we see that the following theorem expresses precisely that the rings of power series are henselian.

12.1 Theorem. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be a normalized polynomial of degree $d > 0$ and let β be a zero with multiplicity d_β of the polynomial $z \longmapsto P(0, \dots, 0, z)$. Then there exists a unique normalized polynomial $P^{(\beta)} \in \mathcal{O}_{n-1}[z_n]$ which divides P and such that* HENS

$$P^{(\beta)}(0, \dots, 0, z) = (z - \beta)^{d_P}.$$

Moreover

$$P = \prod_{P(0,\dots,0,\beta)=0} P^\beta.$$

(Here β runs through the zeros of $z \mapsto P(0, \dots, 0, z)$.)

For the proof of this theorem we need three lemmas:

12.2 Lemma. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be an **irreducible** normalized polynomial with the property $P(0) = 0$. Then P is a Weierstrass polynomial.* irnorm

Proof. By the preparation theorem we have $P = UQ$ with a Weierstrass polynomial and a unit U. We know that U is a polynomial. But P is irreducible. We obtain $U = 1$ and $P = Q$. □

12.3 Lemma. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be an irreducible normalized polynomial of degree $d > 0$. Then* irrnu

$$P(0, \dots, 0, z) = (z - \beta)^d$$

with a suitable complex number β.

Proof. Let β be a zero of the polynomial $z \mapsto P(0, \dots, 0, z)$. We rearrange P as polynomial in $z_n - \beta$ and obtain by 12.2 a Weierstrass polynomial in $\mathcal{O}_{n-1}[z_n - \beta]$. □

12.4 Lemma. *Let P, Q be two normalized polynomials in $\mathcal{O}_{n-1}[z]$. The polynomials $p(z) = P(0,\dots,0,z)$, $q(z) = Q(0,\dots,0,z)$ are assumed to be coprime. (This means that have no common zero.) Then P and Q generate the unit ideal ,* einId

$$(P, Q) = \mathcal{O}_{n-1}[z_n].$$

Proof. The proof will use the theorem of Cohen Seidenberg: The ring polynomial in one variable over a field is a principal ideal ring. Therefore

$$(p, q) = \mathbb{C}[z].$$

We obtain that P and Q together with the maximal ideal $\mathfrak{m}_{n-1} \subset \mathcal{O}_{n-1}$ generate the unit ideal,

$$(P, Q, \mathfrak{m}_{n-1}) = \mathcal{O}_{n-1}[z_n].$$

Now we consider the natural homomorphism

$$\mathcal{O}_{n-1} \longrightarrow \mathcal{O}_{n-1}[z_n]/(P, Q).$$

This ring extension is module-finite. This follows immediately if one applies the Euclidean algorithm to one of the polynomials P, Q. The theorem of Cohen Seidenberg deals with module finite ring extensions. We give here a formulation which is not the standard one but usually a lemma during the proof:

Let A be a noetherian local ring and $A \to B$ a ring homomorphism such that B is an A-module of finite type. We assume that B is different from the zero ring ($1_B \neq 0_B$). Then there exists a proper ideal in B which contains the image of the maximal ideal of A.

(One can take the ideal which is generated by the image of the maximal ideal of A. The problem is to show that this is different form B.)

We continue the proof of 12.4. We want to show that P and Q generate the unit ideal. We give an indirect argument and assume that this is not the case. Then by Cohen Seidenberg we obtain that the image of $\mathfrak{m}_{n-1}$ in $\mathcal{O}_{n-1}[z_n]/(P, Q)$ does not generate the unit ideal. This means the same that $(P, Q, \mathfrak{m}_{n-1})$ is not the unit ideal, which gives a contradiction. □

Proof of theorem 12.1. Let P be a normalized polynomial of degree $d > 0$ in $\mathcal{O}_{n-1}[z_n]$. We decompose P into a product of irreducible normalized polynomials

$$P = P_1 \cdot \cdot P_m.$$

From 12.3 we obtain

$$P_i(0,\dots,0,z) = (z - \beta_i)^{d_i} \quad (1 \leq i \leq m).$$

The numbers β_i are the zeros of the polynomial $P(0,\dots,0,z)$. There is no need that the β_i are pairwise distinct. But we can collect the P_i for a fixed zero and multiply them together. □

We need a further little lemma from algebra:

Let R be a UFD-domain and a, b two coprime elements. The natural homomorphism

$$R/(ab) \longrightarrow R/(a) \times R/(b)$$

is injective. It is an isomorphism if a and b generate the unit ideal.

We apply this to thorem 12.1 and obtain:

12.5 Proposition. *(We use the notations of 12.1.) The natural homomorphism* hensis

$$\mathcal{O}_{n-1}[z_n]/(P) \xrightarrow{\sim} \prod_{\beta} \mathcal{O}_{n-1}[z_n]/(P^{(\beta)})$$

is an isomorphism.

We recall the the P^{β} are Weierstrass polynomials in the ring $\mathcal{O}_{n-1}[z_n - \beta]$. From the division theorem we obtain

$$\mathcal{O}_{n-1}[z_n]/(P^{(\beta)}) = \mathbb{C}\{z_1, \dots, z_{n-1}, z_n - \beta\}/(P^{(\beta)}).$$

Now we can conclude from the Hensel property the following generalization of the division theorem for normalized polynomials instead of Weierstrass polynomials:

12.6 Proposition. *(We use the notations of 12.1.) The natural homomorphism* Hensis

$$\mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]/(P) \xrightarrow{\sim} \prod_{\beta} \mathbb{C}\{z_1, \dots, z_{n-1}, z_n - \beta\}/(P^{(\beta)})$$

is an isomorphism. This remains true if one replaces $P^{(\beta)}$ by the power series expansion of P in $(0, \dots, 0, \beta)$.

The last statement uses the decomposition $P = \prod_{\gamma} P^{(\gamma)}$ and the fact that all factors besides the considered $P^{(\beta)}$ do not vanish at $(0, \dots, 0, \beta)$ and hence define units in $\mathbb{C}\{z_1, \dots, z_{n-1}, z_n - \beta\}$.

13. A special case of Remmert's mapping theorem

Let $U \subset \mathbb{C}^n$ be an open domain. We consider a coherent system of ideals

$$\mathfrak{a} = (\mathfrak{a}_a)_{a \in U}, \quad \mathfrak{a}_a \subset \mathcal{O}_{U,a}.$$

13.1 Definition. *The **zero locus** of a coherent system $\mathfrak{a}$ of ideals is the of all $a \in U$ such that $\mathfrak{a}_a$ is different from the unit ideal.* cohI

In the case that $\mathfrak{a}$ is finitely generated the zero locus is precisely what we called earlier the associated analytic set. This leads as to a general notion of am analytic set.

13.2 Definition. *Let $U \subset \mathbb{C}^n$ an open subset. A subset $X \subset U$ is called a closed analytic subset if it is the zero locus of a coherent system of ideals.* Dasg

Now we assume that $U = V \times \mathbb{C}$ with a polydisk $V \subset \mathbb{C}^{n-1}$. We consider the *projection*

$$\pi : U \longrightarrow V, \quad (z, z_n) \longmapsto z.$$

It may happen that the image of an closed analytic set $X \subset U$ in V is a closed analytic set $Y \subset V$ but this must be not the case. We want to give a sufficient condition where it is the case. The idea is to consider rather coherent systems than analytic sets. So let's assume that X is the zero locus of the coherent system $(\mathfrak{a}_a)$. We expect that in good situations Y is the zero locus of certain coherent system on V. It's not difficult to guess what this system should be.

13.3 Definition. *Let $V \subset \mathbb{C}^{n-1}$ be a polydisk and $\mathfrak{a}$ a coherent system of ideals on $U = V \times \mathbb{C}$. We define for a point $b \in V$ the ideal* projS

$$\mathfrak{b}_b := \mathcal{O}_{V,b} \cap \bigcap_{a \in U,\ \pi(a) = b} \mathfrak{a}_a.$$

and call $\mathfrak{b} := (\mathfrak{b}_b)_{b \in V}$ the projected system.

We recall that the projection π defines a natural inclusion $\mathcal{O}_{V,b} \hookrightarrow \mathcal{O}_{U,a}$ for all a, b with $\pi(a) = b$.

Projections of analytic sets of the above kind can be very bad and similarly the projected systems can be bad and need not to be coherent. But there exist "good" projections:

13.4 Theorem. *Assume that $V \subset \mathbb{C}^{n-1}$ is a polydisk and that $\mathfrak{a}$ is a coherent system on $U = V \times \mathbb{C}$, which can be generated by finitely many functions $f_1, \dots, f_m \in \mathcal{O}(V)[z_n]$. We assume that $P := f_1$ is a normalized polynomial. Then the projected system $\mathfrak{b}$ is coherent.* GRAU

Additional remark. *If X is the zero locus of $\mathfrak{a}$, then $Y = \pi(X)$ is the zero locus of $\mathfrak{b}$. In particular, $\pi(X)$ is a closed analytic subset of V. The map $\pi : X \to Y$ is finite.*

Proof of 13.4. The proof will use Oka's coherence theorem and the Hensel property of rings of power series. The ideal $\mathfrak{a}_a$ is the unit ideal if $P(a) \neq 0$. For every $b \in V$ the number of $a \in U$ with $\pi(a) = b$ and $P(a) = b$ is finite. Therefore $\mathfrak{b}_b$ is the intersection of *finitely many* ideals:

$$\mathfrak{b}_b := \mathcal{O}_{V,b} \cap \bigcap_{\pi(a)=b,\ P(a)=0} \mathfrak{a}_a.$$

The ideal $\mathfrak{b}_b$ contains 1 if and only this is the case for all $\mathfrak{a}_a$, $\pi(a) = b$. We see that the additional remark will follow automatically from the coherence of $\mathfrak{b}$.

We want to consider the ideal

$$\mathcal{I}_b \subset \mathcal{O}_{V,b}[z_n]/(P_b),$$

which is generated by the $f_1, \dots, f_n$ (more precisely by their images). We have to consider this ideal also as $\mathcal{O}_{V,b}$-module. It is of finite type over this ring, more precisely it is generated as module over this ring by the elements

$$f_i z_n^j \qquad (1 \leq i \leq m, \quad 0 \leq j < d).$$

This uses the Euclidean algorithm, which gives an isomorphism

$$\mathcal{O}_{V,b}^d \xrightarrow{\sim} \mathcal{O}_{V,b}[z_n]/(P_b).$$

A vector $(H_0, \dots, H_d)$ is mapped to $\sum H_j z_n^j$. We take the inverse image of $\mathcal{I}_b$ and get a submodule

$$\mathcal{M}_b \subset \mathcal{O}_{V,b}^d.$$

From the given generators we see that the system $\mathcal{M} = (\mathcal{M}_b)_{b \in V}$ is finitely generated hence coherent on V. This system is closely related to our projected ideals $\mathfrak{b}_b$:

Claim. *The projected ideal $\mathfrak{b}_b$ is precisely the inverse image of $\mathcal{I}_b$ with respect to the natural map*

$$\mathcal{O}_{V,b} \longrightarrow \mathcal{O}_{V,b}[z_n]/(P_b).$$

We assume for a moment that the claim is proved. Then $\mathfrak{b}$ can be considered as inverse image of the coherent system $\mathcal{M}$. But Oka's coherence theorem (11.8) then implies that $\mathfrak{b}$ is coherent. So it remains to prove the claim:

Proof of the claim. In this proof the Hensel property of rings of power series will enter. We have to make further use of our normalized polynomial $P \in \mathcal{O}(V)[z_n]$,

$$P = z_n^d + P_{d-1} z_n^{d-1} + \dots + P_0.$$

Its coefficients P_j are holomorphic functions on V. We will use the power series expansion $(P_j)_b \in \mathcal{O}_{V,b}$ for varying points $b \in V$. We have to consider the image of P in $\mathcal{O}_{V,b}[z_n]$,

$$P_b = z_n^d + (P_{d-1})_b z_n^{d-1} + \dots + (P_0)_b \in \mathcal{O}_{V,b}[z_n].$$

We also have to use the ring $\mathcal{O}_{V,b}[z_n]/(P_b)$. The Hensel property of rings of power series gave us important information for this ring. Applying 12.6 we obtain a natural isomorphism*)

$$\mathcal{O}_{V,b}[z_n]/(P_b) \xrightarrow{\sim} \prod_\beta \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)}).$$

Here β runs over the zeros $P(b,\beta)=0$. The elements $P_b^{(\beta)} \in \mathcal{O}_{V,b}$ come from the "Hensel decomposition"

$$P_b = \prod_\beta P_b^{(\beta)}, \qquad P_b^{(\beta)}(b, z_n) = (z_n - \beta)^{d_\beta}.$$

We determine the image of $\mathcal{I}_b$ under this isomorphism. For this we use the simple fact that an ideal $\mathfrak{c} \subset A \times B$ in the cartesian product of two rings always is the direct product of two ideals, $\mathfrak{c} = \mathfrak{a} \times \mathfrak{b}$, where $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ are the projections of $\mathfrak{c}$. Using this and the definition (13.4) of $\mathfrak{a}$ we see:

The image of the ideal $\mathcal{C}_b$ in $\prod_\beta \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)})$ is the direct product of the ideals $\bar{\mathfrak{a}}_{(b,\beta)}$, which mean the images of $\mathfrak{a}_{(b,\beta)}$ in $\mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)})$.

We have to determine the inverse image of this ideal under the natural map

$$\mathcal{O}_{V,b} \longrightarrow \prod_\beta \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)}).$$

The claim states that this inverse image is the projection ideal $\mathfrak{b}_b$. But this inverse image is the intersection of the inverse images of $\bar{\mathfrak{a}}_{(b,\beta)}$ under

$$\mathcal{O}_{V,b} \longrightarrow \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)}).$$

But $P_b^{(\beta)}$ is contained in $\mathfrak{a}_{(b,\beta)}$ (s. 12.6). Therefore it is the same to take the inverse image of $\mathfrak{a}_{(b,\beta)}$ under

$$\mathcal{O}_{V,b} \longrightarrow \mathcal{O}_{U,(b,\beta)}.$$

This is $\mathcal{O}_{V,b} \cap \mathfrak{a}_{(b,\beta)}$ and the intersection of all of then is $\mathfrak{b}_b$. □

---

*) In 12.6 the result has been formulated only for $b=0$ which is no loss of generality.

14. Cartan's Coherence Theorem

There is a second basic coherence theorem. Oka contributes this theorem to Cartan, but as Grauert and Remmert pointed out in there book „Coherent analytic sheaves“, the essential parts of the proof are already in Oka's papers. We give three different formulations for Cartan's theorem:

14.1 Cartan's coherence theorem. *Let $\mathfrak{a} = (\mathfrak{a}_a)_{a \in U}$ be a coherent system* CAR
of ideals on an open domain $U \subset \mathbb{C}^n$. Then its radical

$$\operatorname{rad} \mathfrak{a} := (\operatorname{rad} \mathfrak{a}_a)_{a \in U}$$

is coherent too.

let $X \subset U$ be a closed analytic subset. The vanishing ideal system $\mathfrak{A}_X$ is the system of ideals $\mathfrak{A}_a$, $a \in U$ which consists of all elements from $\mathcal{O}_{U,a}$, which vanish in a small neighborhood of a on X. If a is not in X then $\mathfrak{A}_a = \mathcal{O}_{U,a}$. For this one has to use that X is closed in U. A second form of Cartan's theorem is:

14.2 Cartan's coherence theorem. *Let $X \subset U$ be a closed analytic subset* CART
of an open set $U \subset \mathbb{C}^n$. The vanishing ideal system $\mathfrak{A}$ is coherent.

To see the equivalence one has to have in mind that the zero locus of a coherent ideal system $\mathfrak{a}$ is a closed analytic set and that by the nullstellensatz the radical of $\mathfrak{a}$ is the complete vanishing ideal system $\mathfrak{A}$. One also has to use the trivial fact the every analytic set locally is the zero locus of a coherent system. Another formulation is

14.3 Cartan's coherence theorem. *Let $\mathfrak{a}$ be coherent system of ideals.* CARTA
The set of all points a such that $\mathfrak{a}_a = \operatorname{rad} \mathfrak{a}_a$ is open.

We show that 14.3 implies 14.2. Let $a \in U$ a point. The ideal $\operatorname{rad} \mathfrak{a}_a$ is finitely generated. Therefore there exists a coherent system $\mathfrak{b}$ on an open neighborhood $a \subset V \subset U$ auch that $\mathfrak{b}_a = \operatorname{rad} \mathfrak{a}_a$ and $\mathfrak{a}_b \subset \mathfrak{b}_b \subset \operatorname{rad} \mathfrak{a}_b$. Now 14.3 implies that in a full neighborhood $\mathfrak{b}_b = \operatorname{rad} \mathfrak{a}_b$. The conclusion 14.2 $\Rightarrow$ 14.3is also clear. One uses the fact that two coherent systems which agree in a point agree in a full neighborhood.

The rest of this section is dedicated the proof of Cartan's theorem. We need some preparations:

In a first step we give a reduction. We can assume that the origin is contained U and that $\mathfrak{a}_0 = \operatorname{rad} \mathfrak{a}_0$. We have to prove that $\mathfrak{a}_a = \operatorname{rad} \mathfrak{a}_a$ in a full neighborhood of 0. We want to show that it is enough to treat the case of a prime ideal $\mathfrak{a}_0$. For this we use again the fact that any reduced ideal is the intersection of finitely many prime ideals.

$$\mathfrak{a}_0 = \mathfrak{p}_0^{(1)} \cap \ldots \cap \mathfrak{p}_0^{(m)}.$$

We can extend the $\mathfrak{p}_0^{(j)}$ into a coherent system ${}_a^{(j)}$ on a small neighborhood of 0. From our assumption we know that the $\mathfrak{p}_a^{(j)}$ are reduced (in a small neighborhood). We also know from Oka's coherence theorem that the intersection system $\mathfrak{p}_a^{(1)} \cap \ldots \cap \mathfrak{p}_a^{(m)}$ is coherent. This intersection system and $\mathfrak{a}$ agree in the origin and hence in a full neighborhood,

$$\mathfrak{a}_a = \mathfrak{p}_a^{(1)} \cap \ldots \cap \mathfrak{p}_a^{(m)}.$$

Using the trivial fact that the intersection of reduced ideals is reduced we obtain that the $\mathfrak{a}_a$ are reduced.

From now on we assume that $0 \in U$ and that

$$\mathfrak{P} := \mathfrak{a}_0$$

is a prime ideal. We will show that $\mathfrak{a}_a$ is reduced in a neighborhood of 0. We need some preparations for the proof:

An element a of a ring R is called non-zero-divisor if multiplication with a

$$R \longrightarrow R, \quad x \longmapsto ax,$$

is injective.

14.4 Lemma. *Let $\mathfrak{a}$ be a coherent system on an open set $U \subset \mathbb{C}^n$ and let $f \in \mathcal{O}(U)$ be an analytic function on U. The set of all points $a \in U$ such that the germ f_a is a non-zero-divisor in $\mathcal{O}_{U,a}$ is open* nNof

Proof. We denote the map "multiplication by a" by

$$m_f : \mathcal{O}_{U,a} \longrightarrow \mathcal{O}_{U,a}.$$

The element f_a is non-zero-divisor if and only if

$$m_f^{-1}(\mathfrak{a}_a) = \mathfrak{a}_a.$$

From Oka's coherence theorem we know that the system $\big(m_f^{-1}(\mathfrak{a}_a)\big)_{a \in U}$ is coherent. The coincidence set of two coherent systems is open. □

After this preparations the proof of Cartan's theorem runs as follows. Recall that $0 \in U$ and that $\mathfrak{P} = \mathfrak{a}_0$ is a prime ideal. We have to show that $\mathfrak{a}_a$ is reduced in a full neighborhood of 0. We distinguish the two "alternatives".

1. *Alternative.* $\mathfrak{P} = (P)$ is a principal ideal. The element P is a prime element, especially square free. The theory of the *discrimant* gave us that there exists a small polydisk around 0 in which P converges and such that P_a is square free in this polydisk. Coherence gives us that $\mathfrak{a}_a = (P_a)$ in a full neighborhood. But a principal ideal generated ba a square free element is reduced. What we see that in the case of hypersurfaces the properties of the discriminant imply Cartan's theorem. □

2. *Alternative.* This case is more involved. We will have to use the special case od Grauert's projection theorem. As usual we can assume that $\mathfrak{P} = \mathfrak{a}_0$ is z_n-general and that

$$\mathcal{O}_{n-1}/\mathfrak{p} \longrightarrow \mathcal{O}_n/\mathfrak{P} \qquad (\mathfrak{p} := \mathcal{O}_{n-1} \cap \mathfrak{P})$$

have the same field of fractions. The ideal $\mathfrak{P}$ is finitely generated,

$$\mathfrak{P} = (Q_1, Q_2, \dots, Q_m).$$

We can assume that $Q := Q_1$ is a Weierstrass polynomial and then by the division theorem that all Q_i are polynomials over $\mathcal{O}_{n-1}$. We can take U in the form $U = V \times (-r, r)$, where $V \subset \mathbb{C}^n$ is a polydisk around 0. We can assume that the coefficients of the Q_j converge in V and that the zeros of the polynomial $z \mapsto Q(b, z)$ for all $b \in V$ have absolute value $< r$. From the special case of Grauert's projection theorem we obtain that the system

$$\mathfrak{b}_b = \mathcal{O}_{V,b} \cap \bigcap_{a=(b,\beta),\ Q(a)=0} \mathfrak{a}_a$$

is coherent on V. Because Q is a Weierstrass polynomial we have

$$\mathrm{b}_0 = \mathfrak{p}.$$

We want to prove Cartan's theorem by induction on n. Therefore we can assume that all the projected ideals $\mathfrak{b}_b$ are reduced. We will make use of the natural homomorphism

$$\mathcal{O}_{V,b}/\mathfrak{b}_b \longrightarrow \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

It is quite clear that that this homomorphism is an injection. In the case $a = 0$ this is the homomorphism

$$\mathcal{O}_{n-1}/\mathfrak{p} \longrightarrow \mathcal{O}_n/\mathfrak{P}.$$

Now we make use of the basic fact that the fields of fractions of both rings agree. We find elements

$$A, B \in \mathcal{O}_{n-1}, \quad A \notin \mathfrak{p}, \quad Az_n - B \in \mathfrak{P}.$$

We can assume that A and B converge in V and furthermore because of coherence

$$A_b(z_n - a) - B_b \in \mathfrak{a}_a \qquad (a = (b, \beta) \in U).$$

We have to combine this fact that $\mathcal{O}_{U,a}/\mathfrak{a}_a$ is a module of finite type*) over $\mathcal{O}_{V,b}/\mathfrak{b}_b$. More precisely it is generated by the powers

$$(z_n - a_n)^\nu, \quad 0 \le \nu < d,$$

where d is the z_n-degree of Q. Now we consider the analytic function $f := A^d$ on U. The germ f_0 defines a non-zero element of $\mathcal{O}_{U,0}/\mathfrak{P}$ and hence non-zero-divisor, because this ring is an integral domain. Because of the coherence result 14.4 we can assume that the multiplication map $m_f : \mathcal{O}_{U,a}/\mathfrak{a}_a \to \mathcal{O}_{U,a}/\mathfrak{a}_a$ is injective of all a. This map is no ring homomorphism but it is good enough to test nilpotency: First we collect all points $a = (b, \beta)$ over a given b and consider

$$m_f : \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a \longrightarrow \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

The construction of A shows that the image of m_f is already contained in the subring

$$\mathcal{O}_{V,b}/\mathfrak{b}_b \hookrightarrow \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

The proof of Cartan's theorem now can be completed as follows: Let $C \in \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a$ be a nilpotent element, $C^k = 0$. Then $m_f(C^k) = f_a C^k = 0$. But this implies $(f_a C)^k = 0$. We recall that $m_f(C) = f_a C$ is contained in the subring $\mathcal{O}_{V,b}/\mathfrak{b}_b$. But this ring is reduced (by our induction hypothesis). Hence $m_f(C) = 0$. But m_f is injective (!) and we obtain $C = 0$. Hence the ring $\prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a$ is free of nilpotents and the same is true for each of its factors. This completes the proof of Cartan's coherence theorem. $\square$

Because of the importance of this theorem we formulate again the decisive consequence:

14.5 Theorem. *Every analytic set can be written locally as the set of common zeros of a finite system of analytic functions* vollV

$$f_1, \dots, f_m : U \longrightarrow \mathbb{C} \qquad (U \subset \mathbb{C}^n \ \textit{open}),$$

such that the germs in any point $a \in U$ generate the ***full vanishing ideal*** *in $\mathcal{O}_{U,a}$.*

---

*) This true because $Q_a \in \mathfrak{a}_a$ is a normalized polynomial, hence z_n-general, hence the product of a unit and a Weierstrass polynomial of degree $\le d$.

15. The singular locus

Let $X \subset U \subset \mathbb{C}^n$ be a closed non-empty analytic subset. In this section we denote by $\mathfrak{a} = (\mathfrak{a}_a)_{a\in U}$ the full vanishing system ($\mathfrak{a}_a = \operatorname{rad}\mathfrak{a}_a$) and we use also the notation

$$\mathcal{O}_{X,a} = \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

So this is a nolpotentfree analytic algebra. The dimension of X at a is defined to be

$$\dim_a X = \dim \mathcal{O}_{X,a} \quad \text{(Krull dimension)}.$$

We have $\dim_a X \le n$. So we can define

$$\dim X = \max_{a\in X} \dim_a X.$$

The set X is called pure dimensional if $a \mapsto \dim_a X$ is constant on X.

15.1 Definition. *A point $a \in X$ is called a regular point of the closed analytic set $X \subset U$ if – after replacing U by some open neighborhood of a in U (and X by its intersecton with this neighborhood) – there exists a biholomorphic map $f : U \to V$, $V \subset \mathbb{C}^n$ open, such that $f(a) = 0$ and* Drp

$$f(X) = \{b \in V; \quad b_{d+1}, \dots, b_n = 0\}.$$

We then get an isomorphism of analytic algebras

$$\mathcal{O}_d \cong \mathcal{O}_n/(z_{d+1} = \cdots = z_n) \xrightarrow{\sim} \mathcal{O}_{X,a}.$$

So we see

$$\dim_a X = d.$$

The phrase *"after replacing U by some open neighborhood of a in U (and X by its intersecton with this neighborhood)"* would occur frequently in what follows. To simplify notation we will replace this by *"after shrinking U"*.

One hint to the correctness of our definition is:

15.2 Proposition. *Let $f : U \to \mathbb{C}$ be a holomorphic function on some open connected subset $U \subset \mathbb{C}^n$. We assume that f doesn't vanish identically. Then the closed subspace defined by "$f = 0$" is of pure dimension $n - 1$.* HypDim

This is an application of the theorem of Cohen Seidenberg. □

We want to study the local behavior of the dimension $\dim_a X$ for varying a.

15.3 Lemma. *Let (X, a) be a pointed complex space. After shrinking U we have* dimHAL

$$\dim_b X \le \dim_a X \quad \text{for all} \quad b \in X.$$

Proof. The proof uses noether normalization: We use Remark 9.8. If one translates it into the language of complex spaces, one obtains that the following can be assumed.

1) X is a closed analytic subspace of a polydisk $U = V \times (-r, r)$ where $V = (-\varepsilon, \varepsilon)^{n-1}$. The point a is the origin 0.
2) The ideal $\mathfrak{A}_0$ contains a Weierstrass polynomial $Q \in \mathcal{O}_{n-1}[z_n]$. The coefficients of Q converge in Q and that $Q_a \in \mathcal{A}_a$ for all $a \in U$
3) There is a commutative diagram of holomorphic maps

$$\begin{tikzcd} U \arrow[r] & V \\ X \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

where the first row is the natural projection.
4) $\mathfrak{b}_0 = \mathcal{A}_0 \cap \mathcal{O}_{n-1}$.

The polynomial Q can be developed in any point $a \in U$. One obtains a polynomial in $\mathbb{C}\{z_1 - a_1, \dots, z_{n-1} - a_{n-1}\}[z_n - a_n]$. This needs not to be a Weierstrass polynomial but it is still a normalized polynomial. This is sufficient to show that the ring homomorphism

$$\mathcal{O}_{V,f(a)} \longrightarrow \mathcal{O}_{U,a}$$

is a module finite extension. As a consequence

$$f_a^* : \mathcal{O}_{Y,f(a)} \longrightarrow \mathcal{O}_{X,a}$$

is module-finite for all $a \in X$. This homomorphism needs not to be surjective but from Cohen Seidenberg we obtain still

$$\dim \mathcal{O}_{Y,f(a)} \ge f_a^*(\dim \mathcal{O}_{Y,f(a)}) = \mathcal{O}_{X,a}.$$

For $a = 0$ the homomorphism is injective, i.e.

$$\dim \mathcal{O}_{Y,0} = f_a^*(\dim \mathcal{O}_{Y,0}) = \mathcal{O}_{X,0}.$$

We will proof Lemma 15.3 by induction on n and can therefore assume

$$\dim_0 Y \ge \dim_b Y \qquad (b \in Y).$$

We obtain

$$\dim_0 X = \dim_0 Y \ge \dim_{f(a)} Y \ge \dim_a X,$$

which completes the proof of lemma 15.3. □

We give a typical application of coherence:

15.4 Lemma. *Let (X, a) be a pointed complex space and let $f \in \mathcal{O}_X(X)$. We assume that the germ f_a is a non-zero divisor in $\mathcal{O}_{X,a}$. Then, after shrinking U f_b is not a zero divisor in $\mathcal{O}_{X,b}$ for $b \in X$* duN

Corollary. *The zero locus*

$$Y := \{\, x \in U; \quad f(x) = 0 \,\}$$

is thin in U.

Proof. (Thin means that Y contains no nonempty open subset of U.) We consider the map that is induced by multiplication with f. It can be considered as a map $\mathcal{O}_X$-linear map of sheaves $\mathcal{O}_X \to \mathcal{O}_X$. By assumption the stalk of the kernel at a is zero. By coherence this remains true in a full neighborhood. So in this neighborhood f_b is a non-zero divisor.

In particular, the germ f_b is non-zero for $b \in U$. Hence in any neighborhood of of b there exist points which belong to Y but not to X. This shows that Y is thin in U. □

15.5 Lemma. *Let Y be a closed complex subspace of the complex space X and let $a \in Y$ be a point. We assume* intDue

a) *$\mathcal{O}_{X,a}$ is an integral domain.*
b) *The homomorphism $\mathcal{O}_{X,a} \to \mathcal{O}_{Y,y}$ is not injective.*

Then there exists an open neighborhood $a \in U \subset \mathbb{C}_n$, such that $Y \cap U$ is thin in $X \cap U$.

One can assume that there exists $f \in \mathcal{O}_X(X)$ on X whose germ in a is not contained in the defining ideal of (Y, a). Since $\mathcal{O}_{X,a}$ is an integral domain, f_a can not be a zero divisor. Now we can apply Lemma 15.4. □

15.6 Proposition. *Let a be a point in a complex space X such that $\mathcal{O}_{X,a}$ is an integral domain. Then, after shrinking U, the analytic set X is pure dimensional.* PureInt

Proof. We can assume that $0 \in X \subset \mathbb{C}^n$ is defined by a prime ideal $\mathfrak{P}$. We use induction by n. we can assume $a = 0$. We distinguish the "two alternatives".

1\. *Alternative.* $\mathfrak{P}$ is a principal ideal. Then we can use the theory of hypersurfaces.

2\. *Alternative.* $\mathfrak{P}$ is not a principal ideal. We can assume (15.3) $\dim_a X \leq \dim_0 X$ for all $a \in X$ and by induction $\dim_b Y = \dim_0 Y$ for all $b \in Y$. Let now $a \in X$ be an arbitrary point. Because T is thin, we find in any neighborhood of a a point $x \in X - T$. Because of 15.3 we can assume $\dim_x X \leq \dim_a X$. We obtain

$$\dim_0 X \geq \dim_a X \geq \dim_x X = \dim_{f(x)} Y = \dim_0 Y = \dim_0 X. \qquad \square$$

An important result of Krull dimension theory is:

15.7 Proposition. *Let $Y \subset X \subset U$ be analytic sets in Z and let $a \in Y$ be a point such that $\mathcal{O}_{X,a}$ is an integral domain. Assume* agR

$$\dim_a Y \geq \dim_a X.$$

After shrinking U the sets X and Y agree.

We have seen that it is often useful to reduce statements about radical ideals to prime ideals. This is possible because every radical ideal is the intersection of finitely many prime ideals. We describe the geometric counterpart of this algebraic fact in more detail:

Local irreducible components

Let R be a noetherian ring. A prime ideal $\mathfrak{p}$ which contains a given ideal $\mathfrak{a}$ is called *minimal* with this property, if any prime ideal $\mathfrak{q}$, $\mathfrak{a} \subset \mathfrak{q} \subset \mathfrak{p}$, agrees with $\mathfrak{p}$. A refinement of the already used statement about radical ideals is:

Let $\mathfrak{a}$ be an ideal in a noetherian ring R. There exist only finitely many minimal prime ideals containing $\mathfrak{a}$. Their intersection is $\operatorname{rad}\mathfrak{a}$. *Every prime ideal that contains $\mathfrak{a}$ contains one of the minimals.*

Now we consider the geometric counter part of this decomposition: Let $X \subset U$ be a closed analytic set. We want to study local properties of X at a given point $a \in X$. Since $\mathcal{O}_{X,a}$ is reduced, the zero ideal is a radical ideal. We can write it as the intersection of pairwise distinct minimal prime ideals

$$(0) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m.$$

After shrinking U we can assume that there are closed analytic sets $X_j \subset X$ whose vanishing ideals at a are $\mathfrak{p}_j \subset \mathcal{O}_{X,a}$. Again after shrinking U we can assume

$$X = X_1 \cup \ldots \cup X_m.$$

We call the X_j the local irreducible components of X at a. They are unique up to ordering in an obvious local sense.

15.8 Lemma. *Let (X, a) be a pointed analytic set and* dimIR

$$X = X_1 \cup \ldots \cup X_m$$

be a decomposition into local irreducible components of X at a. Then

$$\dim_a X = \max_{1 \leq j \leq m} \dim_a X_j.$$

Let $Y \subset X$ be an analytic subset which contains a and such that $\mathcal{O}_{Y,a}$ is integral. After shrinking U, the set Y is contained in one of the components X_j.

Proof. The dimension of X at a is defined by means of sequences of prime ideals in $\mathcal{O}_{X,a}$. Let $\mathfrak{a} \subset \mathcal{O}_n$ be the vanishing ideal of X at a. The chains of prime ideals in $\mathcal{O}_{X,a}$ correspond to chains

$$\mathfrak{a} \subset \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_m \subset \mathcal{O}_n.$$

The ideal $\mathfrak{p}_0$ must contain one of the minimal prime ideals containing $\mathfrak{a}$. This proofs the statement about the dimension. The last statement is also clear because the vanishing ideal of Y in a must contain one of the minimal prime ideals containing $\mathfrak{a}$. $\square$

Now we are in the state to prove a main result of local complex analysis:

15.9 Theorem. *The singular locus S of a closed analytic subset $X \subset\subset U$ is also a closed analytic subset of U. It is thin in X.* HAUPT

Proof. Since the statement is of local nature we can replace X be a small open neighborhood of a given point. Therefore we can assume that $X = X_1 \cup \ldots \cup X_m$ is a decomposition into local irreducible components at a. We can assume that the X_i are pure dimensional. The points the intersection of two different X_i are singular points since the local rings there are not integral domains. Hence the singular locus of X is the union of the pairwise intersections and the singular loci of the X_i. Since the finite union of closed analytic subsets is analytic we reduced 15.9 to the pure dimensional case.

In the pure dimensional case we will make use of a differential criterion of regular (singular) points: This rests on the implicit function theorem. One version of it states:

Let X be the zero set of m holomorphic functions $f_1, \ldots, f_m$ on some open subset $U \subset \mathbb{C}^n$. Assume that the (complex) Jacobian matrix $J(f, a)$ has rank r at some point $a \in X$. Then a is a regular point of X and $\dim_a X = n - r$.

There is an immediate consequence:

15.10 Lemma. *Let $X \subset \mathbb{C}^n$ be an analytic set that is defined by analytic equations* ifa0

$$f_1(z) = \cdots = f_m(z) = 0$$

in some open neighborhood $0 \in U \subset \mathbb{C}^n$. Let $a \in X$ be a point. The rank r of the Jacobian of $f = (f_1, \ldots, f_m)$ at a is $r \le n - d$, where $d = \dim_a X$. In the case $r = n - d$ the point a is regular.

Proof of the lemma. We can choose r of the functions f_i whose Jacobi matrix has rank r at a. We can assume that $f_1, \ldots, f_r$ is this system. The set of zeros of this system is a analytic set $\tilde{X}$ that is regular and of dimension $n - r$ at a. Since $X \subset \tilde{X}$ we have $d \le n - r$ or equivalently $r \le n - d$. When equality holds X and $\tilde{X}$ agree close to a. Hence X is regular in a like $\tilde{X}$. $\square$

The converse of 15.10 is not true in general. Consider for example the equation $z^2 = 0$ in $\mathbb{C}$. The dimension d is zero but the rank r of the Jacobi matrix at $a = 0$ is 0. Hence the equation $d + r = n$ is false. The reason is that $z^2 = 0$ is the false description. One should better use the equation $z = 0$. The correct converse of 15.10 is:

15.11 Lemma. *Let $X \subset \mathbb{C}^n$ be an analytic set that is defined by analytic* JacCor
equations

$$f_1(z) = \cdots = f_m(z) = 0$$

in some open neighborhood $0 \in U \subset \mathbb{C}^n$. Let $a \in X$ be a point. Assume that the germs of the f_i generate the full vanishing ideal of X in $\mathcal{O}_{\mathbb{C}^n,a}$. Then a is a regular point of X if and only if the Jacobi matrix $J(f,a)$ has the correct rank $n - \dim_a X$.

Proof. It remains to proof that the condition is necessary. So let's assume that a is regular. Due to the implicit function theorem ??? we can assume that X is given by equations $z_{d+1} = \ldots = z_n$=0. For these equations the rank condition is trivial. But we may have different equations. From the assumption about the vanishing ideal we know that both generate the same ideal. Hence the statement follows from

15.12 Lemma. *Let $P = (P_1, \ldots, P_m)$ and $Q = (Q_1, \ldots, Q_l)$ be two systems* eqRa
of power series which generate the same ideal in $\mathcal{O}_n$. Then the Jacobians of P and Q at the origin have the same rank.

The easy proof is left to the reader. □

Now we are able to prove the main result 15.9. We reduced already to the case of a pure dimension case $d = \dim X$. We can assume that X is defined inside some open subset $U \subset \mathbb{C}^n$ as zero set of a finite number of holomorphic functions $f_1, \ldots, f_n$. We choose some point $a \in X$. We can replace U be a smaller neighborhood since the question is od local nature. Since $\mathcal{O}_a$ is noetherian we can assume that $(f_1, \ldots, f_n)_a$ is a radical ideal in the point a. By Cartan's coherence theorem this then is true in a full neighborhood. We can assume that this is true in U. Now the singular locus is described as set of all $z \in U$ such $f_i(z) = 0$ and such that the rank of $J(f,z)$ is smaller than $r = n - d$. This means that all determinants of $r \times r$-matrices vanish. Hence the singular locus can be defined by a finite set of analytic equations. □

Chapter II. Local theory of complex spaces

1. The general notion of a complex space

We introduce the general notion of a complex space in the sense of Grothendieck.

1.1 Definition. *A ringed space $(X, \mathcal{O}_X)$ is a topological space together with a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_X$.* `DefG`

1.2 Definition. *A morphism* `MorC`

$$(f, \varphi) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

between ringed spaces is a pair, consisting of a continuous map $f : X \to Y$ and a homomorphism $\varphi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of $\mathbb{C}$-algebras.

Recall that $f_*\mathcal{O}_X$ is the sheaf $(f_*\mathcal{O}_X)(V) = \mathcal{O}_X(f^{-1}(V))$ with obvious restriction maps. Hence a homomorphism $\varphi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is just a collection of homomorphisms

$$\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V))$$

which is compatible with restrictions.

It is clear that the identity map is a morphism and how one composes two morphisms $(f, \varphi) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$, $(g, \psi) : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$. So we can talk about the *category of ringed spaces.* (It seems to be better to talk about "algebred spaces" instead of "ringed spaces". But this sounds strange.)

Let $(X, \mathcal{O}_X)$ be a ringed space and let $U \subset X$ be an open set. Then $(U, \mathcal{O}_X|U)$ is a ringed space too and there is a natural morphism $(U, \mathcal{O}_X|U) \to (X, \mathcal{O}_X)$.

Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{J} \subset \mathcal{O}_X$ be an ideal sheaf. Then we can consider the support $Y = \operatorname{supp}(\mathcal{O}_X/J)$. Let us assume that Y is closed. (This is the case if $\mathcal{O}_X$ and $\mathcal{J}$ are coherent.) Then we can consider the ringed space

$$(Y, \mathcal{O}_Y), \quad \mathcal{O}_Y = \mathcal{O}_X|Y.$$

We call this a *closed ringed subspace* of $(X, \mathcal{O}_X)$. There is a natural morphism

$$(Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X).$$

On the level of topological spaces it is the natural embedding $i : Y \to X$ where Y carries the induced topology. The map of sheaves

$$\mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$$

is defined in an obvious way. One uses that there is a canonical isomorphism $i_*\mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{J}$.

Let $U \subset \mathbb{C}^n$ be an open domain and let $f_1, \dots, f_m$ be analytic functions on U. We consider the ideal sheaf $\mathcal{J}$ generated by $f_1, \dots, f_m$ in $\mathcal{O}_U$. The support of the sheaf $\mathcal{O}_U/\mathcal{J}$ is the set X of joint zeros of the f_i. We can consider the ringed space

$$\mathcal{O}_X = (\mathcal{O}_U/\mathcal{J})|X.$$

We call $(X, \mathcal{O}_X)$ a *model space*. There is a natural morphism (a closed embedding) of ringed spaces

$$(X, \mathcal{O}_X) \longrightarrow (U, \mathcal{O}_U).$$

1.3 Definition. *A complex space $(X, \mathcal{O}_X)$ is a ringed space which is locally isomorphic to a model space. A morphism between complex spaces is simply called a holomorphic map.* MS

If $U \subset X$ is an open subspace, then $(U, \mathcal{O}_X|U)$ is a complex space too. We call it an open analytic subspace. A morphism $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces is called an open embedding if there exists an open subset $V \subset Y$ such that f factors through an isomorphism

$$(X, \mathcal{O}_X) \xrightarrow{\sim} (V, \mathcal{O}_Y|V).$$

The composition of two open embeddings is an open embedding.

1.4 Remark. *Let $(X, \mathcal{O}_X)$ be a complex space and $\mathcal{J} \subset \mathcal{O}_X$ a coherent ideal sheaf. The support Y of $\mathcal{O}_X/\mathcal{J}$ is a closed subset and $(Y, \mathcal{O}_Y)$ where $\mathcal{O}_Y = (\mathcal{O}_X/\mathcal{J})|Y$ is complex space too.* ComS

Proof. We can assume that $(X, \mathcal{O}_X)$ is a model space, $X \subset U \subset \mathbb{C}^n$ defined by finitely many holomorphic functions $f_1, \dots, f_m$ on the open subset $U \subset \mathbb{C}^n$. So X is their zero set and $\mathcal{O}_X = (\mathcal{O}_U/(f_1, \dots, f_m))|X$. Here $(f_1, \dots, f_m)$ denotes the ideal sheaf generated by the f_i. Now we have to consider an ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$. We can assume that $J = (g_1, \dots, g_k)$ is finitely generated too.

We call such a space that is defined through a coherent ideal sheaf a closed complex (or analytic) subspace. A morphism $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of complex spaces is called a closed embedding if there exists a closed complex subspaces $(Z, \mathcal{O}_Z)$ of $(Y, \mathcal{O}_Y)$ such that f factors through an isomorphism

$$(X, \mathcal{O}_X) \xrightarrow{\sim} (Z, \mathcal{O}_Z).$$

The composition of two closed embeddings is a closed embedding.

2. Complex spaces and holomorphic functions

2.1 Definition. *A holomorphic function on a complex space is a morphism* Dhf

$$(f,\varphi):(X,\mathcal{O}_X)\longrightarrow(\mathbb{C},\mathcal{O}_{\mathbb{C}}).$$

To every holomorphic function we can associate the image of 1 of the homomorphism

$$\mathcal{O}_{\mathbb{C}}(\mathbb{C})\longrightarrow\mathcal{O}_X(X).$$

This is a distinguished global section. It is easy to show the following result.

2.2 Remark. *Let $(X,\mathcal{O}_X)$ be a complex space. The holomorphic functions on it are in one-to-one correspondence to the global sections in $\mathcal{O}_X(X)$.* Rhfg

Let $(X,\mathcal{O}_X)$ be a complex space. Recall that $\mathcal{O}_{X,a}$ are local rings with maximal ideal $\mathfrak{m}_a$ and that the composition

$$\mathbb{C}\longrightarrow\mathcal{O}_{X,a}\longrightarrow\mathcal{O}_{X,a}/\mathfrak{m}_a$$

is an isomorphism.

Let $f\in\mathcal{O}_X(X)$ be a global section of the structure sheaf of a complex space and let $x\in X$ be a point. We can consider the germ f_x and take its coset mod $\mathfrak{m}(\mathcal{O}_{X,x})$. This is a number which we denote by $f(x)$. In this way we get a usual function

$$\tilde f:X\longrightarrow\mathbb{C},\quad \tilde f(x):=f(x).$$

A look at the definition of model spaces shows that $\tilde f$ is continuous. Hence we have constructed a homomorphism of algebras

$$\mathcal{O}_X(X)\longrightarrow\mathcal{C}_X(X).$$

The same can be done for open subsets. We can read this as map of sheaves of $\mathbb{C}$-algebras

$$\mathcal{O}_X\longrightarrow\mathcal{C}_X.$$

There are two basic results about this homomorphism.

Let R be a ring. The nilradical $\mathfrak{n}$ is the set of all nilpotent elements a ($a^n=0$ for some natural number). It is easy to see that $\mathfrak{n}$ is an ideal. Let $\mathcal{O}$ be a sheaf of rings on a topological space X. The nilradical of $\mathcal{O}$ is the sheaf $\mathcal{J}\subset\mathcal{O}$ generated by $\mathfrak{n}(\mathcal{O}(U))$. Concretely this is

$$\begin{aligned}\mathcal{J}(U)&=\big\{f\in\mathcal{O}(U);\quad f_a \text{ nilpotent in } \mathcal{O}_a \text{ for all } a\in U\big\}\\&=\big\{f\in\mathcal{O}(U);\quad \exists\, U=\bigcup U_i \text{ such that } f|U_i \text{ nilpotent in } \mathcal{O}(U_i)\big\}\end{aligned}$$

2.3 Theorem (Rückert). *Let $(X,\mathcal{O}_X)$be a complex space. The kernel $\mathcal{J}$ of the natural homomorphism* ThRue

$$\mathcal{O}_X\longrightarrow\mathcal{C}_X$$

is the nilradical of $\mathcal{O}_X$.

2.4 Theorem (Cartan). *Let $(X, \mathcal{O}_X)$ be a complex space. The nilradical is coherent.* TheCa

A complex space $(X, \mathcal{O}_X)$ is called *reduced* if the natural map $\mathcal{O}_X \to \mathcal{C}_X$ is injective. By Rückert's theorem this means that the nilradical is zero. For a reduced complex space we can consider $\mathcal{O}_X$ as a subsheaf of $\mathcal{C}_X$, i.e. the sections are functions. Reduced complex spaces are also called complex spaces in the sense of Serre. We consider them as full subcategory of the category of complex spaces in the sense of Grothendieck. Notice that a morphism (f, φ) : $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is determined by f. The map $\varphi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is just given by pulling back functions.

Let $X = (X, \mathcal{O}_X)$ be a complex space. Since the radical J is coherent, we can consider the closed analytic subspace related to $\mathcal{J}$. The support of $\mathcal{O}_X/\mathcal{J}$ is the whole X. So this subspace is just

$$X_{\text{red}} := (X, \mathcal{O}_X/J).$$

This defines a functor from the category of complex spaces into the category of complex spaces in the sense of Serre.

Why nilpotents?

Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/m$. The R-module $\mathfrak{m}/\mathfrak{m}^2$ carries a natural structure as k-vector space. This vector sapce is finite dimensional and so is its dual

$$\operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k).$$

Let now $(X, \mathcal{O}_X)$ be a complex space and let $a \in X$. We then can consider the finite dimensional $\mathbb{C}$-vector space

$$T_aX := \operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_a/\mathfrak{m}_a^2, \mathbb{C}).$$

This is called the tangent space.

Consider the topological space p_n consisting of one point and equip it with the sheaf that is associated to the $\mathbb{C}$-algebra $\mathbb{C}^n$ (pointwise multiplication, $\mathbb{C} \to \mathbb{C}^n$ the diagonal embedding. We claim that this is a complex space. To see this we consider the complex plane $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ and the ideal sheaf $\mathcal{J}$ generated by z^{n+1}. The associated complex space is isomorphic to p_n. The associated reduced complex space is the p_1.

2.5 Lemma. *Let $(X, \mathcal{O}_X)$ be a complex space. The morphisms $p_2 \to X$ are in one-to one correspondence with the elements of the tangent space.* MorInf

2.6 Definition. *A subset $A \subset X$ of a complex space $(X, \mathcal{O}_X)$ is called a closed analytic subset if there exists a coherent sheaf of ideals $J \subset \mathcal{O}_X$ such that* Dass

$$Y = \operatorname{supp}(\mathcal{O}_X/\mathcal{J}).$$

The sheaf $\mathcal{J}$ is not unique. A possible choice is the full vanishing ideal sheaf.

2.7 Remark. *Let A, B be two closed analytic subsets of a complex space X. Then $A \cup B$ and $A \cap B$ are closed analytic subsets.* Rabcs

3. Fibre products of complex spaces

Let X, Y be two objects in a category. The direct product of X, Y is a triple $(X \times Y, p, q)$ consisting of an object $X \times Y$ and two morphisms $p : X \times Y \to X$, $q : X \times Y \to Y$ such that the natural map

$$\operatorname{Mor}(X \times Y, Z) \longrightarrow \operatorname{Mor}(X \times Z) \times \operatorname{Mor}(X, Z)$$

is bijective for all objects Z. It is well-known and easy to show that the direct product is unique up to canonical isomorphism in the obvious sense. On says that a category admits direct products if the direct product for two arbitrary objects exists. For example in the category of topological spaces direct products exist. They are given by the usual cartesian product (equipped with the product topology).

3.1 Proposition. *In the category of complex spaces direct products exist. They are compatible with the forgetful functor from the category of complex spaces into the category of topological spaces.* Pdpcs

We will not give the proof in all details. But we will describe several tools which lead to a proof.

1) The first is a gluing principle for sheafs. Assume that $X = \bigcup U_i$ is an open covering of a topological space. Assume also that for each i there is given a sheaf F_i on U_i and for each pair (i, k) of indices there is given an isomorphism $h_{ij} : F_i|(U_i \cap U_j) \to F_j|(U_i \cap U_j)$ with the conditions

   $$h_{ik} = h_{ij} \circ h_{jk} \qquad \text{on } U_i \cap U_j \cap U_j.$$

   Then there exists a sheaf F and a system of isomorphisms $h_i : F|U_i \to F_i$ with the properties

   $$h_{ik} = h_i h_k^{-1} \qquad \text{on } U_i \cap U_k.$$

2) Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be complex spaces such that their direct product $(X \times Y, \mathcal{O}_{X \times Y})$ exists. Assume that $U \subset X$, $V \subset Y$ are open subsets. Then the direct product of $(U, \mathcal{O}_X|U)$ and $(V, \mathcal{O}_Y|V)$ exists and can be identified with
$$(U \times V, \mathcal{O}_{X \times Y}|U \times V).$$

3) Let X, Y be two model spaces which are closed in open subsets $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ and which are defined through holomorphic functions $f_1, \ldots, f_\nu$ on U and $g_1, \ldots, g_\mu$ on V. Then the direct product of the complex spaces X, Y exists. It can be identified with the model space in $U \times V$ defined through the holomorphic functions $f_i(x)g_k(y)$.

The diagonal

Due to the universal property the two identity maps $X \to X$ induce a holomorphic map
$$(X, \mathcal{O}_X) \longrightarrow (X \times X, \mathcal{O}_{X \times X})$$
which we call the diagonal map. Usually we will write this as
$$\Delta \longrightarrow X \times X.$$

3.2 Lemma. *Let $(X, \mathcal{O}_X)$ be a Hausdorff complex space. Then the diagonal* Lddm
map
$$\Delta \longrightarrow X \times X$$
is a closed embedding.

Inverse images of complex spaces

Let $f : X \to Y$ be a continuous map of topological spaces and let $Y' \subset Y$ be a subspace equipped with the induced topology. Then one can consider the inverse image $X' \subset X$ and equip it also with the induced topology. This is what we call the inverse image in the category of topological spaces.

Let $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a holomorphic map of complex spaces. Let $V \subset Y$ be an open subset. Then we can consider the inverse image $U = f^{-1}V$. Both, U and V can be equipped with the restricted structures $\mathcal{O}_U = \mathcal{O}_X|U$ and $\mathcal{O}_V = \mathcal{O}_Y|V$. We get two complex spaces $(U, \mathcal{O}_U)$ and $(V, \mathcal{O}_V)$ and a natural holomorphic map $(U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$. We call $(U, \mathcal{O}_U)$ the inverse image of $(V, \mathcal{O}_V)$.

We describe a similar construction for closed complex subspaces. So let $\mathcal{J}_Y \subset \mathcal{O}_Y$ be a coherent ideal sheaf and let $(Y', \mathcal{O}_{Y'})$ be the associated closed subspace. There is a canonical injection $(Y', \mathcal{O}_{Y'}) \to (Y, \mathcal{O}_Y)$. We define an ideal sheaf in $\mathcal{O}_X$. For this we consider the maps $\mathcal{O}_{Y,f(a)} \longrightarrow \mathcal{O}_{X,a}$. We denote by $\mathcal{J}_{Y,f(a)}\mathcal{O}_{X,a}$ the ideal in $\mathcal{O}_{X,a}$ that is generated by the image of $\mathcal{J}_{Y,f(a)}$. Then we define for open $U \subset X$

$$\mathcal{J}_X(U) = \{\, f \in \mathcal{O}_X(U); \quad f_a \in \mathcal{J}_{Y,f(a)}\mathcal{O}_{X,a} \text{ for all } a \in U \,\}.$$

This is an ideal sheaf. We claim that it is coherent. For this we can assume that $\mathcal{J}_Y$ is finitely generated by $f_1, \dots, f_n$. Then we can consider the images $F_1, \dots, F_n$ with respect to the map $\mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$. Clearly $\mathcal{J}_X$ is generated by $F_1, \dots, F_n$. So we can consider the closed complex subspace $(X', \mathcal{O}'_X)$ defined by $\mathcal{J}_X$. This is called the inverse subspace.

Coincidence spaces

Let $f, g : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be two holomorphic maps between complex Hausdorff spaces. The set $C := \{(x, y) \in X \times Y,\ f(x) = g(x)\}$ is a closed subset. It is natural ask whether it can be equipped with a structure as closed complex subspace. For this we consider the map $X \to X \times X$ that is induced by the two maps f, g. We consider the diagonal Δ as closed complex subspace in $X \times X$ and take the inverse image in X. This is a closed complex subspace of X whose underlying topology is the coincidence set C equipped with the induced topology.

Fibred products of complex spaces

Let $\mathcal{A}$ be a category and let S be an object. We can define a new category $\mathcal{A}_S$. Its objects are morphisms $X \to S$ in $\mathcal{A}$. A morphism between such objects is just a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

One says that fibred products in $\mathcal{A}$ exist if direct products in all categories $\mathcal{A}_S$ exist. In the case that $\mathcal{A}$ has a final element S (this means that $\mathrm{Hom}(X, S)$ consists of precisely one element), then $\mathcal{A}_S$ equals $\mathcal{A}$.) We denote the fibred product of two $X \to S$, $Y \to S$ by

$$X \times_S Y \longrightarrow S$$

and the two structure morphisms by

$$X \times_S Y \longrightarrow X, \quad X \times_S Y \longrightarrow Y.$$

The category of complex spaces has a final object (one point equipped with the constant sheaf $\mathbb{C}$). Hence the following theorem is a generalization of Proposition 3.1.

3.3 Theorem. *In the category of complex spaces fibred products exist. They are compatible with the forgetful functor from the category of complex spaces over $(S, \mathcal{O}_S)$ into the category of topological spaces over S.* Tdpcs

4. Germs of complex spaces

We consider the category of complex spaces. A pointed complex space (X, a) is a complex space with a distinguished point $a \in X$. We can consider also the category of pointed complex spaces. Morphisms are morphisms of complex spaces that map the distinguished point to the distinguished point.

Let (X, a), (Y, b) be two punctured complex spaces. We are interested in holomorphic maps $(U, a) \to (X, b)$ where U is an open neighborhood of a in U. Let $(U_1, a) \to (X, b)$ and Let $(U_1, a) \to (X, b)$ be two such holomorphic maps. We call them equivalent if there exists an open neighborhood $a \in U \subset U_1 \cap U_2$ such that the restrictions of the two maps are equal (as morphisms of complex spaces). We denote the equivalence class by

$$(X, a) - -- > (Y, b).$$

So a dashed arrow is represented by a holomorphic map $(U, a) \to (Y, b)$ where U is an open neighborhood of a. The dashed arrows define a category. This is the *category of germs of complex spaces*. So the objects are pointed complex spaces. Notice that two pointed complex spaces (X, a), (Y, b) are isomorphic in this category if and only if there exist open neighborhoods $a \subset U \subset V$ and $b \subset V \subset Y$ such that (U, a) and (V, b) are isomorphic as pointed complex spaces.

Let $(X, a) \to (Y, b)$ be two pointed complex spaces and let $f : (U, a) \to (Y, b)$ be a holomorphic map that maps a to b. Then we can consider

$$f_a^* : \mathcal{O}_{Y,b} \longrightarrow \mathcal{O}_{U,a}.$$

Since $\mathcal{O}_{U,a}$ and $\mathcal{O}_{X,a}$ are canonically isomorphic we can read this as homomorphism

$$\mathcal{O}_{Y,b} \longrightarrow \mathcal{O}_{X,a}.$$

It depends only on the equivalence of f. So we get a contravariant functor

category of germs of complex spaces $\longrightarrow$ category of analytic algebras.

4.1 Theorem. *The category of germs of complex spaces is dual to the category of analytic algebras.* DefDu

5. Finite maps

We recall Definition I.9.6.

A continuous map $f : X \to Y$ between locally compact Hausdorff spaces is called finite if is proper and the fibres are finite sets.

For example closed embeddings are finite.

5.1 Lemma. *Let $f : X \to Y$ be a finite map between locally compact* Lfl
topological Hausdorff spaces and let $a \in X$. For every neighborhoods $a \in X' \subset X$, $f(a) \in Y' \subset Y$ there exist open neighborhoods $a \in U \subset X'$, $f(a) \subset V \subset Y'$ such that
a) *$f(U) \subset V$.*
b) *$U \to V$ is finite.*
c) *$f^{-1}(f(a)) \cap U = \{a\}$.*

Proof. We will make use of Lemma I.9.11, 2).

5.2 Definition. *Let $B \to A$ be a homomorphism of analytic algebras.* Dgr
*A **geometric realization** is a holomorphic map of pointed complex spaces $(X, a) \to (Y, b)$ together with isomorphisms $B \to \mathcal{O}_{Y,y}$, $A \to \mathcal{O}_{X,x}$ such that the diagram*

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y,b} & \longrightarrow & \mathcal{O}_{X,a} \end{array}$$

commutes.

It is clear that every homomorphism of analytic algebras admits a geometric realization.

We used the notion "geometric realization" already in Chapt. I, Sect. 8 (compare Remark I.9.8). There is no risk of confusion.

Lemma 5.1 has the following consequence.

5.3 Lemma. *Let $A \to B$ and $B \to C$ be two homomorphisms of analytic* Labc
algebras such that each of them can be realized by a finite holomorphic map. Then the composition $A \to C$ can also be realized by a finite holomorphic map.

We give a simple example how finite maps come up in complex analysis.

5.4 Lemma. *Let $V \subset \mathbb{C}^{n-1}$ be open and let $P \in \mathcal{O}(V)[z_n]$ be a normalized* LvPn
polynomial. We denote the zero set of P by $X \subset V \times \mathbb{C}$. The natural projection $\pi : X \to V$ is finite.

Proof. The fibres are finite. Hence it remains to show that the projection is proper. Let $K \subset V$ be a compact set. Continuity of the roots shows that there exists a bounded set $B \subset \mathbb{C}$ such that $\pi^{-1}(K) \subset K \times B$. Its closure and hence $\pi^{-1}(K)$ is compact. □

We already mentioned that for a ring homomorphism $A \to B$ there are two important conditions of finiteness.

a) B is finitely generated as A-algebra.
b) B is finitely generated as A-module.

Clearly a) implies b), but b) is much stronger. In the following a finite homomorphism $A \to B$ is to be understood as in b).

We are interested to describe all homomorphisms $B \to A$ of analytic algebras that admit a geometric realization which is finite.

Every homomorphism of an analytic algebra $A \to B$ can be embedded into a commutative diagram

$$\begin{tikzcd} A \arrow[r] & B \\ \mathcal{O}_n \arrow[u] \arrow[r, "f"] & \mathcal{O}_m \arrow[u, "g"'] \end{tikzcd}$$

where the vertical arrows are surjective. This diagram can be modified as follows.

$$\begin{tikzcd} A \arrow[r] & B \\ \mathcal{O}_n \arrow[u] \arrow[r, "F"] & \mathcal{O}_{n+m} \arrow[u, "G"'] \end{tikzcd}$$

Here $F : \mathcal{O}_n \to \mathcal{O}_{n+m}$ is the natural embedding ($F(z_i) = z_i$) and G is given by

$$G(z_i) = g(f(z_i)) \text{ for } i = 1 \dots n \quad \text{and} \quad G(z_i) = 0 \text{ for } i > n.$$

5.5 Remark. *Each homomorphism $A \to B$ of analytic algebras can be* Rhae
extended to a commutative diagram

$$\begin{tikzcd} A \arrow[r] & B \\ \mathcal{O}_n \arrow[u] \arrow[r] & \mathcal{O}_{n+m} \arrow[u] \end{tikzcd}$$

where the vertical arrows are surjective an where $\mathcal{O}_n \to \mathcal{O}_{n+m}$ is the natural embedding that comes from the projection $\mathbb{C}^{n+m} \to \mathbb{C}^n$.

We will apply this remark several times. Sometimes it is possible to reduce to the case $m = 1$ with the help of induction on m. For this one proceeds as follows. Denote the kernels of $\mathcal{O}_n \to A$, $\mathcal{O}_{n+m} \to B$ by $\mathfrak{a}, \mathfrak{b}$. From the injectivity follows

$$\mathfrak{a} = \mathfrak{b} \cap \mathcal{O}_n.$$

We can define immediate ideals

$$\mathfrak{a}_{n+j} = \mathfrak{b} \cap \mathcal{O}_{n+j} \qquad (\mathfrak{a}_n = \mathfrak{a},\ \mathfrak{a}_{n+m} = \mathfrak{b})$$

and use them to define intermediate algebras

$$A_{n+j} = \mathcal{O}_{n+j}/\mathfrak{a}_{n+j}.$$

This gives a commutative diagram

$$\begin{tikzcd}
A_n \arrow[r] & A_{n+1} \arrow[r] & \cdots \arrow[r] & A_{n+m} \\
\mathcal{O}_n \arrow[r] \arrow[u] & \mathcal{O}_{n+1} \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{O}_{n+m} \arrow[u]
\end{tikzcd} \qquad (A_n = A,\ A_{n+m} = B).$$

The vertical arrows are surjective. The arrows in the first row are injective. The second row consists of natural embeddings.

5.6 Lemma. *Consider a diagram* Lfnj

$$\begin{tikzcd}
A \arrow[r] & B \\
\mathcal{O}_n \arrow[r] \arrow[u] & \mathcal{O}_{n+m} \arrow[u]
\end{tikzcd}$$

as in Remark 5.5. The homomorphism $A \to B$ is finite if and only if for each $j \in \{1, \dots, m\}$ there exists a z_{n+j}-general element in the kernel of $\mathcal{O}_{n+m} \to B$.

Proof. Assume that the z_{n+j}-general elements exist. We have to show that $A \to B$ is finite. We can argue by induction on m and hence reduce to the case $m = 1$. In this case there exists a Weierstrass polynomial in the kernel of $\mathcal{O}_{n+m} \to B$. The claim then follows from the division theorem.

To prove the converse we can assume that there is no z_{n+m}-general element in the kernel of $\mathcal{O}_{n+m} \to B$. But then the image of z_{n+m} in B cannot be integral over A. $\square$

5.7 Remark. *A finite homomorphism $B \to A$ of analytic algebras can be realized by a closed embedding $(X, a) \to (Y, b)$ if and only if it is surjective.* Rres

Proof. For a closed embedding $(X,a)\to(Y,b)$ the homomorphism $\mathcal{O}_{Y,b}\to\mathcal{O}_{X,a}$ is surjective. This follows immediately from the definition of a closed embedding.

We prove the converse. Let $A\to B$ be surjective. Any homomorphism $A\to B$ of analytic algebras can be splitted in two homomorphisms $A\to A_1\to B$ where $A\to A_1$ is surjective and $A_1\to B$ is injective. If we want to study geometric realizations, the case of injective $A\to B$ is essential. So let $A\to B$ be injective. We extend it to

$$\begin{tikzcd} A \arrow[r] & B \\ \mathcal{O}_n \arrow[u] \arrow[r] & \mathcal{O}_{n+m} \arrow[u] \end{tikzcd}$$

5.8 Theorem. *Every finite homomorphism $B\to A$ of analytic algebras can be realized by a finite holomorphic map $(X,a)\to(Y,b)$.* Tfhr

Proof. We can assume that there is a commutative diagram

$$\begin{tikzcd} B \arrow[r] & A \\ \mathcal{O}_{n-1} \arrow[u] \arrow[r] & \mathcal{O}_n \arrow[u] \end{tikzcd}$$

where the second row is the natural injection $(z_i\mapsto z_i)$. We denote the kernels of the vertical arrows by $\mathfrak{b}$, $\mathfrak{a}$. Since $B\to A$ is finite there exists an integral equation of the image of z_n in A over B. This means that there exists a normalized polynomial $P\in\mathcal{O}_{n-1}[z_n]$ whose image in A is zero. Hence it is contained in $\mathfrak{a}$. We consider the ideal $(\mathfrak{b},P)\subset\mathfrak{a}$ that is generated by (the image of) $\mathfrak{b}$ and P). The homomorphism $\mathcal{O}_n/(\mathfrak{b},P)\to\mathcal{O}_n/\mathfrak{a}$ is surjective and can be realized by a closed embedding. Therefore it is sufficient to assume that $\mathfrak{a}=(\mathfrak{b},P)$. We can extend $\mathfrak{b}$ to a coherent ideal sheaf $\mathcal{B}$ on an open neighborhood $0\subset V\subset\mathbb{C}^{n-1}$. We also can assume that $P\in\mathcal{O}(V)[z_n]$. Then consider the ideal sheaf $\mathcal{A}=(\mathcal{B},P)$ whose germ at 0 is $\mathfrak{a}$. Now we get a geometric realization $X\to Y$ together with a commutative diagram

$$\begin{tikzcd} V\times\mathbb{C} \arrow[r] & V \\ X \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

The vertical arrows are closed embeddings. We denote by $N(P)$ the zero locus of P. Then the above diagram reads as

$$\begin{tikzcd} V\times\mathbb{C} \arrow[r] & N(P) \\ X \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

We know that the first row is finite. The vertical arrows are closed embeddings. The first row is finite. Then the second row is finite too. □

We prove a converse theorem in a rather strong form.

5.9 Theorem. *Let $f : (X, a) \to (Y, b)$ be a holomorphic map between pointed complex spaces. Assume that a is an isolated point of its fibre $f^{-1}(f(a))$. Then the homomorphism $\mathcal{O}_{Y,b} \to \mathcal{O}_{X,a}$ is finite.* Tipf

Proof. Consider a diagram

$$\begin{tikzcd} A \arrow[r] & B \\ \mathcal{O}_n \arrow[u] \arrow[r] & \mathcal{O}_{n+m} \arrow[u] \end{tikzcd} \quad , \quad A = \mathcal{O}_{Y,b},\ B = \mathcal{O}_{X,a},$$

as in Remark 5.5. We argue indirectly and assume that the homomorphism $A \to B$ is not finite. Then there exists a j just that the kernel of $\mathcal{O}_{n+m} \to A$ contains no z_{n+j}-general element. We can assume $j = m$. Then the image of $P(0, \dots, 0, z_{n+m})$ is zero in A. But then the points $(0, \dots, z_{n+m})$ give points in the fibre. So a is not an isolated point of its fibre $f^{-1}(f(a))$.

5.10 Theorem. *A holomorphic map $f : X \to Y$ is locally finite at a if and only if the corresponding map of analytic algebras $\mathcal{O}_{Y,f(a)} \to \mathcal{O}_{X,a}$ is finite.* LocF

6. Grauert's coherence theorem for finite maps

Let $f : X \to Y$ be a continuous map between topological spaces and let F be sheaf of sets on X. Then the direct image sheaf f_*F is defined through $(f_*F)(V) = F(f^{-1}(V)$. Let $a \in X$, $b = f(a)$ and let $b \subset V \subset Y$ be an open neighborhood of b. Then there is a commutative diagram

$$\begin{tikzcd} (f_*(F))(V) \arrow[d] & = & F(f^{-1}(V)) \arrow[d] \\ (f_*F)_b \arrow[rr] & & F_a \end{tikzcd}$$

or, a little more general. Let $a_1, \dots, a_n$ be points in the fibre over b. Then there is a natural commutative diagram

$$\begin{tikzcd} (f_*(F))(V) \arrow[d] & = & F(f^{-1}(V)) \arrow[d] \\ (f_*F)_b \arrow[rr] & & F_{a_1} \times \cdots \times F_{a_n} \end{tikzcd}$$

6.1 Lemma. *Let $f : X \to Y$ be a finite map between locally compact spaces and let $b \in Y$ be some point. For each sheaf of sets F on X the natural map* Lnib

$$(f_*F)_b \longrightarrow \prod_{a \in X;\ f(a)=b} F_a$$

is a bijection.

Corollary *The functor $F \mapsto f_*F$ from the category of sheaves of abelian groups on X into that on Y is exact.*

An important result of Grauert states.

6.2 Theorem. *Let $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a finite holomorphic map between complex spaces. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Then the direct image $f_*\mathcal{M}$ is a coherent $\mathcal{O}_Y$-module.* GrauFi

We recall that $f_*\mathcal{M}$ a priori is a $f_*\mathcal{O}_X$ module. But the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ equips it with a structure as $\mathcal{O}_Y$ module.

Proof of Theorem 6.2. In a first step we assume that the theorem is true in the case $\mathcal{M} = \mathcal{O}_X$. We show that then it is true in general. So let $\mathcal{M}$ be coherent. Since coherence is a local property, we can assume that $\mathcal{M}$ is the cokernel of a homomorphism of free sheaves,

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{M} \longrightarrow 0.$$

Since f_* is exact, we get that $f_*\mathcal{M}$ is the cokernel of $f_*\mathcal{F} \to f_*\mathcal{G}$. This shows that $f_*\mathcal{M}$ is coherent.

Chapter III. Algebraic tools

1. Sets and classes

Besides sets their exist classes. Every set is also a class but not conversely. Imagine classes which are not sets as oversized sets which are to big to deserve to be called sets. The main example is the class of all sets. One can do with classes all one does usually with sets besides one exception. Let A be a class and let $E(a)$ be for each $a \in A$ a property that can be true ore false. One would like to define the class of all $a \in A$ for which $E(a)$ is true. But this class usually does not exist. It exists always if A is a set. In this case it is a set too.

2. Categories

A category $\mathcal{A}$ consists of

1) a class whose elements a re called *objects*.

2) For each two objects $A, B \in \mathcal{A}$ there is associated a set $\mathrm{Mor}(A, B)$. Its elements are called morphisms.

3) For each three objects $A, B, C \in \mathcal{A}$ there is associated a map

$$\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \longrightarrow \mathrm{Mor}(A, C), \quad (f, g) \longmapsto g \circ f.$$

It is called the composition of morphisms.

4) For each object A there is a distinguished element $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ called the identity.

There are obvious axioms: the composition is associative in an obvious sense. The identities are neutral in the sense

$$f \circ \mathrm{id}_A = f, \quad \mathrm{id}_A \circ g = g, \qquad (f \in \mathrm{Mor}(A, B),\ g \in \mathrm{Mor}(B, A)).$$

Typical examples of categories are

The category of sets (objects are sets and morphisms are just maps).

The category of groups (objects are groups and morphisms are just homomorphisms).

The category of topological spaces (objects are topological spaces and morphisms are continuous maps).

3. Functors

Let $\mathcal{A}$, $\mathcal{B}$ be two categories. A functor $F : \mathcal{A} \to \mathcal{B}$ consists of

1) A map F that associates to each object $A \in \mathcal{A}$ an object $F(A) \in \mathcal{B}$

2) For each to objects $A, B \in \mathcal{A}$ there is given a distinguished map, also denoted by F

$$F : \mathrm{Mor}(A, B) \longrightarrow \mathrm{Mor}(F(A), F(B)).$$

Again there are obvious axioms. The identity map goes to the identity map. Let A, B, C be three objects. Then the diagram

$$\begin{tikzcd} \mathrm{Mor}(A,B) \arrow[rr] \arrow[dr] & & \mathrm{Mor}(B,C) \arrow[dl] \\ & \mathrm{Mor}(A,C) & \end{tikzcd}$$

is commutative.

An example: we consider the category $\mathcal{A}$ whose objects are open subsets $U \subset \mathbb{R}^n$ with a distinguished point $a \in U$. Morphisms are (totally) differentiable maps which map the distinguished point to the distinguished point. Let $\mathcal{B}$ be the category whose objects are $\mathbb{R}^n$ for $n \geq 0$ and whose morphisms are linear maps. We define a functor $F : \mathcal{A} \to \mathcal{B}$. For a pointed set (U, a) with open $U \subset \mathbb{R}^n$ we define $F(U, a) = \mathbb{R}^n$. For a morphism $f : (U, a) \to (V, b)$ we define $F(f)$ to be the linear map that is associated to the functional matrix. The chain rule says that this defines a functor.

4. Equivalent categories

We assume that the reader knows the definition of a category $\mathcal{A}$ and of a (covariant) functor $F : \mathcal{A} \to \mathcal{B}$. If $G : \mathcal{B} \to \mathcal{C}$ is another functor, the composition $G \circ F$ us defined. This composition is associative. There is the identity functor $\mathcal{A} \to \mathcal{A}$. Two functors $F, G : \mathcal{A} \to \mathcal{B}$ are called isomorphic if there can be chosen of each X in $\mathcal{A}$ an isomorphism $F(X) \to G(X)$ such that for all morphisms $X \to Y$ in $\mathcal{A}$ the diagram

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & G(X) \arrow[d] \\ F(Y) \arrow[r] & G(Y) \end{tikzcd}$$

commutes.

A category $\mathcal{A}$ is called a subcategory of $\mathcal{B}$ if every object of $\mathcal{A}$ is an object of $\mathcal{B}$ and if for all objects $X, Y \in \mathcal{A}$ one has $\mathrm{Hom}_{\mathcal{A}}(X,Y) \subset \mathrm{Hom}_{\mathcal{B}}(X,Y)$. It is called a *full subcategory* if always equality holds.

4.1 Definition. *A functor $F : \mathcal{A} \to \mathcal{B}$ is called an equivalence of categories if the following two conditions hold:* Dfae

1) *For any two objects $X, Y \in \mathcal{A}$ the natural map*

$$\mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(F(X), F(Y))$$

is bijective.

2) *For each object $Y \in \mathcal{B}$ there exists an object $A \in \mathcal{B}$ such that Y and $F(X)$ are isomorphic.*

One can ask whether there is an inverse functor. Here is a problem with the axiom of choice. In many cases inverse functors exist in the following sense.

4.2 Definition. *Let $\mathcal{A}$, $\mathcal{B}$ be categories. Two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are called inverse to each other if the functors $G \circ F$ and $F \circ G$ are isomorphic to the identity functor on $\mathcal{A}$ and $\mathcal{B}$.* Dfi

Then F and G are equivalence of categories. One calls G an inverse of F. (Notice that it is not uniquely determined.)

5. Modules and ideals

All rings which we consider are assumed to be commutative and with unit elements. Ring homomorphisms are assumed to map the unit element into the unit element. A module M over a ring A is an abelian group together with a map $A \times M \to M$, $(a,m) \longmapsto am$, such that the usual axioms of a vector space are satisfied including $1_A m = m$ for all $m \in M$. The notion of linear maps, kernel, image of a linear map are as in the case of vector spaces. But in contrast to the case of vector spaces, a module has usually no basis. A module which admits a basis is called free. A finitely generated free module is isomorphic to R^n.

If $M \subset N$ is a submodule, then the factor group N/M carries a structure of an A-module.

Recall that an ideal $\mathfrak{a}$ in a Ring R is an abelian subgroup such that $ra \in \mathfrak{a}$ for $r \in R$ and $a \in \mathfrak{a}$. Hence an ideal is nothing but an R-submodule of R. The factor $R/\mathfrak{a}$ is not only a R-module but carries a structure as ring such that $R \to R/\mathfrak{a}$ is a ring homomorphism.

An ideal is called finitely generated if it is finitely generated as module. This means that there are elements $a_1, \dots, a_n$ such that $\mathfrak{a} = Ra_1 + \cdots + Ra_n$.

One writes $\mathfrak{a} = (a_1, \dots, a_n)$. The product $\mathfrak{ab}$ of two ideals is the set of all finite sums $\sum_i a_i b_i$ with $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{b}$. Ideal multiplication is associative. Especially powers of ideals are defined.

All what we have said about exact sequences of abelian groups is literarily true for A-modules.

6. Divisibility

We recall some basic notions of divisibility in rings. Let R be ring (commutative and with unit). An element $a \in R$ of a ring is called a unit if the equation $ax = 1_R$ is solvable in R. Then the solution is unique. The set R^* of units is a group under multiplication. A ring is called an integral domain if $ab = 0 \Rightarrow a = 0$ or $b = 0$.

6.1 Definition. *Let R be an integral domain. An element $a \in R - R^*$ is called* UZer

a) ***indecomposable****, if one has*

$$a = bc \quad \Longrightarrow \quad b \text{ or } c \text{ is a unit}$$

b) ***prime element,*** *if*

$$a|bc \quad \Longrightarrow \quad a|b \text{ or } a|c$$

($a|b$ means that the equation $b = ax$ is solvable in R). Notice that units are not prime elements.)

Of course prime elements are indecomposable, but usually the converse is false.

Example. Let $R = \mathbb{C}[X]$ be the polynomials ring in one variable over $\mathbb{C}$ and R_0 the sub-ring of all polynomials without linear term. The element X^3 is indecomposable in R_0 but not a prime: $X^3 | X^2 \cdot X^4$.

6.2 Definition. *The integral domain R is called* ***factorial*** *or* ***UFD-ring,*** *if the following two conditions are satisfied:* ZPEr

1) *Each element $a \in R - R^*$ can be written as product of finitely man indecomposable elements.*
2) *Each indecomposable element is prime.*

In factorial rings the decomposition into primes is unique in the following sense: Let

$$a = u_1 \cdots u_n = v_1 \cdots v_m$$

be two decompositions of $a \in R - R^*$ into primes. Then one has

a) $m = n$.

b) There exists a permutation σ of the digits $1, \dots, n$, such that

$$u_\nu = \varepsilon_\nu v_{\sigma(\nu)}, \quad \varepsilon_\nu \in R^* \quad \text{for } 1 \le \nu \le n.$$

It is easy to prove this by induction.

Examples for factorial rings.

1) Each field is factorial.
2) $\mathbb{Z}$ is factorial
3) By an important *Theorem of Gauss* the polynomial ring $R[z_1, \dots, z_n]$ over a factorial ring is factorial too.

6.3 Theorem of Gauss. *The polynomial ring $R[z_1, \dots, z_n]$ over a factorial ring is factorial too.* SGZ

7. The discriminant

The discriminant should be treated in an course of basic algebra: We just recall the basic facts. One constructs for each natural number n a polynomial Δ_n of n variables over the ring $\mathbb{Z}$ of integers. Using this universal polynomial one defines for any normalized polynomial

$$P = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

over a ring R the discriminant

$$d(P) := \Delta_n(a_0, \dots, a_{n-1}) \in R.$$

The basic fact about the discriminant is: Assume that R is factorial. Then P is square free if and only if $d(P) \neq 0$.

We just give a comment. In the case $R = \mathbb{C}$ a polynomial is square free if and only if it has no double zero. The discriminant of the quadratic polynomial $X^2 + bX + c$ is $b^2 - 4c$.

8. Noetherian rings

In commutative algebra there is a basic notion of noetherian ring. A ring R (commutative and with unit) is called noetherian, if any ideal $\mathfrak{a} \subset R$ is finitely generated. Noetherian rings have the basic property that a sub-module of a finitely generated module is finitely generated. It is trivial that the factor ring of a noetherian ring is noetherian. The Hilbert basis theorem states the following: The polynomial ring $R[X_1, \dots, X_n]$ over a noetherian ring is noetherian. Hence every finitely generated R-algebra is noetherian.

A ring R is called local if it is not the zero ring and if the set of all non-units $\mathfrak{m}$ is an ideal. Then the factor ring $R/\mathfrak{m}$ is a field. A homomorphism $A \to B$ of local rings is call local, if the maximal ideal of A is mapped into the maximal ideal of B. A field K is a local ring, $\mathfrak{m} = \{0\}$. The ring $\mathbb{Z}$ of integers is not a local ring, since the units are just the elements ± 1. Similarly the ring of polynomials $K[X_1, \dots, X_n]$ $(n \geq 1)$ is not a local ring. The basic example for us the ring $\mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$ of convergent power series. Elements with a non zero constant term are invertible. Elements with zero constant term are not invertible. Obviously they form an ideal $\mathfrak{m}_n$. The residue field $\mathcal{O}_n/\mathfrak{m}_n$ is isomorphic to $\mathbb{C}$. More precisely the composition of the natural homomorphisms

$$\mathbb{C} \longrightarrow \mathcal{O}_n \longrightarrow \mathcal{O}_n/\mathfrak{m}_n$$

is an isomorphism. We can use this isomorphism to identify $\mathbb{C}$ and $\mathcal{O}_n/\mathfrak{m}_n$.

In this connection we want to mention another algebraic result. Let M be a module over a ring R

8.1 Lemma von Nakayama. *Let M be a finitely generated module over a local ring R with maximal ideal $\mathfrak{m}$. Assume $\mathfrak{m}M = M$. Then $M = 0$.* LemNak

There is a rather obvious application:

8.2 Lemma. *Let R be a noetherian local ring R and $r_1, \dots, r_n$ elements of the maximal ideal. Assume that their cosets mod $\mathfrak{m}^2$ generated $\mathfrak{m}/\mathfrak{m}^2$ as R-module. Then they generate $\mathfrak{m}$.* HomGenLoc

For the prove one applies the lemma of Nakayama to $\mathfrak{m}/(r_1, \dots, r_n)$. We also mention that $\mathfrak{m}/\mathfrak{m}^2$ is not only an R-module but an $R/\mathfrak{m}$ module in a natural way. Hence it is vector space over the field $R/\mathfrak{m}$. A subset of $\mathfrak{m}/\mathfrak{m}^2$ is an R-submodule if and only if it as $R/\mathfrak{m}$-module.

8.3 Krull intersection theorem, first version. *Let R be a local noetherian ring. The intersection of all powers of the maximal ideal is zero.* KrullI

The intersection theorem has an important consequence for noetherian local rings:

8.4 Lemma. *Let $f, g : A \to B$ be two local homomorphisms between noetherian local rings. Assume that there exist generators $a_1, \dots, a_n$ of the maximal ideal of A such that $f(a_i) = g(a_i)$. Then $f = g$.* HomMaxDef

There is second version of Krull's intersection theorem:

8.5 Krull intersection theorem, second version. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Assume that M is a finitely generated R-module. Then for each submodule $N \subset M$ one has* KrullIntS

$$N = \bigcap_{\nu=1}^{\infty} (N + \mathfrak{m}^\nu M).$$

If one applies this version to $M = R$ and $N = 0$, one obtains the first version.

Finiteness properties for algebras

Recall that a an algebra is just a fing homomorphism $\varphi : A \to B$. Then B is called an A-algebra. One can consider B as A-module by $ab =: \varphi(a)b$. An algebra homomorphism $B \to C$ of A-algebras is just a ring homomorphism that is also A-linear.

There are two basic finiteness properties for algebras $A \to B$. The first is: B is finitely generated as A-algebra. This means that there exists a surjective homomorphism of A-algebras of the polynomial ring $A[X_1, \dots, X_n]$ to B. This means that there are finitely many elements $b_1, \dots, b_n$ such that any element of B can be expressed as a polynomial with coefficients in A. There is another much more restrictive finiteness condition: B is finitely generated as A-module. This means that there exist finitely many elements $b_1, \dots, b_n$ such that $B = Ab_1 + \cdots + Ab_n$. We call a ring extension $A \to B$ finite, if this second stronger condition is satisfied. A ring extension $A \to B$ is called integral, if any element $b \in B$ satisfies an equation

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0, \qquad a_i \in A.$$

Be aware. This notion of "integral" has nothing to do with "integral domain". Notice that the highest coefficient is one. It is a basic fact that finite extensions are integral. More precisely, a ring extension is finite if and only if it is integral and if it is finitely generated as algebra. The usual noether normalization theorem in commutative algebra states the following:

If $K \to A$ is a finitely generated algebra over a field K then there exist a subalgebra $A_0 \subset A$ such that A is a finite over A_0 and such that A_0 is isomorphic as K-algebra to a polynomial ring $K[X_1, \dots, X_n]$. The number n is unique. It equals the so-called Krull dimension of A.

An ideal $\mathfrak{p}$ in a ring R is called a prime ideal if $R/\mathfrak{p}$ is an integral domain. Concretely this means

$$ab \in \mathfrak{p} \Longrightarrow a \in \mathfrak{p} \quad \text{or} \quad b \in \mathfrak{p}.$$

The Krull dimension $\dim A$ is a basic notion of commutative algebra. It is defined for any commutative ring with unity and can be an integer ≥ 0 or ∞. By definition is the Supremum of all n such that there exists a chain of prime ideals

$$\mathfrak{p}_0 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} \mathfrak{p}_n.$$

The basic facts about the Krull dimension are:

8.6 Proposition. *Let R be a local noetherian ring such the maximal ideal can be generated by n elements. Then* $\dim R \leq n$. KruPro

The rings $K[z_1, \dots, z_n]$, $K[[z_1, \dots, z_m]]$ (where K is a field) and the ring $\mathbb{C}\{z_1, \dots, z_n\}$ have Krull dimension n.

A maximal chain of prime ideals in all three cases is

$$0 \subset (z_1) \subset \ldots \subset (z_1, \dots, z_n).$$

This shows that the dimension is $\geq n$. That the dimension equals n follows in the case $\mathbb{C}\{z_1, \dots, z_n\}$ from the first part.

8.7 Theorem of Cohen Seidenberg. *If $A \subset B$ is an integral ring extension of noetherian rings then* $\dim A = \dim B$. CohSeid

An important result of Krull dimension theory is:

8.8 Proposition. *Let R be a noetherian local ring and $a \in R$ a non-zero divisor. Then* KrDimDown

$$\dim R/(a) = \dim R - 1.$$

Corollary. *If $\mathfrak{a}$ is an ideal which contains a non-zero divisor then*

$$\dim R > \dim R/\mathfrak{a}.$$

Recall that a ring R is called an integral domain if $ab = 0 \Rightarrow a = 0$ or $b = 0$. We recall that each integral domain is contained in a field K as subring. One can achieve that K consists of all a/b, $a, b \in R$, $b \neq 0$. Such a field is called a field of fractions. A field of fractions is uniquely determined up to canonical isomorphism in an obvious way. Hence one talks about "the" field of fractions.

A special case of the so-called primary decomposition in noetherian ring states:

8.9 Proposition. *Every proper radical ideal in a noetherian ring is the intersection of finitely many prime ideals.* PrimInt

Recall that an ideal is called a radical ideal if $a^n = 0$ for some natural number implies $a = 0$. Prime ideals of course are radical ideals. The intersection of radical ideals is a radical ideal.

Chapter IV. Sheaves

1. Presheaves

1.1 Definition. *A presheaf F (of abelian groups) on a topological space X is a map which assigns to every open subset $U \subset X$ an abelian group $F(U)$ and to every pair U, V of open subsets with the property $V \subset U$ a homomorphism* DPG

$$r_V^U : F(U) \longrightarrow F(V)$$

such that for three open subsets U, V, W with the property $W \subset V \subset U$

$$r_W^U = r_W^V \circ r_V^U$$

holds:

Example: $F(U)$ is the set all continuous functions $f : U \to \mathbb{C}$ and $r_V^U(f) := f|V$ (restriction).

Many presheaves generalize this example. Hence the maps r_V^U are called "restrictions" in general and one uses the notation

$$s|V := r_V^U(s) \qquad \text{for} \quad s \in F(U).$$

The elements of $F(U)$ sometimes are called "sections" of F over U. In the special case $U = X$ they are called "global" sections.

1.2 Definition. *Let X be a topological space. A homomorphism of presheaves* DAP

$$f : F \longrightarrow G$$

is a family of group homomorphisms

$$f_U : F(U) \longrightarrow G(U),$$

such that the diagram

$$\begin{array}{ccc} F(U) & \longrightarrow & G(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & G(V) \end{array}$$

commutes for every pair $V \subset U$ of open subsets, i.e. $f_U(s)|_G V = f_V(s|_F V)$.

It is clear how to define the identity map $\mathrm{id}_F : F \to F$ of a presheaf and the composition $g \circ f$ of two homomorphims $f : F \to G$, $g : G \to H$ of presheaves.

There is also a natural notion of a sub-presheaf $F \subset G$. Besides $F(U) \subset G(U)$ for all U one has to demand, that the restrictions are compatible. This means:

The canonical inclusions $i_U : F(U) \to G(U)$ *define a homomorphism* $i : F \to G$ *of presheaves.*

When $f : F \to G$ is a homomorphism of presheaves, the images $f_U(F(U))$ define a sub-presheaf of G. We call it the *presheaf-image* and denote it by

$$f_{\mathrm{pre}}(F).$$

It is also clear that the kernels of the maps f_U define a sub-presheaf of F. We denote it by $\mathrm{Kernel}(f : F \to G)$. When F is a sub-presheaf of G then one can can consider the factor groups $G(U)/H(U)$. It is clear how to define restriction maps to get a presheaf $G/_{\mathrm{pre}}F$. We call this presheaf the *factor-presheaf*.

Since we have defined Kernel and Image we can also introduce the notion of a *preasheaf-exact sequence.* A sequence $F \to G \to H$ is presheaf-exact if and only if $F(U) \to G(U) \to H(U)$ is exact for all U. What we have said about exact sequences of abelian groups carries literarily over to presheaf-exact sequences of presheaves of abelian groups.

2. Germs and Stalks

let F be a presheaf on a topological space X und let $a \in X$ be a point. We consider pairs (U, s), where U is an open neighpourhood of a and $s \in F(U)$ a section over U. Two pairs (U, s), (V, t) are called equivalent, if there exists an open neighborhood $a \in W \subset U \cap V$, such that $s|W = t|W$. This is an equivalence relation. The equivalence classes

$$[U, s]_a := \left\{ (V, t); \quad (V, t) \sim (U, s) \right\}$$

are called *germs* of F in the point a. The set of all germs

$$F_a := \left\{ [U, s]_a, \quad a \subset U \subset X, \ s \in F(U) \right\}$$

is the so-called *stalk* of F in a. The stalk carries a natural structure as abelian group. One defines

$$[U, s]_a + [V, t]_a := [U \cap V, s|U \cap V + t|U \cap V]_a.$$

We use frequently the simplified notation

$$s_a = [U, s]_a.$$

For every open neighborhood $a \in U \subset X$ there is an obvious homomorphism

$$F(U) \longrightarrow F_a, \quad s \longmapsto s_a.$$

A homomorphism of presheaves $f : F \to G$ induces natural mappings

$$f_a : F_a \longrightarrow G_a \qquad (a \in X).$$

The image of a germ $[U, s]_a$ is simply $[U, f_U(s)]_a$. It is easy to see that this is well-defined.

2.1 Remark. *Let $F \to G$ and $G \to H$ be homomorphism of presheaves and let $a \in X$ be a point. Assume that every neighborhood of a contains a small open neighborhood U such that $F(U) \to G(U) \to H(U)$ is exact. Then $F_a \to G_a \to H_a$ is exact.* Hpk

Corollary. *if $F \to G \to H$ is presheaf-exact then $F_a \to G_a \to H_a$ is exact for all a.*

If F is a preasheaf on X, one can consider for each open subset $U \subset X$

$$F^{(0)}(U) := \prod_{a \in U} F_a.$$

The elements are families $(s_a)_{a \in U}$ with $s_a \in F_a$. There is now coupling between the different s_a. Hence $F^{(0)}(U)$ usually is very giantly.

For open sets $V \subset U$, one has an obvious homomorphism $F^{(0)}(U) \to F^{(0)}(V)$. Hence we obtain a preasheaf $F^{(0)}$ together with a natural homomorphism

$$F \longrightarrow F^{(0)}.$$

3. Sheaves

3.1 Definition. *A presheaf F is called **sheaf** if the following conditions are satisfied:* DG

(G1) *When $U = \bigcup U_i$ is an open covering of an open subset $U \subset X$ and if $s, t \in F(U)$ are sections with the property $s|U_i = t|U_i$ fur alle i, then $s = t$.*

(G2) *When $U = \bigcup U_i$ is an open covering of an open subset $U \subset X$ und if $s_i \in F(U_i)$ is a family of sections with the property*

$$s_i|U_i \cap U_j = s_j|U_i \cap U_j \quad \textit{fur alle } i, j,$$

then there exists a section $s \in F(U)$ with the property $s|U_i = s_i$ for all i.

(G3) *$F(\emptyset)$ is the zero group.*

Clearly the presheaf of continuous functions is a sheaf, since continuity is a local property. An example of a presheaf F, which usually is not a sheaf is the presheaf of constant functions with values in $\mathbb{Z}$ ($F(U) = \{f : U \to \mathbb{Z},\ f$ constant$\}$). But the set of *locally constant* functions with values in $\mathbb{Z}$ is a sheaf.

By a subsheaf of a sheaf F we understand a sub-presheaf $G \subset F$ which is already a sheaf. If F, G are presheaves then a homomorphism $f : F \to G$ of presheaves is called also a homomorphism of sheaves.

3.2 Remark. *Let $F \subset G$ be a sub-presheaf. We assume that G (but not* Eug *necessarily F) is a sheaf. Then there is a smallest subsheaf $\tilde{F} \subset G$ which contains F. For an arbitrary point $a \in X$ the induced map $f_a : F_a \to \tilde{F}_a$ is an isomorphism.*

It is clear, that $\tilde{F}(U)$ has to be defined as set of all $s \in G(U)$, such that:

There exists an open covering $U = \bigcup U_i$, such that $s|U_i$ is in the image of $F(U_i) \to G(U_i)$ for all i.

This is equivalent with:

The germ s_a is in the image of $F_a \to G_a$ for all $a \in U$.

3.3 Definition. *Let $F \to G$ be a homomorphism of sheaves. The sheaf-image* Bpg *$f_{sheaf}(F)$ is the smallest subsheaf of G, which contains the presheaf-image-$f_{pre}(F)$.*

We have to differ between two natural notions of surjectivity.

3.4 Definition. Sis

1) *A homomorphism of presheaves $f : F \to G$ is called* ***presheaf-surjective*** *if $f_{pre}(F) = G$.*
2) *A homomorphism of sheaves $f : F \to G$ is called* ***sheaf-surjective*** *if $f_{sheaf}(F) = G$.*

Wenn F and G both are sheaves then sheaf-surjectivity and presheaf-surjectivity are different things. We give an example which will be basic:

Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}$, hence $\mathcal{O}(U)$ is the set of all holomorphic functions on an open subset U. This a sheaf of abelian groups (under addition). Similarly we consider the sheaf $\mathcal{O}^*$ of holomorphic functions without zeros. This is also a sheaf of abelain groups (under multiplication). The map $f \to e^f$ defines a sheaf homomorphism

$$\exp : \mathcal{O} \longrightarrow \mathcal{O}^*.$$

The map $\mathcal{O}(U) \to \mathcal{O}^*(U)$ is not always surjective. For example for $U = \mathbb{C}^*$ the function $1/z$ is not in the image. Hence exp is not presheaf-surjective. But it is know from complex calculus that $\exp : \mathcal{O}(U) \to \mathcal{O}^*(U)$ is surjective if U is simply connected, for example for a disc U. Since a point admits arbitrarily small neighborhoods which are discs, it follows that exp is sheaf-surjective.

3.5 Remark. *A homomorphism of sheaves $f : F \to G$ is sheaf-surjective if* gH
and only if the maps $f_a : F_a \to G_a$ are surjective for all $a \in X$.

Fortunately the notion "injective" doesn't contain this difficulty.

3.6 Remark. *Let $f : F \to G$ be a homomorphism of sheaves. The kernel in* KiG
the sense of presheaves is already a sheaf.

Hence we don't have to distinguish between presheaf-injective and sheaf-injective and also not between preasheaf-kernel and sheaf-kernel.

3.7 Remark. *A homomorphism of sheaves $f : F \to G$ is injective if and only* hH
if the maps $f_a : F_a \to G_a$ are injective for all $a \in X$.

A homomorphism of presheaves $f : F \to G$ (sheaves) is called an isomorphism if all $F(U) \to G(U)$ are isomorphisms. Their inverses then define a homomorphism $f^{-1} : G \to F$.

3.8 Remark. *A homomorphism of sheafs $F \to G$ is an isomorphism if and* AGb
only if $F_a \to G_a$ is an isomorphism for all a.

For presheaves this is false. As counter example on can take for F the presheaf of constant functions and for G the sheaf of locally constant functions.

It is natural to introduce the notion of sheaf-exactness as follows:

3.9 Definition. *A sequence $F \to G \to H$ of sheaf homomorphims is sheaf-* Dse
exact at G , if the the kernel of $G \to H$ and the sheaf-image of $F \to G$ agree.

Generalizing 3.5 and 3.7 one can easily show:

3.10 Proposition. *A sequence $F \to G \to H$ is exact if and only if $F_a \to$* Pee
$G_a \to H_a$ is exact for all a.

Our discussion so far has obviously one gap: Let $F \subset G$ be subsheaf of a sheaf G. We would like to have an exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

The sheaf H should be the factor sheaf of G by F. But up to now we only defined the factor-presheaf $G/_{\mathrm{pre}}F$ which usually is no sheaf. In the next section we will give the correct definition for a factor sheaf $G/_{\mathrm{sheaf}}F$.

4. The generated sheaf

For a presheaf F we introduced the monstrous presheaf

$$F^{(0)}(U) = \prod_{a\in U} F_a.$$

Obviously $F^{(0)}$ is a sheaf. Sometimes its is called the "Godement-sheaf" or the "associated flabby sheaf". There is a natural homomorphism

$$F \to F^{(0)}.$$

We can consider its presheaf-image and then the smallest subsheaf which contains it. We denote this sheaf by $\hat{F}$ and call it the "generated sheaf" by F. There is a natural homomorphism

$$F \to \hat{F}.$$

From the construction follows immediately

4.1 Remark. *Let F be a presheaf. The natural maps* ePi

$$F_a \xrightarrow{\sim} \hat{F}_a$$

are isomorphisms.

A homomorphism $F \to G$ of presheaves induces a homomorphism $F^{(0)} \to G^{(0)}$. Clearly $\hat{F}$ is mapped into $\hat{G}$.

4.2 Remark. *Let $f : F \to G$ be a homomorphism of presheaves. There is a natural homomorphism $\hat{F} \to \hat{G}$, such that the diagram* UEg

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ \hat{F} & \longrightarrow & \hat{G} \end{array}$$

commutes.

When F is already a sheaf then $F \to F^{(0)}$ is injective. Then the map of F into the presheaf image is an isomorphism. This implies that the presheaf image is already a sheaf.

4.3 Remark. *Let F be a sheaf. Then $F \to \hat{F}$ is an isomorphism.* FiF

If F is a sub-presheaf of a sheaf G, then the induced map $\hat{F} \to \hat{G} \cong G$ is an isomorphism $\hat{F} \to \tilde{F}$ between $\hat{F}$ and the smallest subsheaf $\tilde{F}$ of G, wich contains F.

We identify $\tilde{F}$ and $\hat{F}$.

Factor sheaves and exact sequences of sheaves

Let $F \to G$ be a homomorphism of presheaves. We introcuced already the factor preshaf $G/_{\text{pre}}F$, which asociates to an open U the factor group $G(U)/F(U)$. Even if both F and G are sheaves this will usually not a shesf. Hence we define the factor sheaf as the sheaf generated by the factor-presheaf.

$$G/_{\text{sheaf}}F := \widehat{G/_{\text{pre}}}F.$$

This called the factor-sheaf. Since we are interested mainly in sheaves, we will write usually for a homomorphism for sheaves $f : F \to G$:

$$\begin{aligned} G/F &:= G/_{\text{sheaf}}F && \text{(factor sheaf)} \\ f(F) &:= f_{\text{sheaf}}(F) && \text{(sheaf image)} \end{aligned}$$

Notice that there is no need to differ between sheaf- and presheaf-kernel. When we talk about an exact sequence of sheaves

$$F \longrightarrow G \longrightarrow H$$

we usually mean "sheaf exactness". All what we have said about exactness properties of sequences of abelian groups is literally true for sequences of sheaves. For example: A sequence of sheaves $0 \to F \to G$ (0 denotes the zero sheaf) is exact if and only of $F \to G$ is injective. A sequence of sheaves $F \to G \to 0$ is exact if and only if $F \to G$ is surjective (in the sense of sheaves of course). A sequence of sheaves $0 \to F \to G \to H \to 0$ is exact if and only if there is an ismomorphism $H \cong G/F$ which identifies this sequence with

$$0 \longrightarrow F \longrightarrow G \longrightarrow G/F \longrightarrow 0.$$

4.4 Remark. *Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves. Then for open U the sequence* ESf

$$0 \to F(U) \to G(U) \to H(U)$$

is exact.

Corollary. *The sequence*

$$0 \to F(X) \to G(X) \to H(X)$$

is exact.

Usually $G(X) \longrightarrow H(X)$ is not surjective as the example

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O} \overset{f \mapsto e^{2\pi i f}}{\longrightarrow} \mathcal{O}^* \longrightarrow 0$$

shows. Cohomology theory will measure the absence the right exactness. The above sequence will be part of a long exact sequence

$$0 \to F(X) \to G(X) \to H(X) \longrightarrow H^1(X, F) \longrightarrow \cdots$$

5. Direct and inverse image of sheaves

Let $f : X \to Y$ a continuous map of topological spaces and F a pre-sheaf on X. Then $(f_*F)(V) := F(f^{-1}(V))$ with natural restriction maps is a pre-sheaf on Y. It is a sheaf if F is a sheaf. We call it the direct image sheaf. For a point $a \in X$ there is an obvious map $(f_*F)_{f(a)} \to F_a$. Let $X \to Y$ be a closed embedding. This means that the image is closed and that $X \to Y$ is a topological map. In this case the above map induces a bijection $(f_*F)_{f(a)} \cong F_a$.

We use some simple facts about sheaves. Let F be a sheaf on a topological space. We know the trivial procedure of restricting F to an open subset.

We recall shortly the definition of $F|Y$. We need it only in a very special situation. Let X be a topological space and let $Y \subset X$ be a closed subspace. Let F be sheaf of abelian groups. Assume that $F|(X - Y) = 0$. This means that the support of F is contained in Y. Then we can define

$$(F|Y)(V) = \varinjlim F(U)$$

where U runs through all open subsets $U \subset X$. All homomorphisms in this direct system are isomorphisms. This implies that the natural homomorphisms

$$(F|Y)(V) \longrightarrow F(U) \qquad (U \subset X \text{ open, } U \cap Y = V)$$

are isomorphisms. This also says that there is a natural isomorphism

$$(F|Y)(U \cap Y) = F(U) \qquad (U \subset X \text{ open}).$$

We can read them as canonical isomorphisms

$$i_*(F|Y) \cong F.$$

Let conversely G be a sheaf of abelian groups on Y. Then we can consider the direct image i_*G. This sheaf vanishes on $X - Y$, so we can define $(i_*G)|Y$. We have

$$(i_*G)|Y(U \cap Y) = (i_*G)(U) = G(U \cap V).$$

This can be read as a canonical isomorphism

$$(i_*G)|Y \cong G.$$

5.1 Remark. *Let $Y \subset X$ a closed subset of a topological space. We denote by $\mathcal{A}$ the category of sheaves of abelian groups on X which vanish on $X - Y$ (as full subcategory of the category of all sheaves of abelian groups on X) and by $\mathcal{B}$ the category of sheaves of abelian groups on Y. These categories are equivalent where the equivalence is given by two inverse functors* Rce

$$\mathcal{B} \longrightarrow \mathcal{A}, \quad G \longmapsto i_*G,$$
$$\mathcal{A} \longrightarrow \mathcal{B}, \quad F \longmapsto F|Y.$$

The same Remark holds for sheaves of rings or $\mathbb{C}$-algebras.

There is a more general procedure to restrict sheaves to an arbitrary subset $Y \subset X$ (equipped with the induced topology). Even more general, one can define for a continuous map $f : Y \to X$ the inverse image $f^{-1}F$ of a sheaf F on X. First one considers the presheaf

$$G(V) = \varinjlim F(U)$$

where U runs through all open subsets of X that contain $f(V)$. Then one defines $f^{-1}F$ to be its generated sheaf. If $U \subset X$ is open an $\iota : U \to X$ is the canonical injection then $\iota^{-1}F$ can be identified with the restriction. Hence we can use the notation $F|Y = \iota^{-1}F$ for any subset $Y \subset X$, equipped with the induced topology. Again ι denotes the natural injection.

5.2 Lemma. *Let X be a topological space and $Y \subset X$ a closed subspace. Let* LabF *F be a sheaf on X such that $F|(X - Y)$ is zero and let $\iota : Y \to X$ the natural injection. Then there is a natural isomorphism*

$$\iota_*(F|Y) \xrightarrow{\sim} F.$$

More precisely, the functor $F \mapsto F|Y$ defines an equivalence between the category of sheaves on Y and the category of sheaves on X whose restriction to U vanishes.

We use some simple facts about sheaves. Let F be a sheaf on a topological space. We know the trivial procedure of restricting F to an open subset. There is a more general procedure to restrict sheaves to an arbitrary subset $Y \subset X$ (equipped with the induced topology). Even more general, one can define for a continuous map $f : Y \to X$ the inverse image $f^{-1}F$ of a sheaf F on X. First one considers the presheaf

$$G(V) = \varinjlim F(U)$$

where U runs through all open subsets of X that contain $f(V)$. Then one defines $f^{-1}F$ to be its generated sheaf. If $U \subset X$ is open an $\iota : U \to X$ is the canonical injection then $\iota^{-1}F$ can be identified with the restriction. Hence we can use the notation $F|Y = \iota^{-1}F$ for any subset $Y \subset X$, equipped with the induced topology. Again ι denotes the natural injection.

5.3 Lemma. *Let X be a topological space and $Y \subset X$ a closed subspace. Let* LabF *F be a sheaf on X such that $F|(X - Y)$ is zero and let $\iota : Y \to X$ the natural injection. Then there is a natural isomorphism*

$$\iota_*(F|Y) \xrightarrow{\sim} F.$$

More precisely, the functor $F \mapsto F|Y$ defines an equivalence between the category of sheaves on Y and the category of sheaves on X whose restriction to U vanishes.

6. Sheaves of rings and modules

A sheaf of A-modules is a sheaf F of abelian groups such that every $F(U)$ carries a structure as A-module and such the the restriction maps $F(U) \to F(V)$ for $V \subset U$ are A-linear. A homomorphism $F \to G$ is called A-linear if all $F(U) \to G(U)$ are so. Then kernel and image carry natural structures of sheafs of A-modules. Also the stalks carry such a structure naturally. Hence the whole canonical flabby resolution is a sequence of sheafs of A-modules.

There is a refinement of this construction: By a sheaf of rings $\mathcal{O}$ we understand a sheaf of abelian groups such that every $\mathcal{O}(U)$ is not only an abelian group but a ring and such that all restriction maps $\mathcal{O}(U) \to \mathcal{O}(V)$ are ring homomorphisms. Then the stalks $\mathcal{O}_a$ carry a natural ring structure such that the homomorphisms $\mathcal{O}(U) \longrightarrow \mathcal{O}_a$ (U is an open neighborhood of a) are ring homomorphisms.

By an $\mathcal{O}$-module we understand a sheaf $\mathcal{M}$ of abelian groups such every $F(U)$ carries a structure as $\mathcal{O}(U)$-module and such that the restriction maps are compatible with the module structure. To make this precise we give a short comment. Let M be an A-module and N be a module over a different ring B. Assume that a homomorphism $r : A \to B$ is given. A homomorphism $f : M \to N$ of abelian groups is called compatible with the module structures if the formula

$$f(am) = r(a)f(m) \qquad (a \in A,\ m \in M)$$

holds. An elegant way to express this is as follows. We can consider N also as an module over A by means of the definition $an := r(a)n$. Sometimes this A-module is written as $N_{[r]}$. Then the compatibility of the map f simply means that it is an A-linear map

$$f : M \longrightarrow N_{[r]}.$$

Usually we will omit the subscript $[r]$ and simply say that $f : M \to N$ is A-linear.

If $\mathcal{M}$ is an $\mathcal{O}$-module then the stalk $\mathcal{M}_a$ is naturally an $\mathcal{O}_a$-module. An $\mathcal{O}$-linear map $f : \mathcal{M} \to \mathcal{N}$ between two $\mathcal{O}$-modules is a homomorphism of sheaves of abelian groups such the maps $\mathcal{M}(U) \to \mathcal{N}(U)$ are $\mathcal{O}(U)$ linear. Then the Kernel and image also carry natural structures of $\mathcal{O}$-modules.

Another standard construction of commutative algebra carries over to the case of modules over sheaves.

An $\mathcal{O}$-submodule $\mathcal{P} \subset \mathcal{M}$ is an sub-sheaf of abelian groups such that $\mathcal{P}(U)$ is an $\mathcal{O}(U)$-submodule of $\mathcal{M}(U)$ for every open U. Then the natural inclusion $\mathcal{P} \hookrightarrow \mathcal{M}$ is $\mathcal{O}$-linear. The factor sheaf $\mathcal{M}/\mathcal{N}$ carries a natural structure as $\mathcal{O}$-module. An ideal sheaf in $\mathcal{O}$ is just an $\mathcal{O}$-submodule of $\mathcal{O}$ (which can be considered as $\mathcal{O}$-module in the obvious way). The factor sheaf of $\mathcal{O}$ by an ideal sheaf carries a natural structure as sheaf of rings.

Let $\varphi : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{O}$-linear map and $\mathcal{P} \subset \mathcal{N}$ an $\mathcal{O}$-submodule. Then $\varphi^{-1}(\mathcal{P})$ is defined in the naiv way: $\varphi^{-1}(\mathcal{P})(U) := \varphi_U^{-1}(\mathcal{P}((U))$. This is already a sheaf, actually an $\mathcal{O}$-submodule of $\mathcal{M}$.

Since for every open subset $U \subset X$ we have a ring homomorphism $\mathcal{O}(X) \to \mathcal{O}(U)$ all $\mathcal{M}(U)$ can be considered as $\mathcal{O}(X)$-modules. Hence a $\mathcal{O}$-module can be considered as sheaf of $\mathcal{O}(X)$-modules.

Let $\mathcal{O}$ be a sheaf of rings. There is the notion of an $\mathcal{O}$-module $\mathcal{M}$. This is a sheaf of abelian groups together with a homomorphism of sheaves of abelian groups

$$\mathcal{O} \times \mathcal{M} \longrightarrow \mathcal{M}$$

such that the induced maps

$$\mathcal{O}(U) \times \mathcal{M}(U) \longrightarrow \mathcal{M}(U)$$

equip $\mathcal{M}(U)$ with a structure as $\mathcal{O}(U)$-module. Here we use the obvious definition for the direct product of (pre-)sheaves.

$$(\mathcal{M} \times \mathcal{N})(U) = \mathcal{M}(U) \times \mathcal{N}(U).$$

This definition can be extended to more than one factor and one can define

$$\mathcal{M}^n = \mathcal{M} \times \cdots \times \mathcal{M}.$$

There is an obvious notion of an $\mathcal{O}$-linear map $\mathcal{M} \to \mathcal{N}$ of $\mathcal{O}$-modules. So we can talk about the category of $\mathcal{O}$-modules.

This category has the same exactness property as the category of abelian groups. One can define the kernel and the sheaf image in this category and one can define direct products. We also mention that the stalk $\mathcal{M}_a$ of an $\mathcal{O}$-module carries a natural structure as $\mathcal{O}_a$-module.

In the following we will understand by an exact sequence of $\mathcal{O}$-moduls a sheaf exact sequence and we use the notations

$$f(\mathcal{M}) := f_{\text{sheaf}}(\mathcal{M}) \quad \text{and} \quad \mathcal{M}/\mathcal{N} = \mathcal{M}/_{\text{sheaf}}\mathcal{N}.$$

(These are $\mathcal{O}$-modules.)

7. Finitely generated sheaves

Let $\mathcal{M}$ be an $\mathcal{O}$-module and let $\mathcal{O}^m \to \mathcal{M}$ be an $\mathcal{O}$-linear map. There is an induced map $\mathcal{O}(X)^m \to \mathcal{M}(X)$. Hence there are m distinguished global sections $s_1, \dots, s_m \in \mathcal{M}(X)$ (the images of the elements of the standard basis $e_1, \dots, e_m$ of $\mathcal{O}(X)^m$). These global sections determine the map, since for any open $U \subset X$ an arbitrary section of $\mathcal{O}^m$ can be written in the form $s = f_1 e_1|U + \cdots + f_m e_m|U$. The image of this section is $f_1 s_1|U + \cdots + f_m s_m|U$. Conversely we obtain an $\mathcal{O}$-linear map through this formula for any choice of global sections $s_1 \dots, s_m$. This shows:

7.1 Lemma. *There is a natural one to one correspondence between $\mathcal{O}$-linear maps $\mathcal{O}^m \to \mathcal{M}$ and m-tuples of global sections of $\mathcal{M}$.* FrtoM

An $\mathcal{O}$-module is called finitely generated if there is a surjective map of $\mathcal{O}$-modules $\mathcal{O}^m \to \mathcal{M}$. Surjectivity of course is understood in the sense of sheaves. So this means that $\mathcal{O}_a^m \to \mathcal{M}_a$ is surjective for each point $a \in X$.

The support of a sheaf F of abelian groups, rings, algebras is defined as

$$\text{support} F := \{a \in X;\ F_a \neq 0\}.$$

7.2 Lemma. *Let $\mathcal{M}$ be a finitely generated $\mathcal{O}$-module. The support of $\mathcal{M}$ is a closed subset.* SuppCl

Proof. We show that the complement of the support is open. Let a be a point such that $\mathcal{M}_a = 0$. Consider generators $s_1, \dots, s_m$ of $\mathcal{M}$. The germs $(s_i)_a$ are zero. Hence there exists an open neighborhood U such that all $s_i|U = 0$. This shows $\mathcal{M}_b = 0$ for all $b \in U$. □

7.3 Lemma. *Let $\mathcal{M}, \mathcal{N}$ be two finitely generated submodules of an $\mathcal{O}$-module $\mathcal{P}$. Let a be a point such that $\mathcal{M}_a \subset \mathcal{N}_a$. Then there exists an open neighborhood $a \in U$ such that $\mathcal{M}|U \subset \mathcal{N}|U$.* SubCon

Proof. Take generators $s_1, \dots, s_m$ of $\mathcal{M}$ and $t_1, \dots, t_n$ of $\mathcal{N}$. Express the germs $(t_i)_a$ by the $(s_j)_a$. Since there are only finitely coefficients involved, these equations extend to a small open neighborhood of a. □

A similar argument gives:

7.4 Lemma. *Let $\mathcal{M} \to \mathcal{N}$ be an $\mathcal{O}$-linear map of finitely generated $\mathcal{O}$-modules. Let a be a point such that $\mathcal{M}_a \to \mathcal{N}_a$ is surjective. Then there exists an open neighborhood U such that $\mathcal{M}|U \to \mathcal{N}|U$ is surjective.* SurUmg

Lifting of maps

A very simple fact of commutative algebra says. Let $M \to N$ be a surjective R-linear map of R-modules and let $R^n \to N$ be a linear map too. Then there exists a lift $R^n \to M$. Denote the images of the standard basis $e_1, \dots, e_n$ in N by $b_1, \dots, b_n$ and take pre-images a_i in M. Then map e_i to a_i.

To get an analogue for sheaves, we consider a surjective $\mathcal{O}$-linear map $\mathcal{M} \to \mathcal{N}$ of $\mathcal{O}$-modules and an $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{N}$. Now we get a problem since the map $\mathcal{M}(X) \to \mathcal{N}(X)$ needs not to be surjective. So we can not repeat the above argument. We only can say:

7.5 Lemma. *Let $\mathcal{M} \to \mathcal{N}$ be a surjective $\mathcal{O}$-linear map and $\mathcal{O}^n \to \mathcal{N}$ also an $\mathcal{O}$-linear map. For each point a there exists an open neighborhood U and an $\mathcal{O}|U$-linear map such the diagram* LiftLoc

$$\begin{tikzcd}
\mathcal{O}|U^n \arrow[d] \arrow[dr] & \\
\mathcal{M}|U \arrow[r] & \mathcal{N}|U
\end{tikzcd}$$

commutes.

8. Coherent sheaves

Let us recall a basic property of noetherian rings R. Let M be a finitely generated module, i.e. there exists a surjective R-linear map $R^n \to M$. Then the kernel K of this map is finitely generated as well. Hence there exists an exact sequence $R^n \xrightarrow{\varphi} R^m \to M$. The map φ determines $M \cong R^n/\mathrm{Im}(\varphi)$. The map φ just given by a matrix with m rows and n columns. This is the way how computer algebra can manage computations for finitely generated modules over noetherian rings as polynomial rings. Serre found a weak substitute for $\mathcal{O}$-modules.

8.1 Definition. *A sheaf of rings $\mathcal{O}$ is called* ***coherent*** *if for any open subset $U \subset X$ and any $\mathcal{O}|U$-linear map $\mathcal{O}^n|U \to \mathcal{O}^m|U$ the kernel is locally finitely generated.* DCoh

Recall that an $\mathcal{O}$-module $\mathcal{M}$ is called locally finitely generated if there exists an open covering $X = \bigcup_i U_i$ such that $\mathcal{M}|U_i$ is a finitely generated as $\mathcal{O}_X|U_i$-module for all i.

8.2 Definition. *Let $\mathcal{O}$ be a coherent sheaf of rings. An $\mathcal{O}$-module $\mathcal{M}$ is called coherent if for every point there exists an open neighborhood U and an exact sequence* CohMod

$$\mathcal{O}|U^n \longrightarrow \mathcal{O}|U^m \longrightarrow \mathcal{M}|U \longrightarrow 0.$$

Of course $\mathcal{O}$ considered as $\mathcal{O}$-module is coherent. Just consider $0 \to \mathcal{O} \to \mathcal{O} \to 0$.

An $\mathcal{O}$-module is called a (finitely generated) free sheaf if it is isomorphic to $\mathcal{O}^m$ for suitable m. It is called locally free if every point admits an open neighborhood such that the restriction to it is free. A locally free sheaf is also called a vector bundle. For trivial reasons a (finitely generated) free sheaf over a coherent sheaf of rings is coherent. Since coherence is a local property, every vector bundle is coherent. The property "coherent" is stable under standard constructions. The proves are not difficult. We will keep them short:

First we treat some special cases for free $\mathcal{O}$-modules. A first trivial observation is that the image of an $\mathcal{O}$-linear map $\mathcal{O}^p \to \mathcal{O}^q$ is coherent. The next observation is that the intersection $\mathcal{M} \cap \mathcal{N}$ of two coherent subsheaves $\mathcal{M}, \mathcal{N}$ of $\mathcal{O}^n$ is coherent. (The intersection $\mathcal{M} \cap \mathcal{N}$ is defined in the naive sense as presheaf and turns to be out a sheaf, more precisely an $\mathcal{O}$-module.) The idea is to write the intersection as a kernel. We explain the principle for individual modules $M, N \subset R^n$ of finite type over a ring R: Let $F : R^p \to R^n$, $G : R^q \to R^n$ be linear maps and let M, N be their images. We denote by K the kernel of the linear map

$$R^{p+q} \longrightarrow R^n, \quad (a, b) \longmapsto F(a) - G(b).$$

The image of K under the map

$$R^{p+q} \longrightarrow R^n, \quad (a, b) \longmapsto F(a)$$

is precisely the intersection $M \cap N$.

The last observation is the following. Let $\mathcal{O}^p \to \mathcal{O}^q$ be $\mathcal{O}$-linear and let $\mathcal{M} \subset \mathcal{O}^q$ be coherent. We claim that its inverse image in $\mathcal{O}^p$ is coherent. We explain again the algebra behind this result. Let $F : R^m \to R^l$ be a R-linear map and $N \subset R^l$ be an R-module of finite type. We assume that $F(R^m) \cap N$ is finitely generated. Then there exists a finitely generated submodule $P \subset R^m$ such that $F(P) = F(R^m) \cap N$. We also assume that the kernel K of F is finitely generated. It is easily proved that $F^{-1}(N) = P + K$ and we obtain that the inverse image is finitely generated.

These observations carry over to arbitrary coherent $\mathcal{O}$-modules.

8.3 Lemma. *Let $\mathcal{M} \to \mathcal{N}$ be an $\mathcal{O}$-linear map of coherent sheaves. The image sheaf is coherent.* ImCohCoh

Corollary. *A locally finitely generated sub-sheaf of a coherent sheaf is coherent.*

Proof. It is sufficient to show that the image of a map $\mathcal{O}^m \to \mathcal{M}$ is coherent. By definition of coherence it is sufficient to show that the kernel $\mathcal{K}$ is locally finitely generated. We can assume that there exists an exact sequence

$$\mathcal{O}^p \longrightarrow \mathcal{O}^q \longrightarrow \mathcal{M} \longrightarrow 0.$$

Since $\mathcal{O}^q \to \mathcal{M}$ is surjective we can assume (use Lemma 7.5) that there exists a lift $\mathcal{O}^m \to \mathcal{O}^q$ such that the diagram

$$\begin{tikzcd}
\mathcal{O}^m \arrow[r] \arrow[dr] & \mathcal{M} \\
& \mathcal{O}^q \arrow[u] \\
& \mathcal{O}^p \arrow[u]
\end{tikzcd}$$

commutes. Take the image of $\mathcal{O}^p \to \mathcal{O}^q$ and then its pre-image in $\mathcal{O}^m$ It is easy to check that this is the kernel $\mathcal{K}$. □

8.4 Lemma. *The kernel of a map $\mathcal{M} \to \mathcal{N}$ of coherent sheaves is coherent.* KeCohCoh

Proof. Because of Lemma 8.3 we can assume that $\mathcal{M} \to \mathcal{N}$ is surjective. We choose presentations

$$\mathcal{O}^a \longrightarrow \mathcal{O}^b \longrightarrow \mathcal{M}, \quad \mathcal{O}^c \longrightarrow \mathcal{O}^d \longrightarrow \mathcal{N}.$$

We can assume that there is commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{N} \arrow[r] & 0 \\
& & \mathcal{O}^b \arrow[u] \arrow[r, "\varphi"] & \mathcal{O}^d \arrow[u] & \\
& & \mathcal{O}^a \arrow[u] \arrow[r] & \mathcal{O}^c \arrow[u, "\psi"'] &
\end{tikzcd}$$

The existence of φ follows from Lemma 7.5 (after replacing X by a small open neighborhood of a given point). The existence of $\mathcal{O}^a \to \mathcal{O}^c$ is trivial. Then we get a natural surjection $\varphi^{-1}(\psi(\mathcal{O}^c)) \to \mathcal{K}$. □

8.5 Lemma. *The cokernel $\mathcal{N}/\varphi(\mathcal{N})$ of a map $\varphi : \mathcal{M} \to \mathcal{N}$ of coherent sheaves is coherent.* KoCohCoh

Proof. We can assume that $\mathcal{N}$ is a sub-sheaf of $\mathcal{M}$ and that φ is the canonical injection. We can assume that a commutative diagram with exact columns exists:

$$\begin{tikzcd}
& 0 & 0 & & \\
0 \arrow[r] & \mathcal{N} \arrow[u] \arrow[r] & \mathcal{M} \arrow[u] \arrow[r] & \mathcal{M}/\mathcal{N} \arrow[r] & 0 \\
& \mathcal{O}^b \arrow[u] \arrow[r] & \mathcal{O}^d \arrow[u] & & \\
& \mathcal{O}^a \arrow[u] \arrow[r] & \mathcal{O}^c \arrow[u] & &
\end{tikzcd}$$

It is easy to construct from this diagram an exact sequence

$$\mathcal{O}^b \oplus \mathcal{O}^c \longrightarrow \mathcal{O}^d \longrightarrow \mathcal{M}/\mathcal{N} \longrightarrow 0. \qquad \square.$$

8.6 The two of three lemma. *Let $\mathcal{O}$ be a coherent sheaf of rings and* TwoThree

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

an exact sequence of $\mathcal{O}$-modules. Assume that two of them are coherent. Then the third is coherent too.

Proof. All what remains to show is that $\mathcal{M}_2$ is coherent if $\mathcal{M}_1, \mathcal{M}_2$ are. We can assume that there is a commutative diagram

$$\begin{tikzcd}
& & & 0 & \\
0 \arrow[r] & \mathcal{M}_1 \arrow[r] & \mathcal{M}_2 \arrow[r] & \mathcal{M}_3 \arrow[u] \arrow[r] & 0 \\
& & & \mathcal{O}^q \arrow[u] \arrow[ul, "\alpha"] & \\
& & & \mathcal{O}^p \arrow[u] &
\end{tikzcd}$$

We use this to produce a map

$$\mathcal{M}_1 \oplus \mathcal{O}^q \longrightarrow \mathcal{M}_2, \quad (x, y) \longmapsto x - \alpha(y).$$

It is easy to check that this map is surjective. The kernel is defined by $x = \alpha(y)$. Hence it can be identified with the part of $\mathcal{O}^q$ that is mapped into $\mathcal{M}_1$ under α. But this precisely the kernel of $\mathcal{O}^q \to \mathcal{M}_3$ hence the image of $\mathcal{O}^p$. We get an exact sequence

$$\mathcal{O}^p \longrightarrow \mathcal{M}_1 \oplus \mathcal{O}^q \longrightarrow \mathcal{M}_2 \longrightarrow 0.$$

This shows that $\mathcal{M}_2$ is coherent (use Lemma 8.5).). □

8.7 Lemma. *The intersection of two coherent subsheaves of a coherent sheaf is coherent.* SubSc

Proof. One uses the fact that intersections can be constructed as kernels. Let $\mathcal{M}, \mathcal{N} \subset \mathcal{X}$ be two submodules of an $\mathcal{O}$-module $\mathcal{X}$. Then $\mathcal{M} \cap \mathcal{N}$ is isomorphic to the kernel of $\mathcal{M} \times \mathcal{N} \to \mathcal{X}$, $(a, b) \mapsto a - b$. □

8.8 Remark. *Let $\mathcal{M}$ be a coherent $\mathcal{O}$-module. Then the support of $\mathcal{M}$ is a closed subset.* SupC

Proof. We show that the set of all a such that $\mathcal{M}_a = 0$ is open. We can assume that $\mathcal{M}$ is finitely generated by sections $s_1, \dots, s_n$. If there germs at a are zero then $s_1, \dots, s_n$ are zero in a full neighbourhood of a. □

We collect some of the permanence properties of coherent sheaves.

8.9 Proposition. PointEx

1) *Let $\mathcal{M}, \mathcal{N}$ be two coherent sub-sheaves of a coherent sheaf. Assume $\mathcal{M}_a \subset \mathcal{N}_a$ for some point a. Then there exists an open neighborhood U such that $\mathcal{M}|U \subset \mathcal{N}|U$.*
2) *Let $\mathcal{M}, \mathcal{N}$ be two coherent subsheaves of a coherent sheaf. Assume $\mathcal{M}_a = \mathcal{N}_a$ for some point a. Then there exists an open neighborhood U such that $\mathcal{M}|U = \mathcal{N}|U$.*
3) *Let $f, g : \mathcal{M} \to \mathcal{N}$ be two $\mathcal{O}$-linear maps between coherent sheaves such that $f_a = g_a$ for some point a. Then there exists an open neighborhood U such that $f|U = g|U$.*
4) *Let $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ be $\mathcal{O}$-linear maps of coherent sheaves and a a point. The following two conditions are equivalent:*
   a) *The sequence $\mathcal{M}_a \to \mathcal{N}_a \to \mathcal{P}_a$ is exact.*
   b) *There is an open neighborhood U such that the sequence $\mathcal{M}|U \to \mathcal{N}|U \to \mathcal{P}|U$ is exact.*

Proof.

1) Use that $\mathcal{M}_a \subset \mathcal{N}_a$ is equivalent to $\mathcal{N}_a = \mathcal{M}_a \cap \mathcal{N}_a$ $(= (\mathcal{M} \cap \mathcal{N})_a$.

2) follows from 1).

3) Consider the kernel of $f - g$.

4) Consider the image $\mathcal{A}$ of $\mathcal{M} \to \mathcal{N}$ and the kernel $\mathcal{B}$ of $\mathcal{N} \to \mathcal{P}$. Both are coherent. We can assume that they are finitely generated. From assumption we know $\mathcal{A}_a = \mathcal{B}_a$. □

8.10 Proposition. *Let $\mathcal{M}, \mathcal{N}$ coherent $\mathcal{O}$-modules and $\mathcal{M}_a \to \mathcal{N}_a$ an $\mathcal{O}_a$-linear map. There exists an open neighborhood U and an extension $\mathcal{M}|U \to \mathcal{N}|U$ as $\mathcal{O}|U$-linear map.* ExtPtoU

Additional remark. *By Proposition 8.9 this extension is unique in the obvious local sense.*

Proof. We can assume that there is a surjective $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{M}$. We consider the composed map $\mathcal{O}_a^n \to \mathcal{M}_a \to \mathcal{N}_a$. It is no problem to extend $\mathcal{O}_a^n \to \mathcal{N}_a$ to an open neighborhood $\mathcal{O}|U^n \to \mathcal{N}|U$. We can assume that U is the whole space. The kernel of $\mathcal{O}_a^n \to \mathcal{M}_a$ is contained in the kernel of $\mathcal{O}_a^n \to \mathcal{N}_a$. Since the kernels are coherent this extends to a full open neighborhood U. Hence we get a factorization $\mathcal{M}|U \to \mathcal{N}|U$. □

8.11 Lemma. *Let $\mathcal{O}_X$ be a coherent sheaf of rings on a topological space X. Let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent sheaf of ideals. Let Y be the support of $\mathcal{O}_X/\mathcal{J}$. Then the restriction of $\mathcal{O}_X/\mathcal{J}$ to Y is a coherent sheaf of rings $\mathcal{O}_Y$. The category of coherent $\mathcal{Y}$-modules is equivalent to the category of coherent $\mathcal{O}_X$ modules which are annihilated by $\mathcal{J}$.* CohSub

Proof. This is an application of Lemma ???. It is easy to see that $\mathcal{O}_X/J|Y$ is a sheaf Let $\mathcal{M}$ be an $\mathcal{O}_X$-module which is annihilated by $\mathcal{J}$. (This means $J(U)\mathcal{M}(U) = 0$ for all open U). The support of $\mathcal{M}$ is contained in Y. Then $\mathcal{M}|Y$ is defined and carries a natural structure as $\mathcal{O}_Y$-module. The rest is clear. □

□

References

[Fi] Fischer, G.: *Complex Analytic Geometry,* Lecture Notes in Mathematics **538**, Springer-Verlag Berlin Heidelberg New York (1976)

[GR] Grauert, H., Remmert, R.: *Coherent Analytic Sheaves,* Grundlehren der mathematischen Wissenschaften **265**, Springer Verlag, Berlin Heidelberg New-York Tokyo (1984)

[Ho], Houzel, C.: *Géométrie analytique locale, IV,* Séminaire Henri Cartan, tome **13**, n^0 2, exp. n^0 21, p. 1–15, Paris (1960–1961)

Index