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Kaehler manifolds

Real and complex manifolds, vector bundles
Hodge theory, Kodaira's embedding theorem

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Introduction

This book contains a focused introduction into the theory of Kähler manifolds. The main result is Kodaira's embedding theorem which characterizes compact complex manifolds that are biholomorphic equivalent to a projective algebraic manifold. The necessary and sufficient condition is the existence of a positive holomorphic line bundle. This is a a great generalization of the classical result that a complex torus $\mathbb{C}^n/L$ is projective algebraic if and only there exists a positive Hermitian form on $\mathbb{C}^n$ which is integral on $L \times L$. Another special case of Kodaira's embedding theorem is the fact that all compact Riemann surfaces are projective algebraic.

The proof of the embedding theorem is founded on the study of holomorphic vector bundles on complex manifolds. From the beginning, we consider them as sheaves and already our introduction to differentiable am complex manifolds is sheaf theoretic. In the appendices (Chapter V und VI) we give a complete introduction into the theory of sheaves and their cohomology. The cohomology groups are introduced through the Godement resolution (canonically flabby resolution) and not through Čech cohomology as in many other approaches. This is easy and has the further advantage that this approach works also in modern algebraic geometry. We also need some Čech cohomology but here it is sufficient to treat the first Čech cohomology group which is very simple.

The contents of the book are as follows. In the first chapter we give a quick introduction into real and complex manifolds and into vector bundles. Here real manifolds means what is often called differentiable and complex manifolds are the usual complex analytic manifolds. This introduction is given sheaf theoretic. Vector bundles can be treated via the transition functions. This gives the link to other approaches as for example by means of the bundle spaces.

In the second chapter we start with the calculus of differentiable forms on differentiable and complex manifolds. In the case of complex manifolds the space of alternating differential forms can be decomposed into to (p, q)-types. This is fundamental since this decomposition reflects the complex structure of the manifold. The lemma of de Rham characterizes the cohomology groups $H^q(X, \mathbb{R})$ as cohomology groups of the de Rham complex, a certain complex of differential forms. In the complex case the sheaf Ω_X of holomorphic differential forms comes into the game. Here the groups $H^q(X, \bigwedge^p \Omega)$ are treated. The de Rham complex has to be replaced by the Dolbeault complex. The proof rests on the lemmas of de Rham and Dolbeault which we formulate without proof.

In Chapter III we treat the Hodge theory, first for compact Riemannian manifolds. It states that $H^q(X, \mathbb{R})$ is isomorphic to the space of harmonic p-forms $\mathcal{H}^q(X)$. There is a generalization to compact complex manifold. Here the cohomology groups $H^q(X, \bigwedge^p \Omega)$ are isomorphic to certain spaces $\mathcal{H}^{p,q}(X)$ of harmonic forms. This theory can be generalized to $H^q(X, \bigwedge^p \Omega \otimes_{\mathcal{O}_X} \mathcal{M})$ where $\mathcal{M}$ is a holomorphic vector bundle. The Hodge theorem makes use of the main result about linear elliptic differential equations. We formulate the result but do not prove it.

Chapter IV contains the main results. We introduce Kähler manifolds. Kähler manifolds are special complex manifolds which admit an embedding

$$H^q(X, \bigwedge^p \Omega) \longrightarrow H^{p+q}(X, \mathbb{C}).$$

So there is a link between real and complex Hodge theory. A highlight of this chapter is the proof of Kodaira's vanishing theorem which prepares the decisive tool for the proof of the embedding theorem.

The book ends with Appendices, Chapter V and VI. Chapter V contains an introduction into sheaf theory and Chapter VI into their cohomology.

Chapter I. Real and complex manifolds

1. Geometric spaces

Let X be a topological space. In these notes we denote by $\mathcal{C}_X$ the sheaf of all *real* valued continuous function. We will often consider complex valued functions. The sheaf of all of them can be identified with $\mathcal{C}_X \otimes_{\mathbb{R}} \mathbb{C}$.

1.1 Definition. *A (real or complex) geometric space $(X, \mathcal{O}_X)$ is a topological space X together with a subsheaf of rings of $\mathcal{O}_X \subset \mathcal{C}_X$ or $\mathcal{O}_X \subset \mathcal{C}_X \otimes_{\mathbb{R}} \mathbb{C}$. We assume that the constant functions (with values in $\mathbb{R}$ or $\mathbb{C}$) are contained in $\mathcal{O}_X$ and we assume that for a function $f \in \mathcal{O}_X(U)$ without zeros the function $1/f$ is also contained in $\mathcal{O}_X(U)$.*

These assumptions have consequences for the ring of germs

$$\mathcal{O}_{X,a} = \varinjlim \mathcal{O}(U) \qquad (a \in U \subset X \text{ open}).$$

The set $\mathfrak{m}_{X,a}$ of all elements whose germ vanish at a is an ideal. It is obviously the unique maximal ideal. Hence $\mathcal{O}_{X,a}$ is a local ring. The natural map

$$\mathbb{C} \longrightarrow \mathcal{O}_{X,a}/\mathfrak{m}_{X,a} \qquad (\mathbb{R} \longrightarrow \mathcal{O}_{X,a}/\mathfrak{m}_{X,a})$$

in the complex case (and similarly in the real case) is an isomorphism of fields.

1.2 Definition. *A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of geometric spaces is a continuous map $f : X \to Y$ with the following additional property. If $V \subset Y$ is open and $g \in \mathcal{O}_Y(V)$ then $g \circ f$ is contained in $\mathcal{O}_X(f^{-1}(V))$.*

Quite trivial facts are:

The composition of two morphisms is a morphism.
The identical map $(X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ is a morphism.

A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of geometric spaces is called an isomorphism if f is topological and if $f^{-1} : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is also a morphism. This means that the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(f(U))$ are naturally isomorphic.

Let $U \subset X$ be an open subset of a geometric space $(X, \mathcal{O}_X)$. We can define the restricted geometric structure $\mathcal{O}_X|U$ by

$$(\mathcal{O}_X|U)(V) := \mathcal{O}_X(V) \qquad (V \subset U \text{ open}).$$

It is clear that the natural embedding $i : (U, \mathcal{O}_X|U) \hookrightarrow (X, \mathcal{O}_X)$ is a morphism and moreover that a map $f : Y \to U$ from a geometric space $(Y, \mathcal{O}_Y)$ into $(U, \mathcal{O}_X|U)$ is a morphism if and only if $i \circ f$ is a morphism.

2. Vector bundles

Let R be a ring (commutative and with unit). We denote the set of all $m \times n$-matrices (m rows and n columns) by $R^{m\times n}$. The standard action

$$R^{m\times n} \times R^n \longrightarrow R^n$$

can be described as follows. Write the elements $a \in R^n$ as columns. Then $A(a)$ is the matrix product $A \cdot a$.

We consider a topological space X and a sheaf $\mathcal{O}_X$ of rings. We call the pair $(X, \mathcal{O}_X)$ a ringed space. We allways assume that $\mathcal{O}_X(U)$ is not the zero ring for arbitrary non-empty U. For example geometric spaces are ringed spaces with this property. We are interested in $\mathcal{O}_X$-modules $\mathcal{M}$. Recall that these are sheaves of abelian groups such that $\mathcal{M}(U)$ carries for open $U \subset X$ a structure as $\mathcal{O}_X(U)$-module such that the restriction maps are compatible with this module structure. An example is $\mathcal{O}_X$ or, more generally, $\mathcal{O}_X^n$ for natural numbers n are $\mathcal{O}_X$-modules. An $\mathcal{O}_X$-module $\mathcal{M}$ is called free if it is isomorphic (as $\mathcal{O}_X$-module) to $\mathcal{O}_X^n$ for suitable n. This n is uniquely determined if $\mathcal{O}_X(U)$ are not all zero rings. We call it the rank of $\mathcal{M}$. An $\mathcal{O}_X$-module is called *locally free* if every point $a \in X$ admits an open neighbourhood such U that $\mathcal{M}|U$ is free as $\mathcal{O}_X|U$-module. The rank is independent of the choice of U and is called the rank of $\mathcal{M}$ at a. This is a locally constant function, hence constant if X is connected. We say that $\mathcal{M}$ has rank n if it has rank n everywhere. By a vector bundle on a ringed space we just understand a locally free sheaf.

Transition functions

Let $\mathcal{M}$ be a vector bundle of rank n on $(X, \mathcal{O}_X)$. Let $X = \bigcup U_i$ be an open covering such that $\mathcal{M}|U_i$ is free. Choose isomorphisms

$$\varphi_i : \mathcal{M}|U_i \xrightarrow{\sim} (\mathcal{O}_X|U_i)^n.$$

We restrict them to $U_i \cap U_j$ and obtain then an isomorphism

$$h_{ij} : (\mathcal{O}_X|U_i \cap U_j)^n \xrightarrow{\sim} (\mathcal{O}|U_i \cap U_j)^n, \quad h_{ij}{=}\varphi_i\varphi_j^{-1}.$$

This isomorphism is determined by its action on the global sections (Lemma V.7.1) and hence given by a matrix g_{ij} in

$$\mathcal{O}_X(U_i \cap U_j)^{n\times n}.$$

This is a system of transition functions in the following sense.

2.1 Definition. *Let $(X, \mathcal{O}_X)$ be a ringed space. A system of transition functions (of degree n) consists of an open covering $X = \bigcup U_i$ and a system of matrices*

$$g_{ij} \in \mathrm{GL}(n, \mathcal{O}_X(U_i \cap U_j))$$

such that

$$g_{ij} g_{jk} g_{ki} = E \quad \textit{(unit matrix)} \quad \textit{on} \quad U_i \cap U_j \cap U_k.$$

So we have seen that one can associate to a vector bundle of rank n a system of transition functions. This system is not uniquely determined. It depends on the choice of the covering and on the choice of the local trivializations φ_i. We say that a system of transition functions that comes through the above construction from a vector bundle (of pure degree) is associated to this vector bundle. (The notion "function" reflects the fact that in many applications $\mathcal{O}_X$ is a sheaf of functions.)

Next we describe a reverse construction. We assume now that a system of transition functions $X = \bigcup U_i$, g_{ij} is given. We want to associate a sheaf $\mathcal{M}$. First we define the global sections $\mathcal{M}(X)$. They consist of systems $f_i \in \mathcal{O}(U_i)^n$ such that

$$f_i = g_{ij} f_j \quad \text{on} \quad U_i \cap U_j.$$

For arbitrary open U we can do the same. We just restrict everything to the covering $U = \bigcup (U_i \cap U)$. It is easy to check that this is a sheaf and, even more, it is a $\mathcal{O}_X$-module. So we obtain the following result.

2.2 Lemma. *Let X be a topological space, $\mathcal{O}_X$ a sheaf of rings and let $X = \bigcup U_i$, $g_{ij} \in \mathrm{GL}(n, \mathcal{O}(U_i \cap U_j))$ be a system of transition functions. The associated sheaf $\mathcal{M}$ is a vector bundle of rank n. If the system is associated to some vector bundle $\mathcal{N}$, then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*

We call two systems of transition functions equivalent if the associated vector bundles are isomorphic. So we see that the set of all isomorphy classes of vector bundles is in bijection with the set of equivalence classes of systems of transition functions. (By the way, this shows also that the totally of all isomorphy classes of vector bundles is a set.)

Some constructions

The sheaf $\mathcal{O}_X$ is a vector bundle of rank 1. Vector bundles of rank 1 are called also line bundles. Let $\mathcal{M}, \mathcal{N}$ be two vector bundles. Then $\mathcal{M} \times \mathcal{N}$ is a vector bundle too. In the same way one can define the product of finitely many vector bundles. The sheaf $\mathcal{O}_X^n$ is called the trivial bundle of rank n. We also can consider the tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ of two vector bundles Recall that we have a natural map

$$\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U) \longrightarrow (\mathcal{M}_{\mathcal{O}_X} \mathcal{N})(U).$$

This is an isomorphism for small enough U (in the sense that $\mathcal{M}|U$ and $\mathcal{N}|U$ are free.) More generally one can define the product of a finite system of vector bundles. In particular, the tensor product of two vector bundles is a vector bundle too. If $\mathcal{M}$, $\mathcal{N}$ is of rank m, n then the tensor product has rank mn. One also can define the tensor product of a finite system of vector bundles. The usual commutativity and associativity rules for the tensor product hold. Finally one can define the exterior powers $\bigwedge^m \mathcal{M}$ of a vector bundle. If $\mathcal{M}$ is of rank n then this exterior power is a vector bundle of degree $\binom{n}{m}$. The case $p = n$ is of particular importance. Here one obtains a line-bundle that sometimes is called the determinant

$$\det \mathcal{M} := \bigwedge^n \mathcal{M}.$$

Fibres of a vector bundle

Let $\mathcal{M}$ be a vector bundle on the ringed space $(X, \mathcal{O}_X)$. Assume that all stalks $\mathcal{O}_{X,a}$ are local rings. This means that there is a unique maximal ideal $\mathfrak{m}_{X,a}$. It consists of all non-units. We denote the residue field by $K_a = \mathcal{O}_{X,a}/\mathfrak{m}_{X,a}$. For any $\mathcal{O}_X$-module we can consider the K-vector space

$$\mathcal{M}(a) = \mathcal{M}_a/\mathfrak{m}_{X,a}\mathcal{M}_a = \mathcal{M}_a \otimes_{\mathcal{O}_{X,a}} K_a.$$

We call this space the fibre of $\mathcal{M}$ at a. If $\mathcal{M}$ is locally free, then $\mathcal{M}_a$ is a K_a-vector space of dimension n. For any open neighborhood $a \in U$ we have natural maps

$$\mathcal{M}(U) \longrightarrow \mathcal{M}_a \longrightarrow \mathcal{M}(a).$$

Recall that the image of $s \in \mathcal{M}(U)$ in $\mathcal{M}_a$ is called the *germ* of s in a and is denoted frequently by s_a. We can also consider the image in $\mathcal{M}(a)$. We call this the *value* of s at a and denote it by $s(a)$. Usually the values $s(a)$, $a \in U$, do not determine the section s. If for example $\mathcal{M} = \mathcal{O}_X$ and s is nilpotent, then all the values are 0, since in a field there are no nilpotents besides 0. For geometric spaces the situation is better. For example the following is trivially true.

2.3 Remark. *Assume that $(X, \mathcal{O}_X)$ is a geometric space. Let $\mathcal{M}$ be a vector bundle on $(X, \mathcal{O}_X)$. Then a section $s \in \mathcal{M}(U)$, $U \subset X$ open, is uniquely determined by its values $s(a)$, $a \in X$.*

Proof. Since this is a local question we can reduce to the case $\mathcal{M} = \mathcal{O}_X^n$. Here the statement is trivial. □

If $\mathcal{M} \to \mathcal{N}$ is a $\mathcal{O}_X$-linear map of vector bundles and $\mathcal{M}_a \to \mathcal{N}_a$ the induced map of stalks. For trivial reason we have $\mathfrak{m}_{X,a}\mathcal{M}_a \to \mathfrak{m}_{X,a}\mathcal{N}_a$. Hence we get a map $\mathcal{M}(a) \to \mathcal{N}(a)$.

2.4 Remark. *Let $(X, \mathcal{O}_X)$ be a geometric space and let $f : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{O}_X$-linear map of vector bundles. Then f is an isomorphism if an only if the induced map $\mathcal{M}(a) \to \mathcal{N}(a)$ in the fibres is an isomorphism.*

Proof. We have to show that $\mathcal{M}_a \to \mathcal{N}_a$ is an isomorphism if $\mathcal{M}(a) \to \mathcal{N}(a)$ is so. The injectivity follows from Remark 2.3. Hence we have to show that $\mathcal{M}_a \to \mathcal{N}_a$ is surjective if $\mathcal{M}(a) \to \mathcal{N}(a)$ is so. We can assume $\mathcal{M} = \mathcal{O}_X^n$, $\mathcal{N} = \mathcal{O}_X^m$. The map f than can be considered as a $m \times n$-matrix of functions on X. It is easy to restrict the claim to the case $n = m$. Hence we assume this. We than get that the matrix $f(a)$ is an invertible matrix of complex numbers. We can assume that f has no zero on X. Then we can define f^{-1}. □

It is a good device to pursue constructions with vector bundles first along the fibres because this makes the linear algebra background clearer. Let $\mathcal{M}$, $\mathcal{N}$ be two vector bundles and $\mathcal{M}(a)$, $\mathcal{N}(a)$ their fibres at a point a. Then fibres of $\mathcal{M} \times \mathcal{N}$ are $M_a \times N_a$, the fibres of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ are $M_a \otimes_{K_a} N_a$. The fibres of $\mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ are $\mathrm{Hom}_K(\mathcal{M}(a), \mathcal{N}(a))$ and the fibres of $\bigwedge^m \mathcal{M}$ are $\bigwedge^m \mathcal{M}(a)$. The fibre of the dual bundle $\mathcal{M}$ equals the dual vector space of $\mathcal{M}(a)$.

Let V, W be two finite dimensional vector spaces over a field K. Assume that there is given a bilinear map $\beta : M \times N \to K$. Then we obtain a natural map $M \to N^*$ that sends m to the linear form $\ell_m(x) = \beta(m, x)$. The pairing is called non-degenerated if $M \to N^*$ is an isomorphism. Then the natural map $N \to M^*$ is also an isomorphism as a dimension argument shows. This generalizes to vector bundles $\mathcal{M}, \mathcal{N}$. It is clear what a $\mathcal{O}_X$-bilinear form $\mathcal{M} \times \mathcal{N} \to \mathcal{O}_X$ means. And it is clear how it induces an $\mathcal{O}_X$-linear map $\mathcal{M} \to \mathcal{N}^*$. Again we call the pairing non-degenerated of this is an isomorphism.

2.5 Lemma. *Let $(X, \mathcal{O}_X)$ be a geometric space and let $f : \mathcal{M} \times \mathcal{N} \to \mathcal{O}_X$ be an $\mathcal{O}_X$-bilinear map of vector bundles. It is non-degenerated if and only if it is fibre wise non-degenerated.*

In Remark V.7.2 we introduced also $\mathscr{M}\!ult(\mathcal{M}_1, \ldots, \mathcal{M}_n, \mathcal{N})$ for $\mathcal{O}_X$-modules and in the case $\mathcal{M}_1 = \cdots = \mathcal{M}_n = \mathcal{M}$ the module $\mathscr{A}\!lt(calM, \ldots, \mathcal{M}, \mathcal{N})$. Both are vector bundles if $\mathcal{M}_i$ and $\mathcal{N}$ are.

2.6 Remark. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be vector bundles. The natural $\mathcal{O}_X$-linear maps*

$$\mathcal{M}_1^* \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{M}_n^* \longrightarrow \mathscr{M}\!ult(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n, \mathcal{O}_X).$$

and

$$\bigwedge^n \mathcal{M}^* \longrightarrow \mathscr{A}\!lt(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n, \mathcal{O}_X)$$

are isomorphisms.

A final comment to the tensor product. The tensor product of two $\mathcal{O}_X$-modules ($\mathcal{O}_X$ any sheaf of rings) is the *generated sheaf* of the presheaf $U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$. The reader should not be scared from the construction "generating" because the following two facts.

a) There is the general rule

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})|U = \mathcal{M}|U \otimes_{\mathcal{O}_X|U} \mathcal{N}|U.$$

b) When $\mathcal{M}$ and $\mathcal{N}$ are free then $U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$ is already a sheaf.

This shows that for two vector bundles $\mathcal{M}, \mathcal{N}$ the rule

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})(U) = \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$$

holds for small open subsets U.

3. The tangent bundle

Tangents always are related to differentiation. There is an algebraic notion of derivation. Let K by a ring and $K \to R$ be a K-algebra and let M be an R-module. A K-derivation $D : R \to M$ is K-linear map such that $D(ab) = aD(b) + bD(a)$ for all $a, b \in R$. The set $\mathrm{Der}_K(R, M)$ of all K-derivations is an R-module in the obvious way. Let $(X, \mathcal{O}_X)$ be a ringed space. Assume a little more, namely that $\mathcal{O}_X$ is a sheaf of K-algebras and let $\mathcal{M}$ be an $\mathcal{O}_X$ module. By definition, a K-derivation $D : \mathcal{O}_X \to \mathcal{M}$ is a K-linear map of sheaves such that $D(U)$ is a derivation for all open U. We denote the set of all derivations by

$$\mathrm{Der}_K(\mathcal{O}_X, \mathcal{M}).$$

Similarly to Hom this construction can be sheafified. So we get a sheaf, actually an $\mathcal{O}_X$-module.

$$\mathscr{D}\!er_K(\mathcal{O}_X, \mathcal{M})$$

with the property

$$\mathscr{D}\!er_K(\mathcal{O}_X, \mathcal{M})(U) = \mathrm{Der}_K(\mathcal{O}_X|U, \mathcal{M}|U).$$

We are interested in the case $\mathcal{M} = \mathcal{O}_X$. Then we consider

$$\mathcal{T}_X = \mathscr{D}\!er_K(\mathcal{O}_X, \mathcal{O}_X).$$

There are many geometric situations in which this sheaf is locally free and serves as tangent bundle. Then the dual bundle $\mathcal{T}^*$ is called the cotangent bundle or the bundle of differentials and the sections of the bundle

$$\mathcal{T}_X \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{T}_X^* \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{T}_X^*$$

are called mixed tensors. Let A, B be two vector fields. Then $A \circ B$ is usually no vector field but $[A, B] := A \circ B - B \circ A$ is.

3.1 Definition. *The alternating $\mathcal{O}_X$-bilinear map*

$$\mathcal{T}_X \times \mathcal{T}_X \longrightarrow \mathcal{T}_X, \quad (A, B) \longmapsto [A, B],$$

is called the Lie bracket.

It is not the goal of these notes to develop a general theory of ringed spaces. So we switch now to sheaves of differentiable and holomorphic functions.

4. Calculus

We recall some basic facts of calculus. For sake of simplicity, we will use the notion "differentiable" in the sense of "$\mathcal{C}^\infty$-differentiable". Sometimes, for example for curves, we use the notation *smooth* instead of differentiable. We collect some basic facts.

A function

$$f : D \longrightarrow \mathbb{R}, \quad D \subset \mathbb{R}^n \text{ open},$$

is called differentiable if all partial derivatives of arbitrary order exist and are continuous. We denote by $\mathcal{C}^\infty(D) = \mathcal{C}^\infty(D, \mathbb{R})$ the set of all these functions. In the case $n = 1$ one can take as domains of definition besides open sets also arbitrary (open, half-open, closed) intervals. But this not a new concept since one can show the following fact.

Let be $f : I \to \mathbb{R}^n$ a differentiable function on some interval I, then there exists a differentiable function on an open interval $J \supset I$ which extends f.

Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open subsets. A map $f : U \to V$ can be decomposed into its m components,

$$f(x) = (f_1(x), \dots, f_m(x)), \quad f_i : U \longrightarrow \mathbb{R} \ (1 \le i \le m).$$

We denote by $\mathcal{C}^\infty(U, V)$ the set of all maps whose components are contained in $\mathcal{C}^\infty(U)$.

The matrix of partial derivatives of f at a point $a \in U$ we denote by

$$J(f, a) = \begin{pmatrix} \partial f_1/\partial x_1 & \dots & \partial f_1/\partial x_n \\ \vdots & & \vdots \\ \partial f_m/\partial x_1 & \dots & \partial f_m/\partial x_n \end{pmatrix}(a).$$

We recall the chain rule. If

$$U \xrightarrow{f} V \xrightarrow{g} X, \quad U \subset \mathbb{R}^n \text{ open}, \ V \subset \mathbb{R}^m \text{ open}, \ X \subset \mathbb{R}^p,$$

are differentiable, then the composition is differentiable too and one has

$$J(g \circ f, a) = J(g, f(a)) \cdot J(f, a) \quad \text{(matrix product)}.$$

We recall the theorem of invertible functions.

4.1 Theorem of invertible functions. *Let*

$$\varphi : D \longrightarrow \mathbb{R}^n, \quad D \subset \mathbb{R}^n \textit{ open},$$

be differentiable and let $a \in D$ be a point for which the Jacobi-matrix $J(\varphi, a)$ is invertible, then there exists an open neighborhood $a \in U \subset D$ such that its image $V = \varphi(U)$ is open as well and such that the restriction of φ induces a diffeomorphism

$$\varphi : U \xrightarrow{\sim} V.$$

(For sake of simplicity we use often for the restriction of a map the same letter, as long as it is clear what is meant.) A *diffeomorphism* is a bijective map between two open subsets of $\mathbb{R}^n$ which is differentiable in both directions.

The theorem of implicit functions looks like a generalization of the theorem of invertible functions. There are several versions. We will formulate them when we need them. Here we just mention that all are consequences of the theorem of invertible functions and the following lemma.

4.2 Lemma. *Let $a \in U \subset \mathbb{R}^n$ be some point in an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ be a differentiable map such that the Jacobi-matrix $J(f, a)$ has rank m. Then there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^{n-m}$ such that the Jacobi-matrix of the function $F(x) = (f(x), L(x))$ has invertible Jacobi matrix at a.*

Complex calculus

We will frequently identify $\mathbb{C}$ and $\mathbb{R}^2$ by means of

$$z = x + \mathrm{i}y \longleftrightarrow (x, y)$$

and more generally $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ by means of

$$(z_1, \dots, z_n) \longleftrightarrow (x_1, y_1, \dots, x_n, y_n).$$

A $\mathbb{C}$-linear endomorphism of $\mathbb{C}^n$ is given by a complex $n \times n$-matrix in the usual way. The same linear map can be considered as $\mathbb{R}$-linear and then is given by a real $2n \times 2n$-matrix $\tilde{A}$. This matrix is obtained from A if one replaces each entry a by

$$\tilde{a} = \begin{pmatrix} \operatorname{Re} a & -\operatorname{Im} a \\ \operatorname{Im} a & \operatorname{Re} a \end{pmatrix}.$$

One has

$$\det \tilde{A} = |\det A|^2 \qquad (\geq 0).$$

The Wirtinger operators are defined as

$$\frac{\partial}{\partial z_\nu}, \ \frac{\partial}{\partial \bar{z}_\nu} : \mathcal{C}^\infty(U, \mathbb{C}) \longrightarrow \mathcal{C}^\infty(U, \mathbb{C})$$

by

$$\frac{\partial f}{\partial z_\nu} := \frac{1}{2}\Big(\frac{\partial f}{\partial x_\nu} - \mathrm{i}\frac{\partial f}{\partial y_\nu}\Big), \qquad \frac{\partial f}{\partial \bar{z}_\nu} := \frac{1}{2}\Big(\frac{\partial f}{\partial x_\nu} + \mathrm{i}\frac{\partial f}{\partial y_\nu}\Big).$$

In the case $n = 1$ one writes d/dz instead of $\partial/\partial z$ (similarly with $\bar{z}$ instead of z). The Wirtinger operators satisfy the usual product law. Hence it is easy to apply them to polynomial expressions in z_ν and $\bar{z}_\nu$. We write down the rules in the case $n = 1$, the generalizations to arbitrary n are quite obvious:

$$\frac{dz^m}{dz} = mz^{m-1}, \qquad \frac{dz^m}{d\bar{z}} = 0.$$

This shows that a polynomial P in the variables $z_\nu, \bar{z}_\nu$ is a polynomial in the variables z_ν alone if and only if $\partial P/\partial \bar{z}_\nu = 0$.

4.3 Definition. *A differentiable function $f \in \mathcal{C}^\infty(U, \mathbb{C})$ on an open subset of $\mathbb{C}^n$ is called* ***complex differentiable*** *or* ***holomorphic*** *or* ***complex analytic*** *if*

$$\frac{\partial f}{\partial \bar{z}_\nu} = 0 \qquad (1 \le \nu \le n).$$

Then $\partial f/\partial z_\nu$ are called the complex derivatives of f.

So for holomorphic functions f we have the Cauchy-Riemann differential equations

$$\frac{\partial f}{\partial z_\nu} = \frac{\partial f}{\partial x_\nu} = -\mathrm{i}\frac{\partial f}{\partial y_\nu}.$$

It is clear that constant functions are holomorphic and that the set

$$\mathcal{O}(U) \subset \mathcal{C}^\infty(U, \mathbb{C})$$

of all holomorphic functions is a subring of the ring of all differentiable functions.

All definitions and statements for differentiable functions in this section can be give literally in the holomorphic world. One just has to replace differentiable by complex differentiable or holomorphic as we prefer here. So for open subsets $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ we can define the set $\mathcal{O}(U, V)$ of holomorphic mappings. We can introduce the complex Jacobian matrix $J_{\mathbb{C}}(f, a)$ which is a complex $m \times n$-matrix. Since we have identified $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. a holomorphic map is also differentiable in the real sense and the real Jacobi matrix $J_{\mathbb{R}}(f, a)$ is a $2m \times 2n$-matrix. Both matrices are related by

$$J_{\mathbb{R}}(f, a) = J_{\mathbb{C}}\widetilde{(f, a)}.$$

The chain rule and the theorem of invertible functions hold also in the obvious complex sense. In Theorem 4.1 one has just to replace "diffeomorphism" by "biholomorphic" (bijective and holomorphic in both directions).

5. Differentiable and complex manifolds

Topological spaces

We use the language of topological spaces. All topological spaces which we consider are assumed to be Hausdorff. Each metric space has an underlying topological space. If X is a topological space, then every subset Y can be equipped with a topology too. It is called the *induced topology.* Here a subset $V \subset Y$ is called open when there exists an open subset $U \subset X$ such that $V = U \cap Y$. In particular, every subset of $\mathbb{R}^n$ inherits a structure as topological space.

In the following we always tacitly assume, if nothing else is explicitly stated, that every topological space is Hausdorff and that it admits a countable basis of topology. The latter means that there exists a system $(U_i)_{i \in I}$ of open subsets with *countable* index set I such that each open subset can be written as union of certain sets in this system. For example $\mathbb{R}^n$ has countable basis of topology. One considers the countable system of all open balls with rational radius and whose centers have rational coordinates. If X is a space with countable basis of the topology, then each subspace (equipped with the induced topology) has the same property.

The advantage of spaces with countable basis of topology is that they admit many real valued continuous functions. For example they admit partition of unity.

We equip $\mathbb{R}^n$ with the sheaf $\mathcal{C}^\infty_{\mathbb{R}^n}$ of all real valued differentiable functions.

5.1 Definition. *A differentiable manifold is a geometric space $(X, \mathcal{C}^\infty_X)$ such that for every point $a \in X$ there exists an open neighborhood $a \in U \subset X$ and an open subset $V \subset \mathbb{R}^n$ such that the geometric spaces $(U, \mathcal{C}_X|U)$ and $(V, \mathcal{C}^\infty_{\mathbb{R}^n}|V)$ are isomorphic. Such an isomorphism*

$$\varphi : (U, \mathcal{C}_X|U) \xrightarrow{\sim} (V, \mathcal{C}^\infty_{\mathbb{R}^n}|V)$$

is called a differentiable chart.

(By a chart on a topological space one understands a topological map of an open subset of X onto an open subset of $\mathbb{R}^n$. Hence on a differentiable manifold certain charts have been distinguished and are called differentiable). For trivial reason $\mathbb{R}^n$ and its open subsets carry a natural structure as differentiable manifold.

A map between two differentiable manifolds $X \to Y$ is called differentiable if it is a morphism of geometric spaces. It is called a diffeomorphism if it is an isomorphism of geometric spaces. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ open sets. A map $U \to V$ is differentiable in the usual sense if it is differentiable in the sense of differentiable manifolds. A differentiable chart is nothing than a diffeomorphism of an open subset of X onto an open subset of $\mathbb{R}^n$.

For two differentiable charts φ, ψ on X, one defines the *chart transformation* by

$$\gamma : \varphi(U_\varphi \cap U_\psi) \longrightarrow \psi(U_\varphi \cap U_\psi), \quad \gamma(x) = \psi(\varphi^{-1}(x)).$$

This is a diffeomorphism. At variance with the strong principles of set theory, we frequently write

$$\gamma = \psi \circ \varphi^{-1}.$$

Of course the chart transformation is only of interest if the intersection $U_\varphi \cap U_\psi$ is not empty. But it is not necessary to assume this since we follow the convention such there exists exactly one map from the empty set into an arbitrary set.

For trivial reason a map $f : X \to Y$ is differentiable if it is continuous and if for every point $a \in X$ there exist differentiable charts

$$\varphi : U_\varphi \to V_\varphi,\ a \in U_\varphi \subset X \quad \text{and} \quad U_\psi \to V_\psi,\ f(a) \in U_\psi \subset Y$$

such that $\varphi \circ f \circ \psi^{-1}$ (defined on $U_\varphi \cap f^{-1}U_\psi$ is differentiable in the usual sense.

The direct product of differentiable manifolds

Let X, Y be two topological spaces. We equip $X \times Y$ with the product topology. A subset in $X \times Y$ is called open if it is the union of "rectangles" $U \times V$ where $U \subset X$, $V \subset Y$ are open subsets. Let $\varphi : U_\varphi \to V_\varphi$ be a chart on X and let $\psi : U_\psi \to V_\psi$ be a chart on Y. Then we can consider the product chart

$$\varphi \times \psi : U_\varphi \times U_\psi \longrightarrow V_\varphi \times V_\psi.$$

Assume now that X, Y are differentiable manifolds. Let φ run through all differentiable charts of X and let ψ run through all differentiable charts on Y.

5.2 Remark. *Assume that X, Y are differentiable manifolds. Then there exists a unique structure of differentiable manifold on $X \times Y$ (equipped with the product topology) such that the product charts of differentiable charts are differentiable.*

It is clear that the two projections $p : X \times Y \to X$ and $q : X \times Y \to Y$ are differentiable. Moreover a map $f : Z \to X \times Y$ of some differentiable manifold Z into $X \times Y$ is differentiable if and only if $p \circ f$ and $q \circ f$ are differentiable.

Submanifolds

5.3 Definition. *A subset $Y \subset X$ of a differentiable manifold is called **smooth** if, for every $a \in Y$, there exists a differentiable chart φ on X where $a \in U_\varphi$, $0 \in V_\varphi$, $\varphi(a) = 0$, such that*

$$\varphi(Y \cap U_\varphi) = \left\{\, x \in V_\varphi, \quad x_{d+1} = \cdots = x_n = 0 \,\right\}$$

for suitable d, $0 \le d \le n$.

The special case $d = n$ is not excluded. Hence open subsets of X are smooth. We equip Y with a differentiable structure. By restriction of φ in Definition 5.3 we obtain a bijective map

$$\varphi_0 : Y \cap U_\varphi \longrightarrow \left\{\, x \in \mathbb{R}^d; \quad (x, 0, \ldots, 0) \in V_\varphi \,\right\}.$$

This is a chart on Y. It is rather clear that there exists a unique structure as differentiable manifold such that these charts are differentiable. In the case that Y is open this agrees with the trivial restricted structure defined above.

The natural inclusion $i : Y \to X$, $i(x) = x$, is differentiable. Even more holds. A map $f : Z \to Y$ of another differentiable manifold Z into Y is differentiable if and only if the composition $i \circ f : Z \to X$ with the natural inclusion is differentiable.

5.4 Definition. *A differentiable map $f : X \to Y$ of differentiable manifolds is called an embedding if $f(X)$ is smooth and if $X \to f(X)$ is diffeomorphic.*

A differentiable map $f : X \to Y$ of differentiable manifolds is called a local embedding at $a \in X$ if there exists an open neighborhood U of a such that the restriction $U \to Y$ is an embedding.

A variant of the theorem of implicit functions says.

A differentiable map $f : X \to Y$ of differentiable manifolds is a local embedding at $a \in X$ if and only if the tangent map at a is injective.

It is not true that a injective $f : X \to Y$ which is a local embedding at all a is an embedding. The problem is that $f(X)$ needs not to be closed. The situation improves if one assumes that X is compact (f proper is enough).

5.5 Lemma. *Let $f : X \to Y$ an injective differentiable map and let X be compact. Assume that f is a local embedding at each point. Then f is an embedding.*

Instead of $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ we can consider $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ where $\mathcal{O}_{\mathbb{C}^n}$ denotes the sheaf of holomorphic functions. This leads to the notion of a *complex manifold* $(X, \mathcal{O}_X)$. As in the differentiable case defines the notion of holomorphic maps between complex manifold, one defines the cartesian product of two complex manifolds and one defines the notion of a complex submanifold. One also defines the notions of (local and global) holomorphic embedding. Lemma 5.5 is true in the holomorphic case.

6. Examples of manifolds

We give some examples of differentiable and complex manifolds. Some constructions are based on the following general construction for geometric spaces. Let $(X, \mathcal{O}_X)$ be a geometric space and let G be a group of automorphisms of $(X, \mathcal{O}_X)$. (An automorphism is an isomorphism of geometric spaces onto itself.) We recall that G induces an equivalence relation on X. Two points a, b are called equivalent if there exists a $g \in \Gamma$ with $g(a) = b$. We denote by $Y := X/\Gamma$ the set of equivalence classes. There is a natural projection $\pi : X \to Y$. We equip Y with the quotient topology. This means that a subset $V \subset Y$ is open if and only if $\pi^{-1}(V)$ is open in X. Then $\pi : X \to Y$ is continuous (which means that inverse images of open sets of Y are open) and also open (which means that images of open sets of X are open). We equip Y with a geometric structure. A function $h : V \to \mathbb{C}$ is called distinguished if and only if $h \circ \pi : \pi^{-1}(V) \to \mathbb{C}$ is distinguished. It is easy to see that this is a geometric structure $\mathcal{O}_Y$. The geometric space obtained in this way is called the quotient space. We use the notation

$$(Y, \mathcal{O}_Y) = (X, \mathcal{O}_X)/G.$$

A good way to look at this structure is as follows: Consider for open $V \subset Y$ the natural map

$$\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\pi^{-1}(V)).$$

It is clear that this map is injective and that its image consists of all G-invariant elements, i.e. of functions $f \in \mathcal{O}_X(\pi^{-1}(V))$ with the property

$$f(g(x)) = f(x) \qquad (\pi(x) \in V,\ g \in G).$$

In a self explaining notation this means

$$\mathcal{O}_Y(V) \cong \mathcal{O}_X(\pi^{-1}(V))^G.$$

(If a group arises as upper index this usually means "taking invariants".)

When X has countable basis of topology then the same is true for Y. But, even when X is a Hausdorff, the quotient Y needs not to be Hausdorff. The condition that Y is separated means that two points $x_1, x_2 \in X$ with different image points in Y admit neighborhoods $U_1, U_1 \subset X$ such that no point of U_1 is equivalent to some point of U_2. Recall that by our assumption X and Y both have to be separated.

The projective space

We give an example. Let $X = \mathbb{R}^{n+1} - \{0\}$ considered as differentiable manifold and let G be the group of all mappings of the form

$$g(x) = ax, \quad a \in \mathbb{R}^*.$$

This group is isomorphic to $\mathbb{R}^*$, an isomorphism is given by $g \mapsto a$. We consider the quotient (as geometric space)

$$P^n(\mathbb{R}) := (\mathbb{R}^{n+1} - \{0\}, \mathcal{C}^\infty)/G.$$

It is called the real projective space. We claim that it is a differentiable manifold. First we notice that $P^n(\mathbb{R})$ is Hausdorff. Even more, it is a compact space since it is the image of the sphere

$$S^n = \{x \in \mathbb{R}^{n+1}, \quad \sum_i x_i^2 = 1\}.$$

Next we prove that $P^n(\mathbb{R})$ is a differentiable manifold. We write the points of $P^n(\mathbb{R})$ in the form $x = [x_0, \dots, x_n]$. Then we consider the open subspace

$$P_i^n(\mathbb{R}) = \{x \in P^n(\mathbb{R}); \quad x_i \neq 0\}.$$

We claim that this subspace is isomorphic as (geometric space) to $\mathbb{R}^n$ (equipped with the sheaf of differentiable functions.) It is sufficient to do this for $i = 0$. Then the isomorphism is given by

$$\mathbb{R}^n \xrightarrow{\sim} P_i^n(\mathbb{R}), \quad (x_1, \dots, x_n) \longmapsto [1, x_1, \dots, x_n].$$

It is clear that this is an isomorphism of geometric spaces.

In the same way we can introduce the complex projective space $P^n(\mathbb{C})$ as complex manifold. It is the quotient of $\mathbb{C}^n - \{0\}$, equipped with the sheaf of holomorphic functions, by the obvious group isomorphic to $\mathbb{C}^*$.

Freely acting groups

By definition, the group G acts freely on X if the map $\pi : X \to X/G$ is locally topological. This is equivalent to the following fact: Every point $a \in X$ contains an open neighborhood U such that two different points of U are inequivalent mod G. Then $V = \pi(U)$ is open in Y and the restriction of π defines a topological map from U onto V. We assume that X carries a geometric structure such that G respects this structure. Then it is clear that the map $(U, \mathcal{O}_X|U) \to (V, \mathcal{O}_{X/\Gamma}|V)$ is an isomorphism of geometric spaces. We obtain:

6.1 Remark. *Let X be a differentiable manifold and Γ a group of diffeomorphisms of X onto itself, which acts freely. Then X/Γ carries also a structure as differentiable manifold. A map $X/\Gamma \to Y$ to another differentiable manifold is differentiable if and only if its composition with the natural projection $X \to X/G$ is differentiable.*

The same is true in the world of complex manifolds.

Important examples are complex tori. Here one considers a lattice $L \subset \mathbb{C}^n$. Then $\mathbb{C}^n/L$ is a complex torus.

Algebraic varieties

Let P be a homogenous complex polynomial in $n+1$ variables $z_0, \dots, z_n$. Homogenous of degree k means $P(tz) = t^k P(z)$. When P vanishes at a point $a \in \mathbb{C}^{n+1}$ it vanishes on the whole $\mathbb{C}a$. Hence we can consider the set of zeros of P on the projective space $P^n(\mathbb{C}) := P(\mathbb{C}^{n+1})$. By definition, a projective algebraic variety is a subset of $P^n(\mathbb{C})$ which can be defined as the set of common zeros of a finite system of homogenous polynomials

$$X = \{\, x \in P^n(\mathbb{C}); \quad P_1(z) = \ldots = P_m(z) = 0 \,\}.$$

It may happen that X is a (complex) smooth submanifold. From the theorem of implicit functions one can deduce that this is the case if the complex functional matrix $(\partial P_i/\partial z_j)$ has rank m at all points of X. Then the dimension of X is $n-m$.

There is a famous *theorem of Chow* which we will not use in these notes but which is behind the scenes:

Every closed complex submanifold of $P^n(\mathbb{C})$ is algebraic.

We give an example. Consider the polynomial

$$P(t, z, w) := t^4 w^2 - 4t^3 z^3 - g_2 t^5 z - g_3 t^6.$$

We assume that g_2, g_3 are arbitrary complex numbers such that $g_2^3 \neq 27 g_3^2$. One can check that this means nothing else but that the cubic polynomial

$4z^3 - g_2 z - g_3$ has no multiple zero. It can be checked that the set of zeros $X(g_2, g_3)$ of P is smooth in $P^2(\mathbb{C})$. It is a so-called elliptic curve. From the theory of elliptic functions follows that $X(g_2, g_3)$ is biholomorphic equivalent to a complex torus $\mathbb{C}/L$ and conversely that every complex torus is biholomorphic to such an elliptic curve. The affine part $X(g_2, g_2) \cap P_0^2\mathbb{C}$ corresponds to $w^2 = 4z^3 - g_2 z - g_3 = 0$.

Chapter II. Differential forms

1. The calculus of differential forms.

Let X be a differentiable manifold. Recall that we introduced the space $\mathrm{Der}_{\mathbb{R}}(\mathcal{C}_X^\infty, \mathcal{C}_X^\infty)$ whose elements are systems of $\mathbb{R}$-linear mappings

$$D : \mathcal{C}^\infty(U) \longrightarrow \mathcal{C}^\infty(U)$$

which are compatible with restriction and which satisfy the product rule $D(fg) = fD(g) + gD(f)$. Now we assume that $X = U$ is an open subset of $\mathbb{R}^n$. Then we have obvious elements

$$Df = \frac{\partial f}{\partial x_\nu}.$$

We denote them simply by $\partial/\partial x_\nu$.

1.1 Proposition. *Let $U \subset \mathbb{R}^n$ open. Then*

$$\mathrm{Der}_{\mathbb{R}}(\mathcal{C}_U^\infty, \mathcal{C}_U^\infty)$$

is a free module over $\mathcal{C}^\infty(U)$ generated by

$$\frac{\partial}{\partial x_\nu}, \quad \nu = 1, \dots, n.$$

Corollary. *For an arbitrary differentiable manifold of dimension n*

$$\mathscr{D}\!\mathit{er}_{\mathbb{R}}(\mathcal{C}_X^\infty, \mathcal{C}_X^\infty)$$

is a vector bundle of rank n.

Proof. Let D be a derivation. We can apply it to the natural projections $p_\nu(x) = x_\nu$. This gives us n functions

$$D_\nu := D(p_\nu).$$

It is sufficient to prove $D = \sum D_\nu \partial/\partial x_\nu$, i.e.

$$(Df)(a) = \sum D_\nu(a) \frac{\partial f}{\partial x_\nu}(a).$$

This identity is true for the function constant 1, since $D(1\cdot 1)=D(1)+D(1)$, hence for constant functions. By definition of D_ν it is true for linear functions and by the product rule for arbitrary polynomials. The rest is an application of Taylors formula (with an explicit remainder term in form of an integral). From this formula follows that f on any convex open neighborhood of a can be written as

$$f(x)-f(a)+\sum_{\nu=1}^{n}\frac{\partial f}{\partial x_\nu}(a)(x_\nu-a_\nu)+\sum_{1\le\mu,\nu\le n}(x_\mu-a_\mu)(x_\nu-a_\nu)h_{\mu\nu}(x),$$

where $h_{\mu\nu}$ are differentiable functions. $\square$

We call

$$\mathcal{T}_X=\mathrm{Der}_{\mathbb{R}}(\mathcal{C}_X^\infty,\mathcal{C}_X^\infty)$$

the (real) tangent bundle of a differentiable manifold. Its sections are called *vector fields*. Its dual bundle is the cotangent bundle, also called bundle of differentials.

$$\mathcal{T}_X^*=\mathscr{H}om_{\mathcal{C}_X^\infty}(\mathcal{T}_X,\mathcal{C}_X^\infty).$$

We also will consider its exterior powers

$$\mathcal{A}_X^p=\bigwedge^p\mathcal{T}_X^*.$$

The sections of this sheaf are called alternating differential forms of degree p. So

$$\mathcal{A}_X^0=\mathcal{C}_X^\infty,\quad \mathcal{A}_X^1=\mathcal{T}_X^*.$$

From Remark I.2.6 follows that there is a natural isomorphism of sheaves

$$\mathcal{A}_X^p\xrightarrow{\sim}\mathscr{A}lt(\mathcal{T}_X\times\cdots\times\mathcal{T}_X,\mathcal{C}_X^\infty).$$

This means that an element $\omega\in\mathcal{A}_X^p(X)$ can be considered as family of alternating multilinear forms (over the ring $\mathcal{C}_X^\infty(U)$)

$$\omega_U:\mathcal{T}_X(U)\times\cdots\times\mathcal{T}_X(U)\to\mathcal{C}_X^\infty(U)$$

that are compatible with restrictions.

We also have a look at the fibres. For this we introduce the vector space

$$T_aX=\mathrm{Der}_{\mathbb{R}}(\mathcal{C}_{X,a}^\infty,\mathbb{R}).$$

It is called the tangent space of X at a. In the special case where $X=U$ is an open subset of $\mathbb{R}^n$ we obtain special elements as

$$\Big[\frac{\partial}{\partial x_i}\Big]_a$$

(taking partial derivatives an evaluating at a. The same proof as of Lemma 1.1 shows that these elements are a basis. Hence T_aX are n-dimensional vector spaces for n-dimensional differentiable manifolds. There is a natural map $\mathfrak{T}_{X,a}\to T_aX$.

1.2 Lemma. *The natural map $\mathfrak{T}_{X,a} \to T_aX$ induces an isomorphism of vector spaces*

$$\mathfrak{T}_{X,a}/\mathfrak{m}_{X,a}\mathfrak{T}_{X,a} \xrightarrow{\sim} T_aX.$$

Hence the tangent space can be identified with the fibre of the tangent bundle.

Proof. Since this is a local question, we can assume that $X = U$ is an open subset of $\mathbb{R}^n$. Then we can use the constructed bases. □

In the same way one sees that the fibre of $\mathcal{A}_X^m$ at a point $a \in X$ can be identified with the vector space

$$\bigwedge^m T_aX^* = \mathrm{Alt}_{\mathbb{R}}(T_aX \times \cdots \times T_aX, \mathbb{R}).$$

1.3 Lemma. *Let $(X, \mathcal{C}_X^\infty)$ be a differentiable manifold. An alternating differential form ω of degree m is uniquely determined through its values*

$$\omega(a) \in \mathrm{Alt}_{\mathbb{R}}(T_aX \times \cdots \times T_aX, \mathbb{R}).$$

How can the differential form reconstructed from its values. This is very simple. A differential form on X has to be evaluated at m vector fields $A_1, \dots, A_m$ (on arbitrary open subsets U and $\omega(A_1, \dots, A_m)$ are functions with the property

$$\omega(A_1, \dots, A_m)(a) = \omega(a)(A_1(a), \dots, A_m(a)).$$

This gives the following result.

1.4 Lemma. *A system of alternating forms*

$$\omega(a) \in \mathrm{Alt}_{\mathbb{R}}(T_aX \times \cdots \times T_aX, \mathbb{R})$$

comes from a differential form if and only if the function

$$\omega(a)(A_1(a), \dots, A_m(a))$$

is differentiable for arbitrary vector fields $A_1, \dots, A_m$ on some open subset.

This differentiability is usually no problem, since it can be tested locally where one can use charts.

Pulling back differential forms

Pulling back differential forms rests on a functoriality property of the tangent space. Let $f : X \to Y$ be a differentiable map of differentiable manifolds. Let $a \in X$. Pulling back functions on Y means to compose them with f. This pull back induces an obvious ring homomorphism $\mathcal{C}^\infty_{Y,f(a)} \to \mathcal{C}^\infty_{X,a}$. This homomorphism induces an obvious map

$$\mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty_{X,a}, \mathbb{R}) \longrightarrow \mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty_{Y,f(a)}, \mathbb{R}).$$

In other words, we obtain a linear map

$$T_aX \longrightarrow T_{f(a)}Y.$$

This is called the *tangent map.* It is compatible with the composition of differentiable maps. One can it consider as abstract version of the Jacobi matrix. In the following we denote by $A^m(U)$ the space of all differentiable alternating forms over an open subset $U \subset X$. It is enough to treat the case $U = X$, because open subsets can be considered as differentiable manifolds as well. We collect the operations obtained so far and add one more:

1. $A^m(X)$ is a module over the ring of differentiable functions. In the case $m = 0$ it equals the ring of differentiable functions. On has $A^m(X) = 0$ for $m < 0$ and $m > \dim X$.
2. There is a "skew product"
$$A^p(X) \times A^q(X) \longrightarrow A^{p+q}(X)$$
In the case $p = 0$ the skew multiplication is simply the standard multiplication with functions.
3. The skew product is associative and skew commutative. The latter means
$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \qquad (\alpha \in A^p(X),\ \beta \in A^q(X)),$$
in particular
$$\omega \wedge \omega = 0 \quad \text{for odd } d \qquad (\omega \in A^m(X)).$$
From the associativity follows that $\omega_1 \wedge \ldots \wedge \omega_m$ is defined.
4. We introduce a new operation. the exterior differentiation. Here we make use of the fact that differential forms can be considered as alternating multilinear forms on vector fields and that vector fields operate on functions.

1.5 Definition. *The exterior differentiation*

$$d : A^m(X) \longrightarrow A^{m+1}(X)$$

is defined by

$$\begin{aligned}(d\omega)(A_1, \ldots, A_{m+1}) := &\sum_{i=1}^{m+1} (-1)^{i+1} A_i \omega(A_1, \ldots, \hat{A}_i, \ldots, A_{m+1}) + \\ &\sum_{i<j} (-1)^{i+j} \omega([A_i, A_j], A_1, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_{m+1}).\end{aligned}$$

One can check that this is alternating and multilinear over the ring of differentiable functions and hence defines a differential form. It can read as a map of sheaf of vector spaces

$$d : \mathcal{A}_X^m \longrightarrow \mathcal{A}_X^{m+1}.$$

This formula will be clearer in the local version.

5. The exterior differentiation is a vector space homomorphism which satisfies

$$d \circ d = 0.$$

6. The following product rules hold: For functions one has

$$d(fg) = fd(g) + gd(f)$$

or more general for differential forms

$$d(\alpha \wedge \beta) = (-1)^p \alpha \wedge d(\beta) + d(\alpha) \wedge \beta \qquad (\alpha \in A^p(X)).$$

As a consequence one has

$$d(\omega_1) = 0, \dots, d(\omega_m) = 0 \Longrightarrow d(\omega_1 \wedge \dots \wedge \omega_m) = 0.$$

A special case is also

$$d(df_1 \wedge \dots \wedge df_m) = 0.$$

7. There is a pull-back map for a differentiable map $f : X \to Y$:

$$f^* : A^m(Y) \longrightarrow A^m(X).$$

In the case $d = 0$ this is the usual composition of maps. The pull-back is a vector space homomorphism and even more there are the following compatibilities:

$$f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta), \quad f^*(d\omega) = df^*(\omega).$$

8. All these constructions are compatible with restrictions to open submanifolds.

A differential form ω on X is known if its restrictions to the members U_i of an open covering is known. Hence the whole calculus is regulated locally. Using charts this means that it is enough to know the calculus for open subsets $U \subset \mathbb{R}^n$. We reformulate the calculus in this case:

Recall that $\partial/\partial x_1, \dots, \partial/\partial x_n$ are basis vector fields on U. Every vector field can be written as linear combinations of them using differentiable functions as coefficients:

1.6 Definition. *For an open subset $U \subset \mathbb{R}^n$ we define the differentials*

$$dx_1, \dots, dx_n$$

by

$$dx_i(\partial/\partial x_j) = \delta_{ij}.$$

What is then

$$(dx_{i_1} \wedge \ldots \wedge dx_{i_p})(\partial/\partial x_{j_1}, \ldots \partial/\partial x_{j_p})?$$

We can assume that $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_p$. Then we get 1 or 0 related to wether $(i_1, \dots, i_p) = (j_1, \dots, j_p)$ or not.

Every differential can be written in the form

$$\omega = f_1 dx_1 + \cdots + f_n dx_n.$$

More generally, every element $\omega \in A^m(U)$ has a unique representation of the form

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_d} f_{i_1, \dots, i_d} dx_{i_1} \wedge \ldots \wedge dx_{i_d}$$

with differentiable functions $f_{\dots}$.

The alternating product is regulated by the conditions that it is distributive and associative and that

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \qquad (\Longrightarrow dx_i \wedge dx_i = 0).$$

The exterior differentiation of a function is given by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

and more general for forms by

$$d \sum_{1 \le i_1 < i_2 < \cdots < i_d} f_{i_1, \dots, i_d} dx_{i_1} \wedge \ldots \wedge dx_{i_d} = \sum_{1 \le i_1 < i_2 < \cdots < i_d} df_{i_1, \dots, i_d} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_d}.$$

Let $V \subset \mathbb{R}^m$ be another open subset and $U \to V$ a differentiable map. The pullback is regulated by

$$f^*(dy_i) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j \qquad (1 \le i \le m).$$

Notice that this follows from the compatibility $f^*(dg) = d(f^*(g))$ applied to the projection $g(y) = y_i$.

2. Differential forms on complex manifolds

We consider a complex manifold $(X, \mathcal{O}_X)$. It looks natural to introduce the *holomorphic tangent bundle*

$$\mathcal{T}_X^{\text{hol}} = \mathscr{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

There is an obvious analogue of Proposition 1.1. For an open subset $U \subset \mathbb{C}^n$ we can consider the partial derivatives $\partial f/\partial z_i$ applied to holomorphic functions f. We denote these holomorphic vector fields by $\partial/\partial z_i$. There is a small problem with this notation. We used this notation already for a Wirtinger operator that acts on complex valued differentiable functions in the real sense. But we know that its restriction to holomorphic functions gives the holomorphic differentiation. Nevertheless, a very careful reader might prefer a more careful notation as $\partial/\partial z_i|\mathcal{O}_X$. We renounce this.

2.1 Proposition. *Let $U \subset \mathbb{C}^n$ open. Then*

$$\operatorname{Der}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$$

is a free module over $\mathcal{O}_U$ generated by the complex derivatives

$$\frac{\partial}{\partial z_\nu}, \quad \nu = 1, \dots, n.$$

Corollary. *For an arbitrary complex manifold of dimension n*

$$\mathscr{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$$

is a vector bundle of rank n.

We call this bundle the *holomorphic tangent bundle* and denote it by

$$\mathcal{T}_X^{\text{hol}} = \mathscr{D}er_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$$

Its dual bundle is called the holomorphic cotangent bundle and is denoted as

$$\Omega_X = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{T}_X^{\text{hol}}, \mathcal{O}_X).$$

The sections of Ω_X are called holomorphic differentials. More generally, the sections of $\bigwedge^p \Omega_X$ are called alternating holomorphic differential forms of degree p.

2.2 Definition. *For an open subset $U \subset \mathbb{C}^n$ we define the holomorphic differentials*

$$dz_1, \dots, dz_n$$

by

$$dz_i(\partial/\partial z_j) = \delta_{ij}.$$

Then every holomorphic differential can be written in the form

$$\omega = f_1 dz_1 + \cdots + f_n dz_n$$

with holomorphic coefficients. More generally, every element $\omega \in \bigwedge^d \Omega(U)$ has a unique representation of the form

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_d} f_{i_1,\dots,i_d} dz_{i_1} \wedge \ldots \wedge dz_{i_d}$$

with holomorphic functions $f_{\dots}$.

The holomorphic tangent bundle is related to the *holomorphic tangent space* that is defined as

$$T_a^{\text{hol}} X = \text{Der}_{\mathbb{C}}(\mathcal{O}_{X,a}, \mathbb{C}).$$

Similar to Lemma 2.3 we have

2.3 Lemma. *The natural map $\mathfrak{T}_{X,a}^{\text{hol}} \to T_a^{\text{hol}} X$ induces an isomorphism of vector spaces*

$$\mathfrak{T}_{X,a}^{\text{hol}} / \mathfrak{m}_{X,a} \mathfrak{T}_{X,a}^{\text{hol}} \xrightarrow{\sim} T_a^{\text{hol}} X.$$

where $\mathfrak{m}_{X,a}$ now denotes the maximal ideal of $\mathcal{O}_{X,a}$. Hence the holomorphic tangent spaces are the fibres of the holomorphic tangent bundle.

So the calculus of holomorphic differential forms on a complex manifold is analogous to the calculus of differential forms on a differentiable manifold. But there are closer relations. We now make use of the fact that a complex manifold $(X, \mathcal{O}_X)$ has an underlying differentiable manifold $(X, \mathcal{C}_X^\infty)$. We leave its precise definition to the reader and give just the hint that one can use holomorphic charts. We are looking for a link between the holomorphic and the differentiable tangent bundle. First we look at the fibres. We have to compare

$$T_a^{\text{hol}} X \quad \text{and} \quad T_a X.$$

The right hand side is a real vector space of dimension $2n$ and the left hand side is a complex vector space of dimension n. So both sides are real vector spaces of dimension $2n$. Hence both vector spaces are isomorphic as $\mathbb{R}$-vector spaces. Is there a natural isomorphism? The answer is yes. We consider the complexified real tangent space

$$\mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty_{X,a}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty_{X,a}, \mathbb{C}).$$

Every such derivation extends $\mathbb{C}$-linearly to $\mathrm{Der}(\mathcal{C}^\infty_{X,a} \otimes_{\mathbb{R}} \mathbb{C}), \mathbb{C})$. This gives an isomorphism

$$\mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty_{X,a}, \mathbb{C}) = \mathrm{Der}_{\mathbb{C}}(\mathcal{C}^\infty_{X,a} \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}).$$

Now we use that $\mathcal{O}_X \subset \mathcal{C}^\infty_X \otimes_{\mathbb{R}} \mathbb{C}$. So we can take the natural restriction to get a natural map

$$T_a X \longrightarrow T_a X \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow T_a^{\mathrm{hol}} X.$$

2.4 Proposition. *Let X be a complex manifold of dimension n. For every point $a \in X$ the natural map*

$$T_a X \xrightarrow{\sim} T_a^{\mathrm{hol}} X$$

defines an isomorphism of real vector spaces. In local coordinates it is given by

$$\frac{\partial}{\partial x_\nu} \longmapsto \frac{\partial}{\partial z_\nu}, \quad \frac{\partial}{\partial y_\nu} \longmapsto \mathrm{i}\frac{\partial}{\partial z_\nu}.$$

Proof. This is a local question. Hence we can assume that $X = U$ is an open subset of $\mathbb{C}^n$. □

We can use the isomorphism in Proposition 2.4 to equip the real tangent space $T_a X$ of a complex manifold with a structure as complex vector space. To avoid confusion we denote by J the multiplication by i with respect to this complex structure. This means

$$J\Big[\frac{\partial}{\partial x_i}\Big]_a = \Big[\frac{\partial}{\partial y_i}\Big]_a, \quad J\Big[\frac{\partial}{\partial y_i}\Big]_a = -\Big[\frac{\partial}{\partial x_i}\Big]_a.$$

The operator J can be extended to an operator on vector fields

$$J : \mathcal{T}_X \longrightarrow \mathcal{T}_X.$$

The local formula now reads as

$$J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \quad J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}.$$

A confusion might come up for the following reason. We can also consider the complexification $T_a X \otimes_{\mathbb{R}} \mathbb{C}$. This is also complex vector space. But multiplication with i and J on $T_a X$ are something different.

By means of the isomorphism $T_a^{\mathrm{hol}}X \cong T_aX$ we get a natural embedding

$$\mathrm{Hom}_{\mathbb{C}}(T_a^{\mathrm{hol}}X, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(T_aX, \mathbb{C}).$$

Assume now that $X = U$ is an open subset of $\mathbb{C}^n$. On the left hand side we can consider $[dz_i]_a$ (the dual basis of $[\partial/\partial z_i]_a$) and on the right hand side we can consider $[dx_i]_a + \mathrm{i}[dx_i]_a$.

Claim. Under the above embedding $[dz_i]_a$ *maps to* $[dx_i]_a + \mathrm{i}[dy_i]_a$. *In particular, this element is* $\mathbb{C}$*-linear.*

Proof. Recall that $[\partial/\partial z_i]_a \in T_a^{\mathrm{hol}}X$ corresponds to $[\partial/\partial x_i]_a \in T_aX$ and $\mathrm{i}[\partial/\partial z_i] \in T_a^{\mathrm{hol}}X$ corresponds to $[\partial/\partial y_i]_a \in T_aX$ Hence we have to show (we omit indices a)

$$dz_i(\partial/\partial z_j) = (dx_i + \mathrm{i}dy_i)(\partial/\partial x_j), \quad dz_i(\mathrm{i}\partial/\partial z_j) = (dx_i + \mathrm{i}dy_i)(\partial/\partial y_j).$$

Both equalities are trivial. $\square$

We want to extend this to differential forms.

Complex valued differential forms

Let X be a differentiable manifold. We can consider complex valued differential forms. These are elements of $A^m(X) \otimes_{\mathbb{R}} \mathbb{C}$. They can be written as $\omega_1 + \mathrm{i}\omega_2$ where $\omega_i \in A^m(X)$. They are sections of the sheaf

$$\mathcal{A}_X^m \otimes_{\mathbb{R}} \mathbb{C}.$$

The fibres of this sheaf are

$$\Big(\bigwedge^m T_{X,a}^*\Big) \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge^m \mathrm{Hom}_{\mathbb{R}}(T_aX, \mathbb{C}).$$

We extend the wedge product

$$\wedge : (\mathcal{A}_X^p \otimes_{\mathbb{R}} \mathbb{C}) \times \mathcal{A}_X^q \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathcal{A}_X^{p+q} \otimes_{\mathbb{R}} \mathbb{C}.$$

$\mathbb{C}$-linearly.

Now we assume that X is a complex manifold. Then the space T_aX has a structure as complex vector space. Hence $\mathrm{Hom}_{\mathbb{C}}(T_aX, \mathbb{C})$ is defined. But this space is the fibre of the sheaf Ω_X of holomorphic differentials. In this way we can get the following result.

2.5 Remark. *Let X be a complex manifold. Then there is a natural embedding*

$$\Omega_X \longrightarrow \mathcal{A}_X^1 \otimes_{\mathbb{R}} \mathbb{C}.$$

In the local case ($X = U$ some open subset of $\mathbb{C}^n$ it maps

$$dz_i \longmapsto dx_i + \mathrm{i}dy_i.$$

We will identify Ω_X with its image in $\mathcal{A}^1_X \otimes_{\mathbb{R}} \mathbb{C}$. Then we can write $dz_i = dx_i + \mathrm{i}dy_i$. More generally we have an embedding

$$\bigwedge^m \Omega_X \longrightarrow \mathcal{A}^m_X \otimes_{\mathbb{R}} \mathbb{C}.$$

We do a little linear algebra.

Let T be a finite dimensional real vector space. (We will apply this to $T = T_aX$.) There are canonical isomorphisms

$$\mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C}), \quad (\bigwedge\nolimits_{\mathbb{R}}^m T^*) \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge\nolimits_{\mathbb{C}}^m (T^* \otimes_{\mathbb{R}} \mathbb{C}).$$

(The subscript under the $\bigwedge$ indicates which ground field has to be taken. In most cases it is clear from the circumstances what is the ground field. Then we can omit this index.) On the vector space $\mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C})$ we have a natural conjugation $L \mapsto \bar{L}$. It is defined by

$$\bar{L}(v) := \overline{L(v)}.$$

(Notice that on an abstract complex vector space complex conjugation is not well-defined. Therefore one needs that the complex vector space is the complexification of a real vector space.) We use the notation

$$A^m = \bigwedge^m \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R}).$$

Then we have

$$A^m \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge^m \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C}).$$

Now we assume that T is a complex vector space. Since it can be considered as real vector space, the previous constructions work. We consider the subspace of $\mathbb{C}$-linear maps

$$\mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C}) \subset \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C}).$$

A dimension consideration shows

$$\mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C}) \oplus \overline{\mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})}.$$

Elements of $\overline{\mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})}$ are so-called $\mathbb{C}$-antilinear maps. They follow the rule $L(Cv) = \bar{C}L(v)$. Nevertheless $\overline{\mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})}$ is a complex vector space. We want to study the Grassmann algebra

$$A^m \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge^m \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C}).$$

So we have to understand the Grassmann algebra of a direct sum $W = A \oplus B$ of two (in our case complex) vector spaces. There is a linear map

$$\bigwedge^p A \otimes \bigwedge^q B \longrightarrow \bigwedge^m W \qquad (m = p + q),$$

which sends

$$(a_1 \wedge \ldots \wedge a_p) \otimes (b_1 \wedge \ldots \wedge b_q) \longmapsto a_1 \wedge \ldots \wedge a_p \wedge b_1 \wedge \ldots \wedge b_q.$$

The image of this map is denoted by

$$\bigwedge\nolimits^{p,q} W \subset \bigwedge\nolimits^{m} W.$$

For example by means of bases it is easy to check: The map

$$\bigwedge\nolimits^{p} A \otimes \bigwedge\nolimits^{q} B \xrightarrow{\sim} \bigwedge\nolimits^{p,q} W$$

is an isomorphism. One has the direct sum decomposition

$$\bigwedge\nolimits^{m} W = \bigoplus_{p+q=m} \bigwedge\nolimits^{p,q} W.$$

In our case we get a decomposition

$$A^m \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=m} A^{p,q}.$$

Here $A^{p,q}$ is generated by all $a_1 \wedge \ldots \wedge a_p \wedge b_1 \wedge \ldots \wedge b_q$ where $a_i \in \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$, $b_i \in \overline{\mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})}$.

We apply this to the (real) tangent space $T = T_a U$ of an open subset $U \subset X$. In $\mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C})$ we consider $dz_i = dx_i + dy_i$ We know that this element is $\mathbb{C}$-linear. So we get

$$dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q} \in A^{p,q}.$$

We extend this to differential forms

A (complex valued) differential form ω is called of type (p, q) if

$$\omega(a) \in \bigwedge\nolimits^{p,q} \mathrm{Hom}_{\mathbb{R}}(T_a X, \mathbb{C})$$

for all points a. We denote by

$$A^{p,q}(X) \subset A^m(X) \otimes_{\mathbb{R}} \mathbb{C}$$

the subspace of all forms of type (p, q). They can be described easily locally.

2.6 Proposition. *Let $U \subset \mathbb{C}^n$ be an open subset. Elements of $A^{p,q}(U)$ have a unique representation in the form*

$$\omega = \sum_{\substack{1 \le i_1 < \cdots < i_p \\ 1 \le j_1 < \cdots < j_q}} f_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q}} dz_{i_1} \wedge \ldots \wedge dz_{ip} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$

Now we obtain the following decomposition.

2.7 Proposition. *There is a decomposition*

$$A^m(X) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=m} A^{p,q}(X).$$

This decomposition induces of course a decomposition of sheaves

$$\mathcal{A}_X^m \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=m} A_X^{p,q}$$

where the sheaves $\mathcal{A}_X^{p,q}$ are defined in the obvious way. The wedge product preserves this graduation, i.e. it defines maps

$$A^{p,q}(X) \times A^{p',q'}(X) \xrightarrow{\wedge} A^{p+p',q+q'}(X).$$

The total derivative d does not preserve the bi-graduation. To remedy this situation we observe that there are projections

$$A^m(X) \longrightarrow A^{p,q}(X) \qquad (n = p + q).$$

2.8 Definition and Remark. *Let X be a complex manifold. The composition of d with the natural projections gives operators*

$$\partial : A^{p,q}(X) \longrightarrow A^{p+1,q}(X), \quad \bar{\partial} : A^{p,q}(X) \longrightarrow A^{p,q+1}(X).$$

These operators satisfy

$$d = \partial + \bar{\partial} \qquad (on\ A^{p,q}(X))$$

and

$$\partial \circ \partial = 0, \quad \bar{\partial} \circ \bar{\partial} = 0, \quad \partial \circ \bar{\partial} = -\bar{\partial} \circ \partial.$$

Finally we mention $\overline{\partial \omega} = \bar{\partial}\bar{\omega}$.

We express the complex calculus in local coordinates: Let $U \subset \mathbb{C}^n$ be an open subset. We recall that $dz_i = dx_1 + \mathrm{i}dy_i$ is of type $(1, 0)$. We also set

$$d\bar{z}_i = dx_i - \mathrm{i}dy_i$$

which is of type $(0, 1)$.

2.9 Proposition. *Let $U \subset \mathbb{C}^n$ be an open subset. The operators ∂ and $\bar\partial$ are given for functions by*

$$\partial f = \sum_{i=1}^{n} \frac{\partial}{\partial z_i}\, dz_i, \quad \bar\partial f = \sum_{i=1}^{n} \frac{\partial}{\partial \bar z_i}\, d\bar z_i$$

and on forms by

$$\bar\partial \sum_{\substack{1\le i_1<\cdots<i_p\\ 1\le j_1<\cdots<j_q}} f_{\substack{i_1,\dots,i_p\\ j_1,\dots,j_q}}\, dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\ldots\wedge d\bar z_{j_q} =$$
$$\sum_{\substack{1\le i_1<\cdots<i_p\\ 1\le j_1<\cdots<j_q}} \bar\partial f_{\substack{i_1,\dots,i_p\\ j_1,\dots,j_q}} \wedge dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\ldots\wedge d\bar z_{j_q}.$$

and similarly for ∂ instead of $\bar\partial$.

We conclude this section with some remark about holomorphic differential forms. We recall that there is a natural embedding

$$\bigwedge^p \Omega_X \longrightarrow \mathcal{A}_X^{p,0}$$

which we can use to identify $\bigwedge^p \Omega_X$ with its image.

2.10 Lemma. *A differential form ω on a complex manifold is holomorphic if and only if it satisfies the following two conditions:*

a) *It is of type $(p,0)$*
b) $\bar\partial\omega = 0$.

In particular, $d\omega = \partial\omega$ for holomorphic forms.

Finally we mention that there is a natural isomorphism

$$\big(\bigwedge^p \Omega_X\big) \otimes_{\mathcal{O}_X} \big(\mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C}\big) \xrightarrow{\sim} \mathcal{A}_X^{p,0}.$$

So the transfer from holomorphic to differentiable differential forms follows the following general construction. Let X be a topological space, $\mathcal{O} \to \mathcal{O}'$ a homomorphism of sheaves of rings and let $\mathcal{M}$ be an $\mathcal{O}$-module. Then $\mathcal{M}\otimes_{\mathcal{O}}\mathcal{O}'$ is an $\mathcal{O}'$-module. It is called the extension of $\mathcal{M}$ to $\mathcal{O}'$. This extension is a vector bundle if $\mathcal{M}$ is. So each vector bundle $\mathcal{M}$ on a complex manifold X induces a vector bundle $\mathcal{M}_{\mathcal{O}_X} \otimes (\mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C})$ on the underlying differentiable manifold.

3. The lemmas of Poincarè and Dolbeault

Let X be a differentiable manifold: The sequence

$$\cdots \longrightarrow A^{m-1}(X) \xrightarrow{d} A^m(X) \xrightarrow{d} A^{m+1} \cdots$$

is called the de-Rham complex. A differential form ω is called *closed* if $d\omega = 0$ and *total* if it is of the form $\omega = d\omega'$. Hence total differential forms are closed. We are interested in converse results. i.e. in conditions under which the de-Rham complex is exact. For this purpose we introduce the *de-Rham cohomology* which measures the non-exactness. We set

$$H^m_{\mathrm{dR}}(X,\mathbb{R}) := \frac{\mathrm{Kernel}(A^m(X) \longrightarrow A^{m+1}(X))}{\mathrm{Image}(A^{m-1}(X) \longrightarrow A^m(X))}.$$

Recall that we can consider differential forms real- and complex-valued. There is also a complex version which leads to

$$H^m_{\mathrm{dR}}(X,\mathbb{C}) := \frac{\mathrm{Kernel}(A^m(X)\otimes_{\mathbb{R}}\mathbb{C} \longrightarrow A^{m+1}(X)\otimes_{\mathbb{R}}\mathbb{C})}{\mathrm{Image}(A^{m-1}(X)\otimes_{\mathbb{R}}\mathbb{C} \longrightarrow A^m(X)\otimes_{\mathbb{R}}\mathbb{C})}.$$

There is not much difference between the real and complex version. This comes from the fact that an exact sequence of $\mathbb{R}$-vector spaces remains exact if one tensors it with $\mathbb{C}$ over $\mathbb{R}$. Hence one has

$$H^m_{\mathrm{dR}}(X,\mathbb{C}) = H^m_{\mathrm{dR}}(X,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} = H^m_{\mathrm{dR}}(X,\mathbb{R}) \oplus \mathrm{i}\, H^m_{\mathrm{dR}}(X,\mathbb{R}).$$

Of course

$$H^m_{\mathrm{dR}}(X,\mathbb{C}) = 0 \quad \text{for} \quad m<0 \text{ and } m > \dim X.$$

In the case $m = 0$ this space can be identified with all functions from $\mathcal{C}^\infty(X)$, which are annihilated by d. These are the locally constant functions. When X is connected we see

$$H^0_{\mathrm{dR}}(X,\mathbb{C}) \cong \mathbb{C} \qquad (\text{and} \quad H^0_{\mathrm{dR}}(X,\mathbb{R}) \cong \mathbb{R}).$$

The higher cohomology groups are involved. A basis result is:

3.1 Lemma of Poincarè. *Let U be an open convex subset of $\mathbb{R}^n$. The sequence*

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0(U) \longrightarrow \cdots \longrightarrow A^n(U) \longrightarrow 0$$

is exact. Here $\mathbb{R} \longrightarrow A^0(U)$ means the map which assigns to a real number the corresponding constant function. Hence $H^0_{\mathrm{dR}}(U,\mathbb{R}) = \mathbb{R}$ and

$$H^m_{\mathrm{dR}}(U,\mathbb{R}) = 0 \quad \textit{for} \quad m > 0,$$

i.e. every closed differential form is total.

Corollary. *Let X be a differentiable manifold of dimension n. The sequence*

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \mathcal{A}_X^0 \longrightarrow \cdots \longrightarrow \mathcal{A}_X^n \longrightarrow 0$$

is an exact sequence of sheaves of $\mathbb{R}$-vector spaces. Hence the sheaf cohomology of the sheaf $\mathbb{R}_X$ of all real valued locally constant functions equals the de-Rham cohomology.

$$H^m(X,\mathbb{R}) = H^m_{\mathrm{dR}}(X,\mathbb{R}).$$

(The same is true for $\mathbb{C}$ instead of $\mathbb{R}$.)

Recall that for an abelian group A we set $H^m(X,A) = H^m(X,A_X)$ where A_X denotes the sheaf of locally constant functions. In this context we mention that for a sheaf F of $\mathbb{R}$-vector spaces there is a canonical isomorphism

$$H^p(X,F)\otimes_{\mathbb{R}}\mathbb{C} \cong H^p(X,F\otimes_{\mathbb{R}}\mathbb{C}).$$

The lemma of Poincarè has a holomorphic version which we don't need but which we formulate just for the sake of completeness.

3.2 Holomorphic lemma of Poincarè. *Let $U \subset \mathbb{C}^n$ be an open convex subset. The sequence*

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}(U) \longrightarrow \bigwedge\nolimits^1 \Omega(U) \longrightarrow \cdots \longrightarrow \bigwedge\nolimits^n \Omega(U) \longrightarrow 0$$

is exact.

The lemma of Poincarè has another complex version which is fundamental for us: The so-called Dolbeault complex or $\bar\partial$-complex on a complex manifold is

$$\cdots \longrightarrow A^{p,q-1}(X) \xrightarrow{\bar\partial} A^{p,q}(X) \xrightarrow{\bar\partial} A^{p,q+1} \xrightarrow{\bar\partial} \cdots$$

Because of $\bar\partial\circ\bar\partial = 0$ we can define the *Dolbeault cohomology*

$$H^{p,q}(X) := \frac{\mathrm{Kernel}(A^{p,q}(X) \longrightarrow A^{p,q+1}(X))}{\mathrm{Image}(A^{p,q-1}(X) \longrightarrow A^{p,q}(X))}.$$

By a polydisc in $\mathbb{C}^n$ we understand the cartesian product of n disks.

3.3 Lemma of Dolbeault. *Let U be a polydisc. The sequence*

$$0 \longrightarrow \bigwedge\nolimits^p \Omega(U) \longrightarrow A^{p,q}(U) \longrightarrow A^{p,q+1}(U) \longrightarrow \cdots \longrightarrow A^{p,n}(U) \longrightarrow 0$$

is exact. Hence $H^{p,0}(U) = \Omega(U)$ and

$$H^{p,q}(U) = 0 \quad \text{for} \quad q > n.$$

Corollary. *Let X be a complex manifold of dimension n. The sequence*

$$0 \longrightarrow \bigwedge\nolimits^p \Omega_X \longrightarrow \mathcal{A}_X^{p,0} \longrightarrow \cdots \longrightarrow \mathcal{A}_X^{p,n} \longrightarrow 0$$

is an exact sequence of sheaves of $\mathbb{C}$-vector spaces. Hence the sheaf cohomology of the sheaf $\bigwedge^p \Omega_X$ equals the Dolbeault cohomology.

$$H^p(X,\bigwedge\nolimits^q \Omega_X) = H^{p,q}(X).$$

Later we will need the following corollary.

3.4 Proposition. *Let $U \subset \mathbb{C}^n$ a polydisc and let*

$$\alpha \in A^{1,1}(U) \cap A^2(U), \quad d\alpha = 0.$$

Then

$$\alpha = \mathrm{i}\partial\bar{\partial} f$$

with a real differentiable function f

Proof. From the lemma of Poincarè we know $\alpha = d\beta$. We decompose $\beta = \gamma + \bar{\gamma}$ with $\gamma \in A^{1,0}$. We have $\partial\gamma = 0$ and, using the lemma of Dolbeault, $\gamma = \partial h$. Set $f = \mathrm{i}(h - \bar{h})$. □

4. Comparison between Čech- and de Rham cohomology

Let X be a differentiable manifold. We denote by $\mathcal{A}^p_{X\,\mathrm{closed}}$ the subsheaf of all closed p forms of $\mathcal{A}^p_X$. The higher cohomology groups of $\mathcal{A}^p_X$ vanish since it is an $\mathcal{C}^\infty_X$-module. But this is not true for the subsheaf of closed forms. From the exact sequence

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \mathcal{C}^\infty_X \longrightarrow \mathcal{A}^1_{X\,\mathrm{closed}} \longrightarrow 0$$

we obtain a combining isomorphism

$$\delta : H^1(X, \mathcal{A}^1_{X,\,\mathrm{closed}}) \xrightarrow{\ \sim\ } H^2(X, \mathbb{R}).$$

We call it the natural isomorphism. Considering the resolutions

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & \mathcal{A}^1_{X\,\mathrm{closed}} & \longrightarrow & \mathcal{A}^1_X & \longrightarrow & \mathcal{A}^2_X & \longrightarrow & \cdots \\ 0 & \longrightarrow & \mathbb{R}_X & \longrightarrow & \mathcal{A}^0_X & \longrightarrow & \mathcal{A}^1_X & \longrightarrow & \mathcal{A}^2_X & \longrightarrow & \cdots \end{array}$$

we obtain a second isomorphism

$$H^1(X, \mathcal{A}^p_{X\,\mathrm{closed}}) \xrightarrow{\ \sim\ } H^2(X, \mathbb{R}) \qquad (= A^2_{\mathrm{closed}}(X)/dA^1(X)).$$

It can be also called a natural isomorphism. From Remark VI.2.8 follows:

4.1 Lemma. *The two natural isomorphisms*

$$H^1(X, \mathcal{A}^1_{X,\,\mathrm{closed}}) \xrightarrow{\ \sim\ } H^2(X, \mathbb{R})$$

agree. The diagram

$$\begin{tikzcd} & A^2_{\mathrm{closed}}(X) \arrow[dl] \arrow[dr] & \\ H^1(X, \mathcal{A}^1_{X,\,\mathrm{closed}}) \arrow[rr] & & H^2(X, \mathbb{R}) \end{tikzcd}$$

is commutative.

We consider an element of $H^2(X,\mathbb{R})$ and represent it by a closed differential form $\omega \in A^2(X)$. Then we consider an open covering $X = \bigcup U_i$ such that $\omega|U_i = d\alpha_i$, $\alpha_i \in A^1(U_i)$. The $\alpha_{ij} = \alpha_i - \alpha_j$ are closed on $U_i \cap U_j$. They define a Čech cocycle and hence an element of $H^1(X, \mathcal{A}^1_{X,\,\mathrm{closed}})$.

4.2 Lemma. *Let ω be a closed element of $A^2(X)$ and let $X = \bigcup U_i$ be an open covering such that $\omega|U_i = d\alpha_i$. Then $\alpha_{ij} = \alpha_i - \alpha_j$ is a Čech cocycle. It represents the image of ω in $H^1(X, \mathcal{A}^1_{X\,\mathrm{closed}})$ (see Lemma 4.1).*

Proof. Use Remark VI.4.4. □

From the exact sequence

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{C}^\infty_X \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow (\mathcal{C}^\infty_X \otimes_{\mathbb{R}} \mathbb{C})^* \longrightarrow 0$$

follows.

4.3 Proposition. *The combining homomorphism*

$$H^1(X, (\mathcal{C}^\infty_X \otimes_{\mathbb{R}} \mathbb{C})^*) \xrightarrow{\sim} H^2(X,\mathbb{Z})$$

is an isomorphism.

Proof. This follows from the long exact cohomology sequence and the fact that the higher cohomology groups of $\mathcal{C}^\infty_X$ (and its complex variant) vanish. □

We need another commutative diagram. There is a natural map of sheaves

$$(\mathcal{C}^\infty_X \otimes_{\mathbb{R}} \mathbb{C})^* \longrightarrow \mathcal{A}^1_{X,\,\mathrm{closed}} \otimes_{\mathbb{R}} \mathbb{C}.$$

It is defined as following. Let f be a differentiable function without zeros. Write locally $f = e^{2\pi \mathrm{i} g}$. The differentials dg then glue. We obtain a map

$$H^1(X, (\mathcal{C}^\infty_X \otimes_{\mathbb{R}} \mathbb{C})^*) \longrightarrow H^1(X, \mathcal{A}^1_{X,\,\mathrm{closed}} \otimes_{\mathbb{R}} \mathbb{C}).$$

4.4 Lemma. *The diagram*

$$\begin{tikzcd}
H^1(X, (\mathcal{C}^\infty_X \otimes_{\mathbb{R}} \mathbb{C})^*) \arrow[r] \arrow[d] & H^1(X, \mathcal{A}^1_{X,\,\mathrm{closed}} \otimes_{\mathbb{R}} \mathbb{C}) \arrow[d] \\
H^2(X,\mathbb{Z}) \arrow[r] & H^2(X,\mathbb{C})
\end{tikzcd}$$

is commutative.

Proof. There is a general result about sheaf cohomology. Let

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & G \arrow[d] & & \\ 0 \arrow[r] & F \arrow[r] \arrow[ur] & F_0 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[dl] & 0 \\ & & G_1 \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

be a commutative diagram of sheaves with exact row and column. Then the induced diagram

$$\begin{tikzcd} H^1(X,F_1) \arrow[r] \arrow[d] & H^1(X,G_1) \arrow[d] \\ H^2(X,F) \arrow[r] & H^2(X,G) \end{tikzcd}$$

is commutative. (The vertical arrows are combining ones.) We leave this a s an exercise to the reader. □

Finally we recall that the canonical map $H^p(X,\mathbb{R}) \to H^p(X,\mathbb{C})$ is injective, even more we have, as we already mentioned, $H^p(X,\mathbb{C}) = H^p(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. But the map $j : H^p(X,\mathbb{Z}) \to H^p(X,\mathbb{R})$ needs not to be injective.

Chapter III. Hodge theory

1. Integration

We need the notion of *orientation* of a finite dimensional real vector space $V \neq 0$. Two bases of V are called orientation compatible if the base transition matrix has positive determinant. The set of all bases decomposes into two classes in which each two are orientation compatible. An orientation of V is the choice of one of the two. They are then called oriented bases. Hence every basis induces an orientation and two bases define the same orientation if they are orientation compatible. The standard orientation of $\mathbb{R}^n$ is defined by the standard basis. Let $L : V \to W$ be an isomorphism of oriented vector spaces. This isomorphism is called orientation preserving if oriented bases are mapped to oriented bases.

1.1 Definition. *An orientation of a differentiable manifold X is a choice of an orientation on each tangent space T_aX, such that the following condition is satisfied: The manifold X can be covered by charts $\varphi : U_\varphi \longrightarrow V_\varphi$ with the property that the tangent maps*

$$T_aU_\varphi \longrightarrow T_{\varphi(a)}V_\varphi = \mathbb{R}^n$$

*are orientation preserving. The charts with this property are called **oriented charts**.*

Let φ, ψ be two oriented charts. Then the functional matrix of $\psi\varphi^{-1}$ has everywhere positive determinant. Let conversely an atlas be given, such that every two charts from this atlas have this property, then there exists an orientation of X, such that the charts of this atlas are orientable.

Integration

Let X be an oriented differentiable manifold of pure dimension n. Let ω be a differential form. The support is defined as

$$\operatorname{supp}(\omega) := \overline{\{\, a \in X; \quad \omega(a) \neq 0 \,\}}.$$

This is the biggest closed subset A such that $\omega|(X - A) = 0$. We denote by $\mathcal{A}^n_c(X)$ the set of compactly supported top-differential forms. The support of an $\omega \in \mathcal{A}^n_c(X)$ is called *small* if there exists an oriented chart φ with $\operatorname{supp}(\omega) \subset U_\varphi$. The form ω can be written in this chart as $f(x)dx_1 \wedge \ldots \wedge dx_n$. We define

$$\int_X \omega := \int_{V_\varphi} f(x)\, dx_1 \ldots dx_n.$$

From the transformation formula for integrals one can see that this definition is independent from the choice of φ. (Here one has to use that the chart transformations have positive Jacobi determinant.)

Using the technique of decomposition of one, one can show:

There exists a unique linear form

$$A^n_c(X) \longrightarrow \mathbb{C}, \quad \omega \longmapsto \int_X \omega,$$

which agrees with the above construction for forms with small support.

This integral can be extended by the standard techniques of integration theory to a large class of even not continuous differential forms and one uses this technique also to define $\int_A \omega$ for subsets $A \subset X$. We need only little of these constructions, for example we will use that $\int_U \omega$ can be defined for open subsets and differential forms $\omega \in A^n(X)$ such that $\operatorname{supp}(\omega) \cap \bar{U}$ is compact.

The theorem of Stokes

Let $U \subset X$ be an open subset of an oriented differentiable manifold. Let a be a boundary point of U. We say that a is a smooth boundary point of U if there is an oriented chart φ around a such that

$$\varphi(U \cap U_\varphi) = \{x \in V_\varphi,\ x_1 < 0\}.$$

Then automatically

$$\varphi(U \cap \partial U_\varphi) = \{x \in V_\varphi,\ x_1 = 0\}.$$

We denote by $\partial_0 U$ the smooth part of the boundary. It is clear that $\partial_0 X$ is a smooth subset and hence a differentiable manifold. A chart around a is given by the restriction of φ when we consider $\{x \in V_\varphi,\ x_1 = 0\}$ as an open subset of $\mathbb{R}^{n-1}$. It can be checked that there is an orientation on $\partial_0 X$ such that these charts are orientable. The theorem of Stokes states:

Let $\omega \in A^{n-1}(X)$ be a differential form of degree $n-1$ such that

$$\operatorname{supp}(\omega) \cap \bar{U}$$

is compact. Then

$$\int_U d\omega = \int_{\partial U} \omega \mid \partial U.$$

A special case says that for compact X and arbitrary $\omega \in A^{n-1}(X)$

$$\int_X d\omega = 0$$

(because one take $U = X$ with $\partial U = \emptyset$.)

We need some more linear algebra.

1.2 Definition. *Let V be an n-dimensional oriented real vector space. An element of $\bigwedge^n V$ is called positive if it is of the form $Ce_1 \wedge \ldots \wedge e_n$, $C > 0$, with an oriented basis.*

It is clear that this definition is independent of the choice of the oriented basis. Another way to express this, is to say that $\bigwedge^n V$ has been oriented. A top-differential form $\omega \in A^n(X)$ is called positive if $\omega(a) \in \bigwedge^n T_a X$ is positive for all a. We write $\omega > 0$. Similarly one defines $\omega \geq 0$. If $x_1, \ldots, x_n$ denotes an oriented chart and if ω corresponds to $f(x)dx_1 \wedge \ldots \wedge dx_n$ in this chart then $\omega \geq 0$ means $f(x) \geq 0$. Hence

$$\int_X \omega \geq 0 \quad \text{if} \quad \omega \geq 0.$$

2. Elliptic differential operators

Let $U \subset \mathbb{R}^n$ be an open subset. We are interested in maps

$$D : \mathcal{C}^\infty(U) \longrightarrow \mathcal{C}^\infty(U)$$

which can be written as finite sum

$$Df = \sum h_{i_1,\ldots,i_n} \frac{\partial^{i_1+\cdots+i_n} f}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}$$

with differentiable coefficients $h_{\ldots} \in \mathcal{C}^\infty(U)$, which are uniquely determined. We call D a local linear differential operator. When D is non-zero, there exists a maximal m such that $h_{i_1,\ldots,i_n}$ is non-zero for some index with $i_1+\cdots+i_n = m$. We call m the degree of this operator and the function on $U \times \mathbb{R}^n$

$$P(x_1, \ldots, x_n, X_1, \ldots, X_n) = \sum_{i_1+\cdots+i_n=m} h_{i_1,\ldots,i_m}(x) X_1^{i_1} \ldots X_n^{i_n}$$

is called the *symbol* of D. This is a homogenous polynomial of degree m for fixed x. The operator D is called *elliptic* if

$$P(x, X) \neq 0 \quad \text{for all} \quad X \in \mathbb{R}^n,\ X \neq (0, \ldots, 0).$$

We need a slight generalization of this. We consider operators

$$D : \mathcal{C}^\infty(U)^p \longrightarrow \mathcal{C}^\infty(U)^q.$$

They can be considered as $p \times q$ matrices as well as the coefficients. The degree now is defined to be the biggest number m such that one of the coefficients of $h_{i_1,\ldots,i_n}$ is different from zero for some index with $i_1 + \cdots + i_n = m$. The symbol now is a $p \times q$ matrix of functions on $U \times \mathbb{R}^n$.

2.1 Definition. *A local linear differential operator*

$$D : \mathcal{C}^\infty(U)^p \longrightarrow \mathcal{C}^\infty(U)^q \qquad (U \subset \mathbb{R}^n \text{ open})$$

is called ***elliptic*** *if $p = q$ and if the symbol $P(x, X)$ is an invertible matrix for every $x \in D$ and every real $X \neq 0$.*

Example. The Laplace operator

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

There are two simple observations:

a) Let $V \subset U \mathrm{subset} \mathbb{R}^\mathrm{n}$ be open subsets and let D be a local linear differential operator on U. Then there is a natural restriction to a local linear differential operator on V.

b) Let $\varphi : U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^n$ and let D be a local linear differentiable operator on U. Then the transported operator to V is a local linear differential operator as well.

Ellipticity is preserved in both cases.

We want to generalize the notion of an elliptic operator to vector bundles on differentiable manifolds. Let $\mathcal{E}, \mathcal{F}$ be two real differentiable vector bundles over a differentiable manifold.

2.2 Definition. *An linear differential operator $D : \mathcal{E} \to \mathcal{F}$ by definition is an $\mathbb{R}$-linear map of sheaves*

$$D : \mathcal{E} \longrightarrow \mathcal{F}$$

with the following property: Let $U \to V$ a differentiable chart and assume that there exists trivializations of $\mathcal{E}|U$ and $\mathcal{F}|U$, then the induced maps $\mathcal{C}^\infty(V)^p \to \mathcal{C}^\infty(V)^q$ are local linear differential operators. D is called elliptic if they are elliptic in the sense of 2.1.

It is sufficient to take all $U = U_\varphi$ from an atlas for each chart one trivialization of $\mathcal{E}|U$ and $\mathcal{F}|V$.

So far we treated the real valued case. But the whole thing can be done over $\mathbb{C}$ as well. We can define $\mathbb{C}$-linear differential operators

$$D : \mathcal{C}^\infty(U, \mathbb{C})^p \longrightarrow \mathcal{C}^\infty(U, \mathbb{C})^q$$

and there symbol $P(x, X)$. It is called elliptic if $p = q$ and if $\det P(x, X)$ is non-zero for all $x \in U$ and $X \in \mathbb{R}^p$, $X \neq 0$. In particular, we can define for complex valued bundles $\mathcal{E}, \mathcal{F}$ the notion of a $\mathbb{C}$-linear differential operator and define when it is elliptic.

One of the basic facts about elliptic operators D on a *compact* differentiable manifold say that the kernel and the cokernel of $D : \mathcal{E}(X) \to \mathcal{F}(X)$ is finite dimensional. (The cokernel of a linear map $L : V \to W$ is defined as $W/L(V)$.) This and a refinement will be formulated as a theorem a little later: Let $\mathcal{E}$ be a vector bundle on a differentiable manifold. As in the case of differential forms one can define the support of a section $s \in \mathcal{E}(U)$. It is called compactly supported if the support is compact. This means that there exists a compact subset $K \subset U$ such that the restriction of s to $U - K$ is zero. We denote by $\mathcal{E}_c(U)$ the space of compactly supported sections.

2.3 Definition. *Let $\mathcal{E}$, $\mathcal{F}$ be two real or complex vector bundles over an oriented differential manifold. Assume that an everywhere positive top form $\omega \in A^n(X)$ has been distinguished. Two linear differential operators*

$$D : \mathcal{E} \longrightarrow \mathcal{F}, \quad D^* : \mathcal{F}^* \longrightarrow \mathcal{E}^*$$

are called ***formally adjoint*** *(with respect to ω if for all sections $s \in \mathcal{E}_c(X)$, $t \in \mathcal{F}_c(X)$ the formula*

$$\int_X \langle Ds, t\rangle\omega = \int_X \langle s, D^*t\rangle\omega$$

holds.

Any bundle map ($= \mathcal{C}^\infty_X$ map) $f : \mathcal{E} \to \mathcal{F}$ can be considered as a linear differential operator ($m = 0$). In this case the dual operator and the formally adjoint operator agree, since the equality of scalar products agree already before integration.

2.4 Remark. *Let X be a differentiable manifold and let $\mathcal{E}$ and $\mathcal{F}$ be real or complex differentiable vector bundles. Assume that an everywhere positive top form $\omega \in A^n(X)$ has been distinguished. Every linear differential operator $D : \mathcal{E} \to \mathcal{F}$ admits a unique formally adjoint $D^* : \mathcal{F}^* \to \mathcal{E}^*$ and this is elliptic when D is elliptic. The formally adjoint of D^* is D.*

Proof. We want to give just the idea of the proof and treat a special case $X = \mathbb{R}$, $\mathcal{E} = \mathcal{F} = \mathcal{C}^\infty_X$, $D = d/dx$ and $\omega = dx$. Then the formula reads as

$$\int_{-\infty}^{\infty} f'(x)g(x)dx = \int_{-\infty}^{\infty} f(x)g'(x)dx$$

(partial integration of functions with compact support). □

2.5 Definition. *Let $\mathcal{E}$ be a real vector bundle on a differentiable manifold X. A Euclidean metric on $\mathcal{E}$ is a $\mathcal{C}^\infty_X$-bilinear map*

$$\mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{C}^\infty_X,$$

such that

$$E_a \times E_a \longrightarrow \mathbb{R}$$

is symmetric and positive definite for all a.

The usage of a Euclidian metric is often to identify $\mathcal{E}$ and $\mathcal{E}^*$. There is an obvious isomorphism of vector bundles $\mathcal{E} \to \mathcal{E}^*$ that is constructed in the same way as the isomorphism $V \to V^*$ for a real Euclidian finite dimensional vector space. There is an analogue for complex bundles. Here we want to use Hermitian forms. A Hermitian form on a complex vector space V is a complex valued pairing $\langle\cdot,\cdot\rangle$ that is $\mathbb{C}$-linear in the first variable and that satisfies $\langle a, b\rangle = \overline{\langle b, a\rangle}$.

2.6 Definition. *Let $\mathcal{E}$ be a complex vector bundle on a differentiable manifold X. A Hermitian form on $\mathcal{E}$ is a $\mathcal{C}_X^\infty$-bilinear map*

$$\mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C}$$

such that

$$E_a \times E_a \longrightarrow \mathbb{C}$$

is Hermitian for all a. In the case that these forms are positive definite we call h a Hermitian metric.

We want to use the metric to identify $\mathcal{E}$ and $\mathcal{E}^*$. But in the Hermitian case we have to be a little more careful. Let V be a finite dimensional complex vector space with a positive definite Hermitian form. And let $V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the dual space. Then we get a bijection $f : V \to V^*$ that sends a to the linear form $x \mapsto \langle a, x\rangle$. But this bijection is only $\mathbb{R}$-linear. It is antilinear in the sense $f(Ca) = \bar{C} f(a)$.

2.7 Remark. *Let $\mathcal{E}$ be a real or complex vector bundle with a Euclidean or Hermitian metric $\langle\cdot,\cdot\rangle$. Then there is a natural $\mathcal{C}_X^\infty$-linear isomorphism*

$$\sharp : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^*, \quad (\sharp s)(t) = \langle t, s\rangle.$$

It is $\mathbb{C}$-antilinear in the complex case.

Assume that we have two real or complex vector bundles $\mathcal{E}, \mathcal{F}$ on the differentiable manifold. We assume that on both a Euclidean or Hermitian metric has been distinguished. Let $D : \mathcal{E} \to \mathcal{F}$ be a linear differentiable operator and let $D : \mathcal{F}^* \to \mathcal{E}^*$ its formally adjoint. Identifying the bundles with their duals, we can read D^* as operator $\mathcal{F} \to \mathcal{E}$. Notice that this operator is $\mathbb{C}$-linear in the Hermitian case. It is a linear differentiable operator too and it is elliptic if D is elliptic. When it is clear which Euclidian or Hermitian metrics are used, we denote this operator by $D^* : \mathcal{F} \to \mathcal{E}$ and call it also the formally adjoint operator. The adjointness formula reads now

$$\int_X \langle Ds, t\rangle_{\mathcal{F}} \omega = \int_X \langle s, D^* t\rangle_{\mathcal{E}} \omega, \quad , s \in \mathcal{F}_c(U) \ t \in \mathcal{E}_c(U).$$

We formulate now without proof a fundamental result of the theory of partial differential equations:

2.8 Theorem. *Let $\mathcal{E}, \mathcal{F}$ be real or complex vector bundles with Euclidian or Hermitian metric on a oriented compact differentiable manifold of (pure) dimension n. Assume that an everywhere positive top form $\omega \in A^n(X)$ has been distinguished. Let $D : \mathcal{E} \to \mathcal{F}$ be an elliptic operator. Then the kernel and cokernel of the map*

$$D : \mathcal{E}(X) \longrightarrow \mathcal{F}(X)$$

are finite dimensional. Moreover

$$\mathcal{E}(X) = \operatorname{Kernel}\bigl(D : \mathcal{E}(X) \to \mathcal{F}(X)\bigr) \;\oplus\; \operatorname{Image}\bigl(D^* : \mathcal{F}(X) \to \mathcal{E}(X)\bigr).$$

3. Real Hodge theory

3.1 Definition. *A Riemannian manifold is a differentiable manifold together with a Euclidean metric g on the tangent bundle.*

If $U \subset \mathbb{R}^n$ is an open subset, than a Riemannian metric on U is given by a $n \times n$-matrix $g(x) = (g_{ik}(x))$ of differentiable functions, which is symmetric and positive definit at every point. (Identify the tangent space with $\mathbb{R}^n$.)

3.2 Lemma. *Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional real Euclidian vector space. It is possible to extend the Euclidian metric to Euclidian metrics to all $\bigwedge^m V$ by the such that the following condition is satisfied:*

$$\langle a_1 \wedge \ldots \wedge a_d, b_1 \wedge \ldots \wedge b_d\rangle = \det\bigl(\langle a_i, b_j\rangle\bigr)_{1\le i,j\le n}.$$

Proof. Define the metric such that this formula is true for all a_i, b_j from an oriented orthonormal basis and then make use from the multilinearity. $\square$

The lemma carries over to a real vector bundle $\mathcal{E}$ on a differentiable manifold that has been equipped with a Euclidian metric (Definition 3.1). Then there exist Euclidian metrics on $\bigwedge^p \mathcal{E}$ such that the formula in Lemma 3.2 holds. We also recall that we have a natural isomorphism $\mathcal{E} \to \mathcal{E}^*$ (Remark 2.7). Hence we get also a Euclidian metric on $\mathcal{E}^*$ and then on $\bigwedge^p \mathcal{E}^*$. We apply this to an oriented Riemannian manifold (X, g). The (real) tangent bundle $\mathcal{T}_X$ then carries a Euclidean metric.

3.3 Lemma. *Let (X,g) be a Riemannian manifold. The vector bundles $\mathcal{A}_X^p$ carry unique Euclidian metrics such that for every open $U \subset X$ we have*

$$\langle \alpha_1 \wedge \ldots \wedge \alpha_p, \beta_1 \wedge \ldots \beta_p \rangle = \det\bigl(\langle \alpha_i, \beta_j \rangle\bigr)_{1 \le i,j \le n} \qquad (\alpha_i, \beta_j \in A^1(U)).$$

We assume now that V is oriented. In the top space $\bigwedge^n V$ we now can choose the unique *positive* element ω with the property $\langle \omega, \omega \rangle = 1$. We call this the *volume element.* The volume element defines an isomorphism

$$\mathbb{R} \xrightarrow{\sim} \bigwedge\nolimits^n V, \quad C \longmapsto C\omega.$$

The dual space of an oriented space is oriented as well (by the dual bases of the oriented bases). Hence we can consider the volume element of the top space $\bigwedge^n (T_a X)^*$. This defines a top-differential form ω on X. We want to compute it in local coordinates. First we describe the isomorphism $\sharp : \mathcal{T}_X \to \mathcal{T}_X^*$ locally. By definition $\langle A, B \rangle := g(A,B) = \sharp(A)(B)$. Hence

$$g_{ik} = \langle \partial/\partial x_i, \partial/\partial x_k \rangle = \langle \sharp \partial/\partial x_i, \partial/\partial x_k \rangle.$$

3.4 Remark. *Let $U \subset \mathbb{R}^n$ be open then the isomorphism $\mathcal{T}_X \cong \mathcal{T}_X^*$ is given through*

$$\partial/\partial x_i \longmapsto \sum_j g_{ij} dx_j, \quad dx_i \longmapsto \sum_j g^{ij} \partial/\partial x_j.$$

Here (g^{ij}) denotes the inverse matrix of (g_{ij})

A consequence is $\langle dx_i, dx_j \rangle = g(dx_i, dx_j) = g^{ij}$.

3.5 Remark. *Let (X,g) be an oriented Riemannian manifold of pure dimension n. The volume element in $\bigwedge^n (T_a X^*)$ defines a top-differential form $\omega \in A^n(X)$. In the case of an open subset $U \subset \mathbb{R}^n$ it is given by*

$$\sqrt{\det g(x)}\; dx_1 \wedge \ldots, \wedge dx_n.$$

This form is called the ***volume form*** *of X. Its integral (it can be infinite) is called the volume of X.*

Proof. From $\langle dx_i, dx_j \rangle = g^{ij}$ we get

$$\langle dx_1 \wedge \ldots \wedge dx_n, dx_1 \wedge \ldots \wedge dx_n \rangle = \det g^{-1}.$$

This gives the proof of the remark. □

The star operator

If V is an oriented Euclidian real vector space of dimension n then we defined an isomorphism $\bigwedge^n V \cong \mathbb{R}$. Hence we obtain a pairing

$$\bigwedge^p V \times \bigwedge^{n-p} V \longrightarrow \mathbb{R}, \quad (A,B) \longmapsto \text{“}A \wedge B\text{”}.$$

This pairing is non-degenerated and induces an isomorphism

$$\bigwedge^p V \xrightarrow{\sim} \Big(\bigwedge^{n-p} V\Big)^*.$$

As we mentioned already, the Euclidean metric extends to the exterior powers. Hence we obtain an isomorphism, the so-called *star operator*

$$\bigwedge^p V \xrightarrow{\sim} \bigwedge^{n-p} V, \quad A \longmapsto *A.$$

It is characterized by

$$\langle *A, B\rangle\, \omega = A \wedge B \qquad \big(A \in \bigwedge\nolimits^p V,\ B \in \bigwedge\nolimits^{n-p} V\big).$$

We compute the star operator by means of an oriented orthonormal basis. $e_1, \dots, e_n$. For a subset a of $\{1,\dots,n\}$ we define $e_a := e_{a_1} \wedge \ldots \wedge e_{a_p}$ where $a_1 < \cdots < a_p$ are the elements of a in their natural order. From the characteristic equation we get

$$*e_a = \varepsilon(a,b) e_b.$$

Here b is the complement of a and $\varepsilon(a,b)$ is the sign of the permutation that brings a, b into the natural ordering. Clearly $\varepsilon(a,b)\varepsilon(b,a) = (-1)^{p(n-p)}$. We deduce $** a = (-1)^{p(n-p)} a$. This construction extends to oriented Riemannian manifolds:

3.6 Remark. *Let (X,g) be an oriented Riemannian manifold of pure dimension n. The star operators for tangent spaces induce an isomorphism*

$$* : A^p(X) \xrightarrow{\sim} A^{n-p}(X).$$

This has the properties

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle \omega,$$

where ω denotes the volume form. We also have

$$** \alpha = (-1)^{p(n-p)} \alpha \quad \text{for} \quad \alpha \in A^p(X).$$

Proof. The map $\mathcal{C}_X^\infty \to \mathcal{A}_X^n$, $f \mapsto f\omega$ is an isomorphism as can be checked locally. Hence the wedge product gives a map $\mathcal{A}_X^p \times \mathcal{A}_X^q \to \mathcal{C}_X^\infty$. This map induces a map $\mathcal{A}_X^p \to (\mathcal{A}_X^q)^*$. The metric gives an isomorphism $(\mathcal{A}_X^q)^* \to \mathcal{A}_X^q$. Both together give the star operator. □

We can integrate top differential forms with compact support on oriented differentiable manifolds.

Let (X, g) be an oriented Riemannian manifold. Then one can define

$$A_c^p(X) \times A_c^p(X) \longrightarrow \mathbb{R}, \quad (\alpha, \beta)_g := \int_X \langle \alpha, \beta \rangle \, \omega = \int_X \alpha \wedge *\beta.$$

This is a symmetric positive definit bilinear form.

The codifferentiation

We define the codifferentiation by

$$d^* : A^{p+1}(X) \longrightarrow A^p(X), \quad d^* = -(-1)^{np} * d * .$$

3.7 Proposition. *The codifferentiation d^* satisfies.*

$$(d\alpha, \beta) = (\alpha, d^*\beta), \quad \alpha \in A_c^p(X), \ \beta \in A_c^{p+1}(X)$$

Proof. One has to use Stoke's formula $\int_X d(\alpha \wedge *\beta) = 0$ and the product rule. □

The operators d and d^* can be defined for any open subset U instead of X. Hence they are operators on vector bundles

$$d : \mathcal{A}_X^p \longrightarrow \mathcal{A}_X^{p+1}, \quad d^* : \mathcal{A}_X^{p+1} \longrightarrow \mathcal{A}_X^p.$$

They are linear differentiable operators.

3.8 Lemma. *The operators*

$$d : \mathcal{A}_X^p \longrightarrow \mathcal{A}_X^{p+1}, \quad d^* : \mathcal{A}_X^{p+1} \longrightarrow \mathcal{A}_X^p$$

are formally adjoint.

(Notice that we have an Euclidian metric on all $\mathcal{A}_X^p$ and a distinguished top differential form ω on X. These are the ingredients to define the formal adjoint operator.)

The Laplace-Beltrami operator

The Laplace-Beltrami operator on an oriented Riemannian manifold X is defined by

$$\Delta : A^p(X) \longrightarrow A^p(X), \quad \Delta = dd^* + d^*d.$$

The simplest case is the Euclidean metric on an open subset $U \subset \mathbb{R}^n$ and the case $p = 0$. Here

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

This example makes plausible:

3.9 Remark. *The Laplace-Beltrami operator can be considered as linear differential operator of the bundle $\mathcal{A}_X^p$ into itself. This operator is elliptic and it is its own adjoint.*

We denote by

$$\mathcal{H}^p(X) = \left\{ \alpha \in A^p(X); \quad \Delta\alpha = 0 \right\}$$

the kernel of Δ. Its elements are called *harmonic forms*. Now we can apply the theory of partial differential equations to conclude in case of a compact oriented Riemannian manifold X:

$$A^p(X) = \mathcal{H}^p(X) \oplus \Delta(A^p(X)).$$

In the case of a compact manifold harmonic forms are easy to characterize:

3.10 Proposition. *A differential form α on a compact oriented Riemannian manifold X is harmonic if and only if*

$$d\alpha = 0 \qquad \text{and} \qquad d^*\alpha = 0.$$

If X is connected then every harmonic function (=zero-form) is constant.

The proof follows from

$$(\Delta\alpha, \alpha) = (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha). \qquad \square.$$

As a consequence of Proposition 3.10 we obtain for $\alpha \in A^p(X)$ a representation

$$\alpha = \alpha_0 + d\beta + d^*\gamma, \qquad \alpha_0 \text{ harmonic}.$$

We apply this to closed forms α. From $d\alpha = 0$ and Proposition 3.10 follows $dd^*\gamma = 0$, hence

$$(d^*\gamma, d^*\gamma) = (\gamma, dd^*\gamma).$$

It follows

$$\alpha = \alpha + d\beta$$

and this is a direct decomposition

$$\mathrm{Kernel}(A^p(X) \longrightarrow A^{p+1}(X)) = \mathcal{H}^p(X) \,\oplus\, \mathrm{Image}(A^{p-1}(X) \longrightarrow A^p(X)).$$

This means that every class of closed forms in $H^p_{\mathrm{dR}}(X, \mathbb{R})$ contains a unique harmonic representant. This means:

3.11 Main theorem of real Hodge theory. *Let X be a compact oriented Riemannian manifold. Then $\mathcal{H}^p(X)$ is contained in the space of closed forms and the natural homorphism*

$$\mathcal{H}^p(X) \xrightarrow{\sim} H^p_{\mathrm{dR}}(X, \mathbb{R})$$

is an isomorphism. These vector spaces are finite dimensional.

Hence the so-called *Betti numbers*

$$b^p(X) := \dim H^p_{\mathrm{dR}}(X, \mathbb{R})$$

are well defined numbers.

As an application we derive the duality theorem. When α is harmonic then for trivial reasons $*\alpha$ is harmonic too. This obviously defines an isomorphism

$$\mathcal{H}^p(X) \xrightarrow{\sim} \mathcal{H}^{n-p}(X).$$

3.12 Poincarè duality. *Let X be a pure n-dimensional compact oriented Riemannian manifold. The pairing $(\cdot, \cdot)$*

$$H^p_{\mathrm{dR}}(X, \mathbb{R}) \times H^{n-p}_{\mathrm{dR}}(X, \mathbb{R}) \longrightarrow \mathbb{R}, \qquad \int_X \alpha \wedge \beta,$$

is non-degenerated, hence $b^p(X) = b^{n-p}(X)$.

Here α, β denote differential forms that represent de-Rham cohomology classes. The integral is independent of their choice due to the theorem of Stokes. The pairing is non-degenerate, since $\int_X \alpha \wedge \beta$ for all β implies $\int_X \alpha \wedge \alpha = \int_X \langle \alpha, \alpha \rangle \omega = 0$, hence $\alpha = 0$. □

4. Complex Hodge theory

Again we start with a little linear algebra. Recall that a Hermitean form h on a complex vector space V is a map $V \times V \to \mathbb{C}$ which is linear in the first variable and such that

$$h(a, b) = \overline{h(b, a)}.$$

The form h is called positive definit if (the real expression) $h(a, a)$ is positive for non-zero a. A positive Hermitian form is called a Hermitian metric. It is clear that

$$g(a, b) := \operatorname{Re} h(a, b) = \frac{1}{2}(h(a, b) + h(b, a))$$

is a bilinear form on the underlying real vector space. Hence a Hermitian metric has an underlying Euclidean metric. One calculates

$$g(a,b) + \mathrm{i}g(a,\mathrm{i}b) = h(a,b).$$

Hence h is determined by g. Conversely this formula defines a Hermitian form for a given real bilinear form if and only if

$$g(a,b) = g(\mathrm{i}a,\mathrm{i}b).$$

It is also interesting to look at the imaginary part of h (or on its negative),

$$h(a,b) = g(a,b) - \mathrm{i}\Omega(a,b).$$

Obviously Ω is a real bilinear form which is alternating,

$$\Omega(a,b) = -\Omega(b,a).$$

It is closely related to g, one checks easily

$$\Omega(a,b) = g(\mathrm{i}a,b).$$

Hence Ω determines h and moreover:

A real alternating bilinear form on V defines a Hermitian form if and only if

$$\Omega(a,b) = \Omega(\mathrm{i}a,\mathrm{i}b).$$

We generalize this to complex vector bundles $\mathcal{E}$ on a differentiable manifold X. Recall (Definition 2.6) that a Hermitian form h on $\mathcal{E}$ is a family of Hermitian forms h_a on E_a which depends differentiably on a. We denote by Ω the imaginary part of h. This is a section of the bundle

$$\bigwedge^2 \operatorname{Hom}_{\mathcal{C}_X^\infty}(\mathcal{E},\mathcal{C}_X^\infty).$$

Let now X be a complex manifold. We apply this to the real tangent bundle $\mathcal{T}_X$. Recall that the real tangent bundle of a complex manifold carries a complex structure. There are reasons to write the multiplication with i by a new letter

$$J : \mathcal{T}_X \longrightarrow \mathcal{T}_X.$$

Let h be a Hermitian form on this bundle, in particular $h(JA,B) = \mathrm{i}h(A,B)$. We can consider the real part $g = \operatorname{Re} h$. This is a symmetric bilinear form on the real tangent bundle. But we can also consider the imaginary part Ω of h. As we have seen this is a section of the bundle

$$\bigwedge^2 \operatorname{Hom}_{\mathcal{C}_X^\infty}(\mathcal{T}_X,\mathcal{C}_X^\infty) = \bigwedge^2 \mathcal{T}_X^*.$$

This means that Ω is a real alternating differential form of degree 2.

4.1 Definition. *Let X be a complex manifold and let h be a Hermitian form on the real tangent bundle (considered as complex bundle). The real differential form*

$$\Omega = -\operatorname{Im} h \in A^2(X)$$

is called the fundamental form of the pair (X, h).

It is easy to verify that Ω is of type $(1,1)$, hence

$$\Omega \in A^2(X) \cap A^{1,1}(X).$$

This can be also seen form the following formulae in local coordinates:

Let $U \subset \mathbb{C}^n$ be an open subset. The real tangent space T_aU is generated as $\mathbb{R}$-vector space by $\partial/\partial x_i, \partial/\partial y_i$. But it has also a complex structure. Multipliplication by i with respect to this structure is given by $J(\partial/\partial x_i) = \partial/\partial y_i$ and $J(\partial/\partial y_i) = -\partial/\partial x_i$. Hence a complex basis of T_aX is given by $\partial/\partial x_i$. We use this basis to define the Hermitian matrix

$$h_{ij} = h\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big).$$

Since h is Hermitian we have $h(J\partial/\partial x_i, \partial/\partial x_j) = \mathrm{i}h(\partial/\partial x_i, \partial x_j)$. In this way we get

$$h\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big) = \mathrm{i}h\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\Big) = -\mathrm{i}h\Big(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_j}\Big) = h\Big(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\Big).$$

The fundamental form can be computed.

4.2 Lemma. *Let $U \subset \mathbb{C}^n$ an open subset and h a Hermitian form on the tangent bundle, given by the Hermitian matrix*

$$h_{ij} = h\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big).$$

Then the corresponding fundamental two form is

$$\Omega = \frac{\mathrm{i}}{2} \sum_{1 \le i,j \le n} h_{ij}\, dz_i \wedge d\bar{z}_j.$$

Proof. We remind our definition how the wedge product is related to alternating multilinear forms (Remark V.6.3): let α, β be two one-forms. Then

$$(\alpha \wedge \beta)(A, B) = \alpha(A)\beta(B) - \alpha(B)\beta(A).$$

It follows

$$dz_i \wedge d\bar{z}_j\Big(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\Big) = \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}.$$

This implies

$$\frac{\mathrm{i}}{2}\sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j\Big(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\Big) = \frac{\mathrm{i}}{2}(h_{\alpha\beta} - h_{\beta\alpha}) = -\operatorname{Im} h_{\alpha\beta}.$$

This proves Lemma 4.2. □

We collect what we have seen so far.

4.3 Remark. *Let X be a complex manifold. Every Hermitian form h on the real tangent bundle (with its complex structure) induces a real differential form $\Omega = -\operatorname{Im} h$ of type $(1,1)$. Conversely every form $\Omega \in A^2(X) \cap A^{1,1}(X)$ is the negative of the imaginary part of a uniquely defined h.*

Our main interest is in positive definite Hermitian forms.

4.4 Definition. *A Hermitian manifold (X, h) is a complex manifold together with a positive definite Hermitian form on the real tangent bundle (considered as complex bundle).*

Since a Hermitian manifold can be considered also as Riemannian manifold ($g = \operatorname{Re} h$) we have also a fundamental form $\omega \in A^{2n}(X)$, and we have a Euclidean metric

$$\mathcal{A}_X^p \times \mathcal{A}_X^p \longrightarrow \mathcal{C}_X^\infty.$$

We extend this to a *Hermitian* pairing

$$(\mathcal{A}_X^p \otimes_{\mathbb{R}} \mathbb{C}) \times (\mathcal{A}_X^p \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C}.$$

(Notice, we could also take the $\mathbb{C}$-linear extension, but we do not.) By restriction we get a positive definite Hermitian pairing

$$\mathcal{A}_X^{p,q} \times \mathcal{A}_X^{p,q} \longrightarrow \mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C}.$$

For the fibres this means the following. The space T_aX carries a Euclidian metric $g = \operatorname{Re} h$. This carries over to a Euclidean metric on $\operatorname{Hom}_{\mathbb{R}}(T_aX, \mathbb{R})$. This extended to a *Hermitian* metric on $\operatorname{Hom}_{\mathbb{R}}(T_aX, \mathbb{C})$ and then we get a Hermitian metric on $A^{p,q} := \bigwedge^m \operatorname{Hom}_{\mathbb{R}}(T_aX, \mathbb{C})$.

4.5 Definition. *Let X be a Hermitian manifold. We denote the natural* ***Hermitian*** *pairing on $\mathcal{A}_X^{p,q}$ by $\langle \cdot, \cdot \rangle$.*

Since X has an underlying structure as Riemannian manifold, we have the star operator, provided we have an orientation. But complex manifolds are always naturally oriented. This comes form the following simple observation. Let $e_1, \dots, e_n$ be a basis of the complex vector space. Then $e_1, \mathrm{i}e_1, \dots, e_n, \mathrm{i}e_n$ is a basis of the underlying real vector spaces. Bases obtained in this way are orientation compatible. (When $f_1, \dots, f_n$ is a second basis, then one has a complex $n \times n$-transition matrix A and a $2n \times 2n$-transition matrix for the corresponding real bases. One has $\det B = |\det A|^2 > 0$.) We use these real bases to define the orientation of V. Now we can consider the operator $* : A^p(X) \to A^{2n-p}(X)$. We extend this to a $\mathbb{C}$-linear operator

$$* : A^p(X) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow A^{2n-p}(X) \otimes_{\mathbb{R}} \mathbb{C}.$$

Here we denote by n the complex dimension of X which is assumed to be pure dimensional.

4.6 Lemma. *The star operator on a Hermitian manifold of pure dimension n preserves the (p,q)-graduation as follows:*

$$A^{p,q}(X) \longrightarrow A^{n-q,n-p}(X).$$

It satisfies $** = (-1)^{p+q}$.

Proof. Let T be a complex vector space of dimension n and let h be a positive definit Hermitian form. In our application $T = T_a X$ is the tangent space of a complex manifold. For this reason we write J for the multiplication by i in T We choose an orthonormal complex basis $E_1, \dots, E_n$ of T with respect to h. A real basis of T is given by

$$E_1, JE_1, \dots E_n, JE_n.$$

This is an orthonormal basis with respect to $g = \operatorname{Im} h$. Its dual basis in $\operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$ is denoted by

$$X_1, Y_1, \dots X_n, Y_n.$$

This is a orthonormal basis of $\operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$ (with respect to the transferred metric g). Then we consider $Z_i = X_i + \mathrm{i}Y_i$, $\bar{Z}_1 = X_1 - \mathrm{i}Y_i$. One checks that $Z_i \in \operatorname{Hom}_{\mathbb{C}}(T, \mathbb{C})$. Even more, $Z_1, \dots, Z_n$ is the dual basis of $E_1, \dots, E_n$. We have that Z_i is of type $(1,0)$ and $\bar{Z}_i$ is of type $(0,1)$. Recall that we extend the Euclidean metric to a Hermitian metric on $\operatorname{Hom}_{\mathbb{R}}(T, \mathbb{C})$. One computes

$$\langle Z_i, Z_j \rangle = \langle \bar{Z}_i, \bar{Z}_j \rangle = 2\delta_{ij}, \quad \langle Z_i, \bar{Z}_j \rangle = 2.$$

Next we express the volume element ω in terms of the $Z_i, \bar{Z}_i$. We have

$$\omega = X_1 \wedge Y_1 \wedge \dots \wedge X_n \wedge Y_n.$$

The formula $Z \wedge \bar{Z} = -2\mathrm{i}X \wedge Y$ shows

$$\omega = \left(\frac{\mathrm{i}}{2}\right)^n Z_1 \wedge \bar{Z}_1 \wedge \dots \wedge Z_n \wedge \bar{Z}_n.$$

The space $A^{p,q} = \bigwedge^{p,q} \operatorname{Hom}_{\mathbb{R}}(T_a X, \mathbb{C})$ is generated by $Z_a \wedge \bar{Z}_b$. Here a, b are subsets of $\{1, \dots, n\}$ and $Z_a = Z_{a_1} \wedge \dots \wedge Z_{a_n}$ where a_i are the elements of a in their natural ordering (similar for $\bar{Z}_b$). One computes

$$\langle Z_a \wedge \bar{Z}_b, Z_\alpha \wedge \bar{Z}_\beta \rangle = \begin{cases} 2^m & \text{if } a = \alpha,\ b = \beta, \\ 0 & \text{else} \end{cases} \qquad (m = \#a + \#b).$$

From this follows that the spaces $A^{p,q} \subset A^m \otimes_{\mathbb{R}} \mathbb{C}$ are pairwise orthogonal. Now the defining equation for this Hermitian scalar product shows

$$*(Z_a \wedge \bar{Z}_b) = 2^{m-n} \mathrm{i}^n \delta_n(a,b) Z_{\bar{b}} \wedge \bar{Z}_{\bar{a}}.$$

Here $\bar{a}$ denotes the complement of a in $\{1, \dots, n\}$ (the same for b). And $\delta_n(a, b)$ denotes the sign of the permutation that brings $Z_a \wedge \bar{Z}_b \wedge \bar{Z}_{\bar{b}} \wedge Z_{\bar{a}}$ into the ordering $Z_1 \wedge \bar{Z}_1 \wedge \cdots Z_n \wedge \bar{Z}_n$. We see that the star operator maps $A^{p,q}$ into $A^{n-q,n-p}$. □

We now define the *complex codifferentiations* as

$$\begin{aligned} \bar{\partial}^* &= - * \partial * = -\bar{*}\bar{\partial}\bar{*} : A^{p,q+1}(X) \longrightarrow A^{p,q}(X), \\ \partial^* &= - * \bar{\partial} * = -\bar{*}\partial\bar{*} : A^{p+1,q}(X) \longrightarrow A^{p,q}(X). \end{aligned}$$

It can be checked that for forms α, β with compact support this operator satisfies

$$(\bar{\partial}\alpha, \beta) = (\alpha, \bar{\partial}^*\beta) \quad \text{where} \quad (\alpha, \beta) := \int_X \alpha \wedge *\bar{\beta}$$

and similarly $\partial\alpha, \beta) = (\alpha, \partial^*\beta)$.

4.7 Lemma. *For $\alpha, \beta \in A^{p,q}(X)$ the formula*

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle \omega$$

holds. The operators $\bar{\partial}$ and $\bar{\partial}^*$ are formally adjoint. The same holds for ∂ and ∂^*.

We define the complex Laplace-Beltrami operators as:

$$\bar{\Box} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : A^{p,q}(X) \longrightarrow A^{p,q}(X)$$

and similarly

$$\Box = \partial\partial^* + \partial^*\partial.$$

We denote by

$$\mathcal{H}^{p,q}(X) \subset A^{p,q}(X)$$

the kernel of $\bar{\Box}$.

The point is that $\bar{\Box}$ is also an elliptic operator. Similar arguments as in the real case show:

4.8 Main theorem of complex Hodge theory. *Let X be a compact Hermitean manifold. Then $\mathcal{H}^{p,q}(X)$ is contained in the space of $\bar{\partial}$-closed forms and the natural homomorphism*

$$\mathcal{H}^{p,q}(X) \xrightarrow{\sim} H^{p,q}(X) := H^q(X, \bigwedge^p \Omega)$$

is an isomorphism. These vector spaces are finite dimensional.

Hence the so-called *Hodge numbers*

$$h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$$

are well defined numbers.

There is also a duality result: Before we formulate it, we introduce a slight generalization of the notion of a non-degenerated pairing. Let V, W be two finite dimensional complex vector spaces. Consider a $\mathbb{R}$-bilinear map $(\cdot,\cdot) : V \times W \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first variable. Then we get a natural $\mathbb{R}$-linear map $V \to \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$, which sends $a \in V$ to the linear form $x \mapsto (a, x)$. We call the pairing non-degenerated, if this map is an isomorphism. Then $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W$.

4.9 Duality. *Let X be a pure n-dimensional compact Hermitean manifold. The integral $\int_X \alpha \wedge *\bar{\beta}$ induces a non-degenerated pairing*

$$\mathcal{H}^{p,q}(X) \times \mathcal{H}^{n-p,n-q}(X) \longrightarrow \mathbb{C},$$

hence $h^{p,q}(X) = h^{n-p,n-q}(X)$.

The proof is similar to the proof of the Poincarè duality 3.12. $\square$

We finally remark that there is also an analogous result for the ∂-complex. One has to replace $\bar{\square}$ by $\square$.

5. Hodge theory of holomorphic bundles

Let $\mathcal{E}$ be a complex vector bundle over a differentiable manifold, i.e. a locally free $\mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C}$-module. Then

$$\mathcal{A}_X^m \otimes_{\mathcal{C}_X^\infty} \mathcal{E}$$

is a complex vector bundle too. Its fibres are

$$\bigwedge^d \mathrm{Hom}_{\mathbb{R}}(T_a X, \mathbb{R}) \otimes_{\mathbb{R}} E_a.$$

Its sections are denoted by $A^m(X, \mathcal{E})$. One calls them *differential forms with values in $\mathcal{E}$* or *differential forms twisted by $\mathcal{E}$.* There is no directly available operator $d : A^m(X, \mathcal{E}) \to A^{m+1}(X, \mathcal{E})$. Nevertheless reasonable operators exist in some circumstances and are called connections of $\mathcal{E}$. We will study them in more detail in Chap. IV, Sect. 3.

We assume now that X is a complex manifold and that $\mathcal{E}$ is a complex vector bundle on the underlying differential manifold. We consider the bundle

$$\mathcal{A}_X^{p,q} \otimes_{(\mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C})} \mathcal{E}$$

with the fibres

$$\bigwedge^{p,q} \mathrm{Hom}_{\mathbb{R}}(T_aX, \mathbb{C}) \otimes_{\mathbb{C}} E_a.$$

Its differentiable sections are denoted by

$$A^{p,q}(X, \mathcal{E}).$$

Again we cannot expect natural operators

$$\bar{\partial} : A^{p,q}(X, \mathcal{E}) \longrightarrow A^{p,q+1}(X, \mathcal{E}).$$

But the situation improves if we assume that $\mathcal{E}$ comes from a *holomorphic* bundle $\mathcal{M}$,

$$\mathcal{E} = \mathcal{M} \otimes_{\mathcal{O}_X} (\mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C}).$$

Then we have

$$\mathcal{A}_X^{p,q} \otimes_{(\mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C})} \mathcal{E} = \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

So in the case that $\mathcal{E}$ comes from a holomorphic bundle $\mathcal{M}$, the elements of $A^{p,q}(X, \mathcal{E})$ can be identified with the global sections of $\mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}$. We will use the notation

$$A^{p,q}(X, \mathcal{M}) := A^{p,q}(X, \mathcal{E}).$$

We want to define an $\mathcal{O}_X$-linear map

$$\bar{\partial} : \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \mathcal{A}_X^{p,q+1} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

Due to the universal property of the tensor product, it is sufficient to define an $\mathcal{O}_X$-bilinear map

$$\mathcal{A}_X^{p,q} \times \mathcal{M} \longrightarrow \mathcal{A}_X^{p,q+1} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

We take

$$(\omega, s) \longmapsto \bar{\partial}\omega \otimes s.$$

The essential point is the $\mathcal{O}_X$-linearity in the first variable. Actually we have

$$\bar{\partial}(f\omega) = f\bar{\partial}\omega \qquad (f \text{ holomorphic}).$$

This is true because $\bar{\partial} f = 0$.

5.1 Proposition. *Let X be a complex manifold and let $\mathcal{M}$ be a holomorphic vector bundle. There is a natural $\mathcal{O}_X$-linear map*

$$\bar{\partial} : \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \mathcal{A}_X^{p,q+1} \otimes_{\mathcal{O}_X} \mathcal{M}$$

It has the property $\bar{\partial} \circ \bar{\partial} = 0$.

Hence we can define the generalized *Dolbeault-cohomology*

$$H^{p,q}(X, \mathcal{M}) := \frac{\mathrm{Kernel}(A^{p,q}(X, \mathcal{M}) \longrightarrow A^{p,q+1}(X, \mathcal{M}))}{\mathrm{Image}(A^{p,q-1}(X, \mathcal{M}) \longrightarrow A^{p,q}(X, \mathcal{M}))}.$$

Our construction showed that there is a sequence of sheaves

$$0 \longrightarrow \bigwedge^p \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \mathcal{A}_X^{p,0} \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}_X^{p,n} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow 0.$$

From the Lemma of Dolbeault follows that this sequence is exact. (This is a local property and $\mathcal{M}$ is locally free.) Hence we obtain.

5.2 Proposition. *Let $(X, \mathcal{O}_X)$ be a complex manifold and let $\mathcal{M}$ be a holomophic vector bundle. There is an isomorphism*

$$H^{p,q}(X, \mathcal{M}) \cong H^q(X, \bigwedge^p \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}).$$

We want to define a Laplace-Beltrami operator in this context. The operator $\bar{\partial}$ is a linear differential operator. It is naturally to look for its formally adjoint operator

$$\bar{\partial}^* : \mathcal{A}_X^{p,q+1} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

Recall that for this we need a Hermitian metric on the bundle $\mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}$.

To get one, we now make the assumption that X carries a Hermitian metric and that also $\mathcal{M}$ carries a Hermitian metric. Recall that then $\mathcal{A}_X^{p,q}$ carries a Hermitian metric. We denote these Hermitian metrics by $\langle \cdot, \cdot \rangle$.

5.3 Remark. *Let X be a complex manifold with Hermitian metric, and let $\mathcal{M}$ be a holomorphic vector bundle with Hermitian matric. Then there exists a unique Hermitian metric on $\mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}$ such that*

$$\langle \alpha \otimes s, \beta \otimes t \rangle = \langle \alpha, \beta \rangle \langle s, t \rangle.$$

Proof. This follows not quite directly from the universal property of the tensor product. Therefore we explain the linear algebra behind it. Let V be a complex vector space. We denote by $\bar{V}$ the following complex vector space. The underlying abelian group is V, but the multiplication by constants is defined by $C \cdot v := \bar{C}v$. Then a Hermitian form $V \times V \to \mathbb{C}$ is nothing else but a $\mathbb{C}$-bilinear form $V \times \bar{V} \to \mathbb{C}$. Let W be a second space with Hermitian form. Then we can consider the map

$$V \times \bar{W} \times V \times \bar{W} \to \mathbb{C}, \quad (a, b, c, d) \longmapsto \langle a, b \rangle \langle c, d \rangle.$$

This is $\mathbb{C}$-multilinear and induces by the universal property of the tensor product a $\mathbb{C}$-linear map $V \otimes_{\mathbb{C}} \bar{W} \otimes_{\mathbb{C}} V \times \bar{W} \to \mathbb{C}$, or, equivalently a linear map $V \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} \bar{V} \otimes_{\mathbb{C}} \bar{W} \to \mathbb{C}$. Again from the universal property of the tensor product we get a natural isomorphism $\bar{V} \otimes_{\mathbb{C}} \bar{W} \xrightarrow{\sim} \overline{V \otimes_{\mathbb{C}} W}$. This gives a Hermitian form on $V \otimes_{\mathbb{C}} W$.

Now we have a Hermitian metric on $\mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}$ and we have a volume form ω. So it makes sense to ask for the formally adjoint of $\bar{\partial}$.

It is natural to imitate the absolute case (absence of $\mathcal{M}$) and to define star- and wedge operators in this more general context.

$$A^{p,q}(X, \mathcal{M}) \xrightarrow{*} A^{n-q,n-p}(X, \mathcal{M}),$$
$$A^{p,q}(X, \mathcal{M}) \times A^{n-p,n-q}(X, \mathcal{M}^*) \xrightarrow{\wedge} A^{n,n}(X) = A^{2n}(X) \otimes_{\mathbb{R}} \mathbb{C}.$$

The star operator is no problem. In the case where $\mathcal{M}$ is absent it was induced by a bundle map and this bundle map can be tensored with $\mathcal{M}$. This means locally $*(\omega \otimes s) = (*\omega) \otimes s$. In a similar way we get the generalized wedge product (locally $(\alpha \otimes s, \beta \otimes \ell) \mapsto \ell(s)\alpha \wedge \beta$.)

When V is a Hermitian vector space with Hermitian form $\langle \cdot, \cdot \rangle$ we get a natural map

$$\sharp : V \longrightarrow V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

It maps an $a \in V$ to the linear form $x \longmapsto \langle x, a \rangle$. This map is an isomorphism of real vector spaces and satisfies $\sharp(Ca) = \bar{C}\sharp(a)$. We call such a map an antilinear map.

Let $A \to B$ and $C \to D$ be two antilinear maps of complex vector spaces, then the linear maps $A \to \bar{B}$ and $C \to \bar{D}$ induce an antilinear map $A \otimes_{\mathbb{C}} C \to B \otimes_{\mathbb{C}} D$.

Using these general remarks, we get an antilinear map

$$\natural := \bar{*} \otimes \sharp : A^{p,q}(X, \mathcal{M}) \longrightarrow A^{n-p,n-q}(X, \mathcal{M}^*).$$

Its inverse is $(-1)^{p+q}\bar{*} \otimes \sharp^{-1}$.

Now can define the operator

$$\bar{\partial}^* = -\natural\, \bar{\partial}\, \natural = -\bar{\natural}\partial\bar{\natural} : A^{p,q+1}(X, \mathcal{M}) \longrightarrow A^{p,q}(X, \mathcal{M})$$

and the generalized Laplace-Beltrami operator

$$\bar{\square} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : A^{p,q}(X, \mathcal{M}) \longrightarrow A^{p,q}(X, \mathcal{M}).$$

As we have seen $\mathcal{A}^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}$ carries a Hermitian metric. One can check

$$\langle \alpha, \beta \rangle \omega = \alpha \wedge \natural\beta.$$

This allows us to verify the following result.

5.4 Proposition. *Assume that X is a compact connected complex manifold with Hermitian metric and that $\mathcal{M}$ is a holomorphic vector bundle with Hermitian metric. The operators*

$$\bar{\partial} : \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \mathcal{A}_X^{p,q+1} \otimes_{\mathcal{O}_X} \mathcal{M}, \quad \bar{\partial}^* : \mathcal{A}_X^{p,q+1} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}$$

are formally adjoint differential operators (similarly ∂ and ∂^*). In particular, $\bar{\square}$ is a formally self adjoint operator from $\mathcal{A}_X^{p,q}$ into itself.

We denote by $\mathcal{H}^{p,q}(X, \mathcal{M}) \subset A^{p,q}(X, \mathcal{M})$ the kernel of $\bar{\square}$. As in the usual case we obtain now the main theorem for the bundle valued Hodge theory.

5.5 Hodge theory for holomorphic vector bundles. *Let X be a compact complex connected manifold with a Hermitian metric and let $\mathcal{E}$ be a holomorphic vector bundle which has been equipped with a Hermitian metric. Then $\mathcal{H}^{p,q}(X,\mathcal{M})$ is contained in the space of $\bar\partial$-closed forms and the natural homomorphism*

$$\mathcal{H}^{p,q}(X,\mathcal{M}) \xrightarrow{\sim} H^{p,q}(X,\mathcal{M})$$

is an isomorphism. These vector spaces are finite dimensional.

Hence the so-called *Hodge-numbers*

$$h^{p,q}(X,\mathcal{M}) := \dim_{\mathbb{C}} H^{p,q}(X,\mathcal{M})$$

are well defined numbers.

There is also a duality result:

5.6 Duality. *Let X be a pure n-dimensional compact Hermitian manifold and $\mathcal{M}$ a holomorphic vector bundle. The integral $\int_X \alpha\wedge\natural\beta$ induces a non-degenerated pairing*

$$\mathcal{H}^{p,q}(X,\mathcal{M})\times\mathcal{H}^{n-p,n-q}(X,\mathcal{M}^*)\longrightarrow\mathbb{C},$$

hence

$$h^{p,q}(X,\mathcal{M}) = h^{n-p,n-q}(X,\mathcal{M}^*).$$

There is a well-known special case $p=0$. In this case on the right hand side the so-called canonical line bundle $\mathcal{K}_X = \bigwedge^n\Omega_X$ appears. If one uses

$$\mathcal{K}_X\otimes_{\mathcal{O}_X}\mathcal{M}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{K}_X).$$

one obtains the famous Serre duality

5.7 Serre duality. *Let X be a pure n-dimensional compact complex manifold and $\mathcal{M}$ a holomorphic vector bundle and $\mathcal{K}_X = \bigwedge^n\Omega_X$ the canonical line bundle. Then there is an isomorphism*

$$H^q(X,\mathcal{M})^*\cong H^{n-q}(X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{K}_X)).$$

We have to remark, that in the formulation of the Serre duality no metrics of X and of $\mathcal{M}$ occur. We mention that using the technique of partition of unity it is easy to show that such Hermitian metrics always exist. Hence the Serre duality 5.7 is true in general.

6. Complex line bundles and their Chern classes

Let $(X, \mathcal{O}_X)$ be a ringed space. We are interested in the set of all isomorphy classes of line bundles $\mathcal{L}$. (We will see in a minute that this is really a set). We denote by $[\mathcal{L}]$ the isomorphy class of $\mathcal{L}$. We define an addition on this set. It comes from the tensor product.

$$[\mathcal{L}_1] + [\mathcal{L}_2] := [\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2].$$

This addition of classes is commutative and associative because of corresponding properties of the tensor product and there is a zero element coming form the trivial bundle $\mathcal{O}_X$. We claim that we have an abelian group. Therefore we have to show that there exists a negative of $[\mathcal{L}]$. Actually

$$-[\mathcal{L}] = [\mathcal{L}^*]$$

where $\mathcal{L}^* = \mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is the dual bundle. We want to attach to an isomorphy class $[L]$ an element of $H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the (multiplicative) sheaf of invertible elements of $\mathcal{O}_X$. For this we consider an open covering $X = \bigcup U_i$ with local trivalizations $h_i : \mathcal{L}|U_i \to \mathcal{O}_X|U$. Recall that this induces transition functions

$$g_{ij} \in \mathcal{O}^*(U_i \cap U_j).$$

The basic observation is now that this is a Čech 1-cocycle for the sheaf $\mathcal{O}_X^*$. It is easy to check and left to the reader that the corresponding cohomology class does only depend on the isomorphy class of $\mathcal{L}$. And even more, one can check that two line bundles that give the same cohomology class are isomorphic. So we have seen the following result.

6.1 Proposition. *The group of all isomorphism classes of line bundles on a ringed space $(X, \mathcal{O}_X)$ is isomorphic to $H^1(X, \mathcal{O}_X^*)$, the isomorphism induced through the transition functions via Čech cohomology.*

Is there an analogous result for vector bundles? The answer is no. The reason is that the transition functions for vector bundles have values in some $\mathrm{GL}(n)$ which is for $n > 1$ a non abelian group. But our general approach to sheaf cohomology works only for sheaves of abelian groups. Hence the theory of line bundles is easier than that of vector bundles. This gets visible if one considers the complex projective space. It can be shown that the group of holomorphic line bundles is isomorphic to $\mathbb{Z}$, but the classification of vector bundles already on $P^2(\mathbb{C})$ is unsolved.

Let now $(X, \mathcal{O}_X)$ be a geometric space. In many cases we have an exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & 0, \\ & & & & f & \longmapsto & e^{2\pi \mathrm{i} f} & & \end{array}$$

for example for complex manifolds, but also for differentiable manifolds if one takes for $\mathcal{O}_X$ the sheaf of all complex valued differentiable functions (compare Proposition II.4.3). In these cases there is a combining homomorphism

$$\delta : H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}).$$

6.2 Definition. *Let $(X, \mathcal{O}_X)$ be a topological space or a differentiable manifold or a complex manifold equipped with the sheaf of complex valued continuous functions or complex valued differentiable functions or holomorphic functions. The* ***Chern class***

$$c(\mathcal{L}) \in H^2(X, \mathbb{Z})$$

of a line bundle is the image of its cohomology class under the combining homomorphism $H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z})$.

It may happen that $H^p(X, \mathcal{O}_X) = 0$ for $p > 0$. Then the long exact cohomology sequence shows that $H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z})$ is an isomorphism. This is the case if $\mathcal{O}_X$ is the sheaf of continuous differentiable functions on a differentiable manifold. This has an interesting application. Let X be a differentiable manifold. Every continuous line bundle carries a differentiable structure. This means the following. Assume that there is a locally free $\mathcal{C}_X$-module $\mathcal{E}$. Then there exists a locally free $\mathcal{C}_X^\infty$-module $\mathcal{F}$ such that $\mathcal{E} = \mathcal{F} \otimes_{\mathcal{C}_X^\infty} \mathcal{C}_X$.

We now assume that X is a complex manifold and that $\mathcal{L}$ is a holomorphic line bundle. We consider the image $j(c(\mathcal{L}))$ under the map

$$j : H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R}) \subset H^2(X, \mathbb{C}).$$

Sometimes we will call the image of $j(c(L))$ in $H^2(X, \mathbb{R})$ or in $H^2(X, \mathbb{C})$ the Chern class of $\mathcal{L}$. The circumstances will show where we consider the Chern class.

This image must be representable by a differential form $\omega \in A^2(X)$ via the de Rham-isomorphism. We are going to define such a differential form explicitly. To construct it, we need a Hermitian metric $h : \mathcal{L} \times \mathcal{L} \to \mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C}$ on $\mathcal{L}$. We know that there exists one. The form ω (but not its class in $H^2(X, \mathbb{R})$) will depend on the choice the Hermitian metric. Let $X = \bigcup U_i$ be an open covering such there exist trivializations $\mathcal{O}_X|U_i \to \mathcal{L}|U_i$, $1 \mapsto s_i$. Then $h_i := h(s_i, s_i) : U_i \to \mathbb{R}$ is a real everywhere positive function. We consider the transition functions $f_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$, $s_i = f_{ij} s_j$. Then we have

$$h_i = |f_{ij}|^2 h_j \quad \text{on} \quad U_i \cap U_j.$$

The point now is that the function $\log |f_{ij}|^2 = \log(f_{ij} \bar{f}_{ij})$ is annihilated by the operator $\partial\bar{\partial}$. This means that the differential forms $\partial\bar{\partial} \log h_i$ glue to a global differential form which we denote by $\partial\bar{\partial}h$. This is a form of type $(1, 1)$ and it is in $\mathrm{i}A^2(X)$, Hence we have

$$\frac{1}{2\pi\mathrm{i}} \partial\bar{\partial} \log h \in A^{1,1}(X) \cap A^2(X).$$

6.3 Definition. *Let X be a complex manifold and $\mathcal{L}$ a holomorphic line bundle equipped with a Hermitian metric h. The differential*

$$\frac{1}{2\pi\mathrm{i}}\partial\bar\partial\log h.$$

*is called the **Chern form** of $(\mathcal{L},h)$.*

This differential form is annihilated by ∂ and by $\bar\partial$, hence also by d. It defines via the de-Rham isomorphism cohomology class in $H^2(X,\mathbb{R})$.

6.4 Proposition. *Let $\mathcal{L}$ be a holomorphic line bundle, equipped with a Hermitian metric h. Then*

$$j(c(L)) = \frac{1}{2\pi\mathrm{i}}\partial\bar\partial\log h \qquad (\textit{considered in } H^2(X,\mathbb{R}))$$

is the cohomology class of the Chern form. As a consequence we obtain

$$j(c(L)) \in j(H^2(X,\mathbb{Z}))\cap H^{1,1}(X).$$

Proof. We make use of the commutative diagram of Lemma II.4.4,

$$\begin{tikzcd}
H^1(X,(\mathcal{C}^\infty_X\otimes_{\mathbb{R}}\mathbb{C})^*) \arrow[r] \arrow[d] & H^1(X,\mathcal{A}^1_{X\,\mathrm{closed}}\otimes_{\mathbb{R}}\mathbb{C}) \arrow[d] \\
H^2(X,\mathbb{Z}) \arrow[r] & H^2(X,\mathbb{C})
\end{tikzcd}$$

We look at the first line via Čech cohomology. The cocycle (f_{ij}) (transition functions of $\mathcal{L}$) is mapped to the cocycle $(1/2\pi\mathrm{i})d\log f_{ij}$. Notice that $\log f_{ij}$ exists only locally as differentiable function but the ambiguity is killed by d. The claim is that the cohomology class of this cocycle is represented by the differential form $(1/2\pi\mathrm{i})\partial\bar\partial\log h$. Recall that this differential form represents an element of $H^2(X,\mathbb{R})$. The lemmas II.4.1 and II.4.2 explain how the image in $H^1(X,\mathcal{A}^1_{X\,\mathrm{closed}}\otimes_{\mathbb{R}}\mathbb{C})$ can be computed in terms of Čech cohomology. We have to write $\partial\bar\partial\log h_i = d\alpha_i$. We can take $\alpha_i = -\partial\log h_i$. Then we have to consider the Čech cocycle

$$\alpha_i - \alpha_j = \partial\log h_i - \partial\log h_j.$$

From the (locally valid) equation $\log h_i = \log h_j + \log f_{ij} + \log\bar f_{ij}$ we get

$$\partial\log h_i - \partial\log h_j = \partial\log f_{ij} = d\log f_{ij}.$$

This finishes the proof of Proposition 6.4. □

7. The cohomology of the complex projective space

We consider the natural map $\mathbb{C}^{n+1}-\{0\} \to P^n(\mathbb{C})$. Recall that $P_i^n\mathbb{C}$ is defined through $z_i \neq 0$. There is a biholomorphic map

$$P_i^n\mathbb{C} \xrightarrow{\sim} \mathbb{C}^n, \quad Z \longmapsto (z_1, \dots, z_n) = \Bigl(\frac{Z_0}{Z_i}, \dots \frac{\widehat{Z_i}}{Z_i}, \dots \frac{Z_n}{Z_i}\Bigr).$$

A basic role will play the function $\log(|Z|_0^2 + \cdots + |Z|_n^2)$ on $\mathbb{C}^{n+1} - \{0\}$. This function does not come from a function on $P^n(\mathbb{C})$. But the following holds.

7.1 Lemma. *The differential form*

$$\partial\bar\partial \log(|Z_0|^2 + \cdots + |Z_n|^2)$$

is the pull-back of a differential form on $P^n(\mathbb{C})$ which on $P_i^n\mathbb{C}$ corresponds to

$$\partial\bar\partial \log(1 + |z_1|^2 + \cdots + |z_n|^2).$$

The differential form in the Lemma can be evaluated explicitly. We use the notation $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$.

7.2 Remark und Definition. *The differential forms*

$$\frac{\mathrm{i}}{2}\sum h_{ij} dz_i \wedge d\bar z_j$$

on $P_i^n(\mathbb{C})$ given by

$$h_{ij} = \frac{1}{(1+|z|^2)^2}\begin{pmatrix} 1+|z|^2-|z_1|^2 & -\bar z_1 z_2 & \cdots & -\bar z_1 z_n \\ -\bar z_2 z_1 & 1+|z|^2-|z|_2^2 & \cdots & -\bar z_1 z_n \\ \vdots & \vdots & \ddots & \vdots \\ -\bar z_n z_1 & -\bar z_n z_2 & \cdots & 1+|z|^2-|z_n|^2 \end{pmatrix}$$

glue to a differential form Ω on $P^n(\mathbb{C})$. It is called the ***Fubini-Study*** *form.*

Recall that Ω determines a unique Hermitian form h on the tangent bundle such that $\Omega = -\operatorname{Im} h$.

7.3 Proposition. *On the projective space there exists a unique Hermitian metric h, the Fubini-Study metric, such that $\Omega = -\operatorname{Im} h$ is the Fubini-Study form.*

Proof. We have to show that the matrix (h_{ij}) in Remark 7.2 is positive definit. □

So we have equipped the projective space with a distinguished Hermitian metric. We introduce now a fundamental holomorphic line bundle $\mathcal{L}$ on the projective space. For this we use the vanishing ideal sheaf (Definition V.9.1).

7.4 Remark and notation. *Let $Y \subset X$ be a (holomorphically) smooth closed subset of the complex manifold $(X, \mathcal{O}_X)$ of pure codimension one ($\dim_a Y = \dim_a X - 1$ for $a \in Y$). The ideal sheaf $\mathcal{J}_Y$ associated to Y is locally free of rank one, i.e. a holomorphic line bundle.*

We denote this line bundle by $\mathcal{J}_Y$.

Proof. It is sufficient to treat the local case where X is an open neighborhood of 0 in $\mathbb{C}^n$ and Y is defined by $z_n = 0$. Then the vanishing ideal sheaf is $z_n\mathcal{O}_U$ which is a line bundle. □

There is a generalization. Let k be an integer. Let X be a complex manifold of dimension n and let $Y \subset X$ be a submanifold of pure codimension one. A holomorphic function $f : X - Y \to \mathbb{C}$ is called *merormorphic of order* $\geq k$ along Y if for every open subset $U \subset X$ and any generator $g \in \mathcal{J}_Y(U)$ of $\mathcal{J}_Y|U$ (this means $\mathcal{J}_Y|U = g\mathcal{O}_X|U$), the function fg^{-k} extends to a holomorphic function on U. It is clear how to sheafify this definition to get a sheaf $\mathcal{J}_Y(k)$. We have

$$\mathcal{J}_Y = \mathcal{J}_Y(1).$$

There is an obvious pairing

$$\mathcal{J}_Y(k_1) \times \mathcal{J}_Y(k_2) \longrightarrow \mathcal{J}_Y(k_1 + k_2).$$

It induces a map

$$\mathcal{J}_Y(k_1) \otimes_{\mathcal{O}_X} \mathcal{J}_Y(k_2) \longrightarrow \mathcal{J}_Y(k_1 + k_2).$$

This map is obviously an isomorphism. This shows that $\mathcal{J}_Y(k)$ and $\mathcal{J}_Y(-k)$ are dual bundles. So we have

$$\mathcal{J}_Y^{\otimes k} = \mathcal{J}_Y(k)$$

first for $k > 0$ but then for all integers by definition.

We apply this to the embedding

$$P^{n-1}(\mathbb{C}) \longrightarrow P^n(\mathbb{C}), \quad [z_1, \dots, z_n] \longmapsto [0, z_1, \dots, z_n].$$

We denote the image by H. This is a smooth submanifold of dimension $n - 1$. The complement of the image is the affine space $P_i^n\mathbb{C}$. The associated line bundle, actually an ideal sheaf, is denoted by $\mathcal{O}(-1)$. So $\mathcal{O}(-1) = \mathcal{J}(1)$, more generally we introduce the tensor powers

$$\mathcal{O}(k) = \mathcal{J}(-k).$$

We call $\mathcal{O}(-1)$ the hyperplane bundle.

There is another important line bundle $\mathcal{K}_X = \bigwedge^n \Omega_X$, the determinant of the sheaf of holomorphic differentials. This bundle can be defined for every complex manifold of pure dimension. It is called the *canonical bundle.*

7.5 Lemma. *The canonical bundle $\mathcal{K}$ on $P^n(\mathbb{C})$ is isomorphic to $\mathcal{O}(-n-1)$.*

Proof. For simplicity we treat the typical case $n = 2$. We consider on $P_0^2(\mathbb{C})$ the holomorphic differential $\alpha = dz_1 \wedge dz_2$, where z_1, z_2 are the standard coordinates. We consider it on $P_1^n\mathbb{C}$. (The same argument works for $P_2^n(\mathbb{C})$.) We denote the standard coordinates on $P_1^2(\mathbb{C})$ by (w_1, w_2). Here α writes as $f(w_1, w_2)dw_1 \wedge dw_2$ where f is a holomorphic function on $w_1 \neq 0$ which corresponds to our hyperplane $P^1(\mathbb{C})$. The chart transformation is $(w_1, w_2) = (1/z_1, z_2/z_1)$. This gives $f(w_1, w_2) = -w_1^{-3}dw_1 \wedge dw_2$. So we see that multiplication gives a pairing

$$\mathcal{K} \times \mathcal{O}(3) \longrightarrow \mathcal{O}.$$

It induces a map $\mathcal{K} \to \mathcal{O}(3)^* = \mathcal{O}(-3)$ that is obviously an isomorphism. □

Next we define a Hermitian metric on $\mathcal{L}$. For this we consider the differentiable function

$$\mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{R}_{>0}, \quad h(Z) = \frac{|Z_0|^2}{|Z_0|^2 + \cdots + |Z_n|^2}.$$

It is invariant under $Z \mapsto \lambda Z$ and hence comes from a differentiable function h on $P^n(\mathbb{C})$. Then we can define the Hermitian form

$$\mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{C}^\infty \otimes_{\mathbb{R}} \mathbb{C}, \quad (f, g) \longmapsto \frac{f\bar{g}}{h}.$$

It is easy to check that it is positive definite. We compute the corresponding differential form. It is enough to do this on $P_0^n\mathbb{C}$. Here it is given by $\omega = -(1/2\pi\mathrm{i})\partial\bar{\partial}\log h$. We obtain:

7.6 Proposition. *The projective space $P^n(\mathbb{C})$ is a Hermitian manifold such that the associated class $\Omega \in H^{1,1}(X)$ equals the Chern class of the bundle $\mathcal{O}(-1)$. In particular, the class Ω is integral, i.e. contained in the image of $H^2(P^n(\mathbb{C}), \mathbb{Z})$.*

This is all what we need in the following. For sake of completeness we describe the complete picture without proof (which is easy if one uses a little algebraic topology):

7.7 Theorem. *Every holomorphic line bundle on $P^n(\mathbb{C})$ is isomorphic to precisely one $\mathcal{O}(n)$. Hence* $\mathrm{Pic}(P^n(\mathbb{C})) = \mathbb{Z}$.

7.8 Theorem. *The natural map*

$$j : H^m(P^n(\mathbb{C}), \mathbb{Z}) \longrightarrow H^m(P^n(\mathbb{C}), \mathbb{R})$$

is injective, the image generates $H^m(P^n(\mathbb{C}), \mathbb{R})$ as vector space. The space $H^m(P^n(\mathbb{C}), \mathbb{Z})$ is zero for odd m (and for $m > 2n$ and $m < 0$). Moreover

$$H^m(P^n(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}, \qquad m \text{ even and } 0 \leq m < 2n.$$

A generating element is $\Omega^m := \Omega \wedge \ldots \wedge \Omega$.

Chapter IV. Kaehler manifolds

1. Effective forms

This section contains some non-standard linear algebra. We make the following assumptions:

1.1 Assumptions. *Let*

$$(V_m)_{m \in \mathbb{Z}}$$

be a sequence of complex vector spaces. Assume that for each m two $\mathbb{C}$-linear maps

$$L : V_m \longrightarrow V_{m+2}, \quad \Lambda : V_{m+2} \longrightarrow V_m$$

are given such that the following two conditions hold.

1) $V_m = \text{Kernel}(\Lambda) \oplus \text{Image}(L)$.
2) *There exists a natural number n such that*

$$V_m = 0 \quad \text{for} \quad m < 0, \quad \text{and} \quad m > 2n$$

and such that

$$[L, \Lambda]u = (n - m)u \quad \text{for} \quad u \in V_m.$$

We call an element $u \in V_m$ *primitive* if $\Lambda u = 0$. Hence every α can be written in the form $\alpha = \alpha_0 + L(\beta)$ with a primitive α_0. Repeating this argument for β we obtain the following result.

1.2 Proposition. *Under the above assumptions every $\alpha \in A^m$ admits a decomposition*

$$\alpha = \sum_{2t\leq m} L^t(\alpha_t)$$

with primitive $\alpha_t \in A^{m-2t}$.

Another result is the following remark.

1.3 Remark. *Assume that $u \in V_m$ is primitive and $m > n$. Then $u = 0$.*

Proof. From the relation 2) in the Assumptions we obtain

$$(\Lambda L^k - L^k \Lambda)u = k(n-k-m+1)L^{k-1}u \quad \text{for} \quad u \in A^m.$$

We apply this relation to a primitive form u. We notice $L^{2n+1-m}u = 0$ because we are out of the range. From the corollary we see $L^{2n-m}u = 0$ because the factor in front $(2n+1-m)n$ is different from zero. We can apply this argument again and repeat this as long the factor in front is different form 0. We come down to $u = L^0 u = 0$ if $m > n$. □

In the rest of this section we treat a basic example. We take up the computations behind Lemma III.4.6. We consider a complex vector space T of dimension n which is equipped with a positive definit Hermitian form $h(A, B)$. In our application T will be the tangent space $T_a X$ of a Hermitian manifold. (Recall that this is a priori the real tangent space which has been equipped with a complex structure.) Compare also with the constructions for differential forms in Chapt. III, Sect. 4.

We consider the space $\mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C})$, which is a complex vector space of complex dimension $2n$. The complex structure comes from the $\mathbb{C}$ in the Hom. A homomorphism h is multiplied by a complex number C through $(Ch)(x) = Ch(x)$. Recall that there is a decomposition

$$\mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C}) \oplus \overline{\mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})}$$

into two complex sub-vector spaces. Here the complex structure of T has to be used of course. We set

$$A^m := \bigwedge\nolimits_{\mathbb{R}}^m \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{R}).$$

Recall that the above decomposition generalizes to

$$A^m \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge\nolimits_{\mathbb{C}}^m \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C}) = \bigoplus_{p+q=m} A^{p,q},$$

where $A^{p,q}$ is generated by elements of the form

$$a_1 \wedge \ldots, \wedge a_p \wedge b_1 \wedge \ldots \wedge p_q, \qquad a_i \in \operatorname{Hom}_{\mathbb{C}}(T, \mathbb{C}) \quad b_j \in \overline{\operatorname{Hom}_{\mathbb{C}}(T, \mathbb{C})}.$$

To be more precise: There is an isomorphism

$$\bigwedge^p \operatorname{Hom}_{\mathbb{C}}(T, \mathbb{C}) \otimes_{\mathbb{C}} \bigwedge^q \overline{\operatorname{Hom}_{\mathbb{C}}(T, \mathbb{C})} \xrightarrow{\sim} A^{p,q}.$$

The space T carries a real symmetric positive definit bilinear form $g = \operatorname{Re} h$. Recall that the real bilinear form g on T induces an $\mathbb{R}$-isomorphism $T \to \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$ and hence, by transport, a symmetric positive definite bilinear form on $\operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$. This form induces real symmetric positive definite bilinear forms $\langle \cdot, \cdot \rangle$ on

$$A^m = \bigwedge^m \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R}).$$

They are extended to positive definite Hermitian forms $\langle \cdot, \cdot \rangle$ on $A^m \otimes_{\mathbb{R}} \mathbb{C}$. We choose a (complex) basis $E_1, \ldots, E_n$ of T which is orthonormal with respect to h. The dual basis in $\operatorname{Hom}_{\mathbb{C}}(T, \mathbb{C})$ is denoted by $Z_1, \ldots, Z_n$, hence $Z_i(e_j) = \delta_{ij}$. A basis of $\overline{\operatorname{Hom}_{\mathbb{C}}(T, \mathbb{C})}$ is given by $\bar{Z}_1, \ldots, \bar{Z}_n$. A real basis of $\operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$ is $X_1, Y_1, \ldots, X_n, Y_n$, where

$$X_i = \operatorname{Re} Z_i, \quad Y_i = \operatorname{Im} Z_i.$$

This basis is oriented and orthonormal with respect to g. Hence the distinguished (volume) element is

$$\omega = X_1 \wedge Y_1, \ldots, X_n \wedge Y_n = \frac{1}{(-2\mathrm{i})^n} Z_1 \wedge \bar{Z}_1 \wedge \ldots \wedge Z_n \wedge \bar{Z}_n.$$

The space $A^{p,q} = \bigwedge^{p,q} \operatorname{Hom}_{\mathbb{R}}(T_a X, \mathbb{C})$ is generated by $Z_a \wedge \bar{Z}_b$. Here a, b are subsets of $\{1, \ldots, n\}$ and $Z_a = Z_{a_1} \wedge \ldots \wedge Z_{a_n}$ where a_i are the elements of a in their natural ordering (similar for $\bar{Z}_b$). One computes

$$\langle Z_a \wedge \bar{Z}_b, Z_\alpha \wedge \bar{Z}_\beta \rangle = \begin{cases} 2^m & \text{if } a = \alpha,\ b = \beta, \\ 0 & \text{else} \end{cases} \qquad (m = \#a + \#b).$$

The star operator $* : \bigwedge^m \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R}) \to \bigwedge^{2n-m} \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{R})$ has been defined by

$$a \wedge *b = \langle a, b \rangle\, \omega.$$

Recall that we extended the star operator $\mathbb{C}$-linearly to $\bigwedge^m \operatorname{Hom}_{\mathbb{R}}(T, \mathbb{C})$. We can restrict it to a Hermitian form on $A^{p,q}$ (compare Definition III.4.5). The extended star operator can be characterized by

$$a \wedge *\bar{b} = \langle a, b \rangle\, \omega.$$

The star operator defines actually an operator

$$* : A^{p,q} \longrightarrow A^{n-p,n-q}$$

as we have seen during the proof of Lemma III.4.6. There we have seen

$$*(Z_a \wedge \bar{Z}_b) = 2^{m-n} \mathrm{i}^n \delta_n(a,b) Z_{\bar{b}} \wedge \bar{Z}_{\bar{a}}.$$

Here $\bar{a}$ denotes the complement of a in $\{1,\dots,n\}$ (the same for b). And $\delta_n(a,b)$ denotes the sign of the permutation that brings $Z_a \wedge \bar{Z}_b \wedge \bar{Z}_{\bar{b}} \wedge Z_{\bar{a}}$ into the ordering $Z_1 \wedge \bar{Z}_1 \wedge \cdots Z_n \wedge \bar{Z}_n$. □

We have to consider the alternating form $\Omega = -\operatorname{Im} h$. Recall that there is a canonical isomorphism

$$\bigwedge^2 \operatorname{Hom}_{\mathbb{R}}(T,\mathbb{R}) \xrightarrow{\sim} \operatorname{Alt}_{\mathbb{R}}(T \times T, \mathbb{R})$$

This extends $\mathbb{C}$-linearly to

$$\bigwedge^2 \operatorname{Hom}_{\mathbb{R}}(T,\mathbb{C}) \xrightarrow{\sim} \operatorname{Alt}_{\mathbb{R}}(T \times T, \mathbb{C}).$$

Hence Ω can be considered as element of $\bigwedge^2 \operatorname{Hom}_{\mathbb{R}}(T,\mathbb{C})$. Actually

$$\Omega \in \bigwedge^{1,1} \operatorname{Hom}_{\mathbb{R}}(T,\mathbb{C}),$$

as for example the formula

$$\Omega = \frac{\mathrm{i}}{2} \sum_{i=1}^{n} Z_i \wedge \bar{Z}_i$$

shows. This element is fundamental in what follows. It defines an operator

$$L : A^{p,q} \longrightarrow A^{p+1,q+1}, \quad L(u) := \Omega \wedge u.$$

It is correlated with the operator

$$\Lambda = *^{-1} L * : A^{p+1,q+1} \longrightarrow A^{p,q}.$$

We also can consider L and Λ as operators

$$L : A^m \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow A^{m+2} \otimes_{\mathbb{R}} \mathbb{C}, \quad \Lambda : A^{m+2} \longrightarrow A^m.$$

1.4 Lemma. *The operators L and Λ are adjoint,*

$$\langle L\alpha, \beta \rangle = \langle \alpha, \Lambda\beta \rangle.$$

Moreover L and Λ are real operators (i,e. they preserve A^m).

Proof. Check this with a real oriented orthonormal basis $U_1, \dots U_{2n}$. □

1.5 Definition. *A form* $\alpha \in A^m$ *is called* ***primitive*** *if* $\Lambda(\alpha) = 0$.

Because of the adjointness we have

$$A^m = \text{Kernel}(\Lambda) \oplus \text{Image}(L) \qquad \text{(orthogonal decomposition)}.$$

The following relation can be checked by means of a basis:

1.6 Lemma. *One has*

$$[\Lambda, L]u = (n - m)u \quad \textit{for} \quad u \in A^m.$$

Remark 1.3 applies to our case:

1.7 Proposition. *Primitive forms of degree* $m > n$ *are zero.*

We will use this in the proof of the Kodaira vanishing theorem.

2. Kaehler metrics

Let (X, h) be a Hermitian manifold of pure (complex) dimension n. Recall that this is a complex manifold such that the tangent bundle is equipped with a positive definit Hermitian metric. Recall that the imaginary part of h can be considered as a differential form. Its negative is

$$\Omega \in A^{1,1}(X), \quad \Omega = \bar{\Omega}.$$

Due to our notations $A^2(X)$ is the space of real differential forms. Hence we can write

$$\Omega \in A^{1,1}(X) \cap A^2(X).$$

Let conversely $\Omega \in A^{1,1}(X) \cap A^2(X)$. Then there exists a unique Hermitian form

$$h(a) : T_aX \times T_aX \longrightarrow \mathbb{C}$$

whose real part equals $-\Omega$. In the case that $h(a)$ is positive definit for all a we obtain a Hermitian metric h on X.

In the special case, where X is an open subset $U \subset \mathbb{C}^n$, the Hermitean metric is given by a Hermitian matrix $h(z) = (h_{\mu\nu})$ and one has

$$\Omega = \frac{\mathrm{i}}{2} \sum h_{\mu\nu}(z) dz_\mu \wedge d\bar{z}_\nu.$$

It may happen that this differential form is closed, for example when $h(z)$ is constant. It turns out that this is a very important property.

2.1 Definition. *A Kähler manifold is a Hermitian manifold such that Ω is closed.*

Usually we consider only *compact* Kähler manifolds. We give some examples:

1) Complex tori (with the standard metric) are Kählerian.
2) The projective space (with standard metric) is Kählerian.
3) Compact Riemann surfaces (with any Hermitian metric) are Kählerian.

We also mention the following. Let Y be a closed complex submanifold of a Kählerian manifold (X, h). The restriction of the Hermitian metric h to Y equips Y with a structure as Kählerian manifold. As a consequence each projective algebraic manifold admits a structure as Kählerian manifold.

2.2 Proposition. *Let X be a Kähler manifold. For every point $a \in X$ there exists a holomorphic chart φ around a which maps $a \in U_\varphi$ to $0 \in V_\varphi$ and such that the corresponding Hermitian matrix $h(z)$, $z \in V_\varphi$, satisfies the following condition:*

$$h_{\mu\nu}(0) = \delta_{\mu\nu}, \quad \frac{\partial h}{\partial x_i}(0) = \frac{\partial h}{\partial y_i}(0) = 0.$$

We mention by the way that this property also implies that X is Kählerian, because this property implies that $d\Omega$ is zero at the point a. But a is arbitrary. The point is that d involves only first partial derivatives. Assume $d\Omega = 0$. We can assume that X is an open set $U \subset \mathbb{C}^n$ and that $a = 0$ is the origin. Recall

$$\Omega = \frac{\mathrm{i}}{2} \sum_{ij} h_{ij} dz_i \wedge d\bar{z}_j.$$

First we use a linear transformation $z \mapsto Az$, $A \in \mathrm{GL}(n, \mathbb{C})$. The matrix h has to be replaced by $\bar{A}' h A$. We use the well-known result of linear algebra that each positive definite Hermitian matrix h can be transformed by such a transformation into the unit matrix. Hence we can assume that $h(0)$ is the unit matrix. This property will be preserved during the rest of the proof.

We introduce the numbers

$$a_{ijk} = \frac{\partial h_{ij}}{\partial z_k}(0), \quad b_{ijk} = \frac{\partial h_{ij}}{\partial \bar{z}_k}(0).$$

Then we have

$$h_{ij} = \delta_{ij} + \sum_k a_{ijk} z_k + \sum b_{ijk} \bar{z}_k + r_{ij},$$

where the remainder r_{ij} and its first partial derivatives vanish at 0. We decompose

$$\Omega = \Omega_{\mathrm{main}} + R,$$

where

$$R = \sum_{ij} r_{ij} dz_i \wedge dz_j.$$

From $d\Omega = 0$ and form $h_{ij} = \bar{h}_{ji}$ one derives the relations

$$a_{ijk} = a_{kji}, \quad b_{ijk} = a_{ikj} \quad \text{and} \quad b_{ijk} = \bar{a}_{jik}.$$

The transformation

$$w_k = z_k + \frac{1}{2} \sum_{i,j=1}^{n} a_{ijk} z_i z_j$$

maps a small open neighborhood of U biholomorphically onto an open neighborhood V. We have to transform Ω into V. We denote the transformed form by $\tilde{\Omega}$. We have $\tilde{\Omega} = \tilde{\Omega}_{\text{main}} + \tilde{R}$ with obvious notation. The form $\tilde{R}$ is without interest, since $\tilde{R}$ and its first derivatives vanish at the origin. This is easily proved by means of the chain rule. So we have to determine $\tilde{\Omega}_{\text{main}}$. A straight forward calculations gives

$$\frac{\mathrm{i}}{2} \sum_{j=1}^{n} dw_j \wedge d\bar{w}_j = \tilde{\Omega}_{\text{main}}.$$

This completes the proof of 2.2. □

We derive some relations between the operators $L\alpha = \Omega \wedge \alpha$ and the derivative operators $\partial, \bar{\partial}$ and the coderivative operators $\partial^*, \bar{\partial}^*$. We prefer the notation L^* instead of Λ for the adjoint operator.

2.3 Theorem. *On a Kähler manifold the following relations hold.*

$$\begin{aligned}
&[L, \partial] = [L, \bar{\partial}] = [L^*, \partial^*] = [L^*, \bar{\partial}^*] = 0, \\
&[L, \partial^*] = \mathrm{i}\bar{\partial}, \quad [L, \bar{\partial}^*] = -\mathrm{i}\partial, \\
&[L^*, \partial] = \mathrm{i}\bar{\partial}^*, \quad [L^*, \bar{\partial}] = -\mathrm{i}\partial^*.
\end{aligned}$$

Proof. These formulae can be checked directly in the case

$$\Omega = \frac{\mathrm{i}}{2} \sum dz_\nu \wedge d\bar{z}_\nu.$$

Because they obtain only first derivatives they follow in general by 2.2. □

There can be derived some relations which also involve second derivatives, (which could not be proven directly by using Proposition 2.2). For example

$$\partial\bar{\partial}^* = \partial(-\mathrm{i}[L^*, \partial]) = -\mathrm{i}\partial L \partial.$$

In the same manner one proves:

2.4 Corollary. *One has*

$$\partial\bar\partial^* = -\bar\partial^*\partial = -\mathrm{i}\bar\partial^* L\bar\partial^* = -\mathrm{i}\partial L^*\partial,$$
$$\bar\partial\partial^* = -\partial^*\bar\partial = \mathrm{i}\partial^* L\partial^* = \mathrm{i}\bar\partial L^*\bar\partial.$$

Recall that the Laplacians on a Hermitian manifold are defined by

$$\Delta = dd^* + d^*d, \quad \square = \partial\partial^* + \partial^*\partial, \quad \bar\square = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

A formal consequence of the above Kähler relations is

2.5 Theorem. *On a Kähler manifold the relations*

$$\Delta = 2\square = 2\bar\square$$

hold.

2.6 Main theorem of Hodge theory for Kähler manifolds. *For a compact Kaehler manifold X one has in addition to III.5.5*

$$H^m(X,\mathbb{C}) \cong \bigoplus_{p+q=m} H^{p,q}(X).$$

Morover

$$H^{p,q}(X) \cong H^{q,p}(X).$$

This implies for the Betti- and Hodge numbers the following relations:

$$b^m = \sum_{p+q=m} h^{p,q}, \quad h^{p,q} = h^{q,p} = h^{n-p,m-q}.$$

There are many important consequences: For example b^m is even for odd m. Moreover $b^1 = 2h^{1,0}$, hence the dimension of the space of holomorphic differentials is a topological invariant.

3. The canonical connection of a holomorphic bundle

We want to carry over part of the Kähler identities to bundle valued differential forms. Let $\mathcal{M}$ be a holomorphic bundle over a complex manifold. Our problem is that we could define (Proposition III.5.1)

$$\bar\partial : A^{p,q}(X, \mathcal{M}) \longrightarrow A^{p,q+1}(X, \mathcal{M})$$

in a naive way, but up to now there is no operator ∂. We will see that there is a natural one if $\mathcal{M}$ carries a Hermitian metric.

It is worthwhile to start with some generalities about connections. We consider a differentiable vector bundle $\mathcal{E}$ over a differentiable manifold X. In principle $\mathcal{E}$ can be thought to be real or complex. For sake of simplicity we restrict to complex bundles which are more important for us. We use the notation

$$\mathcal{A}_X^p(\mathcal{E}) = \mathcal{A}_X^p \otimes_{\mathcal{C}_X^\infty} \mathcal{E}.$$

Recall that we introduced bundle valued differential forms $A^m(X, \mathcal{E})$. They are sections of $\mathcal{A}_X^m \otimes_{\mathcal{C}_X^\infty} \mathcal{E}$. This is the same as

$$\mathcal{A}_X^m(\mathcal{E}) = (\mathcal{A}_X^m \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{(\mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C})} \mathcal{E}.$$

(Compare with the formula $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}$.)

3.1 Definition. *Let $\mathcal{E}$ be differentiable complex vector bundle over a differentiable manifold. A **connection** is a $\mathbb{C}$-linear maps of sheaves*

$$D : \mathcal{E} \longrightarrow \mathcal{A}_X^1(\mathcal{E}),$$

such that

$$D(fs) = df \otimes s + fD(s),$$

where f is a complex valued differentiable function and $s \in \mathcal{E}(U)$.

Connections can be extended to various types of tensors. We only need the following case.

3.2 Lemma. *Let D be a connection on $\mathcal{E}$. There is a unique extension to a family of $\mathbb{C}$-linear maps of sheaves*

$$D : \mathcal{A}_X^m(\mathcal{E}) \longrightarrow \mathcal{A}_X^{m+1}(\mathcal{E})$$

such that

$$D(\omega \otimes s) = d\omega \otimes s + (-1)^m \omega \wedge D(s)$$

for $\omega \in A^p(U)$ and $s \in \mathcal{E}(U)$.

Proof. It is sufficient to prove the existence and uniqueness for small open U, which are contained in the domain of definition of a chart and such that $\mathcal{E}|U$ is trivial. A little calculation shows

$$d(f\omega)\otimes s + f\omega\wedge D(s) = d\omega\otimes fs + \omega\wedge D(fs). \qquad \square$$

We consider

$$D^2 : \mathcal{E} \longrightarrow \mathcal{A}^2_X(\mathcal{E}).$$

A priori this is a $\mathbb{C}$-linear map of sheaves of vector spaces. The basic fact is that this map is not zero in general. But it has the property

$$D^2(fs) = fD^2(s)$$

for differentiable functions f. This implies that D^2 is a bundle map. Hence $R := D^2$ can be considered as en element of a certain Hom-bundle, namely the bundle with fibre

$$\mathrm{Hom}_{\mathbb{C}}\Big(E_a,\ \bigwedge^2 \mathrm{Hom}_{\mathbb{R}}(T_a(X),\mathbb{R})\otimes_{\mathbb{R}} E_a\Big).$$

This can be also written as

$$\mathrm{Hom}_{\mathbb{C}}\Big(E_a,\ \bigwedge^2 \mathrm{Hom}(T_a(X),\mathbb{C})\otimes_{\mathbb{C}} E_a\Big).$$

This can be reinterpreted as follows. Let A be a complex and B be a real vector space (both of finite dimension). Let $A^* = \mathrm{Hom}_{\mathbb{C}}(A,\mathbb{C})$. There are canonical isomorphisms

$$\mathrm{Hom}_{\mathbb{C}}(A, B\otimes_{\mathbb{R}} A) = A^*\otimes_{\mathbb{C}}(B\otimes_{\mathbb{R}} A) = B\otimes_{\mathbb{R}}(A^*\otimes_{\mathbb{C}} A) = B\otimes_{\mathbb{R}}\mathrm{Hom}(A,A).$$

Hence we can consider R as a global section of the bundle with the fibre

$$\bigwedge^2 \mathrm{Hom}_{\mathbb{R}}(T_a(X),\mathbb{C})\otimes_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}}(E_a,E_a).$$

It can be written also as

$$\bigwedge^2 \mathrm{Hom}_{\mathbb{R}}(T_a(X),\mathbb{R})\otimes_{\mathbb{R}}\mathrm{Hom}_{\mathbb{C}}(E_a,E_a).$$

3.3 Definition. *Let D be a connection on a differentiable complex vector bundle $\mathcal{E}$. The* ***curvature form*** *of D is $R = D^2$, which can be considered as bundle valued differential form with values in the bundle*

$$\mathscr{H}\!om(\mathcal{E},\mathcal{E}).$$

(The $\mathscr{H}\!om$ is taken over $\mathcal{C}^\infty_X\otimes_{\mathbb{R}}\mathbb{C}$.)

Consider now a *holomorphic* vector bundle $\mathcal{M}$ over a complex manifold. We denote by

$$\mathcal{E} = \mathcal{M} \otimes_{\mathcal{O}_X} (\mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C})$$

the underlying differentiable vector bundle. It is naturally to set

$$\mathcal{A}_X^m(\mathcal{M}) = (\mathcal{A}_X^m \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathcal{O}_X} \mathcal{M}, \qquad \mathcal{A}_X^{p,q}(\mathcal{M}) = \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

But this is nothing new since standard ruled of the tensor product show

$$\mathcal{A}_X^m(\mathcal{M}) = \mathcal{A}_X^m(\mathcal{E}), \quad \mathcal{A}_X^{p,q}(\mathcal{M}) = \mathcal{A}_X^{p,q}(\mathcal{E}).$$

We have a decomposition

$$\mathcal{A}_X^1(\mathcal{E}) = \mathcal{A}_X^{1,0}(\mathcal{E}) \oplus A_X^{0,1}(\mathcal{E}).$$

This gives us a decomposition of a connection D into two parts,

$$D = D' + D'', \quad D' : \mathcal{E} \to A_X^{1,0}(\mathcal{E}),\ D'' : \mathcal{E} \to A_X^{0,1}(\mathcal{E}).$$

Similarly the extension of D to $A_X^{p,q}(\mathcal{E})$ (Lemma 3.2) decomposes into a sum $D = D' + D''$, where

$$D' : \mathcal{A}_X^{p,q}(\mathcal{E}) \to \mathcal{A}_X^{p+1,q}(\mathcal{E}), \quad D'' : \mathcal{A}_X^{p,q}(\mathcal{E}) \to \mathcal{A}_X^{p,q+1}(\mathcal{E}).$$

We now assume that $\mathcal{M}$ carries a Hermitian metric $\langle \cdot, \cdot \rangle$. Of course this extends to a Hermitian metric on $\mathcal{E}$. Then we can construct a distinguished connection. The idea is to construct D', D'' separately. We have a candidate for D'', namely the naive $\bar\partial$ (Proposition III.5.1). For the definition of the complete D we need the pairing

$$(\mathcal{A}_X^p \otimes_{\mathcal{O}_X} \mathcal{M}) \times (\mathcal{A}_X^q \otimes_{\mathcal{O}_X} \mathcal{M}) \xrightarrow{[\cdot,\cdot]} \mathcal{A}_X^{p+q}$$

which is locally given by

$$[\alpha \otimes s, \beta \otimes t] = \langle s, t \rangle \alpha \wedge \beta.$$

In the case $p = q = 0$ we have $[\cdot, \cdot] = \langle \cdot, \cdot \rangle$ (the Hermitian metric on $\mathcal{E}$). Notice that the squared bracket here has nothing to with a Lie bracket.

3.4 Proposition. *Let $\mathcal{M}$ be a holomorphic vector bundle over a complex manifold X and let $\mathcal{E}$ be the associated differentiable bundle. Assume that $\mathcal{M}$ (hence $\mathcal{E}$) is equipped with a Hermitian metric $\langle \cdot, \cdot \rangle$. Then there exists a unique connection D – called the* ***canonical connection*** *– such that for all sections $s, t \in \mathcal{E}(U)$*

a) $d[s,t] = [Ds, t] + [s, Dt]$ *for* $s, t \in \mathcal{E}(U)$.
b) $D'' = \bar\partial$.

The formulae a) *and* b) *carry over to $\mathcal{A}_X^{p,q}(\mathcal{M})$.*

Proof. It is enough to prove the existence and uniqueness locally. Hence we can assume that $X = U$ is an open subset of $\mathbb{C}^n$ and that $\mathcal{M}$ is free. We choose a basis $e_1, \dots, e_n$ of $\mathcal{M}$. This is a also a $\mathcal{C}_X^\infty \otimes_\mathbb{R} \mathbb{C}$-basis of $\mathcal{E}$. Hence it is enough to define De_i. We set

$$De_i = \sum_j \theta_{ij} \otimes e_j, \quad \theta_{ij} \in A^{1,0}(X).$$

The matrix $\theta = (\theta_{ij})$ is called the connection matrix with respect to the basis $e_1, \dots, e_n$. We have

$$\begin{aligned} d\langle e_i, e_j \rangle = [De_i, e_j] + [e_i, De_j] &= \Big[\sum_k \theta_{ik} \otimes e_k, e_j\Big] + \Big[e_i, \sum_k \theta_{jk} \otimes e_k\Big] \\ &= \sum_k \theta_{ik} \langle e_k, e_j \rangle + \sum_k \bar\theta_{jk} \langle e_i, e_k \rangle. \end{aligned}$$

Comparing types, we get

$$\partial \langle e_i, e_j \rangle = \sum_k \theta_{ik} \langle e_k, e_j \rangle, \quad \bar\partial \langle e_i, e_j \rangle = \sum_k \bar\theta_{ik} \langle e_k, e_j \rangle.$$

We use now the notation $H = (\langle e_i, e_j \rangle)$ and ∂H. The latter means the componentwise application of ∂ to the matrix H.. It is easy to check that the only solution of these equations is

$$\boxed{\theta = (\partial H) H^{-1} \qquad \text{(matrix product)}}$$

This finishes the proof of Proposition 3.4. □

If $\mathcal{L}$ is a line bundle on a ringed space $(X, \mathcal{O}_X)$, then $\mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$ is canonically isomorphic to $\mathcal{O}_X$. Hence for a connection on a holomorphic line bundle over a complex manifold the curvature can be considered as a usual differential form

$$R \in A^2(X).$$

3.5 Proposition. *Let $\mathcal{L}$ be a holomorphic line bundle over a complex manifold X. Assume that $\mathcal{L}$ carries a Hermitian metric H. Let R be the curvature of the canonical connection. Then*

$$R = -\partial\bar\partial \log H,$$

hence $R/2\pi\mathrm{i}$ represents the Chern class of $\mathcal{L}$ (more precisely its image in $H^2(X, \mathbb{R})$).

Proof. Again we can assume that $X = U$ is an open subset of $\mathbb{C}^n$ and that $\mathcal{L}$ is free. Let $s \in \mathcal{L}(X)$ be a basis element of $\mathcal{L}$. Every element of $\mathcal{A}_X^1 \otimes \mathcal{L}$ is of the form $\alpha \otimes s$ with a 1-form α (on some open subset of X). In particular, we can write $D(s) = \theta \otimes s$. (Here the connection matrix is the 1×1-matrix (θ) From the product rule in Lemma 3.2 we get

$$D(Ds) = (d\theta) \otimes s - \theta \wedge (\theta \otimes s) = (d\theta) \otimes s.$$

It follows

$$R = d\theta.$$

During the proof of Proposition 3.4 we computed the connection matrix (framed formula). In our case we get $\theta = \partial \log\langle s, s\rangle$. We obtain

$$R = d\alpha = \bar{\partial}\partial \log\langle s, s\rangle = -\partial\bar{\partial} \log\langle s, s\rangle.$$

This finishes the proof of Proposition 3.5. $\square$

We mention without proof that for any differentiable complex line bundle over an arbitrary differentiable manifold the class of R in $H^2(X, \mathbb{R})$ is the same for all connections.

Proposition 3.5 is very important. It enables to define Chern classes for vector bundles where the direct cohomological approach does not work. We will not need this in this book.

Now we assume that X carries also a Hermitian metric. Recall that then $\mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}$ carries a Hermitian metric. Hence we can ask for the formally adjoint operator D. We know already the formally adjoint of $D'' = \bar{\partial}$ (Proposition III.7.3). It is $\bar{\partial}^* = -\natural\bar{\partial}\natural$. Hence it remains to compute the formally adjoint D'^* of D'. We also have the operator

$$L : \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{L} \longrightarrow \mathcal{A}_X^{p+1,q+1} \otimes_{\mathcal{O}_X} \mathcal{L}$$

and the star operator

$$* : \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{L} \longrightarrow \mathcal{A}_X^{p+1,q+1} \otimes_{\mathcal{O}_X} \mathcal{L}.$$

They act as $L(\alpha \otimes s) = (L\alpha) \otimes s$ and $*(\alpha \otimes s) = (*\alpha) \otimes s$. We also set $L^* = -*L*$ which acts as $L^*(\alpha \otimes s) = (L^*\alpha) \otimes s$.

3.6 Proposition. *Let (X, h) be a Kähler manifold and $(\mathcal{L}, H)$ be a holomorphic line bundle equipped with a Hermitian metric. Let $D = D' + D'' = D + \bar{\partial}$ be the canonical connection (considered on $\mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{L}$). Then*

$$(D')^* = \mathrm{i}(L^*\bar{\partial} - \bar{\partial}L^*) = *\bar{\partial}*\,.$$

Proof. The second relation involves only operators of the form $\alpha\otimes s\longmapsto\beta\otimes s$. Hence it follows from the Hodge relations in the absolute case (Theorem 2.3). So it remains to prove the first relation. Again we can assume that $\mathcal{L}$ is free and that $X=U$ is an open subset of $\mathbb{C}^n$. Let s be a generator. An arbitrary section of $\mathcal{A}_X^{p,q}\otimes_{\mathcal{O}_X}\mathcal{L}$ can be written in the form $\alpha\otimes s$ with a differential form α of type (p,q). From the definition we have

$$D(\alpha\otimes s)=(d\alpha)\otimes s+\alpha\wedge D(s).$$

We write $D(s)=\theta\otimes s$ where θ is the connection "matrix". So the above formula reads

$$D(\alpha\otimes s)=(d\alpha)\otimes s+(\alpha\wedge\theta)\otimes s.$$

We separate into types

$$\begin{aligned}D'(\alpha\otimes s)&=(\partial\alpha)\otimes s+(\alpha\wedge\theta)\otimes s,\\ D''(\alpha\otimes s)&=(\bar\partial\alpha)\otimes s.\end{aligned}$$

We compute the formally adjoint operators. From the definition of the formally adjoint operator it follows easily that the adjoint operators of $\alpha\otimes s\mapsto\partial\alpha\otimes s$ is the operator $\beta\otimes s\mapsto\partial^*\beta\otimes s$ (where ∂^* is the formally adjoint of ∂ in the absolute case). The same is true for $\bar\partial$ instead of ∂. The formally adjoint of the operator $\alpha\otimes s\mapsto\alpha\wedge\theta\otimes s$ is easy, since this is a bundle operator. This implies that the formally adjoint operator can be computed pointwise. We consider now a special point a where $\theta(a)=0$. Then we get

$$\bigl(D'^*(\alpha\otimes s)\bigr)(a)=\bigl((\partial^*\beta)\otimes s)\bigr)(a)=\bigl((*\bar\partial*\alpha)\otimes s)\bigr)(a).$$

We see that the identity $D'=*\bar\partial*$ holds in all points a where $\theta(a)=0$. But this identity is independent from the choice of a generator s. Hence it is sufficient to show that for each point a there exists a generator (in a small neighborhood of a is enough) such that $\theta(a)=0$. To prove this we take a new generator $\tilde s=fs$ where f is a holomorphic function without zeros. The new connection form connects as $\tilde\theta=\theta+d\log f$. We can assume that a holomorphic logarithm $g=\log f$ exists. Since θ is a form of $(1,0)$ its evaluation at a point is given by n constants $C_1,\dots,C_n$. We can take for g a linear function $C_1z_1+\dots+C_nz_n$ and then $f=e^g$ to cancel these constants. This finishes the proof of Proposition 3.6. □

4. Kodaira's vanishing theorem

Let $\omega\in A^{1,1}(X)\cap A^2_{\mathbb{R}}(\mathbb{R})$ be a differential form on a complex manifold. We recall that $\omega(a)$ can be considered as a alternating bilinear form

$$\omega(a):T_a(X)\times T_a(X)\longrightarrow\mathbb{R}.$$

We recall that then there exists a unique Hermitian form on T_aX with imaginary part $\omega(a)$.

4.1 Definition. *A holomorphic line bundle $(\mathcal{L}, H)$ on a complex manifold equipped with a Hermitian metric is called* ***positive****, if the Hermitian form on X corresponding to the Chern form of H is positive definite.*

A line bundle $\mathcal{L}$ is called positive if there exists a Hermitian metric H on $\mathcal{L}$ such that $(\mathcal{L}, H)$ is positive.

We see that a positive $(\mathcal{L}, H)$ is related to a Kähler metric. Hence positive bundles can only exist on Kähler manifolds. We also mention that one can define what it means that $(\mathcal{L}, H)$ is semipositive, even more, it is possible to define what it means that $(\mathcal{L}, H)$ is positive or semipositive at some point.

(Semi-) positivity of a bundle depends only on its class in $\operatorname{Pic} X$. Hence we can talk about (semi-) positive elments of $\operatorname{Pic} X$.

Let $(\mathcal{L}_1, H_1)$, $(\mathcal{L}_2, H_2)$ be two line bundles with Hermitian metrics. Then $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ carries an obvious tensor product metric $H = H_1 \otimes H_2$. One has

$$\partial\bar\partial \log(H_1 \otimes H_2) = \partial\bar\partial \log H_1 + \partial\bar\partial \log H_2.$$

Hence the sum of two semipositive bundles is semipositive. If one of them is positive, then the sum is positive. We also can define $L_1 > L_2$ if $L_1 - L_2 > 0$ (the same with $\geq$).

4.2 Proposition. *Let $\mathcal{L}$ be a positive line bundle over a compact Kähler manifold with Kähler form Ω. Assume that Ω represents the Chern class of $\mathcal{L}$. There exists a Hermitian metric H on $\mathcal{L}$ such that*

$$\Omega = \mathrm{i}\partial\bar\partial \log H.$$

Proof. We choose a Hermitian metric H of $\mathcal{L}$ as in Definition 4.2. Hence we have a metric H on the bundle and a Kähler metric h on X. We define $\Omega_0 = \mathrm{i}\partial\bar\partial \log H$. We know already that Ω_0 and Ω define the same class in $H^2(X, \mathbb{R})$. We want to modify H such they get equal. We can replace H by $e^h H$ with a real differentiable function h. Hence we have to find h such that $\Omega_0 - \Omega = \mathrm{i}\partial\bar\partial h$, This global version of Proposition II.3.4 is proved as follows.

$$\Omega_0 - \Omega = d\eta, \quad \eta \in A^1(X).$$

We decompose $\eta = \alpha + \bar\alpha$ with $\alpha \in A^{1,0}$. From $d\eta \in A^{1,1}(X)$ we obtain $\partial\alpha = 0$ and $\bar\partial\alpha + \partial\bar\alpha = d\eta$. Using Hodge theory we obtain $\alpha = \alpha_0 + \partial f$ with a $\square$-harmonic α_0 and a function f. Now we use in an essential manner that X is Kählerien. The form α_0 is also $\bar\square$-harmonic and hence $\bar\partial\alpha_0 = 0$. This gives $d\eta = \bar\partial\partial f + \partial\bar\partial\bar f$. Set $h = \mathrm{i}(\bar f - f)$. $\square$

4.3 The Kodaira-Nakano vanishing theorem. *Let $\mathcal{L}$ be a positive (holomorphic) line bundle on a compact complex manifold. Then*

$$H^{p,q}(X,\mathcal{L}) = 0 \quad \textit{for} \quad p+q>n.$$

Proof. We choose a Hermitian metric H on the bundle $\mathcal{L}$ and a Hermitian metric h on the manifold X which are tied together in the sense that $\mathrm{i}\partial\bar\partial \log H$ is the imaginary part of h (which can be considered as differential form as we explained). We want to make use of the canonical connection (Proposition 3.4)

$$D = D' + D''.$$

Recall that $D'' = \bar\partial$ is defined in a naive way and that the operator D' depends on the use of the Hermitian metric H and should be considered as substitute for ∂ in the case of the trivial bundle.

We use for simplicity the notation $D' = \partial$ for the rest of this section.

But be aware that $\bar\partial$ is a naive generalization from the absolute case (absence of $\mathcal{L}$) but ∂ depends on the choice of Hermitian metrics on $\mathcal{L}$ and X. We need also an operator ∂^*. We do not want to compute the adjoint operator of ∂. Instead of this we will make use of the following relation.

From Proposition 3.6 we have

$$\partial^* = *\bar\partial* = \mathrm{i}(L^*\bar\partial - \bar\partial L^*).$$

4.4 Basic identity. *Let (X,h) be an Hermitian manifold and let $(\mathcal{L},H)$ be a holomorphic line bundle with Hermitian metric. Then on $\mathcal{A}_X \otimes_{\mathcal{C}_X^\infty} \mathcal{L}$ we have the identities*

$$\partial\bar\partial + \bar\partial\partial = -\mathrm{i}L, \quad \bar\partial^*\partial^* + \partial^*\bar\partial^* = -\mathrm{i}L^*.$$

Proof. This follows from the fact that Ω is the curvature form of D. Recall that the curvature form is defined by D^2. □

The proof of the vanishing theorem now is very short. Let $\omega \in A^{p,q}(X,\mathcal{L})$ be a $\bar\square$-harmonic form. We want to prove that $\omega = 0$ in the case $p+q>n$. It suffices to prove that ω is primitive, $L^*\omega = 0$. We have $\bar\partial\omega = 0$ and $\bar\partial^*\omega = 0$. Using the above formula for ∂^* we obtain

$$\begin{aligned}\big(\partial^*\omega, \partial^*\omega\big) &= \big(\mathrm{i}(L^*\bar\partial - \bar\partial L^*)\omega, \partial^*\omega\big)\\ &= \big(-\mathrm{i}\bar\partial L^*\omega, \partial^*\omega\big) = \big(-\mathrm{i}L^*\omega, \bar\partial^*\partial^*\omega\big)\\ &= \big(-\mathrm{i}L^*\omega, (\bar\partial^*\partial^* + \partial^*\bar\partial^*)\omega\big).\end{aligned}$$

Here $(\cdot,\cdot)$ denotes the scalar produce on $A^{p,q}(\mathcal{L})$ (integrating $\langle\cdot,\cdot\rangle$). Using the basic identity we get

$$\big(\partial^*\omega,\partial^*\omega\big)=-\big(\mathrm{i}L^*\omega,\mathrm{i}L^*\omega\big).$$

The left hand side is ≥ 0, but the right hand side is ≤ 0. Hence both sides have to be zero, in particular $L^*\omega=0$. Hence ω is a primitive form of degree $p+q>n$. Such forms are zero (Proposition 1.7). This proves the vanishing theorem. □

We exhibit a special case of the Kodaira-Nakano vanishing theorem, namely the case $p=n$. Recall that $\mathcal{K}_X=\bigwedge^n\Omega_X$ is called the canonical line bundle.

4.5 Kodaira vanishing theorem. *Let $\mathcal{L}$ be a positive line bundle on the compact complex manifold X. Then*

$$H^q(X,\mathcal{K}_X\otimes_{\mathcal{O}_X}\mathcal{L})=0\quad \textit{for}\quad q>0.$$

5. Blowing up

Let U,V be open subsets of $\mathbb{C}$ and assume that there is a holomorphic map $f:U\to V$. Assume furthermore that there is a point $a\in V$ such that the restriction $f_0:U-f^{-1}(a)\to V-\{a\}$ is biholomorphic. Then f is biholomorphic. Hence $f^{-1}(a)$ consists of only one point. We will not make use of this and leave the proof to the interested reader.

In the case $n>1$ the situation is different. Consider for example the holomorphic map

$$\mathbb{C}\times\mathbb{C}\longrightarrow\mathbb{C}\times\mathbb{C},\quad (z,w)\longmapsto(z,zw).$$

Then each point different from $(0,0)$ on the right hand side has just one inverse image, but the inverse image of $(0,0)$ consists of all $(z,0)$ and is isomorphic to $\mathbb{C}$. One says that this $\mathbb{C}$ is an exceptional fibre. One can refine this construction to get $P^1(\mathbb{C})$ as exceptional fibre.

For this we consider $V=\mathbb{C}^n$. Notice that the $(n-1)$-dimensional complex projective space $P(V)=P^{n-1}(\mathbb{C})$ can be identified with the set of lines (=one-dimensional sub-vector spaces) in V. We just replace $[z]$ by the line $\mathbb{C}z$. We consider the complex manifold $V\times P(V)$. We consider the subset

$$\hat{V}:=\big\{\,(a,L)\in V\times P(V);\quad a\in L\,\big\}$$

and equip it with the induced topology (of the product topology). There are natural projections $p : \hat{V} \to V$ and $q : \hat{V} \to P(V)$. We determine the fibres $p^{-1}(a)$. This fibre consists of precisely one point if $a \neq 0$, namely the point $(a, \mathbb{C}a)$. The fibre over 0 consists of all $(0, L)$ and hence can be identified with the full $P(V)$. We will denote it by $Y = p^{-1}(0)$ and call it the *exceptional fibre.*

Next we will define a complex structure on $\hat{V}$. Recall

$$P_i(V) = \{L = \mathbb{C}a; \quad a_i \neq 0\}.$$

This is an open subset of $P(V)$. The complex structure of $P(V)$ has been made such that

$$\mathbb{C}^{n-1} \xrightarrow{\sim} P_i(V), \quad (z_1, \dots, z_{n-1}) \longmapsto (z_1, \dots, z_{i-1}, 1, z_i, \dots, z_n),$$

is a holomorphic chart. Similarly we define

$$\hat{V}_i = \{(a, L) \in \hat{V}; \quad L \in P_i(V)\}.$$

We may assume $V = \mathbb{C}^n$. We consider in $P^{n-1}(\mathbb{C}) = P(\mathbb{C}^n)$ the open subset $P_i^{n-1}\mathbb{C}$ consisting of all $[z_1, \dots, z_n]$ such that $z_i \neq 0$. We denote by $\hat{V}_i$ its inverse image in $\hat{V}$. We construct a bijective map

$$\mathbb{C}^N \xrightarrow{\sim} \hat{V}_i.$$

$$(z_1, \dots, z_n) \longmapsto \big((z_1 z_i, \dots z_{i-1} z_i, z_i, z_{i+1} z_i, \dots, z_n z_i), [1, z_2, \dots, z_n]\big).$$

It is clear that there exists a unique complex structure on $\hat{V}$ such that these maps are holomorphic charts. The projection $\hat{V} \to V$ clearly is holomorphic. It is also easy to show that $\hat{V}$ is a smooth submanifold of $V \times P(V)$.

It is important to understand the blowing up construction. Therefore we repeat it in the case $V = \mathbb{C}^2$. Then we have to consider two maps

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow \hat{V}, \quad (z, w) \longmapsto \big((z, zw), [1, w]\big), \\ \mathbb{C}^2 &\longrightarrow \hat{V}, \quad (z, w) \longmapsto \big((zw, w), [z, 1]\big). \end{aligned}$$

The image of them is $\hat{V}_1$ and $\hat{V}_2$. The union of both is $\hat{V}$. The inverse image of $\hat{V}_1 \cap \hat{V}_2$ is $\mathbb{C}^* \times \mathbb{C}$ in the first case and $\mathbb{C} \times \mathbb{C}^*$ in the second. The chart transformation is

$$\mathbb{C}^* \times \mathbb{C} \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}^*, \quad (z, w) \longmapsto \Big(zw, \frac{1}{z}\Big).$$

Now we consider the function $H_Y : \hat{V} \to \mathbb{R}_{\geq 0}$ that is the composition of the natural map $\hat{V} \to V$ and the function $|z|^2 := |z_1|^2 + \cdots + |z_n|^2$. This function vanishes on the exceptional fibre Y. We use it to define a Hermitian form on the bundle $\mathcal{J}_Y$ (Remark III.7.4) through

$$\langle f, g\rangle = \frac{f\bar{g}}{H_Y}.$$

At the first glance it is defined only outside Y, but we will see that it extends to $\hat{V}$. To avoid many dots we restrict now to $n = 2$. The general case then should be clear. We look at the Hermitian form on the two charts. Let $a \in \mathbb{C}^* \times \mathbb{C}$ be a point in the first chart and let U some open neighborhood. A section of $\mathcal{J}_Y$ vanishes along $Y \cap U$ and hence is of the form $f = zf'$ with a holomorphic function. So we see that we have a Hermitian form on the first chart and similarly on the second chart. This carries over to arbitrary n. So we have constructed a Hermitian metric on the bundle $\mathcal{J}_Y$ on $\hat{V}$. We compute its Chern form. Again we restrict to $n = 2$ and to the first chart.

$$\langle z, z\rangle = \frac{z\bar{z}}{z\bar{z} + z\bar{z}w\bar{w}} = \frac{1}{1 + w\bar{w}}.$$

So the Chern form (in the first chart) is

$$\frac{1}{2\pi \mathrm{i}}\partial\bar{\partial}\log\frac{1}{1 + w\bar{w}} = -\frac{1}{2\pi \mathrm{i}}\frac{1}{1 + w\bar{w}}dw \wedge d\bar{w}.$$

We are more interested in the dual bundle $\mathcal{J}_Y^*$ we can equip it with the Hermitian form H_Y^{-1}.

5.1 Lemma. *The Chern form bundle of $\mathcal{J}_Y$, considered on the blow up of $\mathbb{C}^n$ in the origin, and equipped with the Hermitian metric H_Y^{-1} is given on the ith chart $\hat{V}_i$ through*

$$\frac{1}{2\pi \mathrm{i}}\sum_{j\neq i}\frac{1}{1 + z_j\bar{z}_j}dz_i \wedge d\bar{z}_j.$$

The associated Hermitian form is positive semidefinite.

We treat another example of a Chern form. We start with some Hermitian metric on the trivial bundle $\mathcal{O}_V$. It is just given by an everywhere positive differentiable function $H : V \to \mathbb{R}_{>0}$, namely $\langle f, g\rangle = Hf\bar{g}$. We can pull back H to a positive function $\hat{H}$ on $\hat{V}$. This defines in the same way a Hermitian metric on $\mathcal{O}_{\hat{V}}$. We recall $\mathcal{O}_{\hat{V}} = \pi^*\mathcal{O}_V$. The Chern form $\hat{\omega}$ agrees with ω outside the exceptional locus. Hence it must be $\hat{\omega} = \pi^*\omega$. This generalizes to holomorphic line bundles $\mathcal{L}$ on V since they are trivial in an open neighborhood of the origin. So we see the following result.

5.2 Remark. *Let $(\mathcal{L}, H)$ be a holomorphic line bundle on V with Hermitian form. This form extends to a Hermitian form $\hat{H}$ on the inverse image $\hat{\mathcal{L}}$ on $\hat{V}$.*

Assume now that $(\mathcal{L}, H)$ is positive. Then $(\hat{\mathcal{L}}, \hat{H})$ is semipositive and definite outside the exceptional fibre. Combining this with Lemma 5.1 we get the following result.

5.3 Lemma. *Let $(\mathcal{L}, H)$ be a positive line bundle on $V = \mathbb{C}^n$. Then*

$$\hat{\mathcal{L}} \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{J}_Y$$

is positive (with respect to the tensor product of the Hermitian forms $\hat{H}$ and H_Y).

Proof. We consider the blow up construction in a chart $(z, w) \mapsto (u, v) = (z, zw)$. We have to pull back a differential form $h_0 du \wedge d\bar{u} + h_1 du \wedge d\bar{v} + \bar{h}_1 dv \wedge d\bar{u} + h_2 dv \wedge d\bar{v}$ whose associated Hermitian 2×2-matrix is positive definit. The pulled back form has been restriced to the exceptional divisor $z = 0$. The result is

$$\begin{pmatrix} h_0 + h_1\bar{w} + \bar{h}_1 w + h_2 w\bar{w} & 0 \\ 0 & 0 \end{pmatrix}.$$

The first diagonal element is positive for all w. By Lemma 5.1 the bundle $\mathcal{J}_Y$ leads to a matrix such that the second diagonal element is positive and the other entries are zero. The sum of the two matrices is positive definite. □

We need a generalization of the blow up construction. Let X be a complex manifold and let $a \in X$ be point. Consider a holomorphic chart $U \to U' \subset V$, $a \mapsto 0$ around a. Denote by $\hat{U}'$ the inverse image of U' in $\hat{V}$. Then consider the disjoint union

$$\hat{X} = (X - U) \cup \hat{U}'$$

There is an obvious structure as complex manifold on $\hat{X}$ such that $\hat{X} \to X$ is holomorphic. The inverse image Y of a is biholomorphic to $P^{n-1}(\mathbb{C})$ and $\hat{X} - Y \xrightarrow{\sim} X - \{a\}$ is biholomorphic. We call $\hat{X}$ the blow-up of X of the point a. It is uniquely determined in an obvious sense.

6. Maps into the projective space

Let $\mathcal{L}$ be a holomorphic line bundle on a compact complex manifold $(X, \mathcal{O}_X)$ Assume that a basis of $\mathcal{L}(X)$

$$s_0, \dots, s_N \in \mathcal{L}(X)$$

has been selected. We consider the set X_0 of all points $x \in X$ such that at least one of the values $s_i(x)$ is different from zero. Recall that $s_i(x)$ is an element of the vector space $\mathcal{L}(x) \in \mathcal{L}_x / \mathfrak{m}_{X,x}\mathcal{L}_x$. The set X_0 is open. Let $U \subset X_0$ be an open subset over which $\mathcal{L}$ is trivial. Choosing a (holomorphic) trivialization $\mathcal{L}_U \cong U \times \mathbb{C}$, we can define $(s_0(x), \dots, s_N(x))$. Changing the trivialization means to multiply $(s_0(x), \dots, s_N(x))$ by a joint factor. Hence the point

$$[s_0(x), \dots, s_N(x)] \in P^N(\mathbb{C})$$

is independent of the choice of the local trivialization. This means that we obtain a map

$$X_0 \longrightarrow P^N(\mathbb{C}).$$

Clearly this is a holomorphic map. The famous Kodaira embedding theorem states that under certain circumstances $X_0 = X$ and moreover that

$$X \longrightarrow P^N(\mathbb{C})$$

is a closed embedding, i.e. a biholomorphic map onto a smooth closed complex submanifold.

6.1 Theorem. *Let X be a compact complex manifold equipped with a positive holomorphic line bundle $\mathcal{L}$. Then there exists a number k_0 suche that for each $k \geq k_0$ and for each a there exists a section $s \in \mathcal{L}^{\otimes k}(X)$ such that $s(a) \neq 0$*

All ideas of the proof of Kodaira's embedding theorem are contained in the proof of this theorem. The rest of this and the following section are dedicated its proof. In the last section we then will complete the proof of the embedding theorem.

For the proof we will make use of the blow up of a point a

$$\hat{X} \longrightarrow X$$

and the pull back $\hat{\mathcal{L}}$ of the line bundle $\mathcal{L}$ to $\hat{X}$. This is a line bundle on $\hat{X}$. The construction of the pull back gives a natural map $\mathcal{L}(X) \longrightarrow \hat{\mathcal{L}}(\hat{X})$.

6.2 Lemma. *Let X be a complex manifold, $\hat{X} \to X$ the blow up in a point and $\mathcal{L}$ a holomorphic line bundle on X The natural map*

$$\mathcal{L}(X) \longrightarrow \hat{\mathcal{L}}(\hat{X})$$

is an isomorphism.

Proof. We can assume that the dimension is > 1. Then we have to apply an elementary result of complex analysis, which we state without proof (absence of isolated singularities in more than one variable):

Let $U \subset \mathbb{C}^n$, $n > 0$ be an open subset and $a \in U$ a point. Every holomorphic function on $U - \{0\}$ extends to a holomorphic function on U.

A consequence of this remark is: Let s by a section of $\mathcal{L}$ over $X - \{a\}$. Then s extends to a global holomorphic section. The implies Lemma 6.2. □

In the following we denote by Y the exceptional fibre of $\hat{X}$ and by $\mathcal{J} \subset \mathcal{O}_{\hat{X}}$ the vanishing sheaf of Y. This is a line bundle. We consider the short exact sequence

$$0 \longrightarrow \mathcal{J}\hat{\mathcal{L}} \longrightarrow \hat{\mathcal{L}} \longrightarrow \hat{\mathcal{L}}/\mathcal{J} \longrightarrow 0.$$

We claim
$$H^0(\hat{X}, \hat{\mathcal{L}}/\mathcal{J}) = \mathbb{C}.$$
This is clear if $\mathcal{L}$ is the trivial bundle $\mathcal{O}_X$, since then
$$\hat{\mathcal{O}}_X/\mathcal{J} = j_*(\mathcal{O}_Y).$$
The global sections are holomorphic functions on Y (Riemann sphere) and hence constant. The general case follows from this special one, since $\mathcal{L}$ is trivial in some open neighborhood of a.

We now apply the long exact cohomology sequence and obtain the exact sequence
$$\mathcal{L}(X) \longrightarrow \mathbb{C} \longrightarrow H^1(X, \mathcal{J}\hat{\mathcal{L}}).$$
It may happen that $H^1(X, \mathcal{J}\hat{\mathcal{L}})$ vanishes. In this case we obtain that $\mathcal{L}(X) \to \mathbb{C}$ is surjective and hence the existence of a non-trivial global section. If one looks how this map is defined on sees:

6.3 Lemma. *Let X be a complex manifold, $\hat{X} \to X$ the blow up in a point a and let $\mathcal{L}$ be a holomorphic line bundle. We denote by $Y \subset \hat{X}$ the exceptional fibre. Assume $H^1(\hat{X}, \mathcal{J}_Y\hat{\mathcal{L}}) = 0$. Then there exists a global section of $\mathcal{L}$ which doesn't vanish at a.*

Let R be a ring, M a free R-module and $\mathfrak{a}$ an ideal of R. The natural map $\mathfrak{a} \otimes_R M \to \mathfrak{a}M$ is an isomorphism. Hence
$$\mathcal{J}\hat{\mathcal{L}} = \mathcal{J} \otimes_{\mathcal{O}_{\hat{X}}} \hat{\mathcal{L}}$$
is the tensor product of two line bundles and hence a line bundle too. This gives us hope that Kodaira's vanishing theorem(applied to $\hat{X}$) will help us to prove the existence of global sections. For this we need positive bundles on the blow up $\hat{X}$.

7. Positive bundles on the blow up

In this section X is a connected compact complex manifold of dimension n and $(\mathcal{L}, H)$ is a positive holomorphic line bundle on X. The blow up of X in some point a is denoted by $\hat{X}$ and $\hat{\mathcal{L}}$ is the inverse image of $\mathcal{L}$ on $\hat{X}$. The exceptional fibre is denoted by $Y \subset \hat{X}$. We want to construct positive bundles on $\hat{X}$. The first candidate could be $\hat{\mathcal{L}}$. But we will see that this bundle is not positive. Nevertheless it carries an (uniquely determined) Hermitian metric $\hat{H}$ that agrees with H outside the exceptional locus. We repeat the construction from Remark 5.2 in our new context. We can assume that $\mathcal{L} = \mathcal{O}_X$. But then the Hermitian metric is just given by a positive function H. We can pull back this function to a function $\hat{H}$ on $\hat{X}$ and use it to define a Hermitian metric on $\mathcal{O}_{\hat{X}} = \pi^*\mathcal{O}_X$. In this way, we get a Hermitian metric $\hat{H}$ on $\hat{\mathcal{L}}$. We want to compute the corresponding Chern form $\hat{\omega} = \partial\bar{\partial}\log\hat{H}$. This must be of course the pull back of the Chern form ω of $\mathcal{L}$.

7.1 Lemma. *There exists a number k_0, independent of the base point a, such that the bundle*

$$k\hat{L} + [Y], \quad k \geq k_0,$$

is positive.

(Recall that $k\hat{L}$ is the image of $\hat{\mathcal{L}}^{\otimes k}$ in the Picard group.)

Proof. We have a Hermitian metric H such that $(\mathcal{L}, H)$ is positive on X. As we have seen this extends to a Hermitian metric $\hat{H}$ on $\hat{\mathcal{L}}$ that is semipositive and positive definite outside the exceptional fibre. We need a Hermitian metric on $\mathcal{L}_Y$. We have one in some open neighborhood of the exceptional fibre (Lemma 5.1). We can use it to construct a Hermitian metric on $\mathcal{L}_Y$ (considered on X) which coincides with this one in some open neighborhood of $Y \subset \hat{X}$. The bundle $\hat{\mathcal{L}} \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{L}_Y$ then can be equipped with the product metric. The Chern form of this product metric has the desired positivity condition on this neighborhood. The complement of this neighborhood is compact. Hence we find a number k_0 such that for all $k \geq k_0$ the bundle $\hat{\mathcal{L}}^{\otimes k} \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{L}_Y$ gets positive on this compact subset and hence everywhere. It is clear that k_0 can be taken constant on some neighborhood of a. A compactness argument shows that it can be chosen independently on a. □

We need the vanishing of the cohomology of $k\hat{L}+[Y]$ for big enough k. (Then if not $\mathcal{L}$ but $\mathcal{L}^k$ admits a global section that does not vanish at a). We want to apply Kodaira's vanishing theorem. Therefore we need that $k\hat{L} + [Y] - K_{\hat{X}}$ is positive. So we have to understand the canonical class $K_{\hat{X}}$ on $\hat{X}$. We make use of the pull back construction for $\mathcal{O}_X$-module and consider $\pi^*\mathcal{K}_X$. We denote its class by $\hat{K}_X$. There should be a relations to the canonical class $\mathcal{K}_{\hat{X}}$ on $\hat{X}$.

7.2 Lemma. *Let $\hat{X}$ the blow up of a complex manifold X at some point a. Then $K_{\hat{X}}$ equals $\hat{K}_X + (n-1)[Y]$.*

Proof. The claim is $\pi^*(\mathcal{K}_X) \cong \mathcal{K}_{\hat{X}} \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{L}_Y^{\otimes(1-n)}$, or, by functoriality,

$$\mathcal{K}_X \cong \pi_*\big(\mathcal{K}_{\hat{X}} \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{L}_Y^{\otimes(1-n)}\big).$$

This means that we have to define for each open $U \in X$ a map

$$\big(\mathcal{K}_{\hat{X}} \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{L}_Y^{\otimes(1-n)}\big)(\pi^{-1}U) \longrightarrow \mathcal{K}_X(U).$$

This map will be induced by sheafifying of a map

$$\mathcal{K}_{\hat{X}}(\pi^{-1}U) \otimes_{\mathcal{O}_{\hat{X}}(\pi^{-1}U)} \mathcal{L}_Y^{\otimes(1-n)}(\pi^{-1}U) \longrightarrow \mathcal{K}_X(U).$$

Instead of this we can construct a bilinear map

$$\mathcal{K}_{\hat{X}}(\pi^{-1}U) \times \mathcal{L}_Y^{\otimes(1-n)}(\pi^{-1}U) \longrightarrow \mathcal{K}_X(U).$$

□

Now we attack $k\hat{L} + [Y] - K_{\hat{X}}$. First we notice that $k\hat{L} - n[Y]$ is positive for big k (namely $k \geq k_0 n$). We write $k = k_1 + k_2$ and get

$$k\hat{L} + [Y] - K_{\hat{X}} = (k_1\hat{L} + (n-1)[Y]) + (k_2\hat{L} - \hat{K}_X) + (1-n)[Y].$$

But $k_2\hat{L} - \hat{K}_X$ is the pull back of the bundle $k_2 L + K_X$. Since X is compact and L is positive this bundle is positive for big k_2. Its pull back remains semipositive. Hence $k\hat{L} + [Y] - K_{\hat{X}}$ is the sum of a positive and a semipositive bundle and hence positive. This proves Theorem 6.1. □

8. The Kodaira embedding theorem

We can now formulate the main result.

8.1 Kodaira's embedding theorem. *Every compact complex manifold which admits a positive line bundle is biholomorphic to a complex submanifold of some projective space.*

The idea is to use the techniques of the previous section and to embed the manifold by means of a positive line-bundle $\mathcal{L}$ into some projective space. It is easy to work out the conditions that $\mathcal{L}$ leads to an embedding. We need a notation.

A section $s \in \mathcal{L}(U)$ of a holomorphic line bundle vanishes at point $a \in \mathcal{L}(U)$ in at least in second order, if its germ s_a is contained in $\mathfrak{m}_{X,a}^2 \mathcal{L}_a$, where $\mathfrak{m}_{X,a} \subset \mathcal{O}_a$ denotes the maximal ideal.

For $\mathcal{L} = \mathcal{O}_X$ this just means that the derivatives of order ≤ 2 of the function s vanish at a.

8.2 Definition. *A holomorphic line bundle $\mathcal{L}$ on a compact connected complex manifold is called* ***strict ample****, if for every point $a \in X$ there exist a global (holomorphic) section s, which doesn't vanish at a and such that the induced map $X \to P^n(\mathbb{C})$ defines a biholomorphic map onto a smooth complex submanifold of $P^n(\mathbb{C})$.*

The line bundle is called ***ample*** *if there is some tensor power $\mathcal{L}^{\otimes m}$ which is strict ample.*

There are three necessary and sufficient conditions for a line bundle to be strict ample.

8.3 Definition. *A holomorphic line bundle $\mathcal{L}$ on a compact complex manifold is strict ample if the following three conditions are satisfied.*

(C1) *For every point $a \in X$ there exist a global (holomorphic) section s, which does not vanish at a.*
(C2) *Point separation: If a, b are two different points, then there exists a global section s with $s(a) = 0$, $s(b) \neq 0$.*
(C3) *Infinitesimal separation: Let $t \in \mathcal{L}(U)$ be a holomorphic section in some neighborhood of a point a. There exists a global section s such that $s - t$ vanishes at a in at least second order.*

The condition (C1) can of course be cancelled. We left it, because it is the basic starting condition. It implies that

$$X \longrightarrow P^m(\mathbb{C}), \quad x \longmapsto [s_0(x), \ldots, s_m(x)]$$

is a everywhere defined holomorphic map where $s_0, \ldots, s_m$ denotes a basis of the space of global sections. The condition (C2) says that this map is injective and (C3) shows that the tangent map is injective everywhere. Because X is compact, the above map is a topological map from X onto its image. Now it is clear that the image is smooth and that X is mapped biholomorphically onto its image. Kodaira's embedding theorem hence follows from the following theorem.

8.4 Theorem. *Positive line bundles are ample.*

The converse is also true but we do not need this. The rest of this section is dedicated the proof of this theorem. In the previous section we formulated already a sufficient cohomological condition for (C1), namely $H^1(\hat{X}, \mathcal{J}_Y\hat{\mathcal{L}}) = 0$ for all blow ups $\hat{X}$ of X in arbitrary point, Y denotes the exceptional fibre (=inverse image of a in $\hat{X}$). It is easy to derive similar conditions for (C2) and (C3). Hence we concentrate to (C1). At the end we indicate how the argument has to modified to get (C2) and (C3).

We change a little bit the notations: Recall that the set of isomorphy classes of holomorphic line bundles is an abelian group $\operatorname{Pic}(X)$. The composition is induced by the tensor product. Following usual conventions, we use the sign "+" for the composition in $\operatorname{Pic}(X)$. We also use the following notations. If $\mathcal{L}$ is a holomorphic line bundle then we denote by L its image in $\operatorname{Pic} X$. If $Y \subset X$ is a smooth submanifold of a connected complex manifold X, then we use the notation $[Y] = [\mathcal{J}_Y]$. So in the new notation

$$[Y] + L \qquad (\text{replaces } \mathcal{J}_Y\hat{\mathcal{L}} = \mathcal{J}_Y \otimes_{\mathcal{O}_{\hat{X}}} \hat{\mathcal{L}}).$$

8.5 Lemma. *A line bundle L is strict ample if the following conditions are satisfied:*

(C1') *Let $\hat{X} \to X$ the blow up of X in a point. Denote by Y the exceptional fibre and by $\hat{L}$ the pull back of L. The first cohomology of $[Y]+\hat{L}$ vanishes.*

(C2') *Let a_1, a_2 be two different points of X and $\hat{X} \to X$ the blow up of X in the two points a, b. Let Y_1, Y_2 be the two exceptional fibres and $\hat{L}$ the pull back of L. The first cohomology of $[Y_1] + [Y_2] + \hat{L}$ vanishes.*

(C3') *Let $\hat{X} \to X$ the blow up of X in a point. Denote by Y the exceptional fibre and by $\hat{L}$ the pull back of L. The first cohomology of $[Y] + [Y] + \hat{L}$ vanishes.*

We settled already the first case, The two others are similar. We skip them.

□

Now the Kodaira vanishing theorem comes into the game. We denote by $K_X \in \mathrm{Pic}(X)$ the image of the canonical bundle of a compact complex manifold X. By the vanishing theorem the first cohomology of a line bundle $\mathcal{L}$ vanishes if $L - K_X$ is positive.

8.6 Lemma. *A line bundle $\mathcal{L}$ is ample if there exists a natural number k_0 such that for $k \geq k_0$ the following conditions are satisfied: In the notations of Lemma 8.5 the following (classes of) bundles*

$$k\hat{L} + [Y] - K_{\hat{X}}, \quad k\hat{L} + [Y_1] + [Y_2] - K_{\hat{X}}, \quad k\hat{L} + 2[Y] - K_{\hat{X}}$$

are positive.

Now we are ready for

The proof of the embedding theorem

We have to prove that alle positive line bundles on X satisfy the conditions of the Lemma. We settled in Theorem 6.1 the first case. The two others are similar. □

Hodge manifolds

We want to give another formulation of the embedding theorem. It rests on the following theorem.

8.7 Theorem. *Let X be a complex manifold and $\alpha \in A^{1,1}(X)$ be a real closed differential form whose class is in the image of $H^2(X, \mathbb{Z})$. Then there exists a holomorphic line bundle together with a Hermitian metric $(\mathcal{L}, h)$ such that*

$$\alpha = \frac{1}{2\pi \mathrm{i}} \partial\bar{\partial} \log h.$$

Proof. One ingredient of the proof is Proposition II.3.4. It says that there exists an open covering $X = \bigcup U_i$ with the property $\alpha|U_i = (1/2\pi\mathrm{i})\partial\bar{\partial} g_i$. Here g_i is a real differentiable function on U_i. Another ingredient is the isomorphism $H^2(X,\mathbb{Z}) \cong H^1(X, (\mathcal{C}_X^\infty \otimes_{\mathbb{R}} \mathbb{C})^*)$. It tells us that every element of $H^2(X,\mathbb{Z})$ is the Chern class of a differentiable line bundle. We assume that it is given through transition functions $f_{ij} : U_i \cap U_j \to \mathbb{C}^*$ with respect to some open covering. We can assume that it is the same covering $X = \bigcup U_i$. Our problem is that the f_{ij} need not to be holomorphic. There is a corresponding element in $H^1(X, \mathcal{A}^1_{X\,\mathrm{closed}})$. This is represented by the cocycle $(1/2\pi\mathrm{i})d\log f_{ij}$. This cocycle corresponds to the differentiable form α. Recall that this cocycle related to α can to be taken as $\partial g_i - \partial g_j$. So we have two equivalent cocycles and we get

$$d\log f_{ij} = (\partial g_i - \partial g_j) + (\gamma_i - \gamma_j).$$

Here γ_i is a closed 1-form on U_i. We can shrink the covering and assume that $\gamma_i = db_i$. Now we replace f_{ij} by the equivalent cocycle $f_{ij}e^{b_i - b_j}$. This cancels the γ_i and we can assume

$$d\log f_{ij} = \partial g_i - \partial g_j.$$

The differential form on the right hand side is of type $(1,0)$. Hence $\bar{\partial}\log f_{ij}$ must vanish. This implies that $\log f_{ij}$ and then f_{ij} is holomorphic. We get $\partial\log f_{ij} = \partial g_i - \partial g_j$ or

$$\log f_{ij} = g_i - g_j + F_{ij}$$

with an antiholomorhic function F_{ij}. Its imaginary part equals the imaginary part of $\log f_{ij}$. Since an antiholomorphic function is determined by its imaginary part up to an additive constant, we obtain $F_{ij} = -\log \overline{f_{ij}} + C_{ij}$ with some constant C_{ij}. This shows

$$\log(|f_{ij}|^2) = g_i - g_j + C_{ij}.$$

We modify f_{ij} and consider

$$\tilde{f}_{ij} = f_{ij}e^{-C_{ij}/2}.$$

This defines also a holomorphic line bundle. This is the line bundle that gives the solution for Theorem 8.7. We have

$$\log(|\tilde{f}_{ij}|^2) = h_j/h_i, \quad h_i = e^{g_i}.$$

So the (h_i) define a Hermitian metric. The pair $(\tilde{L}, h)$ has the desired property. □

8.8 Definition. *A connected compact complex manifold X is called a **Hodge manifold** if there exist a Kähler metric such that the corresponding Kähler class is integral, i.e. in the image of $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$.*

If X is a Hodge manifold, h a corresponding Kähler metric and Ω associated Kähler form. By assumption, its cohomology class is contained in $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$. We know from Theorem 8.7 that there exists a holomorphic line bundle $\mathcal{L}$ whose Chern class equals this cohomology class. This line bundle is positive.

Assume conversely that $\mathcal{L}$ is a positive line bundle on a complex manifold. Then from the Definition 4.1 there exists a Kähler metric with the desired property. This shows the following result.

8.9 Theorem. *A connected compact complex manifold X is projective algebraic if and only if it is a Hodge manifold.*

In this context one should mention that by Chow's theorem every complex submanifold of a projective space is algebraic. Hence Hodge manifolds are "projective algebraic".

Appendices

Chapter V. Sheaves

1. Abelian groups

We assume that the reader is familiar with the notion of an abelian group and homomorphism between abelian groups. If A is a subgroup of an abelian group B, then the factor group B/A is well defined. All what one needs usually is that there is a natural surjective homomorphism $f : B \to B/A$ with kernel A. Let $f : B \to X$ be a homomorphism into some abelian group. Then f factors through a homomorphism $B/A \to X$ if and only if the kernel of f contains A. That f factors means that there is a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & X \\ B/A \arrow[ur] & \end{tikzcd}$$

Let $f : A \to B$ be a homomorphism of abelian groups. Then the image $f(A)$ is a subgroup of B. If there is no doubt which homomorphism f is considered, we allow the notation

$$B/A := B/f(A).$$

1.1 Lemma. *A commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

induces homomorphisms

$$B/A \longrightarrow D/C, \quad C/A \longrightarrow D/B.$$

A (finite or infinite) sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \cdots$$

is called *exact* at B if

$$\text{Kernel}(B \longrightarrow C) = \text{Image}(A \longrightarrow B).$$

It is called exact if it is exact at every place. An exact sequence $A \to B \to C$ induces an injective homomorphism

$$B/A \hookrightarrow C\ .$$

The sequence $0 \to A \to B$ is exact if and only if $A \to B$ is injective. The sequence $A \to B \to 0$ is exact if and only if $A \to B$ is surjective. The sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact if and only if $A \to B$ is injective and if the induced homomorphism $B/A \to C$ is an isomorphism. A sequence of this form is called a *short exact sequence*. Hence the typical short exact sequence is

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0 \qquad (A \subset B).$$

2. Presheaves

We introduce the language of presheaves of abelian groups. This consists mainly of definitions and simple remarks whose proofs are very simple. In many cases they can be left to the reader.

2.1 Definition. *A presheaf F (of abelian groups) on a topological space X is a map which assigns to every open subset $U \subset X$ an abelian group $F(U)$ and to every pair U, V of open subsets with the property $V \subset U$ a homomorphism*

$$r_V^U : F(U) \longrightarrow F(V)$$

such that $r_U^U = \text{id}$ and such that for three open subsets U, V, W with the property $W \subset V \subset U$ the relation

$$r_W^U = r_W^V \circ r_V^U$$

holds.

Example: $F(U)$ is the set of all continuous functions $f : U \to \mathbb{C}$ and $r_V^U(f) := f|V$ (restriction).

Many presheaves generalize this example. Hence the maps r_V^U are called "restrictions" in general and one uses the notation

$$s|V = s|_F V := r_V^U(s) \qquad \text{for} \quad s \in F(U).$$

The elements of $F(U)$ sometimes are called "sections" of F over U. In the special case $U = X$ they are called "global" sections.

2.2 Definition. *Let X be a topological space. A homomorphism of presheaves*

$$f : F \longrightarrow G$$

is a family of group homomorphisms

$$f_U : F(U) \longrightarrow G(U),$$

such that the diagram

$$\begin{array}{ccc} F(U) & \longrightarrow & G(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & G(V) \end{array}$$

commutes for every pair $V \subset U$ of open subsets, i.e. $f_U(s)|_G V = f_V(s|_F V)$.

It is clear how to define the identity map $\mathrm{id}_F : F \to F$ of a presheaf and the composition $g \circ f$ of two homomorphisms $f : F \to G$, $g : G \to H$ of presheaves.

There is also a natural notion of a sub-presheaf $F \subset G$. Besides $F(U) \subset G(U)$ for all U, one has to demand that the restrictions are compatible. This means:

The canonical inclusions $i_U : F(U) \to G(U)$ define a homomorphism $i : F \to G$ of presheaves.

When $f : F \to G$ is a homomorphism of presheaves, the images $f_U(F(U))$ define a sub-presheaf of G. We call it the *presheaf-image* and denote it by

$$f_{\mathrm{pre}}(F).$$

It is also clear that the kernels of the maps f_U define a sub-presheaf of F. We denote it by $\mathrm{Kernel}(f : F \to G)$. When F is a sub-presheaf of G, then one can consider the factor groups $G(U)/F(U)$. It is clear how to define restriction maps to get a presheaf $G/_{\mathrm{pre}}F$. We call this presheaf the *factor presheaf.*

Since we have defined kernel and image, we can also introduce the notion of a *presheaf-exact sequence.* A sequence $F \to G \to H$ is presheaf-exact if and only if $F(U) \to G(U) \to H(U)$ is exact for all U. presheaf-exact sequences of presheaves of abelian groups.

3. Germs and Stalks

Let F be a presheaf on a topological space X and let $a \in X$ be a point. We consider pairs (U, s), where U is an open neighbourhood of a and where $s \in F(U)$ is a section over U. Two pairs (U, s), (V, t) are called equivalent if there exists an open neighborhood $a \in W \subset U \cap V$, such that $s|W = t|W$. This is an equivalence relation. The equivalence classes

$$[U, s]_a := \big\{ (V, t); \quad (V, t) \sim (U, s) \big\}$$

are called *germs* of F at the point a. The set of all germs

$$F_a := \{ [U,s]_a, \quad a \in U \subset X, \ s \in F(U) \}$$

is the so-called *stalk* of F at a. The stalk carries a natural structure as abelian group. One defines

$$[U,s]_a + [V,t]_a := [U \cap V, s|U \cap V + t|U \cap V]_a.$$

We use sometimes the simplified notation

$$s_a = [U,s]_a.$$

For every open neighborhood $a \in U \subset X$ there is an obvious homomorphism

$$F(U) \longrightarrow F_a, \quad s \longmapsto s_a.$$

A homomorphism of presheaves $f : F \to G$ induces natural mappings

$$f_a : F_a \longrightarrow G_a \qquad (a \in X).$$

The image of a germ $[U,s]_a$ is simply $[U, f_U(s)]_a$. It is easy to see that this is well-defined.

3.1 Remark. *Let $F \to G$ and $G \to H$ be homomorphisms of presheaves and let $a \in X$ be a point. Assume that every point a contains arbitrarily small open neighborhoods U such that $F(U) \to G(U) \to H(U)$ is exact. Then $F_a \to G_a \to H_a$ is exact.*

Corollary. *If $F \to G \to H$ is presheaf-exact then $F_a \to G_a \to H_a$ is exact for all a.*

("Arbitrarily small" means that each neighborhood W of a contains a U.) The proof is easy and can be omitted. □

For a sub-presheaf $F \subset G$ the natural homomorphisms $F_a \to G_a$ are injective. Usually we will identify F_a with its image in G_a. In particular, for a homomorphism $F \to G$ of presheaves and a point $a \in X$, $f_a(F_a)$ and $f_{\text{pre}}(F)_a$ both are subgroups of G_a. It is easy to check that they are equal.

$$f_{\text{pre}}(F)_a = f_a(F_a).$$

If F is a presheaf on X, one can consider for each open subset $U \subset X$

$$F^{(0)}(U) := \prod_{a \in U} F_a.$$

The elements are families $(s_a)_{a \in U}$ with $s_a \in F_a$. There is no coupling between the different s_a. Hence $F^{(0)}(U)$ usually is very monstrous.

For open sets $V \subset U$, one has an obvious homomorphism (projection) $F^{(0)}(U) \to F^{(0)}(V)$. Hence we obtain a presheaf $F^{(0)}$ together with a natural homomorphism $F \longrightarrow F^{(0)}$. Each homomorphism $F \to G$ of presheaves induces a homomorphism $F^{(0)} \to G^{(0)}$ such that the diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ F^{(0)} & \longrightarrow & G^{(0)} \end{array}$$

commutes.

4. Sheaves

4.1 Definition. *A presheaf F is called a* ***sheaf*** *if the following conditions are satisfied:*

(G1) *When $U = \bigcup U_i$ is an open covering of an open subset $U \subset X$ and if $s, t \in F(U)$ are sections with the property $s|U_i = t|U_i$ for all i, then $s = t$.*

(G2) *When $U = \bigcup U_i$ is an open covering of an open subset $U \subset X$ and if $s_i \in F(U_i)$ is a family of sections with the property*

$$s_i|U_i \cap U_j = s_j|U_i \cap U_j \quad \text{for all } i, j,$$

then there exists a section $s \in F(U)$ with the property $s|U_i = s_i$ for all i.

(G3) *$F(\emptyset)$ is the zero group.*

The presheaf of continuous functions clearly is a sheaf, since continuity is a local property. An example of a presheaf F, which usually is not a sheaf, is the presheaf of constant functions with values in $\mathbb{Z}$ ($F(U) = \{f : U \to \mathbb{Z},\ f$ constant$\}$). But the set of *locally constant* functions with values in $\mathbb{Z}$ is a sheaf.

By a subsheaf of a sheaf F we understand a sub-presheaf $G \subset F$ which is already a sheaf. If F, G are sheaves, then a homomorphism $f : F \to G$ of presheaves is called also a homomorphism of sheaves.

4.2 Remark. *Let $F \subset G$ be a sub-presheaf. We assume that G (but not necessarily F) is a sheaf. Then there is a smallest subsheaf $\tilde{F} \subset G$ which contains F. For an arbitrary point $a \in X$ the induced map $f_a : F_a \to \tilde{F}_a$ is an isomorphism.*

Proof. It is clear that $\tilde{F}(U)$ has to be defined as set of all $s \in G(U)$ such that there exists an open covering $U = \bigcup U_i$, such that $s|U_i$ is in $F(U_i)$ for all i. This is equivalent with: the germ s_a is in F_a for all $a \in U$, i.e.

$$\tilde{F}(U) = \{s \in G(U);\ s_a \in F_a \text{ for } a \in U\}. \qquad \square$$

We mention the trivial fact that a section $s \in F(U)$ of a sheaf is zero if all its germs $s_a \in F_a$ are zero for $a \in U$.

4.3 Definition. *Let $F \to G$ be a homomorphism of sheaves. The sheaf-image $f_{\text{sheaf}}(F)$ is the smallest subsheaf of G, which contains the presheaf-image $f_{\text{pre}}(F)$.*

We mentioned above the formula $f_{\text{pre}}(F)_a = f_a(F_a)$. Applying Remark 4.2 gives the formula $f_{\text{sheaf}}(F)_a = f_a(F_a)$ for a homomorphism $F \to G$ of sheaves.

We have to differ between two natural notions of surjectivity.

4.4 Definition.

1) *A homomorphism of presheaves $f : F \to G$ is called **presheaf-surjective** if $f_{\mathrm{pre}}(F) = G$.*
2) *A homomorphism of sheaves $f : F \to G$ is called **sheaf-surjective** if $f_{\mathrm{sheaf}}(F) = G$.*

When F and G both are sheaves, then sheaf-surjectivity and presheaf-surjectivity are different things. We give an example which will be basic.

Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}$, hence $\mathcal{O}(U)$ is the set of all holomorphic functions on an open subset U. This is a sheaf of abelian groups (under addition). Similarly, we consider the sheaf $\mathcal{O}^*$ of holomorphic functions without zeros. This is also a sheaf of abelian groups (under multiplication). The map $f \to e^f$ defines a sheaf homomorphism

$$\exp : \mathcal{O} \longrightarrow \mathcal{O}^*.$$

The map $\mathcal{O}(U) \to \mathcal{O}^*(U)$ is not always surjective. For example for $U = \mathbb{C}^{\bullet}$ the function $1/z$ is not in the image. Hence exp is not presheaf-surjective. But it is know from complex calculus that $\exp : \mathcal{O}(U) \to \mathcal{O}^*(U)$ is surjective if U is simply connected, for example for a disk U. Since a point admits arbitrarily small neighborhoods which are disks, it follows that exp is sheaf-surjective.

4.5 Remark. *A homomorphism of sheaves $f : F \to G$ is sheaf-surjective if and only if the maps $f_a : F_a \to G_a$ are surjective for all $a \in X$.*

We omit the simple proof. □

Fortunately, the notion "injective" does not contain this difficulty. For trivial reason the following remark is true.

4.6 Remark. *Let $f : F \to G$ be a homomorphism of sheaves. The kernel in the sense of presheaves is already a sheaf.*

Hence we don't have to distinguish between presheaf-injective and sheaf-injective and also not between presheaf-kernel and sheaf-kernel.

4.7 Remark. *A homomorphism of sheaves $f : F \to G$ is injective if and only if the maps $f_a : F_a \to G_a$ are injective for all $a \in X$.*

A homomorphism of (pre)sheaves $f : F \to G$ is called an isomorphism if all $F(U) \to G(U)$ are isomorphisms. Their inverses then define a homomorphism $f^{-1} : G \to F$.

4.8 Remark. *A homomorphism of sheafs $F \to G$ is an isomorphism if and only if $F_a \to G_a$ is an isomorphism for all a.*

For presheaves this is false. As counter example one can take for F the presheaf of constant functions and for G the sheaf of locally constant functions.

It is natural to introduce the notion of *sheaf-exactness* as follows:

4.9 Definition. *A sequence $F \to G \to H$ of sheaf homomorphisms is* ***sheaf-exact*** *at G if the kernel of $G \to H$ and the sheaf-image of $F \to G$ agree.*

Generalizing the remarks 4.5 and 4.7 one can easily show the following proposition.

4.10 Proposition. *A sequence $F \to G \to H$ is exact if and only if $F_a \to G_a \to H_a$ is exact for all a.*

We indicate the proof. We make use of the mentioned formula $f_{\text{sheaf}}(F)_a = f_a(F_a)$. This shows that we can replace F by its sheaf image in G. Hence we can assume that F is a subsheaf of G and $F \to G$ is the natural injection. We have to show that the exactness of the sequences $F_a \to G_a \to H_a$ implies that F is the kernel of $G \to H$. It is clear that F is contained in the kernel. Hence it suffices to show the following. Let $s \in G(U)$ be an element of the kernel $G(U) \to H(U)$. Then we know that the germs s_a are in the kernel of $G_a \to H_a$. Hence they are contained in F_a. This means that there exists an open covering $U = \bigcup U_i$ such that $s|U_i \in F(U_i)$. The sheaf axiom G2 implies that they glue to an element of $F(U)$. The sheaf axiom G1 then shows that this element agrees with s. $\square$

Our discussion so far has obviously one gap. Let $F \subset G$ be a subsheaf of a sheaf G. We would like to have an exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

The sheaf H should be the factor sheaf of G by F. But up to now we only defined the factor presheaf $G/_{\text{pre}}F$ which usually is not a sheaf. In the next section we will give the correct definition for a factor sheaf $G/_{\text{sheaf}}F$.

5. The generated sheaf

For a presheaf F we introduced the monstrous presheaf

$$F^{(0)}(U) = \prod_{a \in U} F_a.$$

Obviously $F^{(0)}$ is a sheaf. Sometimes it is called the “Godement sheaf” or the “associated flabby sheaf”. There is a natural homomorphism

$$F \longrightarrow F^{(0)}.$$

We can consider its presheaf-image and then the smallest subsheaf which contains it. We denote this sheaf by $\hat{F}$ and call it the “generated sheaf” by F. There is a natural homomorphism

$$F \longrightarrow \hat{F}.$$

From the construction follows immediately

5.1 Remark. *Let F be a presheaf. The natural maps*

$$F_a \xrightarrow{\sim} \hat{F}_a$$

are isomorphisms.

A homomorphism $F \to G$ of presheaves induces a homomorphism $F^{(0)} \to G^{(0)}$. Clearly $\hat{F}$ is mapped into $\hat{G}$. This gives us the following result.

5.2 Remark. *Let $f : F \to G$ be a homomorphism of presheaves. There is a natural homomorphism $\hat{F} \to \hat{G}$, such that the diagram*

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ \hat{F} & \longrightarrow & \hat{G} \end{array}$$

commutes.

When F is already a sheaf, then $F \to F^{(0)}$ is injective. Then the map of F into the presheaf-image is an isomorphism. This implies that the presheaf-image is already a sheaf.

5.3 Remark. *Let F be a sheaf. Then $F \to \hat{F}$ is an isomorphism.*

If F is a sub-presheaf of a sheaf G, then the induced map $\hat{F} \to \hat{G} \cong G$ is an isomorphism $\hat{F} \to \tilde{F}$ between $\hat{F}$ and the smallest subsheaf $\tilde{F}$ of G, wich contains F.

Hence we can identify $\tilde{F}$ and $\hat{F}$.

Factor sheaves and exact sequences of sheaves

Let $F \to G$ be a homomorphism of presheaves. We introduced already the factor presheaf $G/_{\text{pre}}F$ which associates to an open U the factor group $G(U)/F(U)$. Even if both F and G are sheaves this will usually be not a sheaf. Hence we define the factor sheaf as the sheaf generated by the factor presheaf.

$$G/_{\text{sheaf}}F := \widehat{G/_{\text{pre}}}F.$$

Since we are interested mainly in sheaves, we will write usually for a homomorphism of sheaves $f : F \to G$:

$$\begin{aligned} G/F &:= G/_{\text{sheaf}}F && \text{(factor sheaf)} \\ f(F) &:= f_{\text{sheaf}}(F) && \text{(sheaf image)} \end{aligned}$$

Notice that there is no need to differ between sheaf- and presheaf-kernel. When we talk about an exact sequence of sheaves

$$F \longrightarrow G \longrightarrow H,$$

we usually mean "sheaf exactness". All what we have said about exactness properties of sequences of abelian groups is literally true for sequences of sheaves. For example: a sequence of sheaves $0 \to F \to G$ (0 denotes the zero sheaf) is exact if and only if $F \to G$ is injective. A sequence of sheaves $F \to G \to 0$ is exact if and only if $F \to G$ is surjective (in the sense of sheaves of course). A sequence of sheaves $0 \to F \to G \to H \to 0$ is exact if and only if there is an ismomorphism $H \cong G/F$ which identifies this sequence with

$$0 \longrightarrow F \longrightarrow G \longrightarrow G/F \longrightarrow 0.$$

5.4 Remark. *Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves. Then for open U the sequence*

$$0 \to F(U) \to G(U) \to H(U)$$

is exact.

Corollary. *The sequence*

$$0 \to F(X) \to G(X) \to H(X)$$

is exact.

The simple proof can be left to the reader. □

Usually $G(X) \longrightarrow H(X)$ is not surjective as the example

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O} \overset{f \mapsto e^{2\pi i f}}{\longrightarrow} \mathcal{O}^* \longrightarrow 0$$

shows. Here $\mathbb{Z}_X$ denotes the sheaf of locally constant functions with values in $\mathbb{Z}$. Cohomology theory will measure the absence of right exactness. The above sequence will be part of a long exact sequence

$$0 \longrightarrow F(X) \longrightarrow G(X) \longrightarrow H(X) \longrightarrow H^1(X, F) \longrightarrow \cdots$$

6. Some commutative algebra

In the following all rings are assumed to be commutative and with unit element. Modules M always are assumed to have the property $1 \cdot m = m$. Let M, N be two R-modules, we set

$$\mathrm{Hom}(V, W) = \mathrm{Hom}_R(V, W) = \{\, f : V \longrightarrow W; \quad f \ R\text{-linear} \,\}.$$

This is R-module too.

A special case is the dual module

$$M^* = \mathrm{Hom}_R(M, R).$$

The dual module is contravariant, i.e. a linear map $f : M \to N$ induces an obvious linear map $f^* : N^* \to M^*$. A module M is called free if it admits a basis. By definition, a basis here means subset $B \subset M$ that each element of $m \in M$ can be written in a unique way as linear combination

$$m = \sum_{b \in B} r_b b$$

where all but finitely many rb are different from zero. This means that M is isomorphic to a module M^I. It consists of all maps $I \to M$ which are zero outside a finite map. We are mainly interested in finitely generated free modules. They are isomorphic to R^n. If R is not the zero ring, the number n is uniquely determined and called the rank of R. (This is well known for fields and follows then in general. Use the existence of a maximal ideal $\mathfrak{m}$ and consider the field $R/\mathfrak{m}$.) Let $e_1, \dots, e_n$ be a basis of M then one obtains a basis $e_1^*, \dots, e_n^*$ of M^* by

$$e_i^*(e_j) := \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

Hence M and M^* have the same rank. Hence they are isomorphic, but the isomorphim is not canonical, it depends on the choice of bases.

Pairings

Let M, N be R-modules and let

$$M \times N \longrightarrow R, \quad (a, b) \longmapsto \langle a, b \rangle.$$

be a bilinear form (i.e. R-linear in both variables) There are induced two linear maps

$$M \longrightarrow N^*, \qquad N \longrightarrow M^*.$$

For example the first one attaches to an element $a \in M$ the linear form

$$l_a(x) := \langle a, x \rangle \quad (x \in N).$$

There are two important special cases:

In the case

$$M \times M^* \longrightarrow R, \quad \langle a, l \rangle := l(a)$$

one obtains a linear map $M^* \to M^*$ which turns out to be the identity and a linear map

$$M \longrightarrow M^{**}$$

which is more interesting. In the case that M is finitely generated free, one sees that this map is an isomorphism. (This depends heavily on our assumption that M is free and of finite rank.) We say that M and M^{**} are *canonically isomorphic*.

Another interesting case is $M = N$, i.e.

$$M \times M \longrightarrow K.$$

Hence we get two maps $M \to M^*$. We are mainly interested in the case that this pairing is symmetric, then both maps agree: Following properties are equivalent: If $e_1, \dots, e_n$ is a basis, then the so-called Gram-matrix

$$(\langle e_i, e_j \rangle)_{1 \le i,j \le n}$$

is invertible.

If this is the case, we call the pairing *non-degenerated*.

The tensor product

For two modules M, N over R we now want to study bilinear maps $M \times N \to P$ into arbitrary R-modules. There exists a distinguished one which we call the tensor product.

6.1 Definition. *Let M, N be R-modules. There exists a pair, consisting of an R-module $M \otimes_R N$ and a bilinear map $M \times N \to M \otimes_R N$ such that for each bilinear map $M \times N \to P$ into an arbitrary module P there exists a unique linear map $M \otimes_R P$ such that the diagram*

$$\begin{tikzcd}
M \times N \arrow[rr] \arrow[dr] & & M \otimes_R N \arrow[dl] \\
& P &
\end{tikzcd}$$

commutes.

We denote the defining bilinear map by

$$M \times_R N \longrightarrow M \otimes_R N, \quad (a, b) \longmapsto ab.$$

We just indicate the proof of the existence. When we have $M = R^I$ then we can set $M \otimes_R N = N^I$ with an obvious map $M \times N \to N^I$. In general we choose an exact sequence

$$R^J \longrightarrow R^I \longrightarrow M \longrightarrow 0.$$

and define $M \times_R \otimes N$ through the exact sequence

$$N^J \longrightarrow N^I \longrightarrow M \otimes_R N \longrightarrow 0.$$

The tensor product $M \otimes_R N$ is generated by the special elements $m \otimes n$.

If $f : M \to M'$ and $g : N \to N'$ are R-linear maps, then one gets a natural R-linear map

$$f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N', \quad a \otimes b \longmapsto f(a) \otimes g(a).$$

It is clear that this map is uniquely determined by this formula. The existence follows from the universal property applied to the map $(a, b) \mapsto f(a) \otimes g(b)$.

Basic properties of the tensor product

There is a natural isomorphism

$$R \otimes_R M \xrightarrow{\sim} M, \quad r \otimes m \longmapsto rm,$$

and more generally

$$R^n \otimes_R M \xrightarrow{\sim} M^n.$$

As a special case we get

$$R^n \otimes_R R^m \cong R^{n \times m}.$$

This is related also to the formula

$$(M \times N) \otimes_R P \cong (M \otimes_R P) \times (N \otimes_R P) \qquad \text{(canonically)}.$$

There is an obvious generalization. Let $M_1, \dots, M_n$ be a finite system of modules. Then there exists a module $M_1 \otimes_R \cdots \otimes_R M_n$ together with a multilinear map

$$M_1 \times \cdots \times M_n \longrightarrow M_1 \otimes \cdots \otimes M_n$$

in an obvious sense. By means of the universal property one proves easily the associativity of the tensor product: For $1 < a < n$ one has the following isomorphism

$$\begin{gathered}(M_1 \otimes_R \cdots \otimes_R M_a) \otimes (M_{a+1} \otimes \cdots \otimes M_n) \xrightarrow{\sim} M_1 \otimes \cdots \otimes M_n, \\ (n_1 \otimes \cdots \otimes m_a) \otimes (m_{a+1} \otimes \cdots \otimes m_n) \longmapsto m_1 \otimes \cdots \otimes m_n.\end{gathered}$$

and also the commutativity: Let σ be a permutation of the digits $1, \dots, n$. One has the isomorphism

$$\begin{gathered}M_1 \otimes_R \cdots \otimes_R M_n \xrightarrow{\sim} M_{\sigma(1)} \otimes_R \cdots \otimes_R M_{\sigma(n)}, \\ m_1 \otimes \cdots \otimes m_n \longmapsto m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}.\end{gathered}$$

Since the tensor product is associative, it is mostly enough to treat the case $n = 2$. There is also a commutativity rule for the tensor product.

For R-modules $M_1, \dots, M_n$ and an R-module N we denote by

$$\mathrm{Mult}_R(M_1, \dots, M_n, N)$$

the set of all multilinear maps from $M_1 \times \ldots \times M_n$ into N. There exists a natural R-linear map

$$M_1 \otimes_R \cdots \otimes_R M_n \longrightarrow \mathrm{Mult}_R(M_1^* \times \cdots \times M_n^*, R)$$

that sends $m_1 \otimes \cdots \otimes m_n$ to the multilinear form

$$(L_1, \dots, L_n) \longmapsto L_1(m_1) \cdots L_n(m_n).$$

Usually this is no isomorphism. But if the M_i are finitely generated free modules, for example finite dimensional vector spaces over a field, then this is an isomorphism.

6.2 Remark. *Let $M_1, \dots, M_d$ be finitely generated free R-modules. Then the natural map*

$$M_1 \otimes_R \cdots \otimes_R M_n \longrightarrow \mathrm{Mult}_R(M_1^* \times \cdots \times M_n^*, R)$$

is an isomorphism.

Hence for finite dimensional vector spaces one can take this space of multilinear forms as definition of the tensor product. But from standpoint of commutative algebra this is the wrong approach. An interesting case is also the tensor product of d copies of the same module M:

$$M^{\otimes d} := \overbrace{M \otimes_R \cdots \otimes_R M}^{d}.$$

We set additionally

$$M^{\otimes 0} = R, \quad M^{\otimes 1} = M.$$

We consider the submodule C of $M^{\otimes d}$ that is generated by all $m_1 \otimes \ldots \otimes m_d$ such that two of the m_i are equal. We can define

$$\bigwedge^d M := M^{\otimes d}/C.$$

There is a natural alternating multilinear map

$$M^d = M \times \cdots \times M \longrightarrow \bigwedge^d M$$

which we denote by $m_1 \wedge \ldots \wedge m_d$. It satisfies an obvious universal property for alternating multilinear forms of M^d into an arbitrary module P. Here a multilinear form $f : M^d \to P$ is called alternating if $f(m_1, \ldots, m_d) = 0$ if two m_i agree. Then

$$f(m_1, \ldots, m_d) = \operatorname{sgn}(\sigma) f(m_{\sigma(1)}, \ldots, m_{\sigma(d)})$$

for all permutations σ. The R-module of all alternating multilinear maps is denoted by

$$\operatorname{Alt}_R(M, \ldots, M, N).$$

We give an example for an alternating map. The map

$$M^* \times \ldots \times M^* \longrightarrow \operatorname{Alt}(M \times \ldots \times M, R)$$

that sends $(L_1, \ldots, L_d)$ to the multilinear form

$$(x_1, \ldots, x_d) \longmapsto \det((L_i(x_j))$$

is an alternating multilinear form.

6.3 Remark. *There is a natural R-linear map*

$$\bigwedge^d M^* \longrightarrow \operatorname{Alt}(M \times \ldots \times M, R)$$

that sends $L_1 \wedge \ldots \wedge L_d$ to the multilinear form

$$\det((L_i(x_j)).$$

There is a natural map

$$\operatorname{Mult}_R(M \times \ldots \times M, R) \longrightarrow \operatorname{Alt}_R(M \times \ldots \times M, R), \quad f \longmapsto f^{\text{alt}},$$

where

$$f^{\text{alt}}(m_1, \ldots, m_d) = \sum_\sigma \operatorname{sgn}(\sigma) f(m_{i_1}) \cdots f(m_{i_d}).$$

The diagram

$$\begin{array}{ccc} (M^*)^{\otimes d} & \longrightarrow & \operatorname{Mult}(M \times \cdots \times M, R) \\ \downarrow & & \downarrow \text{alt} \\ \bigwedge^d M^* & \longrightarrow & \operatorname{Alt}(M \times \cdots \times M, R) \end{array}$$

is commutative. Of course Alt can also be considered as submodule of Mult. But this fits not quite well in our point of view. The reason is that for an alternating form $f \in \operatorname{Alt}(M \times \cdots \times M, R)$ we have

$$f^{\text{alt}} = d! f.$$

6.4 Remark. *Let $M \cong R^n$ be a finitely generated free R-module and let R be of characteristic 0. Then the natural arrow*

$$\bigwedge^d M^* \longrightarrow \operatorname{Alt}(M \times \cdots \times M, R)$$

is an isomorphism.

7. Sheaves of rings and modules

Let A be a (commutative and unital) ring. A sheaf of A-modules is a sheaf F of abelian groups such that every $F(U)$ carries a structure as A-module and such the the restriction maps $F(U) \to F(V)$ for $V \subset U$ are A-linear. A homomorphism $F \to G$ is called A-linear if all $F(U) \to G(U)$ are so. Then kernel and image carry natural structures of sheafs of A-modules. Also the stalks carry such a structure naturally. Hence the whole canonical flabby resolution is a sequence of sheafs of A-modules. This implies that the cohomology groups also are A-modules.

There is a refinement of this construction: By a sheaf of rings $\mathcal{O}$ we understand a sheaf of abelian groups such that every $\mathcal{O}(U)$ is not only an abelian group but a ring and such that all restriction maps $\mathcal{O}(U) \to \mathcal{O}(V)$ are ring homomorphisms. Then the stalks $\mathcal{O}_a$ carry a natural ring structure such that the homomorphisms $\mathcal{O}(U) \longrightarrow \mathcal{O}_a$ (U is an open neighborhood of a) are ring homomorphisms.

By an $\mathcal{O}$-module we understand a sheaf $\mathcal{M}$ of abelian groups such that every $\mathcal{M}(U)$ carries a structure as $\mathcal{O}(U)$-module and such that the restriction maps are compatible with the module structure. To make this precise we give a short comment. Let M be an A-module and N be a module over a different ring B. Asssume that a homomorphism $r : A \to B$ is given. A homomorphism $f : M \to N$ of abelian groups is called compatible with the module structures if the formula

$$f(am) = r(a)f(m) \qquad (a \in A,\ m \in M)$$

holds. An elegant way to express this is as follows. We can consider N also as an module over A by means of the definition $an := r(a)n$. Sometimes this A-module is written as $N_{[r]}$. Then the compatibility of the map f simply means that it is an A-linear map

$$f : M \longrightarrow N_{[r]}.$$

Usually we will omit the subscript $[r]$ and simply say that $f : M \to N$ is A-linear.

If $\mathcal{M}$ is an $\mathcal{O}$-module, then the stalk $\mathcal{M}_a$ is naturally an $\mathcal{O}_a$-module. An $\mathcal{O}$-linear map $f : \mathcal{M} \to \mathcal{N}$ between two $\mathcal{O}$-modules is a homomorphism of sheaves of abelian groups such the maps $\mathcal{M}(U) \to \mathcal{N}(U)$ are $\mathcal{O}(U)$-linear. Then the kernel and image also carry natural structures of $\mathcal{O}$-modules. Clearly, the canonical flabby resolution of an $\mathcal{O}$-module is naturally a sequence of $\mathcal{O}$-modules.

Since for every open subset $U \subset X$ we have a ring homomorphism $\mathcal{O}(X) \to \mathcal{O}(U)$, all $\mathcal{M}(U)$ can be considered as $\mathcal{O}(X)$-modules. Hence an $\mathcal{O}$-module can be considered as sheaf of $\mathcal{O}(X)$-modules.

Let $\mathcal{M}$, $\mathcal{N}$ be two $\mathcal{O}_X$-modules. We denote by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ the set of all $\mathcal{O}_X$-linear maps $\mathcal{M} \to \mathcal{N}$. This is an $\mathcal{O}_X(X)$-module. The case $\mathcal{O}_X^n \to \mathcal{M}$

is of particular interest. The induced map $\mathcal{O}_X(X)^n \to \mathcal{M}(X)$ distinguishes n elements $s_1, \dots, s_n$ in $\mathcal{M}(X)$.

7.1 Lemma. *The natural map*

$$\mathrm{Hom}(\mathcal{O}_X^n, \mathcal{M}) \xrightarrow{\sim} \mathcal{M}(X)^n$$

is an isomorphism.

Proof. The map $\mathcal{O}_X(U)^n \longrightarrow \mathcal{M}(U)$ is necessarily of the form $(f_1, \dots, f_n) \mapsto \sum f_i s_i | U$. $\square$

More generally, we can consider for every open $U \subset X$

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|U}(\mathcal{M}|U, \mathcal{N}|U).$$

It is clear that this is presheaf. It is easy to check that it is actually a sheaf and moreover an $\mathcal{O}_X$-module. We denote it by

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}).$$

We denote by $\mathcal{O}_X(U)^{p\times q}$ the set of all $p \times q$-matrices with entries from $\mathcal{O}_X(U)$. This is a free $\mathcal{O}_X(U)$-module. From Lemma 5.7.1 we get an

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{O}_X^m) \xrightarrow{\sim} \mathcal{O}_X(X)^{m\times n}.$$

This extends to an isomorphism

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathcal{O}_X^q) \xrightarrow{\sim} \mathcal{O}_X^{q\times p}.$$

There is another construction which rests on the tensor product of modules. Let $\mathcal{M}$, $\mathcal{N}$ be two $\mathcal{O}_X$-modules. The assignment

$$U \longmapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$$

defines a presheaf. This is usually not a sheaf. Hence we consider the generated sheaf and denote it by $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$. Clearly this is an $\mathcal{O}_X$-module. The notion of an $\mathcal{O}_X$-bilinear map $\mathcal{M} \times \mathcal{N} \to \mathcal{P}$ for $\mathcal{O}_X$-modules $\mathcal{M}, \mathcal{N}, \mathcal{P}$ and the following universal property should be clear. For an $\mathcal{O}_X$-bilinear map $\mathcal{M} \times \mathcal{N} \to \mathcal{P}$ of $\mathcal{O}_X$-modules there exists a unique commutative diagram

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{N} \arrow[rr] \arrow[dr] & & \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \arrow[dl] \\
& \mathcal{P} &
\end{tikzcd}$$

with an $\mathcal{O}_X$-linear map $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{P}$.

One also has $\mathcal{O}_X^n \otimes_{\mathcal{O}_X} \mathcal{O}_X^m \cong \mathcal{O}_X^{n\times m}$. The construction of the tensor product is compatible with the restriction to open subsets,

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})|U \cong \mathcal{M}|U \otimes_{\mathcal{O}_X|U} \mathcal{N}|U.$$

This follows from the fact the construction of the generated sheaf $\hat{F}$ is compatible with restriction to open subsets.

Similarly to $\mathscr{H}om$ we can define $\mathscr{M}ult$ and $\mathscr{A}lt$.

7.2 Remark. *Let $\mathcal{M}_1, \dots, \mathcal{M}_n$ be $\mathcal{O}_X$-modules. There is a natural $\mathcal{O}_X$-linear map*

$$\mathcal{M}_1^* \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} \mathcal{M}_n^* \longrightarrow \mathscr{Mult}(\mathcal{M}_1 \times \dots \times \mathcal{M}_n, \mathcal{O}_X).$$

In the case $\mathcal{M}_1 = \dots = \mathcal{M}_n = \mathcal{M}$ there is a natural map

$$\bigwedge^n \mathcal{M} \longrightarrow \mathscr{Alt}(\mathcal{M}_1 \times \dots \times \mathcal{M}_n, \mathcal{O}_X).$$

8. Direct and inverse images

Let $f : X \to Y$ be a continuous map of topological spaces and let F be a presheaf of abelian groups on X. Then one can define for an arbitrary open subset $V \subset Y$

$$(f_*F)(V) := F(f^{-1}(V)).$$

With obvious restriction maps this defines a presheaf f_*F on X. It is called the *direct image*. If F is a sheaf, then f_*F is a sheaf too. A homomorphism of presheaves $F \to G$ induces a natural homomorphism $f_*F \to f_*G$. We also mention that there is a natural homomorphism

$$(f_*F)_{f(a)} \longrightarrow F_a \qquad (a \in X).$$

There is a similar, but not quite so easy construction, which associates to a presheaf G on Y a presheaf, actually a sheaf, $f^{-1}G$ on X. In the case of the identity map we will obtain the generated sheaf $\hat{F}$. If $i : U \hookrightarrow X$ is the canonical inclusion of an open subset and if G is a sheaf on X, then $i^{-1}G$ is naturally isomorphic to the restriction $G|U$. Actually, we will construct a subgroup

$$(f^{-1}G)(U) \subset \prod_{a \in U} G_{f(a)}.$$

By definition, a family $(t_a)_{a \in U}$, $t_a \in G_{f(a)}$, belongs to this subgroup if it is compatible in the following obvious sense. For each $a \in U$ there exists a small open neighborhood $f(a) \in V \subset Y$ and a section $t \in G(V)$ with the property $t_a = [V, t]_{f(a)}$ for all $a \in U$ such that $f(a) \in V$. It is easy to verify that this defines a sheaf $f^{-1}G$. A homomorphism of presheaves $G \to H$ induces a natural homomorphism $f^{-1}G \to f^{-1}H$. Notice that $\mathrm{id}^{-1}G$ equals the generated sheaf. For an open neighborhood $a \in U \subset X$ there is a natural projection homomorphism $(f^{-1}G)(U) \longrightarrow G_{f(a)}$. It induces a homomorphism $(f^{-1}G)_a \to G_{f(a)}$. This is actually an isomorphism.

8.1 Lemma. *Let $f : X \to Y$ be a continuous map and let G be a sheaf on Y. There is a natural isomorphism*

$$(f^{-1}G)_a \xrightarrow{\sim} G_{f(a)}.$$

Let $V \subset Y$ be an open subset. For each $a \in f^{-1}(V)$ we can consider the natural homomorphism $G(V) \to G_{f(a)}$ and collect them to

$$G(V) \longrightarrow \prod_{a \in f^{-1}(V)} G_{f(a)}.$$

The right hand side contains $f^{-1}(G)(f^{-1}(V)) = f_* f^{-1}(G)(V)$. It is easy to check that the image of $G(V)$ is contained in this subgroup. So we obtain a homomorphism $G(V) \to f_* f^{-1}(G)(V)$. It is easy to check that this gives a homomorphism of sheaves.

8.2 Lemma. *Let $f : X \to Y$ be a continuous map and let G be a sheaf on Y. There is a natural homomorphism of sheaves*

$$G \longrightarrow f_* f^{-1}(G).$$

Now we consider a sheaf F on X and we consider the sheaf $f^{-1} f_* F$. Let $U \subset X$ be an open subset. Then $(f^{-1} f_* F)(U)$ is contained in $\prod_{a \in U} (f_* F)_{f(a)}$ which can be identified with $\prod_{a \in U} F_a$. The module $F(U)$ is embedded in this product. It is easy to check that we obtain a homomorphism of $f^{-1} f_* F$ into F.

8.3 Lemma. *Let $f : X \to Y$ be a continuous map and let F be a sheaf on X. There is a natural homomorphism of sheaves*

$$f^{-1} f_* F \longrightarrow F.$$

Assume now that a sheaf F on X and a sheaf G on Y is given. Let $f^{-1}G \to F$ be a homomorphism. It induces a homomorphism $f_* f^{-1} G \to f_* F$, and, making use of Lemma 8.2, we get a homomorphism $G \to f_* F$. Conversely, let $G \to f_* F$ be a homomorphism. It induces $f^{-1}G \to f^{-1} f_* F$ and, by means of Lemma 8.3, we get a homomorphism $f^{-1} G \to F$. It is easy to check that the two construction are inverse. If we denote by $\mathrm{Hom}(F_1, F_2)$ the set of all homomorphisms of one sheaf into another (on the same space), then we can formalize this as follows.

8.4 Proposition. *Let $f : X \to Y$ be a continuous map, let F be a sheaf on X, and let G be a sheaf on Y. There is a natural bijection*

$$\mathrm{Hom}(f^{-1}G, F) \xrightarrow{\sim} \mathrm{Hom}(G, f_*F).$$

If one specializes this formula to $G = f_*F$, then one can consider the identity on the right hand side. The corresponding homomorphism on the left-hand side is that in Lemma 8.3. If one specializes it to $F = f^{-1}G$, the identity on the left hand side corresponds to Lemma 8.2.

Direct and inverse images of modules

We now assume that we have a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of geometric spaces. Recall that we have for each open subset $V \subset Y$ a natural homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$. This can be read as a homomorphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$. In fact, this is a homomorphism of sheaves of rings. (This means that the occurring homomorphisms are homomorphisms of rings and not only of abelian groups.) Using Proposition 8.4 we obtain a homomorphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. It is easy to verify that $f^{-1}\mathcal{O}_Y$ is a sheaf of rings and that this homomorphism is a homomorphism of sheaves of rings. In particular, $\mathcal{O}_X$ carries a natural structure as $f^{-1}\mathcal{O}_Y$-module. Assume that $\mathcal{M}$ is an $\mathcal{O}_X$-module. Then $f_*\mathcal{M}$ carries a natural structure as $f_*\mathcal{O}_X$-module. Using the homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$, we obtain a structure as $\mathcal{O}_Y$-module. We say simply that $f_*\mathcal{M}$ is an $\mathcal{O}_Y$-module if $\mathcal{M}$ is an $\mathcal{O}_X$-module.

The situation for the inverse image is slightly more complicated. Let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. It is no problem to equip $f^{-1}\mathcal{N}$ with a structure as $f^{-1}\mathcal{O}_Y$-module but there is no natural way to get a structure as $\mathcal{O}_X$-module. What we can do is to consider

$$f^*\mathcal{N} := f^{-1}\mathcal{N} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

This carries a natural structure as $\mathcal{O}_X$-module. An $\mathcal{O}_Y$-linear map $\mathcal{N}_1 \to \mathcal{N}_2$ induces an $\mathcal{O}_X$-linear map $f^*\mathcal{N}_1 \to f^*\mathcal{N}_2$. If we denote the set of all $\mathcal{O}_Y$-linear maps between $\mathcal{N}_1$ and $\mathcal{N}_2$ by $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{N}_1, \mathcal{N}_2)$, we obtain the following analogue of Proposition 8.4.

8.5 Proposition. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of geometric spaces, let $\mathcal{M}$ be an $\mathcal{O}_X$-module, and let be $\mathcal{N}$ be a $\mathcal{O}_Y$-module. There is a natural bijection*

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{N}, \mathcal{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{N}, f_*\mathcal{M}).$$

This includes natural homomorphisms

$$\mathcal{N} \longrightarrow f_* f^* \mathcal{N}, \quad f^* f_* \mathcal{M} \longrightarrow \mathcal{M}.$$

8.6 Proposition. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of geometric spaces. There is a natural isomorphism*

$$f^* \mathcal{O}_Y \cong \mathcal{O}_X.$$

Proof. By definition,

$$f^* \mathcal{O}_Y = f^{-1} \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X.$$

An easy consequence is $f^* \mathcal{O}_Y^n \cong \mathcal{O}_X^n$. The constructions f^{-1}, the tensor product and f^* are all compatible with restriction to open subsets, for example $(f^* \mathcal{N})|U \cong f^*(\mathcal{N}|U)$. This proves Proposition 8.6. □

9. Subspaces and sheaves

We are interested in the following situation. Let $(X, \mathcal{O}_X)$ be a geometric space (in our application a complex manifold) and let $Y \subset X$ a closed subset. We are interested in $\mathcal{O}_X$-modules $\mathcal{M}$ with the property $\mathcal{M}|(X - Y) = 0$. We want to show that such sheaves correspond to sheaves on Y. For this we want to define the restriction $\mathcal{N} := \mathcal{M}|Y$. (The following looks natural: consider the natural inclusion $j : Y \to X$ and define $\mathcal{N} = j^{-1}\mathcal{M}$. This is indeed possible, but we prefer another more direct way.) we have to consider an open subset $V \subset X$. This is the intersection $V = U \cap Y$. We want to define

$$\mathcal{N}(V) = \mathcal{M}(U).$$

The problem is of course the uniqueness of this definition. Let U' be another open subset of X with $V = U' \cap Y$. Using the fact that Y is closed in X and that $\mathcal{M}$ vanishes outside Y is it easy to see, that the restriction maps

$$\mathcal{M}(U), \mathcal{M}(U') \xrightarrow{\sim} \mathcal{M}(U \cap U')$$

are isomorphisms. This shows the claimed independence. To get a logical clean definition one should better define

$$\mathcal{N}(V) := \varinjlim \mathcal{M}(U).$$

Now we show that $\mathcal{N}$ and $j^{-1}\mathcal{M}$ can be identified. For this we need a map $\mathcal{N} \longrightarrow j^{-1}\mathcal{M}$ or, equivalently, $j * \mathcal{N} \to \mathcal{M}$. But this is clear.

The construction has a lack. At the moment we do not have a sheaf of rings on $\mathcal{O}_Y$, so $\mathcal{M}|Y$ is just a sheaf of abelian groups. We want to remedy this situation.

9.1 Definition. *Let $(X, \mathcal{O}_X)$ be a geometric space and let $Y \subset X$ be a closed subset. The vanishing ideal sheaf $\mathcal{J}_Y \subset \mathcal{O}_X$ consist of all functions in $\mathcal{O}_X(U)$ which vanish at $Y \cap U$.*

Clearly $\mathcal{J}_X$ is an ideal sheaf (i.e. a sub-module of $\mathcal{O}_X$). Moreover $\mathcal{O}_X/\mathcal{J}_Y$ is a sheaf of rings that vanishes on $X - Y$. We consider $\mathcal{O}_Y := (\mathcal{O}_X/\mathcal{J}_Y)|Y$. This is a sheaf of rings. The elements of $\mathcal{O}_Y$ can be considered as functions and Y. This defines a geometric structure.The natural inclusion $j : (Y, \mathcal{O}_Y) \to (X, \mathcal{X})$ is a morphism of geometric spaces. Notice that for a $\mathcal{O}_X$-module that vanishes outside $X - Y$ the constructions $j^{-1}\mathcal{M}$ and $j^*\mathcal{M}$ agree.

There is a converse construction Let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. Recall that $j_*\mathcal{N}$ is a sheaf on X, actually an $\mathcal{O}_X$-module defined through

$$(j_*\mathcal{N})(U) := \mathcal{N}(U \cap Y).$$

It vanishes outside Y, hence we should consider $j_*(\mathcal{N}|Y)$.

9.2 Proposition. *Let $(X, \mathcal{O}_X)$ be a geometric space and let $j : Y \hookrightarrow X$ be the natural inclusion of a closed subset. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module which vanishes outside Y. There is a natural isomorphism*

$$\mathcal{M} \xrightarrow{\sim} j_*(\mathcal{M}|Y).$$

Moreover there are natural isomorphisms of the stalks

$$\mathcal{M}_a \xrightarrow{\sim} (\mathcal{M}|Y)_a \qquad (a \in Y).$$

Let $\mathcal{N}$ by an $\mathcal{O}_Y$ module. There is a natural isomorphism

$$\mathcal{N} \xrightarrow{\sim} (j_*\mathcal{N})|Y.$$

We treat an example of complex analyis. Let $U \subset \mathbb{C}^n$ an open subset and let $\mathcal{O}_U$ be the sheaf of holomorphic functions. Let $f \in \mathcal{O}_U(U)$ be a holomorphic function on U and assume that it vanishes along the set of all $z \in U$, $z_1 = 0$. Then $f = z_1 g$ with $g \in \mathcal{O}_U(U)$.

9.3 Remark. *Let X be a complex manifold and Y a smooth subset of pure codimension one. The ideal sheaf $\mathcal{J}_Y$ of Y is a line bundle. Let $j : Y \hookrightarrow X$ be the natural injection. There is a natural exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathcal{J}_Y \longrightarrow \mathcal{O}_X \longrightarrow j_*\mathcal{O}_Y \longrightarrow 0.$$

There are various generalizations: Let M be a module over a ring R (commutative and with unit). Let $\mathfrak{a} \subset R$ be an ideal then $\mathfrak{a}M$ is defined as the set of all finite sums of elements of the form am, $a \in \mathfrak{a}$, $m \in M$. This is a submodule of M. Let more generally $\mathcal{O}$ be a sheaf of rings and $\mathcal{M}$ an $\mathcal{O}$-module and $\mathcal{J} \subset \mathcal{M}$ an ideal sheaf. Then $U \mapsto \mathcal{J}(U)\mathcal{M}(U)$ is a presheaf. The generated sheaf is denoted by $\mathcal{J}\mathcal{M}$. This is is a $\mathcal{O}$-submodule of $\mathcal{M}$. It is easy to prove that there is a natural isomorphism

$$\mathcal{J}_a\mathcal{M}_a \xrightarrow{\sim} (\mathcal{J}\mathcal{M})_a.$$

In other words, the sequence

$$0 \longrightarrow \mathcal{J}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/(\mathcal{J}\mathcal{M}) \longrightarrow 0$$

is exact.

10. Vector bundles

Let $(X, \mathcal{O}_X)$ be a ringed space. A vector bundle $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module. The rank of $\mathcal{M}$ is locally constant, hence constant if X is connected. We say that the rank of $\mathcal{M}$ is n if it is constant n. A vector bundle of rank 1 is called a line bundle.

Let $\mathcal{M}, \mathcal{N}$ be two vector bundles. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is a vector bundle. It is a line bundle if both are line bundles. The *dual bundle* of a vector bundle $\mathcal{M}$ is defined as

$$\mathcal{M}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X).$$

It has the same rank as $\mathcal{M}$.

Let $\mathcal{M}, \mathcal{N}$ be two vector bundles. Then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is a vector bundle too. This follows from the fact that the tensor product is compatible with restriction to open subsets and that

$$\mathcal{O}_X^p \otimes_{\mathcal{O}_X} \mathcal{O}_X^q \cong \mathcal{O}_X^{p \times q}.$$

Recall that we have a natural bilinear map $M \times M^* \longrightarrow R$. It induces a linear map $M \otimes_R M^* \to R$. Sheafifying we get an $\mathcal{O}_X$-bilinear map

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}^* \longrightarrow \mathcal{O}_X.$$

This is an isomorphism if $\mathcal{M}$ is a line bundle.

10.1 Remark. *Let $\mathcal{M}_1, \dots, \mathcal{M}_n$ be vector bundles. The natural $\mathcal{O}_X$-linear maps*

$$\mathcal{M}_1^* \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} \mathcal{M}_n^* \longrightarrow \mathscr{Mult}(\mathcal{M}_1 \times \dots \times \mathcal{M}_n, \mathcal{O}_X).$$

$$\bigwedge^n \mathcal{M} \longrightarrow \mathscr{Alt}(\mathcal{M}_1 \times \dots \times \mathcal{M}_n, \mathcal{O}_X)$$

are isomorphisms.

10.2 Proposition. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of geometric spaces. Let $\mathcal{N}$ be a vector bundle on $(Y, \mathcal{O}_Y)$ Then $f^*\mathcal{N}$ is a vector bundle too. The rank is preserved.*

10.3 Lemma. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of geometric spaces, let $\mathcal{N}$ be a vector bundle on Y, and let a be a point in X. There is a natural isomorphism*

$$(f^*\mathcal{N})_a \cong \mathcal{N}_{f(a)} \otimes_{\mathcal{O}_{Y,f(a)}} \mathcal{O}_{X,a}.$$

Proof. It is sufficient to prove this for $\mathcal{N} = \mathcal{O}_Y$. In this case the proof is trivial. □

The Picard group

Let $(X, \mathcal{O}_X)$ be a ringed space. One can show that there exists a *set* of line bundles such that every line bundle is isomorphic to a line bundle of this set. (Look at the transitions functions.) Therefore one can talk from the set $\mathrm{Pic}(X, \mathcal{O}_X)$ of all isomorphy classes of line bundles. There is a composition in this set induced by the tensor product. This composition is commutative and associative as from the corresponding properties of the tensor product follows. The trivial line bundle $\mathcal{O}_X$ defines a neutral element in $\mathrm{Pic}(X, \mathcal{O}_X)$. The inverse comes from the constriction $\mathcal{L}^*$. So we see that $\mathrm{Pic}(X, \mathcal{O}_X)$ has a structure as commutative group.

Chapter VI. Cohomology of sheaves

1. Some homological algebra

A *complex* $A^\bullet$ is a sequence of homomorphisms of abelian groups (parameterized by $\mathbb{Z}$)

$$\cdots \longrightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \longrightarrow \cdots$$

such that the composition of two consecutive is zero, $d_n \circ d_{n-1} = 0$. Usually one omits indices at the d-s and writes simply $d = d_n$ and hence $d \circ d = 0$, which sometimes is written as $d^2 = 0$. The cohomology groups of $A^\bullet$ are defined as

$$H^n(A^\bullet) := \frac{\operatorname{Kernel}(A^n \to A^{n+1})}{\operatorname{Image}(A^{n-1} \to A^n)} \qquad (n \in \mathbb{Z}).$$

They vanish if and only if the complex is exact. Hence the cohomology groups measure the absence of exactness of a complex.

A homomorphism $f^\bullet : A^\bullet \to B^\bullet$ of complexes is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, "f^{n-1}"] & A^n \arrow[r] \arrow[d, "f^n"] & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r] & B^n \arrow[r] & B^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

It is clear how to compose two complex homomorphisms $f^\bullet : A^\bullet \to B^\bullet$, $g^\bullet : B^\bullet \to C^\bullet$ to a complex homomorphism $g^\bullet \circ f^\bullet : A^\bullet \to C^\bullet$. A sequence of complex homomorphisms

$$\cdots \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow \cdots$$

is called exact if all the induced sequences

$$\cdots \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow \cdots$$

are exact. There is also the notion of a short exact sequence of complexes

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0.$$

Here 0 stands for the zero complex ($0^n = 0$, $d^n = 0$ for all n).

A homomorphism of complexes $A^\bullet \to B^\bullet$ induces natural homomorphisms

$$H^n(A^\bullet) \longrightarrow H^n(B^\bullet)$$

of the cohomology groups. These homomorphisms are compatible with the composition of complex-homomorphisms. A less obvious construction is as follows. Let

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

be a short exact sequence of complexes. We construct a homomorphism

$$\delta : H^n(C^\bullet) \longrightarrow H^{n+1}(A^\bullet).$$

Let $[c] \in H^n(C^\bullet)$ be represented by an element $c \in C^n$. Take a pre-image $b \in B^n$ and consider $\beta = db \in B^{n+1}$. Since β goes to $d(c) = 0$ in C^{n+1} there exists a pre-image $a \in A^{n+1}$. This goes to 0 in A^{n+2} (because A^{n+2} is imbedded in B^{n+2} and b goes to $d^2(b) = 0$ there). Hence a defines a cohomology class $[a] \in H^{n+1}(A^\bullet)$. It is easy to check that this class doesn't depend on the above choices.

1.1 Fundamental lemma of homological algebra. *Let*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

be a short exact sequence of complexes. Then the long sequence

$$\cdots \to H^{n-1}(C^\bullet) \xrightarrow{\delta} H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(C^\bullet) \to \cdots$$

is exact.

We leave the details to the reader. □

There is a second lemma of homological algebra which we will need.

1.2 Lemma. *Let*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A^{00} \arrow[r] \arrow[d] & A^{01} \arrow[r] \arrow[d] & A^{02} \arrow[r] \arrow[d] & A^{03} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & A^{10} \arrow[r] \arrow[d] & A^{11} \arrow[r] \arrow[d] & A^{12} \arrow[r] \arrow[d] & A^{13} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & A^{20} \arrow[r] \arrow[d] & A^{21} \arrow[r] \arrow[d] & A^{22} \arrow[r] \arrow[d] & A^{23} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & A^{30} \arrow[r] \arrow[d] & A^{31} \arrow[r] \arrow[d] & A^{32} \arrow[r] \arrow[d] & A^{33} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

be a commutative diagram where all lines and columns are exact besides the first column and the first row (those containing A^{00}). Then there is a natural isomorphism between the cohomology groups of the first row and the first column,

$$H^n(A^{\bullet,0}) \cong H^n(A^{0,\bullet})$$

For $n = 0$ this is understood as

$$\text{Kernel}(A^{00} \longrightarrow A^{01}) = \text{Kernel}(A^{00} \longrightarrow A^{10}).$$

The proof is given by “diagram chasing”. We only give a hint how it works. Assume $n = 1$. Let $[a] \in H^1(A^{0,\bullet})$ be a cohomology class represented by an element $a \in A^{0,1}$. This element goes to 0 in $A^{0,2}$. As a consequence the image of a in $A^{1,1}$ goes to 0 in $A^{1,2}$. Hence this image comes from an element $\alpha \in A^{1,0}$. Clearly, this element goes to zero in $A^{2,0}$ (since it goes to 0 in $A^{2,1}$.) Now α defines a cohomology class $[\alpha] \in H^1(A^{\bullet,0})$. There is some extra work to show that this map is well-defined. $\square$

2. The canonical flabby resolution

A sheaf F is called *flabby* if $F(X) \to F(U)$ is surjective for all open U. Then $F(U) \to F(V)$ is surjective for all $V \subset U$. An example for a flabby sheaf is the

Godement sheaf $F^{(0)}$. Recall that we have the exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)}.$$

We want to extend this sequence. For this we consider the sheaf $F^{(0)}/F$ and embed it into its Godement sheaf,

$$F^{(1)} := \left(F^{(0)}/F\right)^{(0)}.$$

In this way we get a long exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)} \longrightarrow F^{(1)} \longrightarrow F^{(2)} \longrightarrow \cdots$$

If $F^{(n)}$ has been already constructed, then we define

$$F^{(n+1)} := \big(F^{(n)}/F^{(n-1)}\big)^{(0)}.$$

The sheaves $F^{(n)}$ are all flabby. We call this sequence the *canonical flabby resolution* or the *Godement resolution*. Sometimes it is useful to write the resolution in the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & F^{(0)} \arrow[r] & F^{(1)} \arrow[r] & F^{(2)} \arrow[r] & F^{(3)} \arrow[r] & \cdots
\end{tikzcd}$$

Both lines are complexes. The vertical arrows can be considered as a complex homomorphism. The induced homomorphism of the cohomology groups are isomorphisms. Notice that only the 0-cohomology group of both complexes is different from 0. This zero cohomology group is naturally isomorphic F.

Now we apply the global section functor Γ to the resolution. This is

$$\Gamma F := F(X).$$

We obtain a long sequence

$$0 \longrightarrow \Gamma F \longrightarrow \Gamma F^{(0)} \longrightarrow \Gamma F^{(1)} \longrightarrow \Gamma F^{(2)} \longrightarrow \cdots$$

The essential point is that this sequence is no longer exact. We only can say that it is a complex. We prefer to write it in the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \Gamma F \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \Gamma F^{(0)} \arrow[r] & \Gamma F^{(1)} \arrow[r] & \Gamma F^{(2)} \arrow[r] & \Gamma F^{(3)} \arrow[r] & \cdots
\end{tikzcd}$$

The second line is

$$\begin{array}{c}\cdots \longrightarrow 0 \longrightarrow \Gamma F^{(0)} \longrightarrow \Gamma F^{(1)} \longrightarrow \Gamma F^{(2)} \longrightarrow \cdots \\ \uparrow \\ \text{zero position}\end{array}$$

Now we define the cohomology groups $H^\bullet(X, F)$ to be the cohomology groups of this complex:

$$H^n(X, F) := \frac{\operatorname{Kern}(\Gamma F^{(n)} \longrightarrow \Gamma F^{(n+1)})}{\operatorname{Image}(\Gamma F^{(n-1)} \longrightarrow \Gamma F^{(n)})}.$$

(We define $\Gamma F^{(n)} = 0$ for $n < 0$.) Clearly

$$H^n(X, F) = 0 \qquad \text{for} \quad n < 0.$$

Next we treat the special case $n = 0$,

$$H^0(X, F) = \operatorname{Kernel}(\Gamma F^{(0)} \longrightarrow \Gamma F^{(1)}).$$

Since the kernel can be taken in the presheaf sense, we can write

$$H^0(X, F) = \Gamma \operatorname{Kernel}(F^{(0)} \longrightarrow F^{(1)}).$$

Recall that $F^{(1)}$ is a sheaf which contains $F^{(0)}/F$ as subsheaf. We obtain

$$H^0(X, F) = \Gamma \operatorname{Kernel}(F^{(0)} \longrightarrow F^{(0)}/F)$$

This is the image of F in $F^{(0)}$ and hence a sheaf which is canonically isomorphic to F.

2.1 Remark. *There is a natural isomorphism*

$$H^0(X, F) \cong \Gamma F = F(X).$$

If $F \to G$ is a homomorphism of sheaves, then the homomorphism $F_a \to G_a$ induces a homomorphism $F^{(0)} \to G^{(0)}$. More generally, one has an obvious natural homomorphism $F^{(n)} \to G^{(n)}$ for all n. This gives a homomorphism of the Godement resolution. Hence we obtain a natural homomorphism

$$H^n(X, F) \longrightarrow H^n(X, G).$$

If $F \to G \to H$ is an exact sequence, then $F^{(0)} \to G^{(0)} \to H^{(0)}$ is also exact (already as sequence of presheaves). More generally, the following lemma holds.

2.2 Lemma. *Let $F \to G \to H$ be an exact sequence of sheaves. Then the induced sequence $F^{(n)} \to G^{(n)} \to H^{(n)}$ is exact for every n.*

Proof. By a general principle it is sufficient to prove that $F \mapsto F^{(n)}$ maps short exact sequences $0 \to F \to G \to H \to 0$ into short exact sequences $0 \to F^{(n)} \to G^{(n)} \to H^{(n)} \to 0$. The reason is that an arbitray exact sequence $F \xrightarrow{f} G \xrightarrow{g} H$ can be splitted into short exact sequences

$$\begin{aligned}
&0 \longrightarrow \mathrm{Kernel}(f) \longrightarrow F \longrightarrow f(F) \longrightarrow 0,\\
&0 \longrightarrow f(F) \longrightarrow G \longrightarrow g(G) \longrightarrow 0,\\
&0 \longrightarrow g(G) \longrightarrow H \longrightarrow H/g(G) \longrightarrow 0.
\end{aligned}$$

So we start with a short exact sequence $0 \to F \to G \to H \to 0$. The proof can now be given by induction. One needs the following lemma about abelian groups:

Let

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{A}^{00} \arrow[r] \arrow[d] & \mathcal{A}^{01} \arrow[r] \arrow[d] & \mathcal{A}^{02} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A}^{10} \arrow[r] \arrow[d] & \mathcal{A}^{11} \arrow[r] \arrow[d] & \mathcal{A}^{12} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A}^{20} \arrow[r] \arrow[d] & \mathcal{A}^{21} \arrow[r] \arrow[d] & \mathcal{A}^{22} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

be a commutative diagram such that the three columns and the first two lines are exact. Then the third line is also exact.

The proof is easy and can be omitted. □

Before we continue, we need a basic lemma:

2.3 Lemma. *Let $0 \to F \to G \to H \to 0$ be a short exact sequence of sheaves. Assume that F is flabby. Then*

$$0 \longrightarrow \Gamma F \longrightarrow \Gamma G \longrightarrow \Gamma H \longrightarrow 0$$

is exact.

Proof. Let $h \in H(X)$. We have to show that h is the image of an $g \in G(X)$. For the proof one considers the set of all pairs (U, g), where U is an open subset and $g \in G(U)$ and such that g maps to $h|U$. This set is ordered by

$$(U, g) \geq (U', g') \Longleftrightarrow U' \subset U \text{ and } g|U' = g'.$$

From the sheaf axioms follows that every inductive subset has an upper bound. (Take the union of all open subsets which occur in the inductive set.) By Zorns's lemma there exists a maximal (U, g). We have to show $U = X$. If this is not the case, we can find a pair (U', g') in the above set such that U' is not contained in U. The difference $g - g'$ defines a section in $F(U \cap U')$. Since F is flabby, this extends to a global section. This allows us to modify g' such that it glues with g to a section on $U \cup U'$. $\square$

An immediate corollary of Lemma 2.3 states:

2.4 Lemma. *Let $0 \to F \to G \to H \to 0$ an exact sequence of sheaves. If F and G are flabby then H is flabby too.*

Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheafs. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n-1)} & \longrightarrow & G^{(n-1)} & \longrightarrow & H^{(n-1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n)} & \longrightarrow & G^{(n)} & \longrightarrow & H^{(n)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n+1)} & \longrightarrow & G^{(n+1)} & \longrightarrow & H^{(n+1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

From Lemma 2.2 we know that all lines of this diagram are exact. From Lemma 2.3 follows that they remain exact after applying Γ. Hence the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n-1)} & \longrightarrow & \Gamma G^{(n-1)} & \longrightarrow & \Gamma H^{(n-1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n)} & \longrightarrow & \Gamma G^{(n)} & \longrightarrow & \Gamma H^{(n)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n+1)} & \longrightarrow & \Gamma G^{(n+1)} & \longrightarrow & \Gamma H^{(n+1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

can be considered as a short exact sequence of complexes. We can apply Lemma 1.1 to obtain the long exact cohomology sequence:

2.5 Theorem. *Every short exact sequence $0 \to F \to G \to H \to 0$ induces a natural long exact cohomology sequence*

$$\begin{aligned}0 \to \Gamma F \longrightarrow \Gamma G \longrightarrow \Gamma H \xrightarrow{\delta} H^1(X,F) \longrightarrow H^1(X,G) \longrightarrow H^1(X,H)\\ \xrightarrow{\delta} H^2(X,F) \longrightarrow \cdots\end{aligned}$$

The next Lemma shows that the cohomology of flabby sheaves is trivial.

2.6 Lemma. *Let*

$$0 \to F \longrightarrow F_0 \to F_1 \longrightarrow \cdots$$

be an exact sequence of flabby sheaves (finite or infinite). Then

$$0 \to \Gamma F \longrightarrow \Gamma F_0 \to \Gamma F_1 \longrightarrow \cdots$$

is exact.

Corollary. *For flabby F one has:*

$$H^i(X,F) = 0 \quad \textit{for} \quad i > 0.$$

Proof. We use the splitting principle. The long exact sequence can be splitted into short exact sequences

$$0 \longrightarrow F \longrightarrow F_0 \longrightarrow F_0/F \longrightarrow 0, \quad 0 \longrightarrow F_0/F \longrightarrow F_1 \longrightarrow F_1/F_0 \longrightarrow 0, \ldots.$$

From Lemma 2.4 we get that $F_0/F, F_1/F_0, \ldots$ are flabby. The claim now follows from Lemma 2.3. $\square$

A sheaf F is called *acyclic* if $H^n(X,F) = 0$ for $n > 0$. Hence flabby sheaves are acyclic. By an *acyclic* resolution of a sheaf we understand an exact sequence

$$0 \longrightarrow F \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

with acyclic F_i.

2.7 Proposition. *Let $0 \to F \to F_0 \to F_1 \to \cdots$ be an acyclic resolution of F. Then there is a natural isomorphism between the n-the cohomology group $H^n(X,F)$ and the n-th cohomology group of the complex*

$$\begin{array}{c}\cdots \longrightarrow 0 \longrightarrow \Gamma F_0 \longrightarrow \Gamma F_1 \longrightarrow \Gamma F_2 \longrightarrow \cdots\\ \uparrow \\ \textit{zero position}\end{array}$$

Proof. Taking the canonical flabby resolutions of F and of all F_n on gets a diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & F_0 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(0)} & \longrightarrow & F_0^{(0)} & \longrightarrow & F_1^{(0)} & \longrightarrow & F_2^{(0)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(1)} & \longrightarrow & F_0^{(1)} & \longrightarrow & F_1^{(1)} & \longrightarrow & F_2^{(1)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(2)} & \longrightarrow & F_0^{(2)} & \longrightarrow & F_1^{(2)} & \longrightarrow & F_2^{(2)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

All lines and columns are exact. We apply Γ to this complex. Then all lines and columns besides the first ones remain exact. We can apply Lemma 1.2.

□

One may ask what "natural" means in Proposition 2.7. It means that certain diagrams in which this ismomorphism appears are commutative. Since it is the best to check this when it is used we give just one example: Let $0 \to F \to F_0 \to \dots$ be an acyclic resolution. Then we get an acyclic resolution $0 \to F_0/F \to F_0 \to \dots$. This gives us an isomorphism $H^p(X, F_0/F) \to H^{p+1}(X, F)$ for $p > 0$ which we call the natural isomorphism. The exact sequence $0 \to F \to F_0 \to F_0/F \to 0$ gives another (the combining) isomorphism $H^p(X, F_0/F) \to H^{p+1}(X, F)$ which can be called natural with the same right.

2.8 Remark. *Let $0 \to F \to F_0 \to F_1 \to \dots$ be an acyclic resolution. The two natural isomorphism $H^p(X, F_0/F) \to H^{p+1}(X, F)$ agree.*

3. Paracompact spaces

We consider a very special case. We take for $\mathcal{O}$ the sheaf $\mathcal{C}$ of continuous functions. There are two possibilities: $\mathcal{C}_{\mathbb{R}}$ is the sheaf of continuous real-valued and $\mathcal{C}_{\mathbb{C}}$ the sheaf of continuous complex-valued functions. If we write $\mathcal{C}$ we mean one of both. The sheaf $\mathcal{C}$ or more generally a module over this sheaf have over paracompact spaces a property which can be considered as a weakend form of flabbyness.

3.1 Remark. *Let X be a paracompact space and let $\mathcal{M}$ be a $\mathcal{C}$-module on X. Assume that U is an open subset and that $V \subset U$ is an open subset whose closure is contained in U. Assume that $s \in \mathcal{M}(U)$ is a section over U. Then there is a global section $S \in \mathcal{M}(X)$ such that $S|V = s|V$.*

Proof. We choose a continuous real valued function φ on X, which is one on V and whose support is contained in U. Then we consider the open covering $X = U \cup U'$, where U' denotes the complement of the support of φ. On U we consider the section φs and on U' the zero section. Since both are zero on $U \cap U'$ they glue to a section S on X. □

3.2 Lemma. *Let X be a paracompact space and let $\mathcal{M} \to \mathcal{N}$ be a surjective $\mathcal{C}$-linear map of $\mathcal{C}$-modules. Then $\mathcal{M}(X) \to \mathcal{N}(X)$ is surjective.*

Proof. Let $s \in \mathcal{N}(X)$. There exists an open covering $(U_i)_{i \in I}$ of X such that $s|U_i$ is the image of a section $t_i \in \mathcal{M}(U_i)$. We can assume that the covering is locally finite. We take open subsets $V_i \subset U_i$ whose closure is contained in U_i and such that (V_i) is still a covering. Then we choose a partition of unity (φ_i) with respect to (V_i). By Remark 3.1 there exists global sections $T_i \in \mathcal{M}(X)$ with the property $T_i|V_i = t_i|V_i$. We now consider

$$T := \sum_{i \in I} \varphi_i T_i.$$

Since I can be infinite, we have to explain what this means. Let $a \in X$ a point. There exists an open neighborhood $U(a)$ such $V_i \cap U(a) \neq \emptyset$ only for a finite subset $J \subset I$. We can define the section

$$T(a) := \sum_{i \in J} \varphi_i T_i | U(a).$$

The sets $U(a)$ cover X and the sections $T(a)$ glue to a section T. Clearly T maps to s. □

3.3 Lemma. *Let X be a paracompact space and let $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ be an exact sequence of $\mathcal{C}$-modules. Then $\mathcal{M}(X) \to \mathcal{N}(X) \to \mathcal{P}(X)$ is exact too.*

Proof. As mentioned during the proof of Lemma 2.2, it is sufficient to show that a short exact sequence is led to a short exact sequence. The only problem is the surjectivity at the end of the sequence. But this follows from Lemma 3.2. □

Let $\mathcal{M}$ be a $\mathcal{C}$-module over a paracompact space. Then the canonical flabby resolution is also a sequence of $\mathcal{C}$-modules. From 3.3 follows that the resolution remains exact after the application of Γ. We obtain the following proposition.

3.4 Proposition. *Let X be paracompact. Every $\mathcal{C}$-module $\mathcal{M}$ is acyclic, i.e. $H^n(X,\mathcal{M}) = 0$ for $n > 0$.*

4. Čech Cohomology

Here we will consider only the first Čech cohomology group of a sheaf. We have to work with open coverings $\mathfrak{U} = (U_i)_{i\in I}$ of the given topological space X. Let F be a sheaf on X. A *one-cocycle* of F with respect to the covering $\mathfrak{U}$ is a family of sections

$$s_{ij} \in F(U_i \cap U_j), \quad (i,j) \in I \times I,$$

with the following property: for each triple i,j,k of indices one has

$$s_{ik} = s_{ij} + s_{jk} \quad \text{on} \quad U_i \cap U_j \cap U_k.$$

In more precise writing this means

$$s_{ik}|(U_i \cap U_j \cap U_k) = s_{ij}|(U_i \cap U_j \cap U_k) + s_{jk}|(U_i \cap U_j \cap U_k).$$

We denote by $C^1(\mathfrak{U}, F)$ the group of all one-cocycles. Assume that a family of sections $s_i \in F(U_i)$ is given. Then

$$s_{ij} = s_i|(U_i \cap U_j) - s_j|(U_i \cap U_j)$$

obviously is a one-cocycle. We denote it by

$$\delta(s_i)_{i\in I}.$$

A one-cocycle of this form is called a *one-coboundary*. The set of all one-coboundaries is a subgroup

$$B^1(\mathfrak{U}, F) \subset C^1(\mathfrak{U}, F).$$

The (first) Čech cohomology of F with respect to the covering $\mathfrak{U}$ is defined as

$$\check{\mathrm{H}}^1(\mathfrak{U}, F) := C^1(\mathfrak{U}, F)/B^1(\mathfrak{U}, F).$$

A homomorphism of sheaves $F \to G$ induces a homomorphism

$$\check{\mathrm{H}}^1(\mathfrak{U}, F) \longrightarrow \check{\mathrm{H}}^1(\mathfrak{U}, G).$$

Let $f : G \to H$ be a surjective homomorphism of sheaves and $\mathfrak{U} = (U_i)$ an open covering of X. We denote by

$$H_{\mathfrak{U}}(X) = H_{\mathfrak{U},f}(X)$$

the set of all global sections of H with the following property.

For every index i there is a section $t_i \in G(U_i)$ such that $f(t_i) = s|U_i$.

By definition of (sheaf-)surjectivity, for every global section $s \in H(X)$, there exists an open covering $\mathfrak{U}$ with $s \in H_{\mathfrak{U}}(X)$. It follows

$$H(X) = \bigcup_{\mathfrak{U}} H_{\mathfrak{U}}(X).$$

Let $0 \to F \to G \to H \to 0$ be an exact sequence and let $\mathfrak{U}$ be an open covering. There exists a natural homomorphism

$$\delta : H_{\mathfrak{U}}(X) \longrightarrow \check{H}^1(\mathfrak{U}, F),$$

which is constructed as follows. Let be $s \in H_{\mathfrak{U}}(X)$. We choose elements $t_i \in G(U_i)$ which are mapped to $s|U_i$. The differences $t_i - t_j$ come from sections $t_{ij} \in F(U_i \cap U_j)$. They define a one-cocycle $\delta(s)$. It is easy to check that the corresponding element of $\check{H}^1(\mathfrak{U}, F)$ does not depend on the choice of the t_i.

4.1 Lemma. *Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves and let $\mathfrak{U}$ be an open covering. The sequence*

$$0 \to F(X) \longrightarrow G(X) \longrightarrow H_{\mathfrak{U}}(X) \xrightarrow{\delta} \check{H}^1(\mathfrak{U}, F) \longrightarrow \check{H}^1(\mathfrak{U}, G) \longrightarrow \check{H}^1(\mathfrak{U}, H)$$

is exact.

Remark. *This sequence does not extend naturally to a long exact sequence.*

The proof is easy, since all maps are given by explicit formulae. □

This Lemma indicates that Čech cohomology must be related to usual cohomology. Another result in this direction gives the following remark.

4.2 Remark. *Let F be a flabby sheaf. Then for every open covering*

$$\check{H}^1(\mathfrak{U}, F) = 0.$$

Proof. We start with a little remark. Assume that the whole space $X = U_{i_0}$ is a member of the covering. Then the Čech cohomology vanishes (for every sheaf): if (s_{ij}) is a one-cocycle, one defines $s_i = s_{i,i_0}$. Then $\delta((s_i)) = (s_{ij})$. For the proof of Remark 4.2 we now consider the sequence

$$\begin{array}{ccccccccc}
0 \longrightarrow F(X) & \longrightarrow & \prod_i F(U_i) & \longrightarrow & \prod_{ij} F(U_i \cap U_j) & \longrightarrow & \prod_{ijk} F(U_i \cap U_j \cap U_k) \\
s & \longmapsto & (s|U_i) & & & & \\
 & & (s_i) & \longmapsto & (s_i - s_j) & & \\
 & & & & (s_{ij}) & \longmapsto & (s_{ij} + s_{jk} - s_{ik}).
\end{array}$$

We will prove that this sequence is exact. (Then Remark 4.2 follows.) The idea is to sheafify this sequence: For an open subset $U \subset X$ one considers $F|U$ and also the restricted covering $U \cap U_i$. Repeating the above construction for U instead of X one obtains a sequence of sheaves

$$0 \longrightarrow F \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}.$$

Since F is flabby, also $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are flabby. The remark at the beginning of the proof shows that $0 \longrightarrow F(U) \longrightarrow \mathcal{A}(U) \longrightarrow \mathcal{B}(U) \longrightarrow \mathcal{C}(U)$ is exact, when U is contained in some U_i. Hence the sequence is sheaf-exact. From Lemma 2.6 follows that the exactness is also true for $U = X$. $\square$

Let now F be an arbitrary sheaf, $F^{(0)}$ the associated flabby sheaf. We get an exact sequence $0 \to F \to F^{(0)} \to H \to 0$. Let $\mathfrak{U}$ be an open covering. We know that $\check{H}^1(\mathfrak{U}, F^{(0)})$ vanishes (Remark 4.2). From Lemma 4.1 we obtain an isomorphy

$$\check{H}^1(\mathfrak{U}, F) \cong H_{\mathfrak{U}}(X)/F^{(0)}(X).$$

From the long exact cohomology sequence we get for the usual cohomology

$$H^1(X, F) \cong H(X)/F^{(0)}(X).$$

This gives an *injective* homomorphism

$$\check{H}^1(\mathfrak{U}, F) \longrightarrow H^1(X, F).$$

We obtain the following result.

4.3 Proposition. *Let F be a sheaf. Then*

$$H^1(X, F) = \bigcup_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, F).$$

The following commutative diagram shows that the homomorphism δ from Lemma 4.1 and that of general sheaf theory Theorem 2.5 coincide:

4.4 Remark. *For a short exact sequence* $0 \to F \to G \to H \to 0$ *the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H_{\mathfrak{U}}(X) & \stackrel{\delta}{\longrightarrow} & \check{H}^1(\mathfrak{U}, F) \\
 & & \| & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H(X) & \stackrel{\delta}{\longrightarrow} & H^1(X, F)
\end{array}$$

is commutative.

Proof. The Godement resolutions of F, G, H give a short exact sequence of complexes. The groups in the Remark can be expressed explicitly inside this sequence. So the proof can be given by a straight forward computation which can be left to the reader. □

Let $\mathfrak{V} = (V_j)_{j \in J}$ be a refinement of $\mathfrak{U} = (U_i)_{i \in I}$ and $\varphi : J \to I$ a refinement map ($V_\varphi \subset U_i$). Using this refinement map one obtains a natural map

$$\check{H}^1(\mathfrak{U}, F) \longrightarrow \check{H}^1(\mathfrak{V}, F).$$

This shows the following result.

4.5 Remark. *Let* $\mathfrak{V}$ *be an refinement of* $\mathfrak{U}$ *and* $\varphi : J \to I$ *a refinement map. The diagram*

$$\begin{tikzcd}
\check{H}^1(\mathfrak{U}, F) \arrow[rr] \arrow[dr] & & \check{H}^1(\mathfrak{V}, F) \arrow[dl] \\
 & H^1(X, F) &
\end{tikzcd}$$

commutes. As a consequence, it doesn't depend on the choice of the refinement map.

Usually it is of course very difficult to control all open coverings of a topological space. (There is also the logical difficulty that the set of all coverings does not exist. It is easy to overcome this difficulty, we omit it.) But sometimes a single covering is sufficient:

4.6 Theorem of Leray. *Let* F *be a sheaf on* X *and* $\mathfrak{U} = (U_i)$ *an open covering of* X*. Assume that* $H^1(U_i, F|U_i) = 0$ *for all* i*. Then*

$$H^1(X, F) = \check{H}^1(\mathfrak{U}, F).$$

Proof. Since two coverings admit a joint refinement, it is sufficient to prove that $\check{H}^1(\mathfrak{U}, F) \to \check{H}^1(\mathfrak{V}, F)$ is an isomorphism for each refinement $\mathfrak{V}$ of $\mathfrak{U}$. Since the map is injective (use Proposition 4.3), it remains to prove surjectivity. We choose a refinement map $\varphi : J \to I$. We denote the indices in I by $i, j, \dots$ and those of J by $\alpha, \beta, \dots$. Let be $(s_{\alpha,\beta}) \in C^1(\mathfrak{V}, F)$. We consider the covering $U_i \cap \mathfrak{V} := (U_i \cap V_\alpha)_\alpha$ of U_i. From the assumption $\check{H}^1(U_i \cap \mathfrak{V}, F|U_i) = 0$ we get the existence of $t_{i\alpha} \in F(U_i \cap V_\alpha)$ such that

$$s_{\alpha\beta} = t_{i\alpha} - t_{i\beta} \quad \text{on} \quad U_i \cap V_\alpha \cap V_\beta.$$

From this equation follows that

$$t_{i\alpha} - t_{j\alpha} = t_{i\beta} - t_{j\beta} \quad \text{on} \quad U_i \cap U_j \cap V_\alpha \cap V_\beta.$$

Hence these differences glue to a section $T_{ij} \in F(U_i \cap U_j)$,

$$T_{ij} = t_{i\alpha} - t_{j\alpha} \quad \text{on} \quad U_i \cap U_j \cap V_\alpha.$$

Clearly (T_{ij}) is a one-cocycle in $C^1(\mathfrak{U}, F)$. We consider its image $(T_{(\varphi\alpha,\varphi\beta})$ in $C^1(\mathfrak{V}, F)$. It is easy to check that this one-cocycle, and the one we started with $(s_{\alpha\beta})$, defines the same cohomology class: they differ by the one-coboundary $(h_\beta - h_\alpha)$ with $h_\alpha = t_{\varphi\alpha,\alpha} \in F(V_\alpha)$. □

5. Some vanishing results

Let X be a topological space and A an abelian group. We denote by A_X the sheaf of locally constant functions with values in A. This sheaf can be identified with the sheaf that is generated by the presheaf of constant functions. We will write

$$H^n(X, A) := H^n(X, A_X).$$

We mention that these groups under reasonable assumptions (for example for paracompact manifolds) agree with the singular cohomology in the sense of algebraic topology.

5.1 Proposition. *Let U be an open and convex subset of $\mathbb{R}^n$. Then for every abelian group A*

$$H^1(U, A) = 0.$$

This is actually true for all H^n, $n > 0$. The best way to prove this is to use the comparison theorem with singular cohomology as defined in algebraic topology. But we do not want to use this. Therefore we restrict to H^1, where we can give a simple argument.

Proof of 5.1. Every convex open subset of $\mathbb{R}^n$ is topologically equivalent to $\mathbb{R}^n$. Hence it is sufficient to restrict to $U = \mathbb{R}^n$. Just for simplicity we assume $n = 1$. (The general case then should be clear.) We use Čech cohomology. Because of Proposition 4.3 and Remark 4.5 it is sufficient to show that every open covering admits a refinement $\mathfrak{U}$ such that $\check{H}^1(\mathfrak{U}, A_X) = 0$. To show this we take a refinement of a very simple nature. It is easy to show that there exists a refinement of the following form. The index set is $\mathbb{Z}$. There exists a sequence of real numbers (a_n) with the following properties:

a) $a_n \le a_{n+1}$
b) $a_n \to +\infty$ for $n \to \infty$ and $a_n \to -\infty$ for $n \to -\infty$
c) $U_n = (a_n, a_{n+2})$.

Assume that $s_{n,m}$ is a one-cocycle with respect to this covering. Notice that U_n has non empty intersection only with U_{n-1} and U_{n+1}. Hence only $s_{n-1,n}$ is of relevance. This is a locally constant function on $U_{n-1} \cap U_n = (a_n, a_{n+1})$. Since this is connected, the function $s_{n-1,n}$ is constant. We want to show that it is a one-coboundary, i.e. we want to construct constant functions s_n on U_n such that $s_{n-1,n} = s_n - s_{n-1}$ on (a_n, a_{n+1}). This is easy. One starts with $s_0 = 0$ and then constructs inductively $s_1, s_2, \dots$ and in the same way for negative n. □

Consider on the real line $\mathbb{R}$ the sheaf of complex valued differentiable functions $\mathcal{C}^\infty$. Taking derivatives one gets a sheaf homomorphism $\mathcal{C}^\infty \to \mathcal{C}^\infty$, $f \mapsto f'$. The kernel is the sheaf of all locally constant functions, which we denote simply by $\mathbb{C}$. Hence we get an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty \longrightarrow 0.$$

This sequence is exact since every differentiable function has an integral. Hence this sequence can be considered as an acyclic resolution of $\mathbb{C}$. We obtain $H^q(\mathbb{R}, \mathbb{C}) = 0$ for all $q > 0$. For $q = 1$ this follows already from Proposition 5.1. There is a generalization to higher dimensions. For example, a standard result of vector analysis states in the case $n = 2$.

5.2 Lemma. *Let $E \subset \mathbb{R}^2$ be an open and convex subset and let $f, g \in \mathcal{C}^\infty(E)$ be a pair of differentiable functions with the property*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

Then there is a differentiable function h with the property

$$f = \frac{\partial h}{\partial x}, \quad g = \frac{\partial h}{\partial y}.$$

In the language of exact sequences this means that the sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{C}^\infty(E) & \longrightarrow & \mathcal{C}^\infty(E) \times \mathcal{C}^\infty(E) & \longrightarrow & \mathcal{C}^\infty(E) & \longrightarrow 0 \\ f & \longmapsto & \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right) & & & \\ & & (f,g) & \longmapsto & \dfrac{\partial f}{\partial y} - \dfrac{\partial g}{\partial x} & \end{array}$$

is exact. When E is not convex, this sequence needs not to be exact. But since every point in $\mathbb{R}^2$ has an open convex neighborhood, the sequence of sheaves

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{C}_X^\infty \longrightarrow \mathcal{C}_X^\infty \times \mathcal{C}_X^\infty \longrightarrow \mathcal{C}_X^\infty \longrightarrow 0$$

is exact for every open subset $X \subset \mathbb{R}^2$. This is an acyclic resolution and we obtain the following proposition:

5.3 Proposition. *For convex open $E \subset \mathbb{R}^2$ we have*

$$H^i(E, \mathbb{C}) = 0 \quad \textit{for} \quad i > 0.$$

One can of course consider real valued differentiable functions and the same proof shows $H^i(E, \mathbb{R}) = 0$ for $i > 0$.

As an application we prove the following proposition.

5.4 Proposition. *For convex open $E \subset \mathbb{R}^n$ one has*

$$H^2(E, \mathbb{Z}) = 0.$$

Proof. We consider the homomorphism

$$\mathbb{C} \longrightarrow \mathbb{C}^\bullet, \qquad z \longmapsto e^{2\pi \mathrm{i} z}.$$

The kernel is $\mathbb{Z}$. This can be considered as an exact sequence of sheaves for example on an open convex $E \subset \mathbb{R}^n$. A small part of the long exact cohomology sequence is

$$H^1(E, \mathbb{C}^\bullet) \longrightarrow H^2(E, \mathbb{Z}) \longrightarrow H^2(E, \mathbb{C}).$$

Since the first and the third member of this sequence vanish (Propositions 5.1 and 5.3), we get the proof of Proposition 5.4. $\square$

Next we treat an example of complex analysis.

5.5 Lemma of Dolbeault. *Let $E \subset \mathbb{C}$ be an open disk. For every function $f \in \mathcal{C}^\infty(E)$ there exists $g \in \mathcal{C}^\infty(E)$ with*

$$f = \frac{\partial g}{\partial \bar{z}} := \frac{1}{2}\Big(\frac{\partial g}{\partial x} + \mathrm{i}\frac{\partial g}{\partial y}\Big).$$

A proof can be found in [Fo], Satz 13.2. We give a short sketch here. In a first step one restricts to the case where f has compact support. In this case the function g can be constructed as an integral:

$$g(z) = -\frac{1}{\pi}\int_0^{2\pi}\int_0^1 f(z + re^{\mathrm{i}\varphi})e^{-\mathrm{i}\varphi}\,dr\,d\varphi.$$

One can show that $\partial g/\partial \bar{z} = f$. But this is not trivial. One has to make use of the Theorem of Stokes.

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