Lecture on Hodge theory

Eberhard Freitag

Heidelberg SS 2010

Contents

Chapter I. Differential geometry

1. Differentiable manifolds

Topological spaces always are assumed to be locally compact and with countable basis of the topology if nothing else is stated. Such spaces always are metrizable. Hence it would be no restriction to consider only metric instead of topological spaces, but this is unnatural.

A *chart* on a topological space X is a topological map

$$\varphi : U_\varphi \longrightarrow V_\varphi, \quad U_\varphi \subset X,\ V_\varphi \subset \mathbb{R}^n \text{ open}.$$

The number n is called the dimension of the chart. An atlas $\mathcal{A}$ is a set of charts with the property

$$X = \bigcup_{\varphi\in\mathcal{A}} U_\varphi.$$

If φ, ψ are two charts on X then the chart transformation is the topological map

$$\gamma := \psi \circ \varphi^{-1} : \varphi(U_\varphi \cap U_\psi) \longrightarrow \psi(U_\varphi \cap U_\psi).$$

An atlas $\mathcal{A}$ is called *differentiable*, if all chart transformations inside $\mathcal{A}$ are differentiable. Here "differentiable" is always understood in the $\mathcal{C}^\infty$-sense if nothing else is stated. Two differentiable atlases $\mathcal{A}, \mathcal{B}$ are called equivalent if $\mathcal{A} \cup \mathcal{B}$ is differentiable as well. This is an equivalence relation. A differentiable manifold is a topological space that is equipped with a distinguished equivalence class of differentiable atlases. They are called defining atlases. The union of all defining atlases in the equivalence class of an differentiable atlas $\mathcal{A}$ is an differentiable atlas too. It is the unique maximal atlas in the equivalence class of $\mathcal{A}$. We denote it by $\mathcal{A}_{\max}$. The elements of the maximal atlas of a differentiable manifold are called *differentiable charts.*

Sub-manifolds

If $U \subset X$ is an open subset of a differentiable manifold then one can restrict an defining atlas of the differentiable structure of X in an obvious way to X. So every open subset of an differentiable manifold carries a structure as differentiable manifold as well. This observation admits an important generalization:

A subset $Y \subset X$ of a differentiable manifold is called *smooth* at some point $a \in Y$ if there exists a differentiable chart

$$\varphi : U \longrightarrow V, \quad \varphi(Y \cap U) = \{x \in V;\ x_{d+1} = \cdots = x_n = 0\}.$$

If one identifies $\{x \in V;\ x_{d+1} = \cdots = x_n = 0\}$ in the obvious way with an open subset of $\mathbb{R}^d$, one obtains a chart on $Y \cap U$. If Y is smooth at all points than the set of these charts is a differentiable atlas on Y. In this way a smooth subset of a differentiable manifold gets a structure as differentiable manifold too.

Differentiable maps

A continuous map $f : X \to Y$ between differentiable manifolds is called differentiable at a point $a \in X$ if there exist differentiable charts φ on X, $a \in U_\varphi$ and ψ on Y, $f(a) \in U_\psi$ such that $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(a)$. It is clear that this does now depend on the choice of φ, ψ. One calls f differentiable if it is differentiable everywhere. A diffeomorphism is a bijective map $f : X \to Y$ between differentiable manifolds such that f and f^{-1} are differentiable.

The maximal atlas now gets a more natural explanation. First we notice that $\mathbb{R}^n$ is a differentiable manifold, taking the tautological chart id as defining atlas. A chart $\varphi : U \to V$ on X is differentiable (i.e. in the maximal atlas) if and only it is a diffeomorphism. (Both sides are differentiable manifolds.)

2. Tangent space

The tangent space can be defined for arbitrary differentiable manifold in an abstract way:

2.1 Definition. *Let $a \in X$ be a point in a differentiable manifold. A* ***derivation*** *at a is a family of maps*

$$D : \mathcal{C}^\infty(U) \longrightarrow \mathbb{R},$$

where U runs through all open neighborhoods of a such that the following two conditions hold:

1) *It is compatible with restriction.*
2) *It is $\mathbb{R}$-linear.*
3) *It satisfies the product rule*

$$D(fg) = f(a)D(g) + g(a)D(f).$$

"Compatibility with restriction" means $D(f)|V = D(f|V)$. Here $f \in \mathcal{C}^\infty(U)$ and $a \in V \subset U$ is a smaller neighborhood. More cautious persons may write D_U instead of D. Then the formula means $D_U(f)|V = D_V(f|V)$. But we will prefer convenient writing with not too many indices (violating sometimes strong set theoretic conventions).

It is clear how derivations are added and multiplied with constants and it is also clear that the result is a derivation again. Hence we see that the set T_aX of tangent vectors is a vector space. It is also clear that tangent vectors can be pushed forward under a differentiable map $f : X \to Y$. This means that there is a natural map

$$T_af : T_aX \longrightarrow T_{f(a)}Y.$$

It is defined as follows. Let $D \in T_af$ and φ a differentiable function on some open neighborhood of $f(a) \in Y$. We apply D to the function $\varphi \circ f$ and obtain in this way a derivation $T_a(D)$.

2.2 Remark. *The set of tangent vectors T_aX is a vector space. A differentiable map $f : X \to Y$ induces a linear map*

$$T_af : T_aX \longrightarrow T_{f(a)Y}.$$

If $g : Y \to Z$ is a second differentiable map then the chain rule

$$T_a(g \circ f) = T_{f(a)}g \circ T_af$$

holds.

When f is a diffeomorphism then T_af is an isomorphism.

Let $a \in U \subset X$ be an open neighborhood. Then the natural map

$$T_aU \longrightarrow T_aX$$

is an isomorphism.

Usually we will identify T_aU and T_aX. Let $\varphi : U \to V$ be some differentiable chart with $a \in U$. Recall that $V \subset \mathbb{R}^n$ is an open subset. Applying 2.2 to the diffeomorphism φ we obtain an isomorphism

$$T_aX \xrightarrow{\sim} T_{\varphi(a)}(\mathbb{R}^n).$$

Hence, to understand the concept of tangent space it is sufficient to understand the special case $X = \mathbb{R}^n$.

So let $a \in \mathbb{R}^n$ be some point. An obvious derivation is given by

$$f \longmapsto (\partial_i f)(a), \qquad \partial_i := \frac{\partial}{\partial x_i}.$$

We write for this derivation $\partial_i|_a$ or by $(\partial/\partial x_i)|_a$. If we are lazy or if we want to increase the readability we omit $|_a$ and simply write ∂_i or by $\partial/\partial x_i$ for the tangent vector.

2.3 Lemma. *The tangent space $T_a(\mathbb{R}^n)$ is a vector space of dimension n spanned by the basis $\partial_1, \dots, \partial_n$.*

Proof. This result is basic. We give the proof in the case $n = 1$ only to make it as simple as possible. It is no problem to generalize the argument to the case $n > 1$. This is left to the reader. We also can assume $a = 0$. The basic result needed for the proof is the following:

Let $f : (-r - r) \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function with the property $f(0) = 0$. Then $f(x) = xg(x)$ with a $\mathcal{C}^\infty$-function $g : (-r, r) \to \mathbb{R}$.

The proof follows from the formula

$$f(x) = \int_0^x f'(t)dt = x \int_0^1 f'(tx)dt.$$

Let now D be a derivation $D \in T_0\mathbb{R}$. We write $f(x) - f(0) = xg(x)$. Then $g(a) = f'(a)$. Hence

$$f(x) = f(0) + xf'(0) + xh(x), \quad h(0) = 0.$$

From the product rule follows that $D(C) = 0$ for a constant function C (since $D(1) = D(1 \cdot 1) = 2D(1)$). The product role also shows $D(f_1 f_2) = 0$ if $f_1(0) = f_2(0) = 0$. Hence we get

$$D(f) = Cf'(0) \qquad (C = D(x)).$$

This shows $D = C \cdot d/dx$. □

So any tangent vector can be written in the form

$$\sum_{i=1}^n A^i \, \partial_i|_a.$$

It is useful to have the formula

$$Af = \sum A^i \partial_i f(a) = \frac{d}{dt} f(a + tA)$$

in mind that follows form the chain rule. The right hand A here stands for the vector $(A^1, \dots, A^n)$. By definition the right hand side is the derivative of f at a in the direction given by the vector A. Hence the notion of a tangent vector should be considered as an abstract version of the notion of "derivative in a direction".

Vector fields

By a vector field A on some subset $M \subset X$ of a differentiable manifold one simply understands a family $(A_a)_{a \in M}$ of tangent vectors $A_a \in T_a X$. In the case where M is an open subset we want to define what it means that A depends differentiably on a. There are several equivalent possibilities to do this. We can restrict to the case $M = X$ since an open subset can also be considered as differentiable manifold.

2.4 Definition. *A differentiable vector field on a differentiable manifold X is a collection of tangent vectors $A = (A_a)_{a \in X}$, $A_a \in T_aX$, such for all differentiable functions $f : U \to \mathbb{R}$, $U \subset X$ open, the function $U \to \mathbb{R}$ that sends a to $A_a f$ is differentiable.*

This means that a differentiable vector field induces for each open subset $U \subset X$ an operator —also denoted by A—

$$A : \mathcal{C}^\infty(U) \longrightarrow \mathcal{C}^\infty(U).$$

Sometimes it may be necessary to write A_U here instead of U. We simply express this as:

Vector fields operate on differentiable functions.

A very good example for this is the vector field on $\mathbb{R}^n$ (or some open subset).

$$\partial_i := (\partial_i|_a)_{a \in \mathbb{R}^n}.$$

It operates on differentiable functions by $f \mapsto \partial_i f$ and the result is a differentiable function. It is very important that the vector field is determined by the associated operators:

2.5 Lemma. *There is a one-to one correspondence between differentiable vector fields and families of operators*

$$A : \mathcal{C}^\infty(U) \longrightarrow \mathcal{C}^\infty(U), \qquad U \in X \ \textit{open},$$

with the following properties:

1) *It is compatible with restrictions.*
2) *It is $\mathbb{R}$-linear.*
3) *It satisfies the product rule $A(fg) = fA(g) + gA(f)$.*

The proof is trivial: We already associated to a vector field operators. Conversely one associates to a family of operators the tangent vectors

$$A_a f := (Af)(a).$$

Usually we will use the same letter for a a vector field and the associated operators and we jump freely between the two concepts. So

Differentiable vector fields are operators with certain properties.

Of course vector fields can be added and multiplied with real numbers. But even more they can be multiplied with differentiable functions $f \in \mathcal{C}^\infty(X)$. For example in the operator language the definition is $(fA)(g) = fA(g)$. Each vector field on an open subset $U \subset \mathbb{R}^n$ has a unique expression in the form

$$\sum_{i=1}^n A^i \partial_i, \quad A^i \in \mathcal{C}^\infty(U).$$

If $f : X \to Y$ is a diffeomorphism then vector fields can be pushed forward (and pulled back using f^{-1}). For a vector field A on X one defines

$$(f_*A)_b = (T_a f)(A_a) \quad \text{where} \quad a = f^{-1}(b).$$

This gives an identification between differentiable vector fields on X and Y.

Warning. Since the formula for f_*A involves f and f^{-1}, it is not possible to define the push forward (or pull back) for differentiable maps which are not diffeomorphisms.

If $\varphi : U \to V$ is a differentiable chart on a differentiable manifold X and A a differentiable vector field on X, then we can restrict A to U and push forward to V_φ. This gives us a vector field $\sum A^i \partial_i$. The functions A^i are called the components of A with respect to the chart φ. Sometimes it may be necessary to add a label φ to the components and to write A^i_φ.

Of course a differentiable vector field is determined be the knowledge of its components for all differentiable charts (from some defining atlas is enough).

2.6 Lemma. *There is a one-to one relation between differentiable vector fields on X and families*

$$(A^1_\varphi, \dots, A^n_\varphi), \quad A^i_\varphi \in \mathcal{C}^\infty(V_\varphi),$$

where φ runs through all differentiable charts (from a defining atlas is enough) such that the following compatibility condition is satisfied.

If ψ is another chart and $\gamma = \psi \circ \varphi^{-1}$ the chart transformation then

$$A^i_\psi(\gamma x) = \sum_{j=1}^n \frac{\partial \gamma_i}{\partial x_j} A^j_\varphi(x).$$

We skip the proof and mention that for this and other properties it is useful to have the following *gluing property:*

2.7 Remark. *Vector fields can be restricted to open subsets. If $X = \bigcup U_i$ is an open covering and A_i a differentiable vector field on U_i for each label i. Then a differentiable vector field A on X with the condition $A|U_i = A_i$ exists if and only if the gluing condition*

$$A_i|(U_i \cap U_j) = A_j|(U_i \cap U_j)$$

is satisfied.

3. Differentials

Differentials are dual to vector fields. To understand this one should have in mind the dual vector space V^* of a (here real) vector space V. It consists of all linear forms on V.

$$V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}).$$

There is a remarkable map form V into its double dual

$$V \longrightarrow V^{**}, \quad (a \in V) \longmapsto (L_a \in V^*), \quad L_a(f) = f(a).$$

For finite dimensional vector spaces this turns out to be an isomorphism. The essential point is that this isomorphisms doesn't depend on the choice of a basis or anything else. It is "canonical" in the best sense. So one can identify V and V^{**} (but one has to keep in mind how this identification is done).

By a *dual field* on some subset M of a differentiable manifold X one understands a collection

$$\omega = (\omega_a)_{a \in M}, \quad \omega_a \in (T_aX)^*.$$

Again we have to explain for open M what it means that ω depends in a differential way from a. We can assume $M = X$. We make use of the fact that for any vector field A on some open subset $U \subset X$ the function $a \mapsto \omega_a(A_a)$ can be defined. We denote this function simply by $\omega(A)$.

3.1 Definition. *A dual field ω on X is called differentiable if if for each differentiable vector field A on the some open subset the function $\omega(A)$ is differentiable on U.*

Hence differentiable dual fields act on differentiable vector fields and produce differentiable functions. Again this construction has an inverse:

3.2 Proposition. *We denote by $\mathcal{T}(X)$ the space of all differentiable vector fields on X. Differentiable dual fields on X are in one-to-one correspondence with families of operators*

$$\omega : \mathcal{T}(U) \longrightarrow \mathcal{C}^\infty(U)$$

with the following properties:

1) *They are compatible with restriction.*
2) *They are $\mathcal{C}^\infty(U)$-linear, i.e they are additive and satisfy*

$$\omega(fA) = f\omega(A), \qquad f \in \mathcal{C}^\infty(U),\ A \in \mathcal{T}^\infty(U).$$

Proof. We already attached to a dual field the operators. Now we have do the converse. Assume that a family with the properties 1) and 2) is given. We have to define for each point $a \in X$ a linear form $T_aX \to \mathbb{R}$. For this purpose we can replace X by some open neighborhood of a and hence assume that X is diffeomorphic to an open set of $\mathbb{R}^n$. Then it is sufficient to assume that X is an open subset $U \subset \mathbb{R}^n$.

The proof now will follow form an explicit description of dual fields on U. Recall that we have a basis $\partial_1|_a, \dots, \partial_n|_a$ of T_aU. We make use of the notion of a dual basis: The dual basis of T_aU is denoted by $dx_1|_a, \dots, dx_n|_a$. It is defined by the condition

$$dx_i|_a(\partial_j|_a) = \delta_{ij}.$$

Next we consider the dual field

$$dx_i := (dx_i|_a)_{a \in U}.$$

Its evaluation on vector fields is

$$dx_i(\partial_j) = \delta_{ij}.$$

Of course dx_i is a differentiable dual field. Now we come back to our given family ω with the properties 1) and 2) (now on U). We evaluate ω at some differentiable vector field $A = \sum A^i \partial_i$ (on some open subset of U. Now we use the condition 2) in an essential way:

$$\omega(A) = \omega\Big(\sum A^i \partial_i\Big) = \sum A^i \omega(\partial_i).$$

Now it is clear how to define $\omega_a : T_aU \to \mathbb{R}$:

$$\omega_a\Big(\sum C_i \partial_i|_a\Big) = \sum C_i \omega(\partial_i)|_a.$$

The rest should be clear. □

We should keep in mind that this simple proof heavily rests on the condition 2) ($\mathcal{C}^\infty$-linearity). The proof also shows that in the local case ($X = U$ open subset of $\mathbb{R}^n$) every differentiable dual form can be written in the form

$$\omega = \sum_{i=1}^{n} A_i dx_i$$

with differentiable functions A_i. From now on we will call *differentiable dual fields* also *differentials*. As vector fields, differentials can be pushed forward and pulled back with respect to a diffeomorphism. Especially for a differentiable chart φ the differential ω can be considered on V_φ where it gets the form $\sum A_i^\varphi dx_i$. The functions $A_i^\varphi \in \mathcal{C}^\infty$ are the so-called components of ω with respect to the chart φ. There is an analogue of 2.6 for differentials. The formula there has to be replaced by

$$A_i^\varphi(x) = \sum_{j=1}^n \frac{\partial \gamma_j}{\partial x_i} A_j^\psi(\gamma(x)).$$

As we mentioned, differentials as vector fields can be pulled back under diffeomorphisms. But the situation is much better than in the case of vector fields: Differentials can be pulled back under arbitrary differentiable maps $f : X \to Y$. The procedure is quite clear: Let ω be a differential form on Y. We want to define a differential $f^*\omega$ on X. This means that for any point $a \in X$ we have to define a linear map $T_aX \to \mathbb{R}$. We have a linear map $T_{f(a)}Y \to \mathbb{R}$ coming form ω. We also have a map $T_aX \to T_{f(a)}Y$ (the tangent map T_af). Composing both we get the desired map $T_aX \to \mathbb{R}$. It is clear that this dual field is differentiable and it is also clear that this pull back is transitive: For two differentiable maps $f : X \to Y$ and $g : Y \to Z$ and a differential ω on Y the formula

$$(g \circ f)^*\omega = f^*(g^*(\omega))$$

holds. One can express the pull back also in local coordinates: Let

$$f : U \to V, \quad U \subset \mathbb{R}^n,\ V \subset \mathbb{R}^m \text{ open},$$

then

$$f^*(dy_i) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_i.$$

Finally we mention that for an open subset $U \subset X$ the already considered trivial restriction $\omega|U$ and the pull back $i^*(\omega)$ are the same. Here $i : U \to X$ denotes the canonical injection.

4. Alternating differential forms

Let V be a finite dimensional real vector space. We consider the vector space $\mathrm{Mult}^p(V)$ of all multilinear forms

$$M : V \times \cdots \times V \longrightarrow \mathbb{R}.$$

We allow p do be an arbitrary integer with the convention

$$\mathrm{Mult}^p(V) = \begin{cases} \mathbb{R} & \text{if } p = 0, \\ 0 & \text{if } p < 0. \end{cases}$$

There is a natural map

$$\mathrm{Mult}^p(V) \times \mathrm{Mult}^q(V) \longrightarrow \mathrm{Mult}^{p+q}(V), \quad (M, N) \longmapsto M \otimes N$$

defined by

$$(M \otimes N)(a_1, \dots, a_p, b_1, \dots, b_q) = M(a_1, \dots, a_p) N(b_1, \dots, b_q).$$

A multilinear form is called *alternating,* if $M(a_1, \dots, a_p)$ is zero if two of the a_i agree. Then one can show that for any permutation σ of the digits $1, \dots, p$ the formula

$$M(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = \mathrm{sgn}(\sigma) M(a_1, \dots, a_p)$$

holds. We denote the space of alternating forms by $\mathrm{Alt}^p(V)$. There is a natural projection

$$\mathrm{Mult}^p(V) \longrightarrow \mathrm{Alt}^p(V), \quad M \longmapsto M^{\mathrm{alt}}$$

which is defined by

$$M^{\mathrm{alt}}(a_1, \dots, a_p) = \frac{1}{p} \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma)(a_{\sigma(1)}, \dots, a_{\sigma(p)}).$$

The form M is alternating iff $M = M^{\mathrm{alt}}$.

One defines a bilinear map

$$\mathrm{Alt}^p(V) \times \mathrm{Alt}^q(V) \longrightarrow \mathrm{Alt}^{p+q}(V), \quad (A, B) \longmapsto A \wedge B,$$

through the formula

$$A \wedge B = \binom{p+q}{p} (A \otimes B)^{\mathrm{alt}}.$$

The binomial coefficient here is not important. For our purposes it could be skipped. One reason to insert it is just to avoid denominators in formulas. For example in the case $p = q = 1$ one gets

$$A \wedge B = A \otimes B - B \otimes A.$$

Without the binomial factor one would get a denominator 2. The following facts can easily be checked:

The associative law $(A \wedge B) \wedge C = A \wedge (B \wedge C)$ holds. Especially one can define $A_1 \wedge \ldots \wedge A_m$ for arbitrary m.

One has skew-commutativity

$$A \wedge B = (-1)^{pq} B \wedge A, \quad A \in \mathrm{Alt}^p(V),\ B \in \mathrm{Alt}^q(V).$$

If $e_1, \dots, e_n$ is a basis of V^* then the

$$e_{i_1} \wedge \ldots \wedge e_{i_p}, \quad 1 \le i_1 < \cdots < i_p \le n$$

form a basis for $\mathrm{Alt}^p(V)$.

We conclude the multilinear algebra with the remark:

$$\mathrm{Alt}^0(V) = \mathbb{R}, \quad \mathrm{Alt}^1(V) = V^*$$

and we mention

$$\mathrm{Alt}^p(V) = 0 \text{ for } p < 0 \text{ and } p > \dim V.$$

This system of vector spaces together with the alternating product is called a *Grassmann algebra.* Since its first essential part $\mathrm{Alt}^1(V)$ agrees with the dual space, it is called the Grassmann algbra over V^*.

A (covariant) tensor field T on a differential manifold X is a collection $T = (T_a)_{a\in X}$ of elements $A_a \in \mathrm{Mult}^p(T_aX)$. One can evaluate such a tensor field on tuples $(A_1, \dots, A_p)$ of vector fields (defined on some open subset $U \subset X$. The result is function on U, namely

$$a \longmapsto T_a((A_1)_a, \dots, (A_p)_a).$$

We denote this function simply by $T(A_1, \dots, A_p)$. The tensor field is called differentiable if all these functions are differentiable. Similarly to the case of vector fields and differentials one has (compare 3.2):

4.1 Lemma. *Differentiable tensor fields T are in on-to-one correspondence to collections $(T_U)_{U\subset X \text{ open}}$, where T_U is map that associates to each tuple $(A_1, \dots, A_p)$ of differentiable vector fields on U a differentiable function $T_U(A_1, \dots, A_p)$ on U sich the following conditions are satisfied:*

1) *It is compatible with restrictions.*
2) *Is $\mathcal{C}^\infty(U)$-multilinear.*

it should be clear what $\mathcal{C}^\infty(U)$-multilinear means: $T_U(A_1, \dots, A_p)$ is additive in each of the variables and futhermore

$$T_U(f_1A_1, \dots, f_pA_p) = f_1 \cdots f_p T_U(A_1, \dots, A_p) \quad \text{for} \quad f_1, \dots f_p \in \mathcal{C}^\infty(U).$$

Like differentials covariant tensor fields can be pulled back under arbitrary differentiable maps $f : X \to Y$. If $a \in X$ is some point one X one considers the tangent map $T_af : T_aX \to T_{f(a)}Y$. Let T be a tensor field on Y. One defines

$$(f^*T)(A_1, \dots, A_p) := T_{f(a)}(T_af(A_1), \dots, T_af(A_p)).$$

If f is a diffeomorphism, one can use f^{-1} to transform in the other direction. If T is a differentiable tensor field on X then one can consider for each differentialble chart φ the transformed tensor of $T|U_\varphi$ on V_ϕ. We call it T^φ. Using the standard basis we may may define its components by

$$T_{i_1,\dots,i_p} = T^\varphi(\partial_{i_1},\dots,\partial_{i_p}).$$

with respect to a chart transformation $\gamma = \psi\circ\varphi^{-1}$ one has the transformation formula

$$T^\varphi_{i_1,\dots,i_p} = \sum_{\nu_1,\dots,\nu_p} \frac{\partial\gamma_{\nu_1}}{\partial x_{i_1}}\cdots\frac{\partial\gamma_{\nu_p}}{\partial x_{i_p}} T^\psi_{\nu_1,\dots,\nu_p}.$$

(Functions have to be evaluated at corresponding places $x, \gamma(x)$.)

As in the case of differentials each system of differentiable functions $T^\varphi_{i_1,\dots,i_p}$ with this transformation property comes from a differentiable tensor field on X. This leads to the old-fashioned definition:

A tensor is a tensor iff it transforms like a tensor.

It is clear what it means that covariant tensor fields are alternating.

4.2 Definition. *An alternating differential form of degree p (in short p-form) on a differentiable manifold is an alternating differentiable tensor field of degree p.*

Notation. $A^p(X)$ = *space of all p-forms on* X.

Alternating differential forms of degree 1 are just differentials. The special importance of differential forms rests on the existence of the exterior derivative:

4.3 Proposition. *There is a unique way to define for each differentiable manifold X and for each $p\in\mathbb{Z}$ an $\mathbb{R}$-linear map*

$$d : A^p(X) \longrightarrow A^{p+1}(X)$$

such that the following properties are satisfied:

1) *It is compatible with pulling back for a differentiable map $f : X \to Y$,*

$$f^*(d\omega) = d(f^*\omega).$$

2) *On differentiable functions f it is defined as*

$$df(A) = A(f).$$

Here A is some differentiable vector field on an open subset of X.

3) *For differentiable functions $f, f_1,\dots,f_p$ the rule*

$$d(f\,df_1\wedge\ldots\wedge df_p) = df\wedge df_1\wedge\ldots\wedge df_p$$

holds.

Moreover the following rules hold:

a) *It is compatible with wedge product,*

$$d(\alpha \wedge \beta) = (d\alpha) \wedge (d\beta).$$

b) *The rule*

$$d(d(\omega)) = 0$$

always holds.

c) *The product rule*

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta \qquad (\omega \in A^p(X))$$

holds.

Scetch of two proofs. The first proof rests on coordinates. One first studies the local case where X is an open set $U \subset \mathbb{R}^n$. For functions f we have by 3) the formula $df(\partial_i) = \partial_i f$. This means

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i.$$

the rule 4) enforces the definition

$$d \sum_{1 \le i_1 < \cdots < i_p \le n} f_{i_1,\dots,i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p} = \sum_{1 \le i_1 < \cdots < i_p \le n} df_{i_1,\dots,i_p} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

With this definitions one has to verify the proposition in the local case. Then one glue the local definition to a global one (essentially using the transformation invariance 3). Since this has been done in detail in my analysis 3 course we skip details.

In some sense this proof is not satisfactory since it uses coordinates in the old-fashioned way. One may ask whether there is a coordinate free definition. Just for fun we give but we have to confess that it is not very simple. Here is it:

$$\begin{aligned}(d\omega)(A_1, \dots, A_{d+1}) := &\sum_{i=1}^{d+1} (-1)^{i+1} A_i \omega(A_1, \dots, \hat{A}_i, \dots, A_{d+1}) + \\ &\sum_{i<j} (-1)^{i+j} \omega([A_i, A_j], A_1, \dots, \hat{A}_i, \dots, \hat{A}_j, \dots, A_{d+1}).\end{aligned}$$

In the formula the so-called Lie-bracket $[A, B]$ of two vector fields occurs. It uses the interpretation of vector fields as operators and is defined by $[A, B] = A \circ B - B \circ A$. This is an operator acting on differentiable functions on open sets. It is easy to check that it satisfies the product rule (in contrast to $A \circ B$ alone) and hence is an vector field.

5. The de-Rham complex

For a differentiable manifold the de-Rham complex is defined to be the sequence of maps

$$\cdots \longrightarrow A^{p-1}(X) \longrightarrow A^p(X) \longrightarrow A^{p+1}(X) \longrightarrow \cdots$$

We use the notation

$$\begin{aligned} C^p(X) &:= \text{kernel}(d : A^p(X) \longrightarrow A^{p+1}(X)), \\ B^p(X) &:= \text{image}(d : A^p(X) \longrightarrow A^{p+1}(X)). \end{aligned}$$

The elements of $C^p(X)$ are called *closed* ($d\omega = 0$) and the elements of $B^p(X)$ are called *exact.* They are of the form $d\omega'$. Because of $d \circ d = 0$ exact forms are closed. The converse is not always true and it is important to understand this. To measure the difference between exact and closed forms on introduces the de-Rham cohomology groups. (One should better say "de-Rham cohomology vector spaces", but this is unusual.) They are defined as factor space of $C^p(X)$ by the subspace $B^p(X)$.

$$H^p(X, \mathbb{R}) = C^p(X)/B^p(X).$$

The elements of $H^p(X, \mathbb{R})$ are classes of elements from $C^p(X)$. The class $[\omega]$ of an element $\omega \in C^p(X)$ consists of all elements of the form $\omega + d\alpha$, $\alpha \in A^{p-1}(X)$. The set of all classes can be made to a vector space in a natural way. The vector space structure is defined through the fact that the natural projection

$$C^p(X) \longrightarrow H^p(X), \quad \omega \longmapsto [\omega],$$

is a linear map. This linear map is surjective and its kernel is $B^p(X)$. Hence we see: The group $H^p(X, \mathbb{R})$ vanishes if and only of each closed p-form is exact.

Let X be a differentiable manifold of dimension n. (This means that all charts land in $\mathbb{R}^n$.) Then of course $H^p(X, \mathbb{R}) = 0$ for $p > n$. Of course $H^p(X, \mathbb{R}) = 0$ for $p < 0$ is always true. Let's consider the case $p = 0$. Clearly $B^0(X) = 0$ since every form of degree -1 is zero. Hence $H^0(X, \mathbb{R}) = C^0(X)$. The space $C^0(X)$ consists of all functions with $df = 0$. Such functions are locally constant. If we assume that X is connected then the are constant. Hence $H^0(X, \mathbb{R})$ for a connected differentiable manifold can be identified with $\mathbb{R}$.

5.1 Remark. *For each connected differentiable manifold one has*

$$H^0(X, \mathbb{R}) = \mathbb{R}.$$

A basic result is the

5.2 Lemma of Poincaré. *Let $U \subset \mathbb{R}^n$ be an open convex subset. Then*

$$H^p(U,\mathbb{R}) = 0 \quad \textit{for} \quad p > 0.$$

Proof. Let ω be a closed form. We decompose it as

$$\omega = \alpha + \beta \wedge dx_n,$$

where α doesn't contain any term with dx_n. We write

$$\beta = \sum f_a d_a$$

where a are subsets of $\{1,\dots,n-1\}$ that do nor contain n. Integrating with respect to the last variable we find differentiable functions F_a such that $\partial_n F_a = f_a$. Now the difference $\omega - d\sum_a F_a dx_a$ doesn't contain any term in which dx_n occurs. Hence we can assume that in ω no term with dx_n occurs. We write

$$\omega = \sum_a g_a dx_a,$$

where all a are subsets of $\{1,\dots,n-1\}$. Now we use $d\omega = 0$. We obtain $\partial_n g_a = 0$. Hence g_a do not depend on x_n. But now ω can be considered as differential form in one dimension less (on the image of U with respect to the projection map that cancels the last variable) and an induction argument completes the proof. □

Next we want to give an important class of examples for a non-vanishing de-Rham cohomology groups. It rests on the theorem of Stokes and hence on integration of differential forms. We just recall the basic concept.

First one has to introduce the concept of orientation: A differentiable manifold X is called *orientable,* if there exists a defining atlas $\mathcal{A}$ such that all chart transformations inside $\mathcal{A}$ have positive functional determinant everywhere. Two such atlases are called oriented equivalent if their union is oriented as well. An orientation of a differentiable manifold is given by an equivalence class of oriented equivalent atlases (consisting of differentiable charts with respect to the given differentiable structure on X). In this equivalence class there exists a unique maximal (oriented) atlas $\mathcal{A}^+$. The elements of this atlas are called the oriented differentiable charts on X.

Let now X be of dimension n and ω a top form $\omega \in A^n(X)$. We assume that ω has compact support. Then one can define

$$\int_X \omega.$$

We recall the definition. First one considers the case where the support of ω (which has to be defined in an obvious way) is contained in the domain of definition of an oriented differentiable chart φ. Then one defines

$$\int_X \omega := \int_{v_\varphi} f_\varphi(x)dx,$$

where the function $f_\varphi : V_\varphi \to \mathbb{R}$ is the component of ω with respect to this chart. then one has to show that this definition is independent of the choice of φ. This uses the transformation property of the components and the transformation formula for the integral.

For the general case one uses the technique of partition of one. We skip details.

Since we have an oriented manifold it makes sense to say when a top form ω is positive in some point. It just means $f_\varphi(\varphi(a)) > 0$ is positive for an oriented differentiable chart φ with $a \in U_\varphi$. This is independent of the choice of the chart since chart transformations have positive functional determinant in the oriented world.

Using a partition of unity it is easy to show that one any compact oriented differential manifold of dimension n there exists a top form ω which is everywhere positive. Now we assume that X is compact. We have

$$\int_X \omega > 0.$$

Let now be a form α of degree $n-1$. The theorem of Stokes states that for every $(n-1)$-form α one has

$$\int_X d\alpha = 0 \qquad (\alpha \text{ with compact support}).$$

5.3 Proposition. *Let X be a compact oriented differentiable manifold of dimension n. Then*

$$H^n(X, \mathbb{R}) \neq 0.$$

This indicates that the de-Rham cohomology groups are related to the geometry of differentiable manifolds.

Chapter II. Real Hodge theory

1. Riemannian manifolds

Let X be a differentiable manifold. We consider a covariant tensor field g of degree two. We always assume that it is differentiable. Recall that this means that we have a bilinear form

$$T_a X \times T_a X \longrightarrow \mathbb{R}$$

for each point a. Now we are interested in the case where this bilinear form is symmetric and positive definite for all a. (Hence g is no differential form).

1.1 Definition. *A Riemannian metric on a differentiable manifold X is a collection of postive definite bilinear forms*

$$g_a : T_a X \times T_a X \longrightarrow \mathbb{R}$$

that is differentiable at a (i.e. a differentiable tensor field).

A Riemannian manifold is a pair (X, g) consisting of a differentiable manifold and a Riemannian metric on it.

If $X = U$ is some open subset of $\mathbb{R}^n$ than a Riemann metric is nothing else but a symmetric $n \times n$-matrix of differentiable functions that is positive definit at every point.

2. The star operator

The background of the star operator is simple linear algebra. Consider a finite dimensional vector space V together with a positive definite bilinear form $g : V \times V \to \mathbb{R}$. The pair (V, g) is called an euclidian vector space. It is a very important concept that g induces an isomorphism $\sigma : V \to V^*$. To define it one has to associate to a vector $a \in V$ a linear form $l_a : V \to \mathbb{R}$. One simply defines

$$l_a(x) = g(a, x).$$

It is easy to check that this is an isomorphism (using bases). The isomorphism enables to carry over the bilinear form to V^*. It is defined by

$$V^* \times V^* \longrightarrow \mathbb{R}, \quad (x,y) \longmapsto g(\sigma^{-1}x, \sigma^{-1}(y)).$$

We denote this new bilinear form with the letter g^*. Hence

$$g^*(x,y) := g(\sigma^{-1}x, \sigma^{-1}(y)) \quad \text{for} \quad x, y \in V^*.$$

Bilinear forms can be described by matrices using bases: Let $e_1, \dots, e_n$ be an arbitrary basis of V. The the Gram matrix of g is defined as

$$g_{ik} := g(e_i, e_k).$$

This is a symmetric and positive definite matrix. Now let ${e_1}^*, \dots, {e_n}^*$ be the dual basis. We denote the corresponding Gram matrix of g^* by

$$g^{ik} := g^*({e_i}^*, {e_k}^*).$$

2.1 Lemma. *Let g be a euclidian metric on V and $e_1, \dots, e_n$ a basis of V. and (g_{ik}) the corresponding Gram matrix. Similarly let (g^{ik}) be the Gram matrix of g^* with respect to the dual basis. The matrices $(g_{ik}$ and (g^{ik}) are inverse matrices).*

Proof. We denote by E_i the image of e_i under the isopmorphism $V \to V^*$ induced by g. By definition $E_i(e_j) = g(e_i, e_j)$. On the other side the dual basis is defined by $e_i^*(e_j) = \delta_{ij}$. This shows

$$E_i = \sum_{\nu=1}^{n} g_{i\nu} e_\nu^* \quad \text{equivalently} \quad e_i^* = \sum_{\nu=1}^{n} h_{i\nu} E_\nu.$$

Here we denoted by (h_{ik}) the inverse matrix of (g_{ik}). From this formula we can compute $g^{ik} = g^*(e_i^*, e_k^*)$. A sraight forward calculation shows $g^{ik} = h_{ik}$. □

We consider now $V^* = \operatorname{Alt}^1(V)$ as part of the Grassmann algebra. It is natural to ask whether the bilinear form extends to $\operatorname{Alt}^p(V)$ in a natural way. The answer is yes:

2.2 Lemma. *Assume that (V, g) is a finite dimensional Euclidean vector space. There is a unique positive definite bilinear form g^* on $\operatorname{Alt}^p(V)$ with the following property: Let $a_1, \dots, a_p$ and $b_1, \dots, b_p$ be elements of V^* then*

$$g^*(a_1 \wedge \dots, \wedge a_p, b_1 \wedge \dots, \wedge b_p) = \det((g^*(a_i, b_j)).$$

Proof. We choose an orthonormal bases $e_1, \dots, e_n$ of V^*. This means $g(e_i, e_j) = \delta_{ij}$. Then we define the scalar product on $\mathrm{Alt}^p(V)$ such that the basis $e_{i_1} \wedge, \dots, e_{i_p}$, $1 \le i_1 < \dots < i_p$ is orthonormal. Then the formula

$$g^*(e_{i_1} \wedge \ldots \wedge e_{i_p}, e_{j_1} \wedge \ldots \wedge e_{j_p}) = \det((g^*(e_{i_\mu}, e_{j_\nu})))$$

is valid under the condition $i_1 < \dots < i_p$ and $j_1 < \dots < j_p$. But it is also true in the case where two of the indices agree since both sides then are zero. And also the ordering condition can be omitted since both sides pick up the same sign if one reorders the terms. Now consider on $\mathrm{Alt}^{2p}(V)$ the two functions

$$M_1(a_1, \dots, a_p, b_1, \dots, b_p) = g^*(a_1 \wedge \ldots \wedge a_p, b_1 \wedge \ldots \wedge b_p),$$
$$M_2(a_1, \dots, a_p, b_1, \dots, b_p) = \det((g^*(a_i, b_j)).$$

Obviously both are multilinear forms. Since they agree for basis elements, they agree. □

Especially $\mathrm{Alt}^n(V)$ with $n = \dim V$ is a one dimensional euclidian vector space. Hence this space contains precisely two elements of ω with the property $g^*(\omega, \omega) = 1$. It easy to construct them. Let $e_1, \dots, e_n$ be a basis of V. Then

$$\frac{e_1 \wedge \ldots \wedge e_n}{\sqrt{\det(g^*(e_i, e_j))}}$$

has the desired property. We prefer to start with a basis $e_1, \dots, e_n$ and then to use the dual basis $e_1^*, \dots, e_n^*$ as basis for V^*. Using 2.1 we get that

$$\omega = \sqrt{\det(g_{ik})}\, e_1^* \wedge \ldots \wedge e_n^*$$

is an element of euclidian norm 1. As we mentioned already the element ω only depends up to a sign on the choice of the basis $e_1 \dots, e_n$.

2.3 Lemma. *Let $e_1, \dots, e_n$ and $f_1, \dots, f_n$ be two bases of V. Assume the the determinant of the transition matrix (defined by $f_i = \sum_j a_{ij} e_j$) has positive determinant then both bases give the same element ω. In the case where the determinant is negative the element ω changes its sign.*

The proof is straight forward an can be omitted.

To select one of the $\pm\omega$ is a question of orientation: Here it is useful to introduce the notation of orientation for real vector spaces. To do this we call two bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$ orientation equivalent if the transition matrix has positive determinant. By definition an orientation of V is an equivalence class of orientation equivalent bases. In the case $n > 1$ there are two possible orientations. An oriented vector space is a pair consisting of a real finite dimensional vector space and a distinguished orientation. In an oriented vector space it makes sense to talk about oriented bases. By definition these are the bases in the distinguished equivalence class. Using oriented bases one can now define the element $\omega \in \mathrm{Alt}^n(V)$ without sign ambiguity. Using the element we get a natural isomorphism

$$\mathbb{R} \xrightarrow{\sim} \mathrm{Alt}^n(V), \quad t \longmapsto t\omega.$$

Now we use the wedge product

$$\mathrm{Alt}^p(V) \times \mathrm{Alt}^p(V) \longrightarrow \mathrm{Alt}^n(V) \quad \text{for} \quad p+q=n.$$

Using the above isomorphism we get a bilinear form

$$\mathrm{Alt}^p(V) \times \mathrm{Alt}^p(V) \longrightarrow \mathbb{R} \quad \text{for} \quad p+q=n.$$

As usual this induces a linear map

$$\mathrm{Alt}^p(V) \longrightarrow \mathrm{Alt}^q(V)^*.$$

Since we have an euclidian metric on $\mathrm{Alt}^q(V)^*$ we have a natural isomorphism $\mathrm{Alt}^q(V) \cong \mathrm{Alt}^q(V)^*$. Hence we obtain a natural linear map

$$\mathrm{Alt}^p(V) \longrightarrow \mathrm{Alt}^{n-p}(V).$$

This map is called the star operator.

2.4 Remark. *Let V be a finite dimensional oriented vector space together with a distinguished symmetric and positive definit bilinear form. There is a natural linear map*

$$A^p(V) \longrightarrow A^{n-p}(V), \quad \alpha \longmapsto *\alpha.$$

It has the following property and is characterized by it:

$$\alpha \wedge *\beta = g^*(\alpha,\beta)\omega, \quad \textit{for} \quad \alpha,\beta \in \mathrm{Alt}^p(V).$$

The star operator gets a very simple explicit form if one uses an orthonormal basis $e_1,\dots,e_n$. We can assume that it is oriented (eventually changing the sign of one basis element). Then we have $\omega = e_1^* \wedge \ldots \wedge e_n^*$ and it is very simple to show:

2.5 Lemma. *Let $e_1,\dots,e_n$ be an oriented orthonormal basis of V^*. Then*

$$*\left(e_{i_1} \wedge \ldots \wedge e_{i_p}\right) = \pm e_{j_1} \wedge \ldots \wedge e_{j_{n-p}}.$$

Here $j_1,\dots,j_{n-p}$ means just the complementary tuple order in the natural way. The sign in this formula is given by the sign of the permutation that brings $i_1,\dots,i_p,j_1,\dots,j_{n-p}$ in its natural ordering.

From this description one sees:

2.6 Lemma. *The star operator $\mathrm{Alt}^p(V) \to \mathrm{Alt}^{n-p}(V)$ has the property*

$$*(*\alpha) = (-1)^{p(n-p)}\alpha.$$

As a consequence it is an isomorphism.

3. The Laplace Beltrami operator

In the following X oriented differentiable manifold of dimension n. $a \in X$ be a point and φ an oriented chart with $a \in U_\varphi$. We denote by x the variable in V_φ. The chart φ induces a basis of T_aX which we denote by $\partial/\partial x_1, \dots, \partial/\partial x_n$ (using the isomorphism $T_aU_\varphi \to T_{\varphi(a)}V_\varphi$). Let ψ be a second oriented chart. We denote the variable in V_ψ by y and the corresponding basis of T_aX by $\partial/\partial y_1, \dots, \partial/\partial y_n$. The transition matrix between the two basis is given by the Jacobi matrix of the chart transformation $\gamma = \psi \circ \varphi^{-1}$ evaluated at $\varphi(a)$. Since the charts are oriented the determinant of this matrix is positive. Hence the vector space T_aX can be oriented in such a way that the bases above are oriented bases.

As a consequence the element $\omega_a \in \mathrm{Alt}^n(V)$ is defined. If $U \subset \mathbb{R}^n$ is an open subset and if the Riemannian matrix is given by the (symmetric and positive definite) matrix g then the formula for ω at an arbitrary point is given by

$$\omega = \sqrt{\det g(x)}\, dx_1 \wedge \ldots \wedge dx_n.$$

This formula shows that ω_a depends differentiable form a. Hence we obtain a top form

$$\omega \in \mathrm{Alt}^n(X).$$

It is clear that this is positive in the sense that its components with respect to arbitrary oriented charts are positive everywhere. We call ω the volume form of X. Clearly

$$\int_X \omega > 0.$$

This is called the volume of (X, g).

Now let $\alpha \in A^p(X)$ be a p-form on X. For each point we can define $*\alpha_a$. It follows from 2.4 that this depends differentiable from a. Hence we obtain a differential form $*\alpha \in A^{n-p}$. The operator

$$* : A^p(X) \xrightarrow{\sim} A^{n-p}(X)$$

is an isomorphism. It is compatible with multiplication with $\mathcal{C}^\infty$-functions,

$$*(f\alpha) = f * \alpha,$$

and its satisfies

$$* * \alpha = (-1)^{p(n-p)}\alpha.$$

Finally we mention that one obtains ω if one applies $*$ to the function "constant one" and conversely.

Let α be a p-form. Then $*\alpha$ is a $(n-p)$-form and $d(*\alpha)$ is a $(n-p+1)$ form. Applying the star operator again, we get the $(p-1)$-form $*d(*\alpha)$. Up to a sign this operator is the so-called codifferentiation:

3.1 Definition. *The **codifferentiation** is the operator*
$$d^* := (-1)^{n(p+1)} * d* : A^p(X) \longrightarrow A^{p-1}(X).$$

The sign needs an explanation. We give it in the case of a compact X. In this case the theorem of Stokes states
$$\int\limits_X d\alpha = 0 \quad \text{for} \quad \alpha \in A^{n-1}(X).$$
In the following we use the notation
$$A^p(X) \times A^p(X) \longrightarrow \mathcal{C}^\infty(X), \quad \langle \alpha, \beta \rangle = g^*(\alpha, \alpha).$$
Hence $\langle \alpha, \beta \rangle$ is a function. But we can integrate this function to get a pairing, which produces scalars, more precisely:
Let (X, g) be a compact oriented Riemannian manifold. Then one can define
$$A^p(X) \times A^p(X) \longrightarrow \mathbb{R}, \quad (\alpha, \beta) := \int\limits_X \langle \alpha, \beta \rangle \, \omega = \int\limits_X \alpha \wedge *\beta.$$
This is a symmetric positive definit bilinear form.

3.2 Proposition. *The codifferentiation on a compact oriented Riemannian manifold satisfies*
$$(d\alpha, \beta) = (\alpha, d^*\beta), \quad \alpha \in A^{p-1}(X), \ \beta \in A^p(X),$$
hence d^* is the adjoint for d.

Proof. One has to use Stokes formula $\int_X d(\alpha \wedge *\beta) = 0$ and the product rule. □

3.3 Definition. *The Laplace Operator*
$$\Delta : A^p(X) \longrightarrow A^p((X)$$
is defined by
$$\Delta := d \circ d^* + d^* \circ d.$$

It is hard to get explicit formulas in local coordinates and we will not use them. In the case $p = 0$ and for an open domain in $\mathbb{R}^n$ the reader may try to proof
$$\Delta f = \frac{1}{\det g} \sum_{ij} \partial_i (\sqrt{\det g} \, g^{ij}) \partial_j f.$$
If g is the unit matrix (euclidian case) then the formula reduces to the usual Laplace operator.

We denote by
$$\mathcal{H}^p(X) = \left\{ \alpha \in A^p(X); \quad \Delta\omega = 0 \right\}$$
the kernel of Δ. Its elements are called *harmonic forms.*

3.4 Proposition. *A differential form α on a compact oriented Riemannian manifold X is harmonic if and only if*

$$d\alpha = 0 \qquad \textit{and} \qquad d^*\alpha = 0.$$

If X is connected then every harmonic function (=zero-form) is constant.

The proof follows from

$$(\Delta\alpha, \alpha) = (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha)$$

which is a consequence of 3.2. This also shows that Δ is formally self adjoint in the sense

$$(\Delta\alpha, \beta) = (\alpha, \Delta\beta).$$

We recall a simple fact about linear algebra: Let V be a finite dimensional euclidian vector space and $A : V \to V$ a self adjoint linear map, $(Aa, b) = (a, Ab)$. Then the Kernel of A and the image of A obviously are orthogonal. Since the dimension of V equals the dimension of the kernel and the image we get

$$V = \mathrm{kernel}(A) \oplus \mathrm{image}(A).$$

This argument of course breaks down for vector spaces of infinite dimension. Nevertheless:

3.5 Theorem. *Let X be a compact Riemannian manifold. Then*

$$A^p(X) = \mathcal{H}^p(X) \oplus \Delta A^p(X).$$

The space $\mathcal{H}^p(X)$ is finite dimensional.

This is a special case of a general theorem about *elliptic differential equations.* We will give comments about this theorem in an appendix. As a consequence of 3.5 we obtain for $\alpha \in A^p(X)$ a representation

$$\alpha = \alpha_0 + d\beta + d^*\gamma, \qquad \alpha_0 \text{ harmonic.}$$

We apply this to closed forms α. From $d\alpha = 0$ and 3.4 follows $dd^*\gamma = 0$, hence

$$(d^*\gamma, d^*\gamma) = (\gamma, dd^*\gamma).$$

It follows $d\gamma = 0$ and

$$\alpha = \alpha + d\beta$$

This gives a direct decomposition

$$\mathrm{Kernel}(A^p(X) \longrightarrow A^{p+1}(X)) = \mathcal{H}^p \ \oplus\ \mathrm{Image}(A^{p-1}(X) \longrightarrow A^p(X)).$$

This means that every class of closed forms in $H^p_{\mathrm{dR}}(X, \mathbb{R})$ contains a unique harmonic representant:

3.6 Main theorem of real Hodge theory. *Let X be a compact oriented Riemannian manifold. Then $\mathcal{H}^p(X)$ is contained in the space of closed forms and the natural homorphism*

$$\mathcal{H}^p(X) \xrightarrow{\sim} H^p(X, \mathbb{R})$$

is an isomorphism. These vector spaces are finite dimensional.

Hence the so-called it Betti-numbers

$$b^p(X) := \dim H^p(X, \mathbb{R})$$

ere well defined (finite) numbers.

As an application we derive the duality theorem. When α is harmonic then by trivial reasons $*\alpha$ is harmonic too. This obviously defines an isomorphism

$$\mathcal{H}^p(X) \xrightarrow{\sim} \mathcal{H}^{n-p}(X).$$

This shows $b^p = b^{n-p}$. There is a better way to interpret this result. For this we just remind what it means that bilinear map $B : V \times W \to \mathbb{R}$ is non-degenerated. Here V, W are finite dimensional (real) vector spaces. By definition it means that the induced map

$$V \longmapsto W^*, \quad a \longmapsto (b \longmapsto B(a, b)),$$

is an isomorphism. Then V and W must have the same dimension and one can see that the map

$$W \longmapsto V^*, \quad b \longmapsto (a \longmapsto B(a, b))$$

is an isomorphism as well. One just has to check that it is injective. Now we consider for compact X the pairing

$$C(X) \times C^{n-p}(X), \quad (\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta.$$

By Stokes theorem on gets 0 if α or β is closed. Hence there is induced a pairing

$$H^p(X) \times H^{n-p}(X), \quad (\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta.$$

Looking at the harmonic forms it is clear that this pairing is non-degenerate. This gives:

3.7 Poincaré duality. *Let X be a pure n-dimensional compact oriented Riemannian manifold. The pairing $(\cdot,\cdot)$*

$$H^p(X,\mathbb{R}) \times H^{n-p}(X,\mathbb{R}) \longrightarrow \mathbb{R}, \quad (\alpha,\beta) \longmapsto \int_X \alpha \wedge \beta,$$

is non-degenerated and one has

$$b^p(X) = b^{n-p}(X).$$

If X is connected, we have $b^0(X) = b^n(X) = 1$.

An example

Let $L \subset \mathbb{R}^n$ a lattice and $X = \mathbb{R}^n$ the corresponding torus. There is a Riemann metric on X such that the pull-back to $\mathbb{R}^n$ is the standard euclidian metric that is defined by the unit matrix. From the general theory we know that each harmonic function is constant (since $b^0 = 1$). A straight forward calculation shows

$$\Delta \sum f_{i_1,\dots,i_p} dx_{i_1} \wedge \ldots \wedge dx_{ip} = \sum \Delta f_{i_1,\dots,i_p} dx_{i_1} \wedge \ldots \wedge dx_{ip}.$$

Hence a differential form is harmonic if and only if its components is constant. The number of components is $\binom{n}{p}$. So we obtain:

3.8 Proposition. *The Betti numbers of a torus are*

$$b^n = \binom{n}{p}.$$

So Poincaré duality is in concordance with $\binom{n}{p} = \binom{n}{n-p}$.

Chapter III. Complex Hodge theory

1. Holomorphic maps

Let $U \subset \mathbb{C}^n$ be an open subset. A map $f : U \to \mathbb{C}^m$ is called totally complex differentiable if for any point $a \in U$ there exists a $\mathbb{C}$-linear map $A : \mathbb{C}^n \to \mathbb{C}^m$ such that

$$f(z) = f(a) + A(z-a) + r(z), \quad \lim_{z \to a} \frac{r(z)}{\|z-a\|} = 0.$$

Here $\|\cdot\|$ denotes one of the (equivalent) standard norms of $\mathbb{C}^m$. We will identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using

$$(z_1, \dots, z_n) \longleftrightarrow (x_1, y_1, \dots, x_n, y_n).$$

Then a totally complex differentiable map can be considered as totally real differentiable map. From the real theory one knows that A is uniquely determined and we use the usual notation

$$J(f,a) : \mathbb{C}^n \longrightarrow \mathbb{C}^m.$$

But we have two ways to associate to $J(f,a)$ a matrix. Since it is $\mathbb{C}$-linear, it can be described by complex $m \times n$-matrix. Sometimes we denote this matrix by $J_{\mathbb{C}}(f,a)$ and since it is also $\mathbb{R}$-linear we can describe it by a real $(2n) \times (2m)$-matrix which we denote by $J_{\mathbb{R}}(f,a)$. We describe the relation between these two matrices. This is problem in linear algebra. We start with the case $m = n = 1$. A $\mathbb{C}$-linear map $l : \mathbb{C} \to \mathbb{C}$ is given by multiplication with a fixed complex number a $(= l(1))$. If we write $a = \alpha + \mathrm{i}\beta$ then

$$l(z) = az = (\alpha x - \beta y) + \mathrm{i}(\alpha y + \beta x).$$

Hence the real 2-by-2 matrix associated to a is the matrix

$$\tilde{a} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

Let more general $l : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ be a $\mathbb{C}$ linear map and denote A the corresponding complex $n \times m$-matrix. If one replaces each entry a of A by the real

2×2-matrix then one obtains the real $(2n) \times (2m)$-matrix $\tilde{A}$ associated to l if one considers l as $\mathbb{R}$-linear map.

Exercise. Let $m = n$. Then the formula

$$|\det A|^2 = \det \tilde{A} \qquad (\Longrightarrow \det \tilde{A} \geq 0).$$

holds.

It is natural to use for the entries of $J_{\mathbb{C}}(f, a)$ the notation $\partial f_i / \partial z_j(a)$. They are called the complex derivatives. We use the notation

$$f_i = u_i + \mathrm{i} v_i.$$

The corresponding real 2×2-bloc in $J_{\mathbb{R}}(f, a)$ is

$$\frac{\partial f_i}{\partial z_j} \longleftrightarrow \begin{pmatrix} \dfrac{\partial u_i}{\partial x_j} & \dfrac{\partial u_i}{\partial y_j} \\ \dfrac{\partial v_i}{\partial x_j} & \dfrac{\partial v_i}{\partial y_j} \end{pmatrix}.$$

From this description we can see:

1.1 Proposition. *A totally real differentiable function $f : U \to \mathbb{C}^m$ ($U \subset \mathbb{C}^n$ open) is totally complex differentiable if and only if the **Cauchy-Riemann differential equations** hold:*

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial v_i}{\partial y_j}, \quad \frac{\partial u_i}{\partial y_j} = -\frac{\partial v_i}{\partial x_j}$$

and in this case one has

$$\frac{\partial f_i}{\partial z_j} = \frac{\partial u_i}{\partial x_j} + \mathrm{i} \frac{\partial v_i}{\partial x_j}.$$

This shows that complex differentiability is very restrictive property with the consequence that complex analysis is something very special. Without proof we formulated two results that emphasize this difference. (The proofs are given in the case $n = 1$ in any course on complex analysis. The generalization to several variables can be done using the same arguments.

1) Let $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}^n$ open) be complex totally differentiable. Then any point $a \in U$ admits a neighborhood $a \in W \subset U$ such that f can be written in this neighborhood as absolutely convergent power series:

$$f(z) = \sum_{0 \leq \nu_1, \ldots, \nu_n} a_{\nu_1 \ldots \nu_n} (z_1 - a_1)^{\nu_1} \cdots (z_n - a_n)^{\nu_n}.$$

Functions that locally can be expanded into power series often are called *analytic.* For this reason totally complex differentiable functions are called *complex analytic.* They also are called *holomorphic.*

2) **Principle of analytic continuation.** Let f, g be two holomorphic functions on an open connected set $U \subset \mathbb{C}^n$. Assume that f and g agree on some open non empty subset of U. Then f and g agree on the whole U.

3) **Maximum principle.** Let $f : U \to \mathbb{C}$ be a holomorphic function on an open an connected subset of $\mathbb{C}^n$ such that $|f(z)|$ attains a maximum in U. Then f is constant.

We conclude this section with a simple remark:

The *chain* rule is literarily true for holomorphic as for real functions. It simply follows from the real case, since the composite of two $\mathbb{C}$-linear maps is $\mathbb{C}$-linear. Similarly the theorem of invertible functions holds for holomorphic functions. Again this follows from the real case, since the inverse of a bijective $\mathbb{C}$-linear map is automatically $\mathbb{C}$-linear. A formal consequence of the theorem of invertible functions is the theorem of implicit functions, which therefore carries over to the complex case.

2. Complex manifolds

A complex chart on a topological space X is a topological map

$$\varphi : U \longrightarrow V, \quad U \in X,\ V \in \mathbb{C}^n \text{ both open.}$$

Two complex charts are called holomorphically equivalent if the chart transformation is biholomorphic (holomorphic in both directions.) A holomorphic atlas is an atlas of charts such that alle chart transformations inside it are holomorphically compatible. Two such atlases are called equivalent if there union is holomorphic equivalent. A *complex manifold* is a topological space together with an equivalent class of holomorphically equivalent holomorphic atlases. This equivalence class contains a unique maximal atlas. We call this atlas be $\mathcal{O}$ and call its elements *holomorphic charts.*

It is clear how the notion of holomorphic map $f : X \to Y$ between complex manifolds has to be defined. It is also clear that the identity map of a complex manifold is holomorphic and that the composition of holomorphic maps is holomorphic. The map f is called biholomorphic if f is bijective and if f and f^{-1} are holomorphic.

Also the notion of "sub manifold" carries over to the complex case. A first remark in this direction is that open subsets of complex manifolds inherit a structure of complex manifold. Analytic charts $\varphi : U \to V$ turn now out to be nothing else than biholomorphic maps. A subset $Y \subset X$ is called smooth in the complex analytic sense if every point $a \in Y$ admits a chart $\varphi : U \to V$, $a \in U$, such that $\varphi(Y \cap U)$ can be defined by complex linear equations. One can always manage that these equations are $z_{d+1} = \cdots = z_n = 0$. It should be clear that such subsets inherit a structure as complex manifold. Such manifolds are called complex sub-manifolds.

From the theorem of implicit functions follows:

Let $U \subset \mathbb{C}^n$ be an open subset and let $f : U \to \mathbb{C}^m$ be a holomorphic map. Assume that the complex Jacobi matrix of f has rank m at every point of the zero set $\mathcal{N} = \{a \in U;\ f(a) = 0\}$. Then $\mathcal{N}$ is a complex manifold. Its (complex) dimension is $n - m$.

Examples of complex manifolds

A lattice $L \subset V$ of a finite dimensional subgroup is a a subset with the following property: There exists a basise $e_1, \dots, e_n$, such that

$$L = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n.$$

Lattices are subgroups. The associated torus is the factor group $X = V/L$ equipped with the quotient topology. It is easy to equip X with a structure as differentiable manifold. For this an open subset $V \subset V$ is called small, if the projection $V \to X$ is injective. Then the image U is open and the inverse map $U \to V$ is a chart on X. The set of all these charts is a differentiable atlas and defines a structure as differentiable manifold.

Let now V be a finite dimensional complex vector space. By definition a lattice in V is a lattice of the underlying real vector space. The same construction as above shows that $X = V/L$ carries a structure as complex manifold. It is called a complex torus.

A *Riemann surface* is a complex manifold of complex dimension one. They also are called “complex curves”. The simplest examples are open subsets of $\mathbb{C}$. Less trivial examples are complex one dimensional complex tori $\mathbb{C}/L$.

We give another example. We extend $\mathbb{C}$ by some additional symbol ∞ to a set $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We make use of the convention

$$\frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0.$$

W call a subset $U \subset \bar{\mathbb{C}}$ open if the following two conditions are satisfied

1) The intersection $U \cap \mathbb{C}$ is open in the usual sense.

2) Assume $\infty \in U$. Then the set

$$\{z \in \mathbb{C};\quad 1/z \in U\}$$

is open in $\mathbb{C}$ in the usual sense. It is clear that by this definition $\bar{\mathbb{C}}$ gets a compact topological space. It contains $\mathbb{C}$ as open subset and the induced topology on $\mathbb{C}$ is the usual one. A typical neighborhood of ∞ is the set

$$\{z \in \mathbb{C}; \quad |z| > r\} \cup \{\infty\} \qquad (r > 0).$$

each neighborhood of ∞ contains one of this type. It is clear form the definition that the map

$$\sigma : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}, \qquad \sigma(z) = 1/z,$$

is a topological map. We define two charts:

$$\bar{\mathbb{C}} - \{\infty\} \longrightarrow \mathbb{C},\ z \longmapsto z, \qquad \bar{\mathbb{C}} - \{0\} \longrightarrow \mathbb{C},\ z \longmapsto 1/z.$$

The chart transformation

$$\mathbb{C} - \{0\} \longrightarrow \mathbb{C} - \{0\}, \quad z \longmapsto 1/z,$$

is biholomorphic. Hence the two charts define an holomorphic atlas and hence a structure of Riemann surface on $\bar{\mathbb{C}}$.

We want to generalize this example to the case of several variables. This leads to the *complex projective space.*

We denote by $P^n(\mathbb{C})$ the set of all one dimensional subvector spaces of $\mathbb{C}^{n+1}$. If $a \in \mathbb{C}^{n+1}$ is not the zero vector, it generates a one dimensional space $[a] := \mathbb{C}a$. We obtain a surjective map

$$\mathbb{C}^{n+1} - \{0\} \longrightarrow P^n(\mathbb{C}), \quad a \longmapsto [a].$$

We have $[a] = [b]$ if and only if there exist a complex number t with $b = ta$. Hence we can consider alternatively $[a]$ as equivalence class with respect to the equivalence relation

$$a \sim b \Longleftrightarrow b = ta \qquad (a, b \in \mathbb{C}^{n+1} - \{0\}).$$

We equip $P^n(\mathbb{C})$ with the quotient topology. It is not difficult to show that it is a compact topological space. We consider the part

$$U_i := \{[a] \in P^n(\mathbb{C}), \quad a_i \neq 0\}.$$

Notice that the condition a_i does not depend on the choice of the representant. Obviously U_i is an open subset. The map

$$U_i \longrightarrow \mathbb{C}^n, \quad [a_0, \dots, a_n] \longmapsto \Bigl(\frac{a_0}{a_i}, \dots, \widehat{\frac{a_i}{a_i}}, \dots, \frac{a_n}{a_i}\Bigr)$$

is topological. The hat means that the element beyond it should be canceled. The inverse map is given by

$$\mathbb{C}^n \longrightarrow U_i, \quad (z_1, \dots, z_n) \longmapsto [z_1, \dots, z_{i-1}, 1, z_i, \dots, z_n].$$

So the maps $U_i \to \mathbb{C}^n$ are charts that define an atlas. The chart transformations are biholomorphic. Hence we obtain a structure as complex manifold on $P^n(\mathbb{C})$. This is the complex projective space.

Projective manifolds

A polynomial $P \in \mathbb{C}[z_0, \dots, z_n]$ is called homogenous of degree d if

$$P(tz) = t^d P(z) \qquad (t \in \mathbb{C}).$$

This means tht P is of the form

$$P(z) = \sum_{\nu_0+\cdots+\nu_n=0} a_{\nu_0,\dots,\nu_n} z_0^{\nu_0} \cdots z_n^{\nu_n}.$$

Let $[a] \in P^n(\mathbb{C})$. For homogenous P the condition $P(a) = 0$ is independent of the choice of the representant a. Hence we can talk of the zero set of P in $P^n(\mathbb{C})$. A subset $X \subset P^n(\mathbb{C})$ is called *algebraic,* if there exist finitely many homogenous polynomials such that X their joint set of zeros.

Algebraic sets need not to be complex sub-manifold, they can have "singularities". For example $z_0 z_1 z_2 = 0$ defines not a sub-manifold in $P^2(\mathbb{C})$. Very remarkable is the following

2.1 Theorem of Chow. *Let $X \subset P^n(\mathbb{C})$ be a closed complex submanifold. Then X is algebraic.*

2.2 Definition. *A complex manifold X is called* ***projective algebraic****, if there exists for some suitable n a complex submanifold $Y \subset P^n(\mathbb{C})$ that is also an algebraic set and such that X and Y are biholomorphic equivalent.*

Clearly projective complex manifolds are compact.

3. Differential forms on complex manifolds

Complex manifolds can also be considered as (real) differential manifolds (identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$). The real dimension is the double of the complex dimension. Hence the space $\mathcal{C}^\infty(X)$ is defined also for complex manifolds. In this connection it is natural to introduce complex valued differentiable functions and more general complex valued differential forms even in the case of a (real) differentiable manifold X. For this one introduces $\mathrm{Alt}^p_{\mathbb{C}}(V)$ for a real vector space V to be the set $\mathbb{R}$-alternating forms $V \times \dots \times V \to \mathbb{C}$. They are simply of the form $\omega = \omega_1 + \mathrm{i}\omega_2$ with elements $\omega_i \in \mathrm{Alt}^p(V)$. Now $\mathrm{Alt}^p_{\mathbb{C}}(V)$ is a complex vector space in an obvious way. A complex differential form ω is a family $\omega_a \in \mathrm{Alt}^p_{\mathbb{C}}(T_a X)$ that depends differentiable from a. This just means by definition that the real and imaginary part of ω are differentiable. We denote the space of complex valued differential forms by $A^p_{\mathbb{C}}$). They just are of the form $\omega = \omega_1 + \mathrm{i}\omega_2$ with $\omega_i \in A^p(X)$. In the special case $p = 0$ we can identify the elements $A^0_{\mathbb{C}}(X)$ with complex valued functions on X whose real- and imaginary part is differentiable. The space of these function is denoted by $\mathcal{C}^\infty_{\mathbb{C}}(X)$. Using this decomposition it is clear how to define the wedge product and the exterior differentiation. We can define the complex version of the de Rham complex

$$\cdots \longrightarrow A^{p-1}_{\mathbb{C}}(X) \longrightarrow A^p_{\mathbb{C}}(X) \longrightarrow A^{p+1}_{\mathbb{C}}(X) \longrightarrow \cdots$$

and its cohomology groups which now note by $H^p(X,\mathbb{C})$. There is a natural $\mathbb{R}$-linear map

$$H^p(X,\mathbb{R}) \longrightarrow H^p(X,\mathbb{C}).$$

This is injective. We define $H^p(X,\mathbb{R})$ with its image. One has

$$H^p(X,\mathbb{C}) = H^p(X,\mathbb{R}) + \mathrm{i}H^p(X,\mathbb{R}).$$

Hence the Betti numbers are

$$b^p(X) = \dim_{\mathbb{R}} H^p(X,\mathbb{R}) = \dim_{\mathbb{C}} H^p(X,\mathbb{C}).$$

It should be clear that the complex version of the de-Rham complex just contains the same information as the real one. In the following we will use only the complex variant. Hence we change the notation:

From now now on $\mathrm{Alt}^p(V)$ *denotes the space of complex valued* $\mathbb{R}$*-alternating forms on the real vector space* V*, and* $A^p(X)$ *denotes the space of complex valued differential forms and* $\mathcal{C}^\infty(X) = A^0(X)$ *the space of complex valued differentiable functions.*

For the cohomology groups of the (complex) de-Rham complex we keep the notation $H^i(X,\mathbb{C})$.

Decomposition of differential forms

We start with the local case: let $U \subset \mathbb{C}^n$ be an open subset. We introduce

$$dz_i := dx_i + \mathrm{i}dy_i, \quad d\bar z_i := dx_i - \mathrm{i}dy_i.$$

Then we have

$$dx_i = \frac{dz_i + \mathrm{i}d\bar z_i}{2}, \quad dy_i = \frac{dz_i - \mathrm{i}d\bar z_i}{2\mathrm{i}}.$$

3.1 Definition. *A differential form is called of type* (p,q) *if it can be written in the form*

$$\omega = \sum_{\substack{1\le i_1<\cdots<i_p\le n\\ 1\le j_1<\cdots<j_q\le n}} f_{\substack{i_1\ldots i_p\\ j_1\cdots j_q}}\, dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\ldots\wedge d\bar z_{j_q}.$$

The space of all (p,p)*-forms is denoted by*

$$A^{p,q}(U).$$

It is easy to show that the components $f_{\substack{i_1\ldots i_p\\ j_1\cdots j_q}}$ are uniquely determined. So one obtains:

3.2 Lemma. *There is a direct sum decomposition*

$$A^m(X) = \bigoplus_{p+q=m} A^{p,q}(X).$$

Now complex analysis comes into the game:

3.3 Proposition. *Let*

$$\varphi : U \longrightarrow V, \quad U \subset \mathbb{C}^n,\ V \subset \mathbb{C}^m \text{ both open},$$

be a ***holomorphic*** *map. Then (p,q)-types are preserved under pulling back. This means that we get*

$$\varphi^* : A^{p,q}(V) \longrightarrow A^{p,q}(U).$$

Proof. We treat the case $m = n = 1$ that is typical. The general case is essentially the same. We compute $\varphi^*(dz)$. For this we decompose $\varphi = \varphi_1 + \mathrm{i}\varphi_2$ into real and imaginary part. Then (using the coordinates $w = u + \mathrm{i}v$ for V)

$$\varphi^*(dw) = \varphi^*(du) + \mathrm{i}\varphi^*(dv) = (\partial_x\varphi_1 dx + \partial_y\varphi_1 dy) + \mathrm{i}(\partial_x\varphi_2 dx + \partial_y\varphi_2 dy).$$

The Cauchy Riemann differential equation gives $\partial_y\varphi_2 = \partial_x\varphi_1$. Hence we get

$$\varphi^*(dw) = (\partial_z\varphi)\, dz.$$

Here we used the usual notations as $\partial_x = \partial/\partial x$. □

Since φ is a holomorphic function of one variable, one can also use standard notations like

$$\frac{\partial\varphi}{\partial z} = \frac{d\varphi}{dz} = \varphi'.$$

The fact that (p,q)-types are preserved under holomorphic transformations, allows us to generalize the types to arbitrary complex manifolds:

3.4 Definition and Remark. *A differential form $\omega \in A^m(X)$ on a complex manifold is called of type (p,q) if this is the case for all components $\omega_\varphi \in A^m(V_\varphi)$ for any holomorphic chart. It is sufficient to check this for a defining atlas. Especially for open subsets of $\mathbb{C}^n$ one obtains the original notion of (p,q)-type.*

We denote the space of (p,q)-forms on a complex manifold by $A^{p,q}(X)$. Again it is true that

$$A^m(X) = \bigoplus_{p+q=m} A^{p,q}(X).$$

For holomorphic maps $f : X \to Y$ of complex manifolds pilling back gives an operator

$$\varphi^* : A^{p,q}(Y) \longrightarrow A^{p,q}(X).$$

Now a basis fact arises. The (p,q) types are not preserved under exterior differentiation. For example, let $f \in \mathcal{C}^\infty(\mathbb{C})$. Then

$$df = (\partial_x f)dx + (\partial_y f)dy = \frac{\partial_x f - \mathrm{i}\partial_y f}{2} dz + \frac{\partial_x f + \mathrm{i}\partial_y f}{2} d\bar{z}.$$

More generally the product rule in connection with the rule $d \circ d = 0$ implies

$$d(f dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}) = (df) \wedge dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$

This formula shows that d defines an operator

$$d : A^{p,q}(X) \longrightarrow A^{p+1,q}(X) \oplus A^{p,q+1}(X).$$

Hence we can define unique operators

$$\partial : A^{p,q}(X) \longrightarrow A^{p+1,q}(X), \quad \bar{\partial} : A^{p,q}(X) \longrightarrow A^{p,q+1}(X)$$

such that

$$d = \partial + \bar{\partial}.$$

The Wirtinger calculus

The Wirtinger calculus is a very convenient tool to handle the operators $\partial, \bar{\partial}$. It rests on the formula

$$df = \frac{\partial_x f - \mathrm{i}\partial_y f}{2} dz + \frac{\partial_x f + \mathrm{i}\partial_y f}{2} d\bar{z}.$$

This formula implies

$$\partial f = \frac{\partial_x f - \mathrm{i}\partial_y f}{2} dz, \quad \bar{\partial} f = \frac{\partial_x f + \mathrm{i}\partial_y f}{2} d\bar{z}.$$

This formula is valid for arbitrary differentiable functions. In the case of a holomorphic function f it simplifies. Obviously the Cauchy-Riemann differential equations can be written in the form

$$\partial_x f + \mathrm{i}\partial_y f = 0.$$

And the complex derivative of f is

$$\partial_z f = \frac{\partial_x f + \mathrm{i}\partial_y f}{2}.$$

Hence for a holomorphic function the formula simplifies to

$$\partial f = \partial_z f dz.$$

These formula indicate that one should define the operators

$$\frac{\partial}{\partial z_i} = \frac{\partial_x - \mathrm{i}\partial_y}{2} \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_i} = \frac{\partial_x + \mathrm{i}\partial_y}{2}.$$

They act on arbitrary differentiable functions on open subsets $U \subset \mathbb{C}^n$. Such a function is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}_i} = 0$$

and then $\partial f / \partial z_i$ is the usual complex derivative (which justifies the notation.)

3.5 Remark. *let f be a differentiable function on some open subset of $\mathbb{C}^n$. Then*

$$\partial f = \sum_{\nu=1}^{n} \frac{\partial f}{\partial z_\nu} dz_\nu \quad \textit{and} \quad \bar{\partial} f = \sum_{\nu=1}^{n} \frac{\bar{\partial} f}{\partial \bar{z}_\nu} d\bar{z}_\nu.$$

The function f is holomorphic if and only if $\bar{\partial} f = 0$.

4. The Dolbeault complex

For a fixed p we consider the complex

$$\cdots \longrightarrow A^{p,q-1}(X) \xrightarrow{\bar{\partial}} A^{p,q}(X) \longrightarrow A^{p,q+1}(X) \xrightarrow{\bar{\partial}} \cdots.$$

“Complex” means that $\partial \circ \partial = 0$. We then can define

$$\begin{aligned} C^{p,q}(X) &= \text{kernel}(A^{p,q}(X) \longrightarrow A^{p,q+1}(X)), \\ B^{p,q}(X) &= \text{kernel}(A^{p-1,q}(X) \longrightarrow A^{p,q}(X)). \end{aligned}$$

The complex vector spaces

$$H^{p,q}(X) = C^{p,q}(X) / B^{p,q}(X)$$

are the so-called Dolbeault cohomology groups. The numbers

$$h^{p,q} = h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X) \qquad (\leq \infty)$$

are called the Hodge numbers. We will see that they are finite for compact X. Biholomorphic equivalent complex manifolds have the same Hodge numbers. Of course $A^{p,q}(X)$ vanishes if $p < 0$ or $q < 0$ or $p + q > n$. Hence the Hodge numbers can be written in a table which has the form of a diamond, for example in the case of dimension $n = 3$:

$$\begin{array}{ccccccc} & & & h^{00} & & & \\ & & h^{10} & & h^{01} & & \\ & h^{20} & & h^{11} & & h^{20} & \\ h^{30} & & h^{21} & & h^{12} & & h^{03} \\ & h^{31} & & h^{22} & & h^{13} & \\ & & h^{32} & & h^{23} & & \\ & & & h^{33} & & & \end{array}$$

Before we continue we introduce the notion of a *holomorphic differential form.* In the local theory this means that it is of the form

$$\omega = \sum_{i_1 < \cdots < \cdots i_p} f_{i_1,\dots,i_p} dz_{i_1} \wedge \ldots \wedge dz_{i_p}$$

with holomorphic functions $f_{i_1,\dots,i_p}$. This are two conditions:

a) ω is of type $(p, 0)$.

b) One has $\bar{\partial}\omega = 0$.

Observe that the first condition implies that the second is equivalent to the holomorphy of the components. The conditions a) and b) make sense for arbitrary complex manifolds.

4.1 Definition. *A differential form one a complex manifold is called holomorphic if it is of type $(p, 0)$ and if it is in the kernel of $\bar{\partial}$.*

We denote the space of all holomorphic forms of type $(p, 0)$ by

$$\Omega^p(X).$$

Clearly $\Omega^0(X)$ can be identified with the space $\mathcal{O}(X)$ of holomorphic functions.

There is a relation to $H^{p,0}(X)$. Since $A^{p,-1}(X) = 0$ we have $B^{p,0}(X) = 0$. Hence we have

$$H^{p,0}(X) = C^{p,0}(X) = \Omega^p(X).$$

4.2 Remark. *There is a natural isomorphism*

$$\Omega^p(X) \cong H^{p,0}(X).$$

Already the case $p = 0$ is interesting here. One a connected compact complex manifold every holomorphic function is constant by the maximum principle. Hence we see

$$h^{00}(X) = 1 \qquad (X \text{ connected}).$$

There are two complex variants of the Lemma of Poincaré. We formulate them without proof. We will not need them in the rest of the text.

4.3 Holomorphic Lemma of Poincaré. *Let $U \subset \mathbb{C}^n$ an open convex domain. Every holomorphic differential form ω of degree $p > 0$ with the property $d\omega = 0$ can be written in the form $\omega = d\alpha$ with a holomorphic differential form α.*

The proof is similar to the proof of the real Lemma of Poincaré and hence very easy.

By a *polydisk* in $\mathbb{C}^n$ one understands an open subset of the form $U_1 \times \ldots \times U_n$, where U_i are open discs in $\mathbb{C}$.

4.4 Lemma of Dolbeault. *Let ω be a differential form on a polydisk of degree (p, q) with $q > 0$. Assume that $\bar\partial\omega = 0$. Then there exists a differential form α of degree $(p, q-1)$ such that $\omega = \bar\partial\alpha$*

The proof is much more involved as that of the Lemmas of Poincaré. Already the case $n = 1$ is difficult.

5. A complex structure on the real tangent space of a complex manifold

The tangent space of $\mathbb{R}^n$ has been identified with $\mathbb{R}^n$. Hence the tangent space of $\mathbb{C}^n$ can be identified with $\mathbb{C}^n$. This indicates that the tangent space T_aX (taken from the real theory) should have a structure as complex vector space. Notice: The dimension of T_aX is $2n$. Hence if T_aX is equipped with a structure as complex vector space, the complex dimension should be n. For this construction it is convenient to introduce the *holomorphic tangent space* $T_a^{\text{hol}}(X)$ for a point a on a complex manifold. This is analogous to Definition 5.1.

5.1 Definition. *Let $a \in X$ be a point in a complex manifold. A **holomorphic derivation** at a is a family of maps*

$$D : \mathcal{O}(U) \longrightarrow \mathbb{C},$$

where U runs through all open neighborhoods of a such that the following two conditions hold:

1) *It is compatible with restriction.*
2) *It is $\mathbb{C}$-linear.*
3) *It satisfies the product rule*

$$D(fg) = f(a)D(g) + g(a)D(f).$$

The set of holomorphic derivations is a complex vector space. We denote it by $T_a^{\text{hol}}X$. As in the real case a holomorphic map $f : X \to Y$ induces now a $\mathbb{C}$-linear map

$$T_a^{\text{hol}}f : T_a^{\text{hol}}X \longrightarrow T_{f(a)}^{\text{hol}}Y.$$

This is compatible with composition of holomorphic maps and it is an isomorphism for biholomorphic maps. The space $T_a^{\text{hol}}\mathbb{C}^n$ is n-dimensional a (complex) basis is given by the derivations $\partial/\partial z_\nu$, $1 \le \nu \le n$. The proof is the same as in the real case. So for a n-dimensional complex manifold $T_a^{\text{hol}}X$ is a complex vector space of dimension n. We want to compare it with the tangent space T_aX of the real theory. This is a real vector space of dimension $2n$. Hence both tangent spaces have real dimension $2n$ and hence are isomorphic as real vector spaces. We want to define a natural ($\mathbb{R}$-linear) isomorphism

$$T_aX \xrightarrow{\sim} T_a^{\text{hol}}X.$$

For this consider a derivation $A \in T_aX$. Recall that A acts on real valued differentiable functions. We can extend a to complex valued differentiable functions (defined in some open neighborhood) of a by the definition $D(u+\mathrm{i}v) = D(u) + \mathrm{i}D(v)$. Then we can restrict D to holomorphic functions. We call this the natural map.

5.2 Lemma. *The natural map*

$$T_aX \longrightarrow T_a^{\text{hol}}X$$

is an isomorphism of real vector spaces.

Proof. Since both are vector spaces of the same (real) dimension, it is sufficient to show surjectivity. Using a chart this can be reduced to the case $X = \mathbb{C}^n$. But for holomorphic (!) functions we have

$$\frac{\partial}{\partial x_i} \longmapsto \frac{\partial}{\partial z_i} \quad \text{and} \quad \frac{\partial}{\partial y_i} \longmapsto \mathrm{i}\frac{\partial}{\partial z_i}.$$

Since $\partial/\partial z_i$ and $\mathrm{i}\partial/\partial z_i$ generate $T_a^{\text{hol}}\mathbb{C}^n$ as real vector space we get surjectivity. □

Now make use of the fact that $T_a^{\text{hol}}X$ is complex vector space. We can transport this structure to get a structure as complex vector space on T_aX.

5.3 Proposition. *Let X be a complex manifold. The real tangent space T_aX at a point $a \in X$ carries a structure as complex vector space, such that the natural map*

$$T_aX \longrightarrow T_a^{\text{hol}}X$$

is a $\mathbb{C}$-linear isomorphism.

How is the complex structure on T_aX defined concretely? On could think that multiplication by i of a derivation A is defined by $(\mathrm{i}A)(f) = \mathrm{i}A(f)$. But this is nonsense. Of course one can define the operator $\mathrm{i}A$ by this formula. But $\mathrm{i}A$ is no longer contained in the real tangent space T_aX. Hence multiplication with i inside T_aX must be something different. For this reason we use a different notation:

The operator $J : T_aX \to T_aX$ means multiplication with i *with respect to the introduced complex structure,*

So $J(A)$ must be contained in T_aX and has nothing to do with $\mathrm{i}A$ as defined above. In local coordinates it is easy to make J concrete:

5.4 Lemma. *The operator $J : T_a\mathbb{C}^n \to T_a\mathbb{C}^n$ is given by*

$$J(\partial/\partial x_i) = \partial/\partial y_i, \quad J(\partial/\partial y_i) = -\partial/\partial x_i.$$

At the first glance it looks strange that J has to do something like multiplying with i. But notice that one holomorphic functions J is the usual multiplication with i.

Linear algebra background

Using the complex structure on the real tangent space T_aX the decomposition of differential forms can be reconsidered in pure algebraic context. Recall for a real vector space V we meanwhile use the notation

$$\mathrm{Alt}^1(V) = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}).$$

In the case that V is the underlying real vector space of a complex vector space —now called V— this space gets some extra structure. Namely one can consider the subspaces

$$\mathrm{Alt}^{1,0}(V) := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}), \quad \mathrm{Alt}^{0,1}(V) := \overline{\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})}.$$

It is easy to see that

$$\mathrm{Alt}^1(V) = \mathrm{Alt}^{1,0}(V) \oplus \mathrm{Alt}^{0,1}(V).$$

More generally one defines $\mathrm{Alt}^{p,q}(V)$ to be the vector space generated by

$$a_1 \wedge \dots, \wedge a_p \wedge b_1 \wedge \dots b_q, \quad a_i \in \mathrm{Alt}^{1,0}(V),\ b_i \in \mathrm{Alt}^{(0,1)}.$$

Using bases it is easy to show

$$\mathrm{Alt}^m(V) = \bigoplus_{p+q=m} \mathrm{Alt}^{p,q}(V).$$

5.5 Remark. *Let X be a complex manifold. A differential form $\omega \in A^m(X)$ is of type (p,q) iff for every point $a \in X$ one has*

$$\omega_a \in A^{p,q}(T_a X).$$

Proof. This is local problem. Hence X can be assumed to be an open subset of $\mathbb{C}^n$. All what we have to show is that

$$dz_i : T_a U \longrightarrow \mathbb{C}$$

is $\mathbb{C}$-linear. Here $T_a X$ of course carries the complex structure defined by J. Hence we have to show for example

$$dz_i(\partial/\partial x_j) = \mathrm{i} dz_i(J(\partial/\partial x_j)).$$

To see this one has to use $J(\partial/\partial x_j) = \mathrm{i}\partial/\partial y_)$ and the fact that by definition $dx_1, dy_1, \dots, dx_n, dy_n$ is dual to $\partial/\partial x_1, \partial/\partial y_1, \dots, \partial/\partial x_n, \partial/\partial y_n$. This implies $dz_i(\partial/\partial x_j) = \delta_{ij}$ and $dz_i(\partial/\partial y_j) = \mathrm{i}\delta_{ij}$.

The complexified real tangent space

We also can give an algebraic interpretation of the operators $\partial/\partial z_i$ and $\partial/\partial \bar{z}_i$. For this one introduces the complexified tangent space

$$T_a^{\mathbb{C}} X = T_a X + \mathrm{i} T_a X.$$

Its elements are complex valued derivations that can be written (uniquely) in the form $A + \mathrm{i}B$, $A, B \in T_a$. We can apply them to complex valued differentiable functions by $\mathbb{C}$-linear extension. Hence elements of $T_a^{\mathbb{C}} X$ act on complex valued differentiable functions on open neighborhoods of a and they produce complex numbers. In the local case where X is an open subset $U \in \mathbb{C}^n$ we can consider the complex tangent vectors

$$\partial/\partial z_i|_a = \frac{\partial/\partial x_i|_a - \mathrm{i}\partial/\partial y_i|_a}{2}, \quad \partial/\partial \bar{z}_i|_a = \frac{\partial/\partial x_i|_a + \mathrm{i}\partial/\partial y_i|_a}{2}.$$

The operators $\partial/\partial z_i$, $\partial/\partial \bar{z}_i$ now can be considered as complex valued vector fields.

Also the operator J can be extended as $\mathbb{C}$-linear map to $T_a^{\mathbb{C}} X$. The action on $\partial/\partial z_i$, $\partial/\partial \bar{z}_i$ is given by (we omit "$|a$" in the notation)

$$J(\partial/\partial z_i) = \mathrm{i}\partial/\partial z_i, \quad J(\partial/\partial \bar{z}_i) = -\mathrm{i}\partial/\bar{\partial} z_i.$$

So we see:

5.6 Remark. *Let X be a complex manifold. The complexified tangent space $T_a^{\mathbb{C}}$ decomposes as direct sum of two subspaces*

$$T_a^{\mathbb{C}} X = T_a^{1,0} X \oplus T_a^{0,1} X,$$

where J acts on $= T_a^{1,0}X$ by multiplication with i *and on $T_a^{0,1}X$ by multiplication with* $-$i. *In the local case these subspaces are generated by $\partial/\partial z_i$ resp. by $\partial/\partial \bar{z}_i$.*

Recall that we have a natural restriction map

$$T_a^{\mathbb{C}} X \longrightarrow T_a^{\text{hol}} X.$$

Its restriction $T_a X \longrightarrow T_a^{\text{hol}} X$ is an $\mathbb{R}$-linear isomorphism that has been used to define the operator J. Of course the restriction

$$T_a^{1,0} X \longrightarrow T_a^{\text{hol}} X$$

is also an isomorphism. In some sense it looks more natural because J and mutiplication with i are the same on $T_a^{1,0}X$. Notice also that $\partial/\partial z_i \in T^{1,0}X$ maps to $\partial/\partial z_i$ (considered now as operator that acts on holomorphic functions). Hence $T_a^{1,0}X$ can be identified with the T_a^{hol} very naturally and sometimes the holomorphic tangent space is defined to be $T_a^{1,0}X$.

□

6. Hermitian manifolds

We want to generalize the notion of Riemannian manifold to the complex case. We start with some elementary linear algebra. Let V a finite dimensional complex vector space. A *hermitian form* on V is a map

$$h : V \times V \longrightarrow \mathbb{C}$$

with the following properties.

1) It is $\mathbb{C}$-linear in the first variable.
2) It has the property $h(a, b) = \overline{h(b, a)}$.

As a consequence $h(a, a)$ is always real. The hermitian form is called positive definite if $h(a, a) > 0$ for all non-zero a. The standard example of a positive definite hermitian form is $V = \mathbb{C}^n$ and $h(z, w) = \sum z_i \bar{w}_i$. This is essentially the unique example. More precisely: Every positive definite hermitian form admits an orthonormal basis $e_1, \dots, e_n$ in the sense $h(e_i, e_j) = \delta_{ij}$.

The real part

$$g(a, b) = \operatorname{Re} h(a, b)$$

of a hermitian form is a symmetric real bilinear form (on the underlying real vector space). It is positive definite if h is so.

The imaginary part

$$A(a,b) = \operatorname{Im} h(a,b)$$

is an alternating real bilinear form ($A(a,b) = -A(b,a)$). The real part g determines h because of the formula

$$h(a,b) = g(a,b) + \mathrm{i}g(a,\mathrm{i}b).$$

But also A determines g and hence also h, since

$$A(a,b) = g(a,-\mathrm{i}b).$$

Not every real symmetric bilinear form g comes from a hermitian on. A necessary condition is

$$g(a,b) = g(\mathrm{i}a,\mathrm{i}b).$$

But this condition is also sufficient, since then one can check that $h(a,b) = g(a,b) + \mathrm{i}g(a,\mathrm{i}b)$ is hermitian. Similarly one can check that an alternating real bilinear form A is the imaginary part of a hermitian form if and only if $A(a,b) = A(\mathrm{i}a,\mathrm{i}b)$.

6.1 Definition. *A hermitian metric h on a complex manifold is a collection of positive definite hermitian forms h_a on the tangent space T_aX that depend differentiable on a. A hermitian manifold (X,h) is a pair consisting of a complex manifold and a hermitian metric on it.*

It should be clear what differentiable means. For example it is sufficient to demand that $g_a = \operatorname{Re} h_a$ is differentiable in a. Hence a hermitian manifold also can be considered as a Riemannian manifold.

Let $U \subset \mathbb{C}^n$ be an open subset and h an hermitian metric on it. This is just given be a matrix $h_{ij} = h(\partial/\partial z_i, \partial/\partial z_j)$ of differentiable functions that is hermitian and positive definite at every point.

6.2 Lemma. *Let h be a hermitian metric on an open subset $U \subset \mathbb{C}^n$ and $g = \operatorname{Re} h$ the associated Riemannian metric. Then*

$$g(\partial/\partial x_i, \partial/\partial x_j) = g(\partial/\partial y_i, \partial/\partial y_j), \quad g(\partial/\partial x_i, \partial/\partial y_j) = 0$$

and conversely every Riemann metric with this property comes from a hermitian one.

Proof. One just has to use the formula $g(a,b) = g(Ja, Jb)$ and the formula how J acts on the $\partial/\partial x_i$ and $\partial/\partial x_i$. □

This implies that the star operator is defined. Originally the star operator has been defined of real differential forms through the formula

$$\alpha \wedge *\beta = g^*(\alpha, \beta)\omega \qquad (\omega \text{ volume form.}$$

We extend $*$ to complex valued differential forms as $\mathbb{C}$-linear map. Similarly we extend g^* to complex valued forms as $\mathbb{C}$-bilinear map. The the above defining formula remains valid in the complex valued case.

The star operator has special properties if the Riemann metric g is the real part of a hermitian metric. In this case it preserves (p,q)-types in a certain sense:

6.3 Lemma. *Let (X, h) be a hermitian manifold. The star operator preserves the (p,q) in the sense that it defines maps*

$$* : A^{p,q}(X) \longrightarrow A^{n-q,n-p}(X).$$

Proof. We use the linear algebra description. We have to show for a complex vector space with hermitian metric the star operator induces a map

$$\mathrm{Alt}^{p,q}(V) \longrightarrow \mathrm{Alt}^{n-q,n-p}(V).$$

For this it is convenient to use an orthonormal basis $e_1, \dots, e_n$ of V. Obviously then $e_1, \mathrm{i}e_1, \dots, e_n, \mathrm{i}e_n$ then is an orthonormal basis of the underlying real euclidian space. We denote the dual basis in $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ by $A_1, B_1, \dots, A_n, B_n$. Obviously this is a orthonormal basis of $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. It is also a $\mathbb{C}$-basis of $\mathrm{Alt}^1(V) = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, where we use the natural inclusion $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$. We denote the symmetric $\mathbb{C}$-bilinear pairing on $\mathrm{Alt}^1(V)$ that is induced by g simply by

$$\langle A, B \rangle = g^*(A, B).$$

So we have

$$\langle A_i, A_j \rangle = \langle B_i, B_j \rangle = \delta_{ij}, \quad \langle A_i, B_j \rangle = 0.$$

Now we introduce the elements

$$L_i := A_i + \mathrm{i}B_i.$$

They are $\mathbb{C}$-linear and they define a $\mathbb{C}$-basis of $A^{1,0}(V) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Moreover the $\bar{L}_i = A_i - \mathrm{i}B_i$ form a $\mathbb{C}$-basis of $A^{1,0}(V)$. One also checks

$$\langle L_i, L_j \rangle = \langle \bar{L}_i, \bar{L}_j \rangle = 0, \quad \langle L_i, \bar{L}_j \rangle = 2\delta_{ij}.$$

Let $a \subset \{1,\dots,b\}$ be a subset. We define

$$L_a = L_{a_1}\wedge\ldots\wedge L_{a_p},\quad \bar L_a = \bar L_{a_1}\wedge\ldots\wedge \bar L_{a_p},$$

where $a_1 < \cdots < a_p$ are the elements of a in their natural order. We claim

$$\langle L_a\wedge \bar L_b, L_\alpha\wedge\bar L_\beta\rangle \neq 0 \Longrightarrow a=\beta,\ b=\alpha.$$

This is a simple consequence of the formula

$$\langle \alpha_1\wedge\ldots\wedge\alpha_p, \beta_1\wedge\ldots\beta_p\rangle = \det(\langle\alpha_i,\beta_j\rangle).$$

(We derived this formula in the real valued case. The same argument works in the complex valued case.) Using the definition of the star operator one now obtains

$$*(L_a\wedge\bar L_b) = C\cdot L_\alpha\wedge\bar L_\beta,$$

where α is the complement of b and β is the complement of a. This finishes the proof of 6.3. □

7. The complex Laplace Beltrami operator.

We introduced the operator

$$\bar\partial : A^{p,q}(X)\longrightarrow A^{p,q+1}(X).$$

Similar to the operator d^* we are looking for a *complex codifferentiation*

$$\bar\partial^* : A^{p,q}(X)\longrightarrow A^{p,q-1}(X).$$

One can get a natural one as follows. Let $\alpha\in A^{p,q}(X)$. Then $*\alpha\in A^{n-q,n-p}$. We take its complex conjugate

$$\bar *\alpha := \overline{*\alpha}\in A^{n-p,n-q}.$$

Then we differentiate

$$\bar\partial\bar *\alpha\in A^{n-p,n-q+1}$$

and apply the operator $\bar *$ again,

$$\bar\partial^* := -\bar *\,\bar\partial\,\bar * : A^{p,q}(X)\longrightarrow A^{p,q-1}(X).$$

An equivalent definition is

$$\bar\partial^* = -*\,\partial\,*\,.$$

The sign can be explained as follows: It can be checked that for forms with compact support this operator satisfies

$$(\bar\partial\alpha, \beta) = (\alpha, \bar\partial^*\beta) \quad \text{where} \quad (\alpha, \beta) := \int\limits_X \alpha \wedge *\bar\beta.$$

We define the complex Laplace-Beltrami operators as

$$\bar\Box := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : A^{p,q}(X) \longrightarrow A^{p,q}(X)$$

We denote by

$$\mathcal{H}^{p,q}(X) \subset A^{p,q}(X)$$

the kernel of $\bar\Box$.

The point is that $\bar\Box$ is also an elliptic operator. Similar arguments as in the real case show:

7.1 Main theorem of complex Hodge theory. *Let X be a compact Hermitean manifold. Then $\mathcal{H}^{p,q}(X)$ is contained in the space of $\bar\partial$-closed forms and the natural homomorphism*

$$\mathcal{H}^{p,q}(X) \xrightarrow{\sim} H^{p,q}(X)$$

is an isomorphism. These vector spaces are finite dimensional.

Hence the so-called *Hodge-numbers*

$$h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$$

are well defined numbers.

There is also a duality result. The operator $\bar *$ clearly maps harmonic to harmonic forms. hence we get that $\mathcal{H}^{p,q}(X)$ and $\mathcal{H}^{n-p,n-q}(X)$ are isomorphic. Similar to the real case II.3.7 this can be reformulated as in terms of a duality pairing.

7.2 Duality. *Let X be a pure n-dimensional compact hermitian manifold. The integral $\int_X \alpha \wedge \bar *\beta$ induces a non-degenerated pairing*

$$H^{p,q}(X) \times H^{n-p,n-q}(X) \longrightarrow \mathbb{C}$$

in the sense that the induced map

$$H^{n-p,n-q}(X) \longrightarrow \operatorname{Hom}_{\mathbb{C}}(H^{p,q}(X), \mathbb{C})$$

is an (only $\mathbb{R}$-linear) isomorphism. So one has

$$h^{p,q}(X) = h^{n-p,n-q}(X).$$

We finally remark that there is also an analogous result for the complex $\partial : A^{p,q}(X) \to A^{p+1,q}$. Here one defines an Laplace Beltrami operator

$$\Box = \partial\partial^* + \partial^*\partial, \quad \text{where} \quad \partial^* = -\bar{*}\partial\bar{*} = - * \bar{\partial} *,$$

and gets completely analogous results. But there is no need to formulate them, since the two variants of the theory are interchanged by complex conjugation. There is no reason to prefer the $\bar{\partial}$ or the ∂-complex. Finally we mention

$$\partial^* + \bar{\partial}^* = d^*.$$

IV. Kähler manifolds

1. Kähler metrics

We already mentioned that also the imaginary part of a hermitian form is of interest. It is an alternating bilinear form. If (X,h) is a hermitian manifold, then

$$\Omega := \operatorname{Im} h$$

is a (real) alternating differential form of degree two. We compute it local coordinates. So let h be a hermitian metric on some open domain $U \subset \mathbb{C}$. The tangent space $T_a\mathrm{U}$ at some point a has the real basis

$$\partial/\partial x_1, \dots, \partial/\partial x_n, \partial/\partial y_1, \dots, \partial/\partial y_n.$$

We want to use a complex basis. Recall that the complex structure in T_a is defined by an operator J with $J^2 = -\operatorname{id}$ and $J(\partial/\partial x_i) = \mathrm{i}\partial/\partial y_1$. Hence as complex basis we can take the $\partial/\partial x_i$. The matrix

$$h_{ij} = h(\partial/\partial x_i, \partial/\partial x_j)$$

is a positive definite hermitian matrix. We get

$$\begin{aligned} h(\partial/\partial x_i, \partial/\partial x_j) &= h_{ij}, \\ h(\partial/\partial y_i, \partial/\partial y_j) &= h_{ij}, \\ h(\partial/\partial x_i, \partial/\partial y_j) &= -\mathrm{i}h_{ij}, \\ h(\partial/\partial y_i, \partial/\partial x_j) &= \mathrm{i}\bar{h}_{ij}. \end{aligned}$$

Taking the imaginary part A we get

$$\begin{aligned} A(\partial/\partial x_i, \partial/\partial x_j) &= \operatorname{Im} h_{ij}, \\ A(\partial/\partial y_i, \partial/\partial y_j) &= \operatorname{Im} h_{ij}, \\ A(\partial/\partial x_i, \partial/\partial y_j) &= \operatorname{Re} h_{ij}, \\ A(\partial/\partial y_i, \partial/\partial x_j) &= -\operatorname{Re} \bar{h}_{ij}. \end{aligned}$$

Another way to write this is

$$\begin{aligned} A = &\sum_{ij} \operatorname{Im} h_{ij} dx_i \otimes dx_j + \sum_{ij} \operatorname{Im} h_{ij} dy_i \otimes dy_j + \\ &\sum_{ij} \operatorname{Re} h_{ij} dx_i \otimes dy_j - \sum_{ij} \operatorname{Re} h_{ij} dy_i \otimes dx_j. \end{aligned}$$

It is easy to check that this equals

$$A = \sum_{ij} \frac{\mathrm{i}}{2} h_{ij}(dz_i \otimes dz_j - dz_j \otimes dz_i).$$

Using our definition of the wedge product we get the formula

$$\boxed{\Omega = \frac{\mathrm{i}}{2} \sum_{1 \le i,j \le n} h_{ij}\, dz_i \wedge dz_j}$$

In the case of the standard metric of $\mathbb{C}^n$ the components h_{ij} are constant. In this case we have that Ω is a closed form, $d\Omega = 0$. This property has turned out to be very basis. It leads to the following definition:

1.1 Definition. *A compact hermitian manifold is called **Kählerian**, if the form Ω is closed, $d\Omega = 0$.*

We give some examples:

1) Complex tori (with the standard metric) are Kählerian.
2) The projective space (with standard metric) is Kählerian.
3) Compact Riemann surfaces (with any hermitian metric) are Kählerian.

We also mention the following. Let Y be a closed complex submanifold of a Kählerian manifold (X, h). The restriction of the hermitian metric h to Y equips Y with a structure as Kählerian manifold. As a consequence each projective algebraic variety admits a structure as Kählerian manifold.

1.2 Theorem. *A Hermitian metric h is Kählerian if and only if the following is true: For every point $a \in X$ there exists a holomorphic chart $\varphi : U \to V$, $a \in U$, $\varphi(a) = 0$, such that the components h_{ij} of the fundamental form Ω with respect to this chart satisfy:*

$$h_{ij}(0) = \delta_{ij}, \qquad \partial h_{ij}/\partial x_k(0) = \partial h_{ij}/\partial x_k(0) = 0.$$

It is clear that this condition implies $d\Omega = 0$. So we have to prove the converse. Assume $d\Omega = 0$. We can assume that X is an open set $U \subset \mathbb{C}^n$ and that $a = 0$ is the origin. Recall

$$\Omega = \frac{\mathrm{i}}{2} \sum_{ij} h_{ij} dz_i \wedge d\bar{z}_j.$$

First we use a linear transformation $z \mapsto Az$, $A \in \mathrm{GL}(n, \mathbb{C})$. The matrix h has to be replaced by $\bar{A}' h A$. We use the well-known result of linear algebra that each positive definite hermitian matrix h can be transformed by such a transformation into the unit matrix. Hence we can assume that $h(0)$ is the unit matrix. This property will be preserved during the rest of the proof.

We introduce the numbers

$$a_{ijk} = \frac{\partial h_{ij}}{\partial z_k}(0), \quad b_{ijk} = \frac{\partial h_{ij}}{\partial \bar z_k}(0).$$

Then we have

$$h_{ij} = \delta_{ij} + \sum_k a_{ijk} z_k + \sum b_{ijk} \bar z_k + r_{ij},$$

where the remainder r_{ij} and its first partial derivatives vanish at 0. We decompose

$$\Omega = \Omega_{\text{main}} + R,$$

where

$$R = \sum_{ij} r_{ij} dz_i \wedge dz_j.$$

From $d\Omega = 0$ and form $h_{ij} = \bar h_{ji}$ one derives the relations

$$a_{ijk} = a_{kji}, \quad b_{ijk} = a_{ikj} \quad \text{and} \quad b_{ijk} = \bar a_{jik}.$$

The transformation

$$w_k = z_k + \frac{1}{2} \sum_{i,j=1}^{n} a_{ijk} z_i z_j$$

maps a small open neighborhood of U biholomorphically onto an open neighborhood V. We have to transform Ω into V. We denote the transformed form by $\tilde\Omega$. We have $\tilde\Omega = \tilde\Omega_{\text{main}} + \tilde R$ with obvious notation. The form $\tilde R$ is without interest, since $\tilde R$ and its first derivatives vanish at the origin. This is easily proved by means of the chain rule. So we have to determine $\tilde\Omega_{\text{main}}$. A straight forward calculations gives

$$\frac{\mathrm{i}}{2} \sum_{j=1}^{n} dw_j \wedge d\bar w_j = \tilde\Omega_{\text{main}}.$$

This completes the proof of 1.2.

Examples of Kähler manifolds

Usually one describes the Kähler form Ω if one wants to define a Kähler structure. So one has to define a $(1,1)$-form Ω. In local coordinates it is given by a hermitian matrix (h_{ij}).

$$\Omega = \frac{\mathrm{i}}{2} \sum_{ij} h_{ij} dz_i \wedge dz_j.$$

The matrix $h = (h_{ij})$ has to be positive definit for every holomorphic chart (from a defining atlas is enough). Then Ω comes form a hermitian metric. It is Kählerian if $d\Omega = 0$.

Complex tori. Let $L \subset \mathbb{C}^n$ be lattice. We consider the differential form

$$\Omega = \frac{\mathrm{i}}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i.$$

This form is invariant under translations. This implies that there is a differential form on the torus $X = \mathbb{C}^n/L$ whose inverse image on $\mathbb{C}^n$ with respect to the natural projection is Ω. We denote this form on the torus by Ω too. It defines a Kähler metric on the torus.

The projective space. We consider the open subset of $P^n(\mathbb{C})$ that is defined by $z_i \neq 0$ and we consider the chart

$$\varphi_i : U_i \longrightarrow \mathbb{C}^n, \quad (z_0, \dots, z_n) \longmapsto \Big(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i}\Big).$$

We wirite $w_1, \dots, w_n$ for the coordinates in $\mathbb{C}^n$. We consider the differential form

$$\Omega_i = \frac{\mathrm{i}}{2\pi} \partial\bar{\partial} \log\Big(\sum_{i=1}^{n} |w_i|^2 + 1).$$

One can check that there is a differential form Ω on $P^n(\mathbb{C})$ which has the forms Ω_i as components. On also can check that this defines a Käehler metric on $P^n(\mathbb{C})$. This metric is called the *Fubini-Study* metric.

If (X, Ω) is Kähler manifold and if $Y \subset X$ is a complex submanifold then $(Y, \Omega|Y)$ is a Kähler manifold too. As a consequence each projective complex manifold carries a structure as Kähler manifold. For this reason, the Kähler theory is a basic tool for algebraic geometry.

2. The Hodge decomposition

We introduce two basic operators. Let (X, h) be a hermitian manifold and Ω the associated fundamental form:

The *Lefschetz operator* is

$$L : A^{p,q}(X) \longrightarrow A^{p+1,q+1}(X), \quad \alpha \longmapsto \alpha \wedge \Omega.$$

The *dual Lefschetz operator* is

$$\Lambda : A^{p,q}(X) \longrightarrow A^{p-1,q-1}(X), \quad \Lambda = *^{-1} \circ L \circ *.$$

In the Kählerian case there are fundamental commutation rules between the complex derivatives. We recall the notation $[A, B] = A \circ B - B \circ A$.

2.1 Theorem (Kähler relations). *Let (X, h) be a Kähler manifold. Then the follwing relations hold:*

1) $[\bar\partial, L] = [\partial, L] = 0$ *and* $[\bar\partial^*, \Lambda] = [\partial^*, \Lambda] = 0$.
2) $[\bar\partial^*, L] = \mathrm{i}\partial$, $[\partial^*, L] = -\mathrm{i}\bar\partial$ *and* $[\Lambda, \bar\partial] = -\mathrm{i}\partial^*$, $[\Lambda, \partial] = \mathrm{i}\bar\partial^*$.

Since these formulas only involve first derivatives it is sufficient to very them for the standard metric in $\mathbb{C}^n$. This can be done be a straight forward calculation. □

A basic consequence of the Kähler relations is:

2.2 Theorem. *On a Kähler manifold the relation*

$$\Delta = 2\bar\Box$$

holds.

Proof. We have

$$\begin{aligned}\Delta &= (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial) \\ &= \Box + \bar\Box + (\partial\bar\partial^* + \bar\partial^*\partial) + \overline{(\partial\bar\partial^* + \bar\partial^*\partial)}.\end{aligned}$$

We first show

$$(\partial\bar\partial^* + \bar\partial^*\partial) = 0.$$

Actually

$$\mathrm{i}(\partial\bar\partial^* + \bar\partial^*\partial) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = \partial\Lambda\partial - \partial\Lambda\partial = 0.$$

It remains to show $\Box = \bar\Box$. This is done by the same method. One uses the formulae

$$[\Lambda, \bar\partial] = -\mathrm{i}\partial^*, \quad [\Lambda, \partial] = \mathrm{i}\bar\partial^*$$

to express $\Box$and $\bar\Box$ merely by ∂, $\bar\partial$ and Λ. Then the identity gets obvious if one uses also

$$[\bar\partial^*, \Lambda] = [\partial^*, \Lambda] = 0.$$ □

Chapter V. Elliptic differential operators

1. Differential operators

Let $\Omega \subset \mathbb{R}^n$ be an open subset. We are interested in maps

$$D : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega)$$

which can be written as finite sum

$$Df = \sum h_{i_1 \dots i_m} \frac{\partial^{i_1+\cdots+i_m} f}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}$$

with differentiable coefficients $h_{\dots}$. Clearly they are uniquely determined. We call D a *linear differential operator.* When D is non-zero there exists a maximal m such that $h_{i_1,\dots,i_n}$ is non-zero for some index with $i_1 + \cdots + i_n = m$. We call m the degree of this operator and the function on $\Omega \times \mathbb{R}^n$

$$P(x_1, \dots, x_n, X_1, \dots, X_n) = \sum_{i_1+\cdots+i_n=m} h_{i_1,\dots,i_m}(x) X_1^{i_1} \dots X_n^{i_n}$$

is called the *symbol* of D. This is homogenous polynomial of degree m for fixed x. The operator D is called *elliptic,* if it is not zero and if

$$P(x, X) \neq 0 \quad \text{for all} \quad X \neq (0, \dots, 0).$$

2. Oscillating integrals

In the following we denote the euclidian scalar product simply by

$$xy = \sum x_i y_i$$

and by

$$|x| = \sqrt{xx}$$

the euclidian norm.

We fix two natural numbers ν, n. In our applications we will have $\nu = n$ or $\nu = 2n$. Let $\Omega \subset \mathbb{R}^\nu$ be an open subset and $P \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^n)$. For multi indices $\alpha = (\alpha_1, \dots, \alpha_\nu)$ and $\beta = (\beta_1, \dots, \beta_n)$ we use the notation

$$P_\alpha^\beta(x, \xi) := \Bigl(\frac{\partial}{\partial x}\Bigr)^\alpha \Bigl(\frac{\partial}{\partial \xi}\Bigr)^\beta P(x, \xi).$$

2.1 Definition. *let m be a real number. The space $S^m(\Omega, \mathbb{R}^n)$ consists of all functions $P \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^n)$ with the following property: For each pair of multi indices (α, β) there exists a locally bounded function $c_{\alpha,\beta}$ on Ω such that*

$$\bigl|P_\alpha^\beta(x, \xi)\bigr| \le c_{\alpha,\beta}(x)\bigl(1 + |\xi|\bigr)^{m-|\beta|}.$$

Here we use the usual notation $|\beta| = \beta_1 + \cdots + \beta_n$ for multi indices. A locally bounded function is bounded on each compact subset. Hence on could also say that the estimate holds for each compact subset with a constat $c_{\alpha,\beta}$ that may depend on K.

By trivial reason we have

$$S^m(\Omega, \mathbb{R}^n) \subset S^{m'}(\Omega, \mathbb{R}^n) \quad \text{for} \quad m < m'.$$

We define

$$S^\infty(\Omega, \mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S^m(\Omega, \mathbb{R}^n), \quad S^{-\infty}(\Omega, \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S^m(\Omega, \mathbb{R}^n).$$

Example. The function

$$(1 + |\xi|^2)^{-m/2}$$

is contained in $S^m(\Omega, \mathbb{R}^n)$. The Leibniz product rule on $\mathbb{R}^n$ stats

$$\partial^\alpha(fg) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g)$$

where the usual multi-index conventions are used. Hence $\beta \le \alpha$ means $\beta_i \le \alpha_i$ for $1 \le i \le n$ and $\binom{\alpha}{\beta} = \prod \binom{\alpha_i}{\beta_i}$. From this rule one deduces

$$f \in S^m(\Omega, \mathbb{R}^n), \quad g \in S^{m'}(\Omega, \mathbb{R}^n) \Longrightarrow fg \in S^{m+m'}(\Omega, \mathbb{R}^n).$$

In the following we use the notations

$$\mathcal{D}(X) = \mathcal{C}_c^\infty(X), \quad \mathcal{E}(\Omega) = \mathcal{C}^\infty(X)$$

for a differentiable manifold (at the moment always an open subset of some $\mathbb{R}^n$).

Let be $P \in S^m(\Omega, \mathbb{R}^n)$. We introduce the oscillating integral

$$Au = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \int\limits_{\Omega} e^{\mathrm{i}x\xi} P(x,\xi) u(x)\, dx\, d\xi \quad \text{for} \quad u \in \mathcal{D}(\Omega).$$

The problem is that this integral does not exists in the usual sense of a Lebesgue integral though we assume that u has compact support. Of course the inner integral exists without any problem but then we get a problem since after integrating over x there is no need that resulting function in ξ has compact support. Nevertheless we want to make sense of the integral in all cases.

One idea is to integrate first over ξ and then look what happens. Hence we are lead to look at an integral of the kind

$$\int\limits_{\mathbb{R}^n} e^{\mathrm{i}a\xi} P(\xi) d\xi.$$

Here we assume that

$$|\partial^\beta P(\xi)| \le c_\beta (1 + |\xi|)^{m-|\beta|}$$

for all multi-indices β. We just write $S^m(\mathbb{R}^m)$ for this class of functions. (At the moment we don't need the variable $x \in \Omega$.) This integral will not exist in the usual sense, as the example $P(\xi) = \xi^2$ shows already. To get existence of the integral we introduce a cut-off function $\chi : \mathbb{R} \to \mathbb{R}$ that is decreasing and such that

$$\chi(\xi) = 1 \text{ for } \xi \le 1 \quad \text{and} \quad \chi(\xi) = 0 \text{ for } \xi \ge 2.$$

We then have

$$\lim_{\varepsilon \to 0^+} \chi(\varepsilon\xi) = 1 \qquad \text{(pointwise limit)}.$$

The integral $\int_{\mathbb{R}^n} e^{\mathrm{i}a\xi} P(\xi)\chi(\varepsilon\xi) d\xi$ exists for $\varepsilon > 0$ by trivial reason and we can try to take the limit $\varepsilon > 0$. This actually works:

2.2 Lemma. *Let $P \in S^m(\mathbb{R}^n)$ and let $a \neq 0$. Then the limit*

$$\lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^n} e^{\mathrm{i}a\xi} P(\xi)\chi(\varepsilon\xi) d\xi$$

exists.

We will simply write

$$\int_{\mathbb{R}^n} e^{\mathrm{i}a\xi}P(\xi)d\xi := \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} e^{\mathrm{i}a\xi}P(\xi)\chi(\varepsilon\xi)d\xi$$

Since the convergence is caused by the oscillation of the function $e^{\mathrm{i}x\xi}$ will call this kind of integral an *oscillating integral.*

Proof of 2.2. We use partial integration:

$$\int_{\mathbb{R}^n} e^{\mathrm{i}a\xi}P(\xi)\chi(\varepsilon\xi)d\xi = \frac{1}{(\mathrm{i}a)^{|\beta|}}\int_{\mathbb{R}^n} e^{\mathrm{i}a\xi}\partial_\xi^\beta(P(\xi)\chi(\varepsilon\xi))d\xi.$$

Now we use the well known fact from real analysis that the integral

$$\int_{\mathbb{R}^n}(1+|\xi|)^m d\xi$$

exists for $m<-n$. Hence if we choose $|\beta|$ big enough the above integral exists in the usual sense also for $\varepsilon=0$ and one can apply the Lebesgue limit theorem to show

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} e^{\mathrm{i}a\xi}P(\xi)\chi(\varepsilon\xi)d\xi = \frac{1}{(\mathrm{i}a)^{|\beta|}}\int_{\mathbb{R}^n} e^{\mathrm{i}a\xi}\partial_\xi^\beta(P(\xi))d\xi.$$

Especially the integral exists. □

In our application (definition of Au the function P also depends on a variable $x\in\Omega$ and we want to integrate also over x. Lemma 2.2 shows that this is possible if 0 is not contained in Ω. But we must include the case $0\in\Omega$. Since 0 is a single point, it is possible of course that integration over x exists. This needs a refinement of the above argument of partial integration which we are going to explain now in some detail.

In the definition of Au the function P occurs in the combination $P(x,\xi)u(x)$ where u has compact support. As P this product is also contained in $S^m(\Omega,\mathbb{R}^n)$. Hence functions $P\in S^m$ that are compactly supported with respect to x come into our interest:

2.3 Lemma. *Let $P\in S^m(\Omega,\mathbb{R}^n)$. Assume that there exists a compact set $K\subset\Omega$ such that $P(x,\xi)=0$ for $x\notin K$. In the case $m+n<0$ the integral*

$$\int_{\Omega\times\mathbb{R}^n} P(x,\xi)dxd\xi$$

exists in the Lebesgue sense.

Corollary. *In the case $m+n<0$ the oscillating integral exists in the sense of the usual Lebesgue integral.*

Again this follows form the fact that $\int_{\mathbb{R}^n}(1+|\xi|)^m d\xi$ exists for $m < -n$. We don't give more details. □

In the case $m+n<0$ we get from the Lebesgue limit theorem

$$\int\limits_{\Omega\times\mathbb{R}^n} e^{\mathrm{i}x\xi}P(x,\xi)u(x)\,dx\,d\xi = \lim_{\varepsilon\to 0^+}\int\limits_{\Omega\times\mathbb{R}^n} e^{\mathrm{i}x\xi}P(x,\xi)\chi(\varepsilon\xi)u(x)\,dx\,d\xi.$$

Of course the integral

$$\int\limits_{\Omega\times\mathbb{R}^n} e^{\mathrm{i}x\xi}P(x,\xi)\chi(\varepsilon\xi)u(x)\,dx\,d\xi \quad (\varepsilon>0)$$

exists for all m. One can try to take the limit. This actually works:

2.4 Proposition. *Let $P\in S^m(\Omega,\mathbb{R}^n)$ and $u\in\mathcal{D}(\Omega)$. The limit*

$$\lim_{\varepsilon\to 0^+}\int\limits_{\Omega\times\mathbb{R}^n} e^{\mathrm{i}x\xi}P(x,\xi)\chi(\varepsilon\xi)u(x)\,dx\,d\xi$$

exists.

We will denote this limit simply by

$$\int\limits_{\Omega\times\mathbb{R}^n} e^{\mathrm{i}x\xi}P(x,\xi)u(x)\,dx\,d\xi \quad \text{or by} \quad \int\limits_{\mathbb{R}^n}\int\limits_{\Omega} e^{\mathrm{i}x\xi}P(x,\xi)u(x)\,dx\,d\xi$$

and call it an oscillating integral.

Using 2.2 (to be correct one should say that one needs a version of 2.2 that includes a x-dependency) one sees immediately that the limit exists if $0\notin\Omega$. But we definitely want to include the origin. This needs a refinement of the technique of partial integration

Proof of 2.4. Again we want to apply integration by parts but in a refined form: We use differential operators of the form

$$L=\sum_{i=1}^{n} a_i(x,\xi)\frac{\partial}{\partial\xi_i}+\sum_{j=1}^{\nu} b_j(x,\xi)\frac{\partial}{\partial x_j}+c(x,\xi).$$

Here a_i, b_j, c are differential functions. The so-called formal adjoint of L is defined by

$$L'=\sum_{i=1}^{n} \alpha_i(x,\xi)\frac{\partial}{\partial\xi_i}+\sum_{j=1}^{\nu} \beta_j(x,\xi)\frac{\partial}{\partial x_j}+\gamma(x,\xi),$$

where

$$\alpha_i = -a_i, \quad \beta_j = -b_j, \quad \gamma = c - \sum_{i=1}^{n} \frac{\partial a_i}{\partial \xi_i} - \sum_{j=1}^{\nu} \frac{\partial b_j}{\partial x_j}.$$

The meaning of the formal adjoint is as follows: Let f_1, f_2 be differentiable functions on $\Omega \times \mathbb{R}^n$ with compact support then

$$\int\limits_{\Omega\times\mathbb{R}^n} (L' f_1) \cdot f_2 \, dx d\xi = \int\limits_{\Omega\times\mathbb{R}^n} f_1 \cdot L f_2 \, dx d\xi.$$

We only want to consider operators with the property

$$L' e^{\mathrm{i}x\xi} = e^{\mathrm{i}x\xi}.$$

Then we get

$$\int\limits_{\mathbb{R}^n} \int\limits_{\Omega} e^{\mathrm{i}x\xi} P(x,\xi)\chi(\varepsilon\xi)u(x) \, dx \, d\xi = \int\limits_{\mathbb{R}^n} \int\limits_{\Omega} e^{\mathrm{i}x\xi} L(P(x,\xi)\chi(\varepsilon\xi)u(x)) \, dx \, d\xi$$

We can iterate this with the same operator several times to get

$$\int\limits_{\mathbb{R}^n} \int\limits_{\Omega} e^{\mathrm{i}x\xi} P(x,\xi)\chi(\varepsilon\xi)u(x) \, dx \, d\xi = \int\limits_{\mathbb{R}^n} \int\limits_{\Omega} e^{\mathrm{i}x\xi} L^k(P(x,\xi)\chi(\varepsilon\xi)u(x)) \, dx \, d\xi$$

We introduce

$$Q_\varepsilon(x,\xi) = P(x,\xi)\chi(\varepsilon\xi)u(x).$$

This is a function from $S^m(\Omega, \mathbb{R}^n)$ and there exists a compact subset $K \subset \Omega$ such that Q_ε vanishes outside $K \times \mathbb{R}^n$.

Now the idea is to construct L in such a way that

$$L : S^q(\Omega, \mathbb{R}^n) \longrightarrow S^{q-1}(\Omega, \mathbb{R}^n) \quad \text{(for all } q\text{)}.$$

Assume for a moment that this has been done. Then for sufficiently large k the integral

$$\int\limits_{\mathbb{R}^n} \int\limits_{\Omega} e^{\mathrm{i}x\xi} L^k(Q_\varepsilon(x,\xi)) \, dx \, d\xi$$

exists for $\varepsilon = 0$ and their is a good chance that its value at $\varepsilon = 0$ is its limit for $\varepsilon \to 0^+$. So this limit would exist!

So we have to construct an appropriate L.

2.5 Lemma. *There exists an operator*

$$L = \sum_{i=1}^{n} a_i(x,\xi)\frac{\partial}{\partial \xi_i} + \sum_{j=1}^{\nu} b_j(x,\xi)\frac{\partial}{\partial x_j} + c(x,\xi)$$

with the properties

a) $L' e^{\mathrm{i}x\xi} = \mathrm{e}^{\mathrm{i}x\xi}$.

b) $a_i \in S^0, \quad b_j, c \in S^{-1}$.

Proof. For simplicity of notation (but not essentially) we assume $n = \nu = 1$. Then we define

$$\alpha = \frac{1-\chi(|\xi|)}{\mathrm{i}(1+x^2)}x, \quad \beta = \frac{1-\chi(|\xi|)}{\mathrm{i}(1+x^2)\xi}, \quad \gamma = \chi(|\xi|).$$

In the formula for β the denominator ξ appears. But the nominator vanishes in a neighborhood of $\xi = 0$. Hence β can be defined as a differentiable function that vanishes in a neighborhood of $\xi = 0$.

It is no problem that the operator L' fixes $e^{\mathrm{i}x\xi}$. The corresponding operator L is given by the data

$$a = -\alpha, \quad b = -\beta, \quad c = \gamma - \frac{\partial \alpha}{\partial \xi} - \frac{\partial \beta}{\partial x}.$$

For large ξ the function a is independent of ξ. This shows $a \in S^0$. The function b grows as $1/\xi$ and is contained in S^{-1}. Similarly one sees that $c \in S^{-1}$.

□

Now we know that the integral

$$\int_{\mathbb{R}^n}\int_{\Omega} e^{\mathrm{i}x\xi} L^k(Q_\varepsilon(x,\xi))\,dx\,d\xi$$

exists for $\varepsilon \geq 0$ but we still have to prove that as a function of ε its is continuous at $\varepsilon = 0$.

A limit theorem

Just for convenience we replace the limit $\varepsilon \to 0^+$ by (an arbitrary) sequence $\varepsilon_\mu \to 0$. Using the notation of the previous section we set

$$f_\mu(x,\xi) := e^{\mathrm{i}x\xi} L^k(Q_{\varepsilon_\mu}(x,\xi)) \quad \text{and} \quad f = e^{\mathrm{i}x\xi} L^k(Q_0(x,\xi)).$$

Recall that these are functions from S^{m-k} and that we choose k such that $m - k + n < 0$. We know $f_\mu \to f$. It is clear that this convergence is uniform on each compact subset and that this is true also for all derivatives of arbitrary order. Unfortunately this is not enough to prove

$$\lim_{\mu\to\infty} \int\limits_{\Omega\times\mathbb{R}^n} f_\mu(x,\xi)dxd\xi = \int\limits_{\Omega\times\mathbb{R}^n} f(x,\xi)dxd\xi.$$

(The integrals exist because of $m-k+n<0$.) We want to apply the Lebesgue limit theorem and for this one needs other conditions for the kind of the convergence $f_\mu \to f$. For example it would be sufficient to have an estimate (independent of μ) of the form

$$|f_\mu(x,\xi)| \le C(1+\xi)^{m-k}.$$

(Notice that $m+k<n$.) Actually we shall be able to prove a slightly weaker result which is enough for our purpose. We will see:

For each $\varepsilon>0$ there exists an estimate

$$|f_\mu(x,\xi)| \le C(1+\xi)^{m-k+\varepsilon} \qquad (C=C(\varepsilon)).$$

This is enough since we can choose $\varepsilon>0$ such that $m-k+\varepsilon<n$.

The rest of this section is devoted the proof of this estimate. We introduce the following notation:

Let $K\subset\Omega$ be a compact subset and $l\ge 0$ an integer. Then we set for $P\in S^m$:

$$N^m_{K,l}(P) := \sup_{\substack{(x,\xi)\in K\times\mathbb{R}^n\\ |\alpha|+|\beta|\le l}} \left|\frac{P^\beta_\alpha(x,\xi)}{(1+|\xi|)^{m-|\beta|}}\right|.$$

This number is finite due to the definition of S^m. One can use these functions to define a topology on S^m. For our purpose it is enough to explain the resulting notion of convergent sequences:

2.6 Definition. *Let P_μ be a sequence in S^m and P a fixed element in S^m. The sequence P_μ converges to P in S^m if $N^m_{K,l}(P_\mu - P)$ converges to zero for all K and l.*

There are two obvious stability properties:

1) *Assume $P_\mu\to P$ in S^m and $P'_\mu\to P'$ in $S^{m'}$. Then $P_\mu P'_\mu \to PP'$ in $S^{m+m'}$.*

2) Assume $P_\mu\to P$ in S^m. Then $LP_\mu \to LP$ in S^{m-1}.

It is good to understand the convergence $\chi_\mu\to\chi$ with $\chi_\mu(\xi)=\chi(\varepsilon_\mu\xi)$. As we mentioned already this convergence is locally uniform and this is true for all higher derivatives. But it is not true that χ_μ converges to χ uniformly on the whole $\mathbb{R}^n$. The reason for this is that $\chi_\mu-\chi$ takes the value 1 for each μ somewhere. We can consider χ_μ as element of $S^0(\Omega,\mathbb{R}^n)$. But this consideration shows that χ_μ does not converge to χ in S^0. But we can consider χ_μ also as sequence in S^ε for arbitrary $\varepsilon>0$. The following simple lemma is left as an exercise to the reader:

2.7 Remark. *For each $\varepsilon > 0$ the sequence χ_μ converges to χ in S^ε.*

Using this remark and the stability properties we obtain the convergence $f_\mu \to f$ in $S^{m-k+\varepsilon}$. This shows especially

$$|f_\mu(x,\xi) - f(x,\xi| \le C(1+\xi)^{m-k+\varepsilon}$$

Since f are contained in S^{m-n} and hence in $S^{m-n+\varepsilon}$ we get

$$|f_\mu(x,\xi)| \le C(1+\xi)^{m-k+\varepsilon}$$

which allows to apply the Lebesgue limit theorem.

This completes the definition of the oscillating integral and we get two different possible definitions for it:

$$\begin{aligned}&\int\limits_{\mathbb{R}^n}\int\limits_{\Omega} e^{\mathrm{i}x\xi} P(x,\xi)u(x)\,dx\,d\xi \\ &\qquad = \lim_{\varepsilon\to 0^+}\int\limits_{\mathbb{R}^n}\int\limits_{\Omega} e^{\mathrm{i}x\xi} P(x,\xi)\chi(\varepsilon\xi)u(x)\,dx\,d\xi \\ &\qquad = \int\limits_{\mathbb{R}^n}\int\limits_{\Omega} e^{\mathrm{i}x\xi} L^k(P(x,\xi)u(x))\,dx\,d\xi \quad (m-k+n<0).\end{aligned}$$

3. Pseudodifferential operators

From now on Ω is an open subset of $\mathbb{R}^n$. We are interested in the space $S^m(\Omega,\mathbb{R}^n)$ ($\nu = n$). We also are interested $S^m(\Omega\times\Omega,\mathbb{R}^n)$ ($\nu = 2n$). Then for fixed $x\in\Omega$, the function $(y,\xi)\mapsto P(x,y,\xi)$ is contained in $S^m(\Omega,\mathbb{R}^n)$. We can consider the oscillating integral now to produce a function in x:

$$Au(x) = (2\pi)^{-n}\int\limits_{\Omega}\int\limits_{\mathbb{R}^n} e^{\mathrm{i}(x-y)\xi}P(x,y,\xi)u(y)dyd\xi.$$

Here $u\in\mathcal{D}(X)$ (differentiable function with compact support.

3.1 Lemma. *let $P\in S^m(\Omega\times\Omega,\mathbb{R}^n)$. Then for each $u\in\mathcal{D}(\Omega)$ the function Au is differentiable. Hence the oscillating integral defines an operator*

$$L_P : \mathcal{D}(\Omega) \longrightarrow \mathcal{E}(\Omega).$$

Proof. We consider the operator L as in 2.5 but for the domain $\Omega\times\Omega$ instead of Ω. Hence the functions a_j, b_j, c there are defined on $\Omega\times\Omega\times\mathbb{R}^n$. We write the oscillating integral in the form

$$Au(x) = (2\pi)^{-n} \int\limits_{\Omega}\int\limits_{\mathbb{R}^n} e^{\mathrm{i}(x-y)\xi} L^k(P(x,y,\xi)u(y))dyd\xi$$

with a big enough k. We want to differentiate by x. This means that we only have to consider x from some compact subset. Since u has compact support, we can get a compact subset $K\subset\Omega\times\Omega$ such that in the following argument only elements of $(x,y)\in K$ occur. We investigate the derivatives ∂_α with respect to x of the integrand. Using the produce rule for differentiation one derives that there is an estimate by $C(1+\xi)(1+|\xi|)^{|\alpha|+m-k}$. The constant is independent of $(x,y)\in K$. We can k take large enough (for given α) such that the integral exists. Now one can apply the Lebesgue version of the Leibniz rule that allows to interchange integration and taking derivatives. □

3.2 Definition. *An operator $A:\mathcal{D}(\Omega)\longrightarrow\mathcal{E}(\Omega)$ is called a* ***pseudodifferential operator*** *if it is of the form $A=L_P$ for a suitable $P\in S^m(\Omega\times\Omega,\mathbb{R}^n)$ (with a suitable n). The set of all these operators is denoted by $L^m(\Omega)$.*

There is a very important special case where P is independent of y. The elements of $S^m(\Omega\times\Omega,\mathbb{R}^n)$ with this property can be identified with the elements of $S^m(\Omega,\mathbb{R}^n)$. In this case we have

$$Au(x) = (2\pi)^{-n} \lim_{\varepsilon\to 0}\int\limits_{\Omega}\int\limits_{\mathbb{R}^n} e^{\mathrm{i}(x-y)\xi} P(x,\xi)\chi(\varepsilon\xi)u(y)dyd\xi.$$

The integral over y is a Fourier transformation. The function u can be extended by zero to a function of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ and we can write

$$Au(x) = (2\pi)^{-n} \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^n} e^{\mathrm{i}x\xi} P(x,\xi)\chi(\varepsilon\xi)\hat{u}(\xi)d\xi,$$

where $\hat{u}$ denotes the Fourier transform of u. Since the Fourier transform of a compactly supported $\mathcal{C}^\infty$-function is temperated, the integral exists also for $\varepsilon=0$ and the Lebesgue limit theorem allows to interchange the limit and integration. Hence we obtain:

3.3 Lemma. *If the function $P\in S^m(\Omega\times\Omega)$ is independent of y then the associated pseudodifferential operator can be written as*

$$Au(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{\mathrm{i}x\xi} P(x,\xi)\hat{u}(\xi)d\xi$$

which is a usual Lebesgue integral.

Examples of pseudodifferential operators

Let $K(x,y)$ be a differentiable function on $\Omega \times \Omega$. Then one can define for $u \in \mathcal{D}(\Omega)$

$$Au(x) = \int\limits_{\Omega} K(x,y)u(y)dy.$$

It is clear that this function depends differentiably on x. Hence we get an operator

$$A : \mathcal{D}(\Omega) \longrightarrow \mathcal{E}(\Omega).$$

An operator of this kind is called an *integral operator.* The function K is determined by the operator A. It is called the kernel function of A. It is clear that K is determined by A.

3.4 Remark. *Integral operators are pseudodifferential operators. More precisely: The set $L^{-\infty}(\Omega)$ agrees with the set of integral operators.*

Proof. First let A we an integral operator defined by the kernel K. We choose a function $g \in \mathcal{D}(\mathbb{R}^n)$ with the property $\int_{\mathbb{R}^n} g(\xi)d\xi = 1$ and define

$$P(x,y,\xi) = e^{\mathrm{i}(y-x)\xi} g(\xi)K(x,y).$$

Then $P \in S^{-\infty}$ and one easily checks $L_P = A$. This also shows that in contrast to the kernel of an integral operator, the function P is not determined by the corresponding pseudodifferential operator.

Now let $P \in S^{-\infty}$. We want to show that the corresponding pseudodifferential operator is an integral operator. One takes

$$K(x,y) = \int\limits_{\mathbb{R}^n} e^{\mathrm{i}(x-y)\xi} P(x,y,\xi)d\xi.$$

Since $P \in S^{-\infty}$ this integral exists and is a differentiable function. The same argument as in the proof of 3.1 shows that K is differentiable. That the integral operator of K and the pseudodifferential operator L_P agree follows from Fubini's theorem.

3.5 Lemma. *Each differential operator —considered as map $A : \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ is a pseudodifferential operator. More precisely: If A is of degree m then $A \in L^m(\Omega)$.*

Proof. Using multi-indices we can write a linear differential operator A in the form

$$A = \sum_{\alpha} h_\alpha D^\alpha, \quad D = \frac{1}{\mathrm{i}}(\partial_1, \ldots, \partial_n).$$

The Fourier inversion formula for $u \in \mathcal{D}(\Omega)$, considered as function form $\mathcal{D}(\mathbb{R}^n)$ states

$$u(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{\mathrm{i}x\xi} \hat{u}(\xi) d\xi.$$

Differentiating under the integral we get

$$\begin{aligned}(Au)(x) &= (2\pi)^{-n} \sum_\alpha h_\alpha(x) D^\alpha e^{\mathrm{i}x\xi} \hat{u}(\xi) d\xi \\ &= (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{\mathrm{i}x\xi} P(x,\xi) \hat{u}(\xi) d\xi.\end{aligned}$$

Here we use the notation

$$P(x,\xi) = \sum_\alpha h_\alpha(x) \xi^\alpha.$$

This can be considered as element $S^m(\Omega, \mathbb{R}^n)$, where m is the degree of A. We can consider P also as element of $S^m(\Omega \times \Omega, \mathbb{R}^n)$ (independent of y) and we A turns out to be the associated pseudodifferential operator. □

4. Asymptotic expansion of symbols

In this section we consider an open set $\Omega \subset \mathbb{R}^\nu$ with arbitrary ν. We introduce an equivalence relation in S^∞. By definition $P \sim P'$ means $P - P' \in S^{-\infty}$.

We need another notation: Let P_k be a sequence of elements of $S^\infty(\Omega, \mathbb{R}^n)$ and P a further element from this set. By definition the notation

$$P \sim \sum_{k=1}^\infty P_k$$

means:

a) There exists a sequence $m_k \to -\infty$ such that $P_k \in S^{m_k}$.
b) There exists a sequence $m'_k \to -\infty$ such that

$$P - \sum_{j=1}^k P_j \in S^{m'_k}.$$

The function P is not determined by the P_k. If Q is another function with the same property, then $P \sim P'$. We call $P \sim \sum_{k=1}^\infty P_k$ an *asymptotic expansion* of P.

There is a fundamental existence result:

4.1 Theorem. *Let $P_k \in S^{m_k}$ be a sequence such that $m_k \to -\infty$. Then there exists a P such that $P \sim \sum P_k$.*

Proof. It is sufficient to assume $m_k \le -1$ and one also easily reduces to the case $P_k \in S^{-k-1}$. So let's assume this. The idea is to replace P_k be some equivalent $\tilde{P}_k \sim P_k$ such that the sum $\sum_k \tilde{P}_k$ converges in a good sense. For this we consider an increasing function $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ with the property $\varphi(x) = 0$ for $x \le 1$ and $\varphi(x) = 1$ for $x \ge 2$.

Now we choose a sequence λ_k of positive numbers. Then we define

$$\tilde{P}_k(x,\xi) = P_k(x,\xi)\varphi\Big(\frac{|\xi|}{\lambda_k}\Big).$$

Then clearly $\tilde{P}_k - P_k \in S^{-\infty}$. We want to arrange the sequence λ_k in such a way that $\sum \tilde{P}_k$ converges. For this we need an estimate of $\tilde{P}_k$ and its derivatives $\tilde{P}^\beta_{k,\alpha}$.

4.2 Lemma. *For each k, α, β there exists a locally bounded function $d^\beta_{k,\alpha}$ on Ω such that*

$$(1+|\xi|)^{k+|\beta|}|\tilde{P}^\beta_{k,\alpha}(x,\xi)| \le d^\beta_{k,\alpha}(x)\lambda_k^{-1}$$

for all k, α, β, x, ξ.

Proof of 4.2. In the case $\alpha = 0$ and $\beta = 0$ the inequality states

$$(1+|\xi|)^k|P_k(x,\xi)|\varphi\Big(\frac{|\xi|}{\lambda_k}\Big) \le d_k(x)\lambda_k^{-1}.$$

This case is very easy. The left hand side vanishes if $|\xi| \le \lambda_k$ or if $|\xi| \ge 2\lambda_k$. Hence only the range $\lambda_k \le |\xi| \le 2\lambda_k$ is of interest. Since $P_k \in S^{-k-1}$ the left hand side can be compared in this range with $(1+|\xi|)^k(1+|\xi|)^{-1-k}$ and this is $\le \lambda_k^{-1}$.

The general case can be treated in a similar but notational more involved way. One has to apply the product rule to get the derivatives of $\tilde{P}_k$ and use the same method for the pieces which arise. We omit details. □

So far we needed no conditions for the constants λ_k. But now we have to adapt them to get good convergence of $\sum \tilde{P}_k$. For this we choose an exhaustion $\Omega = \bigcup K_k$ by compact subsets such that K_i is contained in the interior of its successor K_{i+1}. We denote the interior of K_i by U_i. Then $\Omega = \bigcup U_i$ and each compact set in Ω is contained in one of the U_i. Now we choose the constants λ_k. We can choose them inductively such the that estimate

$$\boxed{\sum_{|\alpha|+|\beta|\le k}(1+|\xi|)^{k+\beta}|\tilde{P}^{\beta}_{k,\alpha}(x,\xi)|\le 2^{-k}}$$

is valid for all $x\in U_k$, $\xi\in\mathbb{R}^n$ and all α,β with $|\alpha|+|\beta|\le k$. Especially we have $|\tilde{P}_k(x,\xi)|\le 2^{-k}$ for $x\in U_k$. This shows that the series $P=\sum\tilde{P}_k$ converges locally uniformly on $\Omega\times\mathbb{R}^n$. The estimate for the higher derivatives gives that this also true for the series of the higher derivatives. This shows that P is a $\mathcal{C}^\infty$-function.

It remains to show that for all N

$$P-\sum_{k\le N}\tilde{P}_k\in S^{-N}(\Omega\times\mathbb{R}^n).$$

The condition $P-\sum_{j=1}^k P_j\in S^{m'_k}$ is hard to check since it involves all derivatives. Actually this condition can be weakened:

4.3 Proposition. *Let be $P\in\mathcal{E}(\Omega,\mathbb{R}^n)$ and let be $P_k\in S^{m_k}$ such that m_k is a decreasing sequence that tends to $-\infty$. We assume:*

a) *There exist numbers $m_{\alpha,\beta}$ such that*

$$|P^{\beta}_{\alpha}(x,\xi)|\le C_{\alpha,\beta}(x)(1+\xi)^{m_{\alpha,\beta}}$$

with certain locally bounded functions $C_{\alpha,\beta}$.

b) *For all N we have*

$$\Big|P(x,\xi)-\sum_{k=1}^{N}P_k(x,\xi)\Big|\le C_N(x)(1+|\xi|)^{m_N}$$

with locally bounded functions C_N.

Then one has

$$P\in\mathcal{S}^\infty\quad\textit{and}\quad P\sim\sum_{k=1}^{\infty}P_k.$$

Asymptotic expansion of pseudodifferential operators

Integral operators are in some sense trivial operators. Two pseudodifferential operators P_1, P_2 are called equivalent, $P_1 \sim P_2$ if there difference is an integral operator. In other words we consider the factor space

$$L^m(\Omega)/L^{-\infty}(\Omega).$$

We have a natural surjective map

$$S^m(\Omega \times \Omega, \mathbb{R}^n)/S^{-\infty}(\Omega \times \Omega, \mathbb{R}^n) \stackrel{\sim}{\longrightarrow} L^m(\Omega)/L^{-\infty}(\Omega).$$

An element $P \in S^m(\Omega \times \Omega, \mathbb{R}^n)$ is called a *symbol* of a pseudodifferential operator L if $L \sim L_P$.

As for symbols we define: Let A be a pseudo differential operator and A_k a sequence of them. Then $A \sim \sum A_k$ means that $A_k \in L^{m_k}(\Omega)$ with $m_k \to -\infty$ and $A - \sum_{j=1}^{A} P_j \in L^{m'_k}(\Omega)$ with $m'_k \to \infty$. So $P \sim \sum_k P_k$ clearly implies $L_P \sim \sum_k L_{P_k}$ for the associated operators.

4.4 Theorem. *Let $A_k \in L^k(\Omega)$ be a sequence of pseudodifferential operators such that $k \to -\infty$. Then there exists a pseudo differential operator $A \sim \sum A_k$. This operator is unique up to an integral operator.*

This is a reformulation of 4.1. □

5. Pseudo differential operators and distributions

Let $\Omega \subset \mathbb{R}^\nu$ be an open subset. One can introduce a structure as Fréchet space on $\mathcal{E}(\Omega)$. This structure is characterized by the following property. A sequence (f_n) converges in $\mathcal{E}(\Omega)$ then f_n and all its partial derivatives converge uniformly on compact subsets. The usual limit of this sequence then also is the limit in the Fréchet space $\mathcal{E}(\Omega)$. We consider the dual $\mathcal{E}'(\Omega)$. This is the set of all continuous linear forms on $\mathcal{E}(\Omega)$. The elements f $\mathcal{E}'(X)$ are called *distributions with compact support.*

We consider the scalar product

$$\langle f, g \rangle = \int_\Omega f(x)g(x)dx.$$

Here $f, g \in \mathcal{E}(\Omega)$ and at least one of the two has to be in $\mathcal{D}(\Omega)$. This scalar produce enables to define an injective maps

$$\mathcal{D}(\Omega) \longrightarrow \mathcal{E}'(\Omega), \qquad f \longmapsto (g \mapsto \langle f, g \rangle).$$

We identify $\mathcal{D}(\Omega)$ with their images. Hence the elements of $\mathcal{D}(\Omega)$ are considered as special distributions with compact support. If D is a distribution one sometimes writes

$$\langle D, u\rangle := D(u).$$

One also can define $\mathcal{D}'(\Omega)$. One cannot take the induced topology of $\mathcal{E}(\Omega)$ since it may happen that a sequence (f_n) in $\mathcal{D}(\Omega)$ converges with respect to the topology of $\mathcal{E}(\Omega)$ against a function that is not compactly supported. Therefore we consider for a compact subset $K \subset \Omega$ the space

$$\mathcal{D}_K(\Omega) := \{f \in \mathcal{E}(\Omega); \quad \text{support}(f) \subset G\}.$$

It is easy to see that this is a closed subset of $\mathcal{E}(\Omega)$ and hence a Fréchet space. The union of all $\mathcal{D}_K(\Omega)$ is $\mathcal{D}(\Omega)$. A linear form on $\mathcal{D}(\Omega)$ is called continuous if the restrictions to the $\mathcal{D}_K(\Omega)$ are continuous. We denote by $\mathcal{D}'(\Omega)$ the set of all linear forms in this sense. The elements of $\mathcal{D}'(\Omega)$ are called distributions on Ω. We have an obvious embedding

$$\mathcal{E}(\Omega) \longrightarrow \mathcal{D}'(\Omega), \qquad f \longmapsto (g \mapsto \langle f, g\rangle).$$

Hence the elements of $\mathcal{E}(\Omega)$ can be considered as special distributions.

It looks strange that we defined the notion of continuous linear forms on $\mathcal{D}(\Omega)$ without defining a topology on $\mathcal{D}(\Omega)$. Actually one can do this. One can equip $\mathcal{D}(\Omega)$ with the inductive limit topology of the $\mathcal{D}_K(\Omega)$. This means that an subset of $\mathcal{D}(\Omega)$ is called open if and only if its intersection with each $\mathcal{D}_K(\Omega)$ is open. A sequence f_n in $\mathcal{D}(\Omega)$ with respect to this topology converges if it converges in $\mathcal{E}(\Omega)$ and if in addition there exists a compact subset $K \subset \Omega$ such that the supports of all f_n are contained in K. The inclusion map $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$ is continuous. One can show that this topology defines a structure as locally vector space on $\mathcal{D}(\Omega)$. But it is not possible to find a countable system of semi norms that defines the topology. So $\mathcal{D}(\Omega)$ is no Fréchet space. Nevertheless it is complete in the sense that each Cauchy sequence converges.

A linear map $A : \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ is called continuous if the restriction to each $\mathcal{D}_K(\Omega)$ is continuous. For a a continuous linear map the dual map

$$A' : \mathcal{E}'(\Omega) \longrightarrow \mathcal{D}'(\Omega), \quad A'(f)(g) = A(g \circ f)$$

is well defined, One can restrict this map to $\mathcal{D}(\Omega)$. It may happen that the image of $\mathcal{D}(\Omega)$ is contained in $\mathcal{E}(\Omega)$. If this is the case we denote this operator by $A^* : \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$. It is called then the formal adjoint of A. This again is continuous and we can consider its dual again,

$$A^{*\prime} : \mathcal{E}'(\Omega) \longrightarrow \mathcal{D}'(\Omega).$$

It is easy to see that the restriction of $A^{*\prime}$ to $\mathcal{D}(\Omega)$ coincides with A. Hence we have constructed a canonical extensions of the operator A to an operator that is defined on $\mathcal{E}'(\Omega)$. This is the *natural extension of A to distributions.* Usually we denote it by A again,

$$A : \mathcal{E}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$$

is it exists. This can be applied to pseudodifferential operators:

5.1 Proposition. *Pseudodifferential operators $A : \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ are continuous. The formal adjoint operator A^* exists and is a pseudodifferential operator too. As a as a consequence there is a natural extension*

$$A : \mathcal{E}'(\Omega) \longrightarrow \mathcal{D}'(\Omega).$$

More precisely one has: If A is defined by $P \in S^m(\Omega \times \Omega, \mathbb{R}^n)$ then A^* can be defined by $P^*(x, y, \xi) = P(y, x, \xi)$.

This extension allows to characterize integral operators in a nice way:

5.2 Proposition. *A pseudodifferential operator is an integral operator if an only it defines a map $\mathcal{E}'(\Omega) \to \mathcal{E}(\Omega)$.*

Differential operators have the property that they map $\mathcal{D}(X)$ into $\mathcal{D}(X)$ and that they extend to continuous operators $\mathcal{E}(X) \to \mathcal{E}(X)$. Not all pseudodifferential operators have this property. Counter examples can be given by means of integral operators.

5.3 Definition. *A pseudodifferential operator A is called* ***proper*** *if it defines a map $A : \mathcal{D}(\Omega)) \to \mathcal{D}(\Omega)$ and if it extends to a continuous operator $A : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$.*

Of course this extension is unique since $\mathcal{D}(\Omega)$ is a dense subset of $\mathcal{E}(\Omega)$.

We need a criterion for an operator defined by S^m to ne proper: For this we need the concept of the *support of a distribution.* Let $D \subset \mathcal{D}'(\Omega)$ be a distribution. Let $U \subset \Omega$ be an open subset. Using the natural inclusion $\mathcal{D}(U) \hookrightarrow \mathcal{D}(\Omega)$ (extension by zero) and can restrict the distribution to U and obtains a distribution $D|U \in \mathcal{D}'(U)$. When this distribution vanishes we say that D vanishes on U. It can be shown that there is a largest open subset of Ω with this property. Its complement is called the support of D and is denoted by support(D).

let now $P \in S^m(\Omega \times \Omega \times \mathbb{R}^n)$. We consider the following distribution (oscillating integral) $K_P \in \mathcal{D}'(\Omega \times \Omega)$,

$$\langle K_P, w(x,y)\rangle = \int\limits_{\Omega\times\Omega\times\mathbb{R}^n} e^{\mathrm{i}(x-y)\xi} P(x,y,\xi) w(x,y)\,dxdyd\xi$$

In the case $P \in S^{-\infty}$ this agrees with the distribution that is associated to the corresponding kernel function $K(x, y)$. Hence K_P is called the associated kernel distribution.

5.4 Lemma. *let $P \in S^m(\Omega \times \Omega, \mathbb{R}^n)$ and K_P the associated kernel distribution. We restrict to the two projections $p(x, y) = x$, $q(x, y) = y$ to the support of K_P. The following conditions are equivalent:*

a) *The pseudodifferential operator $A = L_P$ is proper.*
b) *The two projections*

$$p, q : \text{support}(K_P) \rightrightarrows \Omega$$

are proper.

Proof. We only will need b) $\Rightarrow$ a). Hence we will prove only this direction. So let's assume that b) is satisfied. We have to prove to facts:

First Fact: Let $K \subset \Omega$ be a compact subset. Then there exists a compact subset $K' \subset \Omega$ such that for $u \in \mathcal{D}(\Omega)$ the following holds:

$$u|K' = 0 \Longrightarrow Au|K = 0.$$

Second Fact: Let $K \subset \Omega$ be a compact subset. Then there exists a compact subset $K' \subset \Omega$ such that for $u \in \mathcal{D}(\Omega)$ the following holds

$$\operatorname{support}(u) \subset K \Longrightarrow \operatorname{support}(Au) \subset K'.$$

We assume that the facts have been proved.

The second second fact already shows that A defines a map $\mathcal{D}_K(\Omega) \to \mathcal{D}_{K'}(\Omega)$. Especially we get a map $\mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$.

It remains to show that we get an extension $\mathcal{E}(X) \to \mathcal{E}(X)$. So let $u \in \mathcal{E}(\Omega)$ and $a \in \Omega$ a point. We have to define $(Au)(a)$. We set $K = \{a\}$ and choose a compact subset K' as in the first fact has been formulated. Then we choose a function $\varphi \in \mathcal{D}(\Omega)$ which is one on K'. We define $Au(a) = A(\varphi u)(a)$. This definition is independent of the choice of φ. Hence we obtain a function Au on Ω. If one repeats the same construction with a compact neighborhood K of a one sees that Au is differentiable. It is easy to show that this extension $A : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ is continuous. Thus 5.4 has been reduced to the two facts.

We prove the first fact. (The proof of the second fact is similar an will be omitted.) We use that the second projection $q : \operatorname{support}(K_P) \to \Omega$ is proper. let $K \subset \Omega$ be a compact subset. We consider its inverse image $q^{-1}(K)$ in $\operatorname{support}(K_P)$ and project it by means of the first projection p to a (compact) subset $K' \subset \Omega$. Now let $u \in \mathcal{D}_K(\Omega)$. We want to show $Au \in \mathcal{D}_{K'}(\Omega)$. So we have to show that Au vanishes on the complement of K. This is equivalent to

$$\int_\Omega \varphi(x) Au(x) dx$$

for each function $\varphi \in \mathcal{D}(\Omega)$ whose support is in the complement of K. We have

$$\int_\Omega \varphi(x) Au(x) dx = \langle K_A, \varphi u \rangle.$$

The support of φu is contained $K \times \operatorname{support}\varphi$ and this set is disjoint to the support of K_P by definition of K'. This finishes the proof. $\square$

As we have seen the pseudodifferential operator L_P is an integral operator if $P \in \mathcal{S}^{-\infty}$. Actually one can weaken this condition:

5.5 Lemma. *Let $P \in S^\infty(\Omega \times \Omega, \mathbb{R}^n)$. Assume that there exists an open subset $U \subset \Omega \times \Omega$ that contains the diagonal and such that the restriction of P to $U \times \mathbb{R}^n$ belongs to $S^{-\infty}(U, \mathbb{R}^n)$. Then the associated pseudodifferential operator L_P is an integral operator.*

Proof. First we treat the case that there exists U in such a way that P vanishes on $U \times \mathbb{R}^n$. From 2.2 we know that the function

$$K(x,y) = \int\limits_{\mathbb{R}^n} e^{\mathrm{i}(x-y)} P(x,y,\xi)d\xi$$

exists (as an oscillating integral). It is easy to show that this is a $\mathcal{C}^\infty$-function and that L_P is the integral operator defined by K (compare with the proof of 3.4). Now we treat the general case.

We consider a function $\psi \in \mathcal{E}(\Omega \times \Omega)$ that is one on the diagonal and has support in U. Then we decompose $P = P_1 + P_2$, where

$$P_1(x,y,\xi) = \psi(x,y)P(x,y,\xi),\ P_2(x,y,\xi) = (1-\psi(x,y))(P(x,y,\xi).$$

by the first step P_2 defines an integral operator. The function P_1 is contained in $S^{-\infty}(\Omega \times \Omega, \mathbb{R}^n)$ by the assumption in 5.5 and therefore also defines an integral operator.

5.6 Proposition. *Each pseudodifferential operator is equivalent to a proper one.*

We consider a closed neighborhood W of the diagonal in $\Omega \times \Omega$ such that both projections $W \rightrightarrows \Omega$ are proper. Then we choose a function $\psi \in \mathcal{E}(\Omega \times \Omega)$ such that its support is contained in W and such that it is one on a full neighborhood of the diagonal. Now we decompose $P = P_1 + P_2$, where

$$P_1(x,y,\xi) = \psi(x,y)P(x,y,\xi),\ P_2(x,y,\xi) = (1-\psi(x,y))(P(x,y,\xi).$$

Obviously the support of P_1 is contained in V. Hence P_1 defines a proper operator (By 5.4). The function P_2 is an integral operator by 5.5. □

6. The calculus of symbols

Let A be a pseudodifferential operator. By definition there exists a function $P \in S^\infty(\Omega \times \Omega, \mathbb{R}^n)$ such that $A = L_P$. We called P a symbol for A. The symbol is not at all unique:

6.1 Lemma. *Let $P(x,y,\xi)$ be a symbol. Then for each multiindex α*

$$(x-y)^\alpha(P(x,y,\xi) \quad \textit{and} \quad \frac{1}{\mathrm{i}^{|\alpha|}}\Big(\frac{\partial}{\partial\xi}\Big)^\alpha P(x,y,\xi)$$

define the same pseudodifferential operator.

The proof can be given by induction on $|\alpha|$. We omit it. □

There is one problem with the symbols It can happen that a symbol P defines an operator L_P that is contained in some $L^m(\Omega)$ but $P \notin S^m(\Omega \times \Omega, \mathbb{R}^n)$. This can be illustrated by the following

6.2 Lemma. *Let $P \in S^m(\Omega \times \Omega, \mathbb{R}^n)$. Assume that all partial derivatives*

$$\Big(\frac{\partial}{\partial y}\Big)^\alpha P(x,y,\xi)$$

of order $|\alpha| \le N$ vanish on the diagonal of $\Omega \times \Omega$. Then $L_P \in L^{m-N}(\Omega)$.

(In the case that P vanishes in a full neighborhood of the diagonal we get $L_P \in L^{-\infty}(\Omega)$. This we already have seen in 5.5.)

Proof of 6.2. We have to use Taylor's formula with explicit remainder term: We keep x and ξ fixed and expand the resulting function in y around the center x. The result is:

$$P(x,y,\xi) = \sum_{|\alpha|\le N}\Big[\Big(\frac{\partial}{\partial y}\Big)^\alpha P(x,y,\xi)\Big]_{y=x} +$$

$$\int_0^1 \frac{(1-t)^{N-1}}{(N-1)!}\sum_{|\alpha|=N}\frac{N!}{\alpha!}(y-x)^\alpha\Big(\frac{\partial}{\partial y}\Big)^\alpha P(x,x+t(y-x),\xi)dt.$$

If we set

$$Q_\alpha(x,y,\xi) = \int_0^1 (1-t)^{N-1}\frac{N}{\alpha!}\Big(\frac{\partial}{\partial y}\Big)^\alpha P(x,x+t(y-x),\xi)dt$$

and if we make use of the assumption in 6.2 we get

$$P(x,y,\xi) = \sum_{|\alpha|=N}(y-x)^\alpha Q_\alpha(x,y,\xi).$$

It is easy to check that Q_α as P lies in S^m. Now we modify the symbol and define

$$Q(x,y,\xi) = \mathrm{i}^{-N}\sum_{|\alpha|=N}\Big(\frac{\partial}{\partial\xi}\Big)^\alpha Q_\alpha(x,y,\xi).$$

We know from 6.1 that P and Q define the same pseudodifferential operator. But now

$$Q \in S^{m-N}(\Omega \times \Omega, \mathbb{R}^n)$$

since derivation by ξ makes the order smaller. This completes the proof of 6.2. □

6.3 Theorem. *Each pseudodifferential operator is equivalent to an operator L_P, where $P(x, y, \xi)$ is a symbol that is independent of y.*

Corollary. *It is also equivalent to a L_P where P is independent of x.*

The functions $P \in S^m(\Omega \times \Omega, \mathbb{R}^n)$ that are independent of x can be identified with the functions $P \in S^m(\Omega, \mathbb{R}^n)$.

Proof of 6.3. We set

$$P_\alpha(x, \xi) = \frac{\mathrm{i}^{-|\alpha|}}{\alpha!} \Big(\frac{\partial}{\partial \xi}\Big)^\alpha \Big[\Big(\frac{\partial}{\partial \xi}\Big)^\alpha P(x, y, \xi)\Big]_{y=x}.$$

Then $P \in S^{m-|\alpha|}(\Omega, \mathbb{R}^n) \subset S^{m-|\alpha|}(\Omega \times \Omega, \mathbb{R}^n)$. From 6.1 and 6.2 follows

$$L_{P(x,y,\xi)} \sim \sum_\alpha L_{P_\alpha(x,\xi)}.$$

(The multi indices are thought to be ordered with increasing $|\alpha|$.) the basic existence theorem 4.1 implies the existence of a $Q \in S^\infty(\Omega, \mathbb{R}^n)$ such that

$$Q(x, \xi) \sim \sum_\alpha P_\alpha(x, \xi).$$

This relation has to been read in $S^\infty(\Omega, \mathbb{R}^n)$. But then it is also true in $S^\infty(\Omega \times \Omega, \mathbb{R}^n)$. We get $L_P = L_Q$.

For the proof of the Corollary one uses the formal adjoint operator which causes a switch between x and y, □

We can express 6.3 in the following way: The map

$$S^m(\Omega, \mathbb{R}^n) \longrightarrow L^m(\Omega)/L^{-\infty}(\Omega)$$

is surjective. Clearly $S^{-\infty}$ is in the kernel of this map. Hence we obtain a (surjective) linear map

$$S^m(\Omega, \mathbb{R}^n)/S^{-\infty}(\Omega, \mathbb{R}^n) \longrightarrow L^m(\Omega)/L^{-\infty}(\Omega).$$

Actually this map is an isomorphism:

6.4 Theorem. *The natural map $P \mapsto L_P$, where $P(x, \xi)$ is considered as element of $\mathcal{S}(\Omega \times \Omega, \mathbb{R})$ that is independent of y, induces an isomorphism*

$$S^m(\Omega, \mathbb{R}^n)/S^{-\infty}(\Omega, \mathbb{R}^n) \longrightarrow L^m(\Omega)/L^{-\infty}(\Omega).$$

During this proof we use the following notation. Let $K \subset \Omega$ be a compact subset. Then $S^m(K, \mathbb{R}^n)$ for $m \in \mathbb{R}$ denotes the set of all $P \in \mathcal{E}(\Omega \times \mathbb{R}^n)$ that satisfy an estimate

$$\left|P_\alpha^\beta(x,\xi)\right| \le C_{\alpha,\beta}\big(1+|\xi|\big)^{m-|\beta|} \qquad (x \in K)$$

and we define

$$S^{-\infty(}(K,\mathbb{R}^n) = \bigcap\nolimits_m S^m(K,\mathbb{R}^n), \quad S^{\infty(}(K,\mathbb{R}^n) = \bigcup\nolimits_m S^m(K,\mathbb{R}^n).$$

Obviously $S^m(\Omega, \mathbb{R}^n)$ is the intersection of all $S^m(K, \mathbb{R}^n)$.

Proof of 6.4. Let $P(x,\xi)$ be a symbol, independent of y, such that $L_P \in L^{-\infty}$. We have to show $P \in S^{-\infty}$. We fix a compact set $K \subset \Omega$ and choose a function $\varphi \in \mathcal{D}(\Omega)$ which is constant 1 on a full neighborhood of K. Let $A = L_P$ the pseudodifferential operator associated to P. For fixed ξ we can apply the operator A to the function $x \mapsto e^{\mathrm{i}x\xi}\varphi(x)$. Then we can define

$$P_\varphi(x,\xi) = e^{-\mathrm{i}x\xi} A(x \mapsto e^{\mathrm{i}x\xi}\varphi(x)).$$

Claim 1. $P_\varphi \in S^{-\infty}(\Omega, \mathbb{R}^n)$.

Claim 2. $P_\varphi - P \in S^{-\infty}(K, \mathbb{R}^m)$.

From the two claims we get $P \in S^{-\infty}(K, \mathbb{R}^m)$. Since this is true for all K we get $S^{-\infty}(\Omega, \mathbb{R}^m)$. This is what we have to prove. The two claims are special cases of more general facts. We treat them in the following Lemma:

6.5 Lemma. *Let A be the pseudodifferential operator defined by the symbol $P(x,\xi) \in S^m(\Omega,\mathbb{R})$ (independent of y). For $\varphi \in \mathcal{D}(\Omega)$ the function*

$$P_\varphi(x,\xi) = e^{-\mathrm{i}x\xi} A(x \mapsto e^{\mathrm{i}x\xi}\varphi(x)).$$

is contained in $S^m(\Omega, \mathbb{R}^n)$. Moreover one has

$$P_\varphi(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\left(\Big(\frac{1}{\mathrm{i}}\frac{\partial}{\partial x}\Big)^\alpha \varphi(x)\right)\Big(\frac{1}{\mathrm{i}}\frac{\partial}{\partial \xi}\Big)^\alpha P(x,\xi).$$

In the situation of the claims we have that φ is constant on some neighborhood of K. Hence on K only the term for $\alpha = 0$ in the sum. This gives the second claim.

Proof of 6.5.

7. Composition of pseudodifferential operators

We are now in the position to consider compositions of pseudodifferential operators. Let $A, B : \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ two pseudodifferential operators. We assume that one of them is proper. Then one can define the composite

$$B \circ A : \mathcal{D}(\Omega) \longrightarrow \mathcal{E}(\Omega).$$

This is clear when A is proper, since then A can be considered as map $A : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$. But it is also clear if B is proper, since then B extends to a map $B : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$. It is natural to ask whether $B \circ A$ is a pseudodifferential operator. First we treat a special case:

7.1 Lemma. *Let A, B be two pseudodifferential operators, one of them proper and the other an integral operator. Then $A \circ B$ is an integral operator too.*

7.2 Theorem. *The composite of two pseudodifferential operators A, B, one of them proper, is a pseudodifferential operator.*
Assume that A can be defined by a symbol $P(x, \xi)$ from that is independent of x and that B can be defined by a symbol $Q(y, \xi)$ that is independent of x then $A \circ B$ can be defined by the symbol $P(x, \xi)Q(y, \xi)$.

Recall (6.4) that we have an isomorphism

$$S^m(\Omega, \mathbb{R}^n)/S^{-\infty}(\Omega, \mathbb{R}^n) \xrightarrow{\sim} L^m(\Omega)/L^{-\infty}(\Omega)$$

where the elements in $S^m(\Omega, \mathbb{R}^n)$ are considered as elements of $S^m(\Omega \times \Omega, \mathbb{R}^n)$ that are independent of y. We denote the inverse map by

$$\sigma : L^m(\Omega)/L^{-\infty}(\Omega) \xrightarrow{\sim} S^m(\Omega, \mathbb{R}^n)/S^{-\infty}(\Omega, \mathbb{R}^n).$$

If P is an pseudodifferential operator we use the notation

$$\sigma(P) := \sigma(P \bmod L^{-\infty}(\Omega)).$$

One can ask how $\sigma(AB)$ can be computed from $\sigma(A)$ and $\sigma(B)$.

7.3 Proposition. *Let A, B be two pseudo-differential operators, one of them proper. We denote by $\sigma_0(A)$, $\sigma_0(B)$, $\sigma_0(AB)$, representatives of the symbols $\sigma(A)$, $\sigma(B)$, $\sigma(AB)$. One has*

$$\sigma_0(AB) \sim \sum_\alpha \frac{\mathrm{i}^{-\alpha}}{\alpha!}\Big(\frac{\partial}{\partial\xi}\Big)^\alpha \Big(\sigma_0(A)\Big(\frac{\partial}{\partial x}\Big)^\alpha \sigma_0(B)\Big).$$

The principal part of the symbol

There is a natural projection

$$S^m(\Omega,\mathbb{R}^n)/S^{-\infty}(\Omega,\mathbb{R}^n) \longrightarrow S^m(\Omega,\mathbb{R}^n)/S^{m-1}(\Omega,\mathbb{R}^n).$$

For an pseudodifferential operator $A \in L^m(\Omega)$ we denote by $\sigma_m(A)$ the image of $\sigma(A)$ with respect to this project. We call $\sigma_m(A)$ the principal part of the symbol of the operator $L \in L^m(\Omega)$. (So it depends on the chosen m.)

7.4 Proposition. *The map $P \longmapsto L_P$ induces a map*

$$\frac{S^m(\Omega,\mathbb{R}^n)}{S^{m-1}(\Omega,\mathbb{R}^n)} \longrightarrow \frac{L^m(\Omega)}{L^{m-1}(\Omega)}.$$

This map is an isomorphism, the inverse is induced by σ_m.

Recall that multiplication induces a map $S^m \times S^{m'} \to S^{m+m'}$. Hence we get a multiplication map

$$\frac{S^m}{S^{m-1}} \times \frac{S^{m'}}{S^{m'-1}} \longrightarrow \frac{S^{m+m'}}{S^{m+m'-1}}.$$

7.5 Theorem. *Let $A \subset L^m(\Omega)$ and $B \subset L^{m'}(\Omega)$ be two pseudodifferential operators. Then*

$$\sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B).$$

8. Elliptic pseudodifferential operators

We introduced in the previous section the multiplication map

$$\frac{S^m}{S^{m-1}} \times \frac{S^{m'}}{S^{m'-1}} \longrightarrow \frac{S^{m+m'}}{S^{m+m'-1}}.$$

A special case is $m' = m$:

$$\frac{S^m}{S^{m-1}} \times \frac{S^{-m}}{S^{-m-2}} \longrightarrow \frac{S^0}{S^{-1}}.$$

In S^0 we have the function "constant one". Its image in $\frac{S^0}{S^{-1}}$ can be denoted the "unit element".

An element $a \in S^m/S^{m-1}$ is called *invertible,* if there exist an element $b \in S^{-m}/S^{-m-1}$ such that ab is the unit element in S^0/S^{-1}.

We give an example. Let $P(x, \xi)$ be a homogenous polynomial of degree m in the variable ξ over the ring $\mathcal{E}(\Omega)$. Using the function We assume that $P(x, \xi) = 0$ can happen only for $\xi = 0$. By means of the function φ

we define

$$Q(x, \xi) = \frac{\varphi(\xi)}{P(x, \xi)} \qquad (:= 0 \text{ if } \xi = 0).$$

Then $Q \in S^{-m}$ and $QP - 1 \in S^{-1}$. Hence Q defines an invertible element in S^m/S^{m-1}.

8.1 Definition. *A pseudodifferential operator A is called elliptic of order m, if $A \in L^m(\Omega)$ and if $\sigma_m(A)$ is invertible in the above sense.*

The example above shows:

8.2 Remark. *Elliptic differential operators are also elliptic pseudodifferential operators.*

Let A by a pseudodifferential operator. By definition a *parametrix* of A is a pseudodifferential operator B such that $A \circ B - \text{id}$ and $B \circ A - \text{id}$ both are integral operators.

8.3 Main theorem for elliptic operators. *Each elliptic pseudodifferential operator admits a parametrix.*

Proof. By assumption we find a proper operator Q' such that

$$PQ' = \text{id} - H, \quad H \in L^{-1}.$$

Since there is no need for H to be proper we decompose

$$H = H' + R, \quad H' \text{ proper}, \quad R \text{ integral operator}.$$

Now we can define powers of H' and we find by the fundamental existence theorem an Q'' with

$$Q'' \sim \sum H'^n.$$

Now $Q = Q'Q''$ has the property $PQ \sim \text{id}$. It is clear that Q also is elliptic. We still have to prove $QP \sim \text{id}$. This is done by a simple algebra argument the image of the set of elliptic operators in $S^\infty/S^{-\infty}$ is a semigroup with unit element. We have shown that each element has a left inverse. From groups theory one knows that this is enough to ensure that it is a group. □

An important application of the existence of a parametrix is the smoothing lemma (should better be called "smoothing theorem"). First we recall that a pseudodofferential $A : \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ extends to $A : \mathcal{E}'(\Omega) \to \mathcal{D}'(\omega)$.

8.4 Smoothing Lemma. *Let A be an elliptic pseudodifferential operator and $f \in \mathcal{E}'(\Omega)$ be a distribution with compact support such that $A(f) \in \mathcal{E}(X)$. Then $f \in \mathcal{D}(X)$.*

There is a variant of 8.4 in the case of a proper elliptic operator $A : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$. In this case we get an extension $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$. Then the following is true:

Let $f \in \mathcal{D}'(\Omega)$ be a distribution such that $Af \in \mathcal{E}(\Omega)$ then $f \in \mathcal{E}(X)$.

Usually one wants to solve equations $Af = g$ where g is given function. The solutions f can be distributions. In this case one talks about weak solutions. More interesting are strong solutions where f is a differentiable function and not only a distribution. The smoothing lemma tells that in the weak solutions are automatically strong in the case of elliptic operators.

Chapter VI. Appendices

In the appendices we consider vector spaces over the field of real numbers. The case of complex vector spaces needs only minor modifications.

1. Topological facts

1.1 Proposition. *Let (X, d) be a metric space such that each sequence admits a comvergent sub-sequence. Then X is compact.*

First we claim that for each $r > 0$ the space X can be covered by finitely many open balls of radius r. We prove this by contradiction and hence assume that this is false for some r. Then one inductively can construct a sequence of balls $U_r(a_n)$ such that a_n is not contained in any of its precursors $U_r(a_\nu)$, $\nu < n$. Then the distance between any two of the mid-points a_n is $\geq r$. But then the sequence (a_n) cannot have a convergent subsequence which contradicts to our assumption about X.

Next we construct a countable dense subset $S \subset X$. For each natural n we cover X by finitely many balls of radius $1/n$. This gives a countable system of balls and their centers define a dense subset of X.

A metric space that admits a countable dense subset has countable basis of the topology. This means that there exists a countable system of open subset such that each open subset is the union of sets of the countable system. One just takes the open balls with rational radii around points of a countable dense set.

It is well-known that a topological Hausdorff space with countable basis of the topology is compact if every sequence admits a convergent sub-sequence.

□

A subset of a metric space is called bounded if it is contained in some ball. For a bounded non-empty set A the diameter can be defined as

$$\operatorname{diam}(A) := \sup\{d(x, y);\ x, y \in A\}.$$

1.2 A variant of the nested interval nesting theorem. *Let X be a complete metric space and $A_1 \supset A_2 \supset \dots$ a descending chain of non-empty closed subsets such that the diameter* $\operatorname{diam}(A_n)$ *tends to zero. Then the intersection $\bigcap A_n$ is not empty.*

Proof. Choose for each A_n an element $a_n \in A_n$ and prove that the sequence (a_n) is a Cauchy sequence. □

1.3 The Baire category theorem. *Let X be a complete metric space (every Cauchy sequence converges). Let $A_1, A_2, \dots$ be a countable system of closed subset such that $X = \bigcup A_n$. Then at least one A_n has an interior point (this means that a full ball is contained in A_n).*

Proof. On constructs inductively a sequence of open balls $U_1 \supset U_2 \supset \dots$ such that $\bar{U}_n \subset X - A_n$ and such that the diameter of $\bar{U}_n$ is $\leq 1/n$. For the construction of U_1 (begin of the induction) one just observes that A_1 must be different from X and hence in $X - A_1$ there exists a closed open ball. Assume that U_n has already been constructed. Of course U_n cannot be contained in A_n. Since A_n is closed, $U_n - A_n$ is open and non empty and we can choose a ball of diameter $\leq 1/n + 1$ inside this set. By the nested interval theorem we find a point $a \in \bigcap \bar{U}_n$. But a cannot by contained in any A_n. □

2. Hilbert spaces

Let H be a (real) vector space. A scalar product $\langle \cdot, \cdot \rangle$ on H is a symmetric positive definit bilinear form. Then

$$\|a\| := \sqrt{\langle a, a \rangle}$$

defines a norm on H and hence a structure as metric space, $d(a, b) := \|a - b\|$. One calls $(H, \langle \cdot, \cdot \rangle)$ a *Hilbert space* if every Cauchy sequence converges. Examples of Hilbert spaces are provided through Radon measures (X, dx). The space $L^2(X, dx)$ is a Hilbert space with the scalar product

$$\langle f, g \rangle = \int_X f(x) g(x) dx.$$

A special example is the Hilbert space l^2, which can be considered as the Hilbert space that is associated to the Radon measure on $X = \mathbb{N}$, equipped with the discrete topology and the integral

$$\int_X f(x) dx := \sum_{n \in X} f(n).$$

Recall that then l^2 consists of all sequences (a_n) such that $\sum |a_n|^2$ converges.

Let $M \subset H$ be a subset of a Hilbert space. We define its orthogonal complement by

$$M^\perp := \{a \in H; \quad \langle a, x \rangle = 0 \text{ for all } x \in M\}.$$

This is a subvector space and it is also clear that it is a closed subset. The starting result in the theory of Hilbert spaces is:

2.1 Proposition. *Let $A \subset H$ be a closed subvector space of H. Then*

$$H = A \oplus A^{\perp}.$$

The proof is not difficult. One has to construct for given $a \in H$ a vector $b \in A$ such that $a - b$ is orthogonal to A. It is easy to see that such a b has the property that it it minimizes the $\|a - x\|$, $x \in A$. This gives the idea of the construction: Choose a sequence $x_n \in A$ such that $\|a - x_n\|$ tends to the infimum of $\|a - x\|$, $x \in A$. One can show that this is a Cauchy sequence and can then consider its limit b.

The Proposition has the following consequence:

2.2 Theorem of Riesz. *Every continuous linear function $L : H \to \mathbb{R}$ on a Hilbert space is of the form $L(x) = \langle a, x\rangle$ with a unique $a \in H$.*

For the proof one applies 2.1 to the kernel of A.

A subset $B \subset H$ of a Hilbert space is called an orthonormal system if

$$\langle a, b\rangle = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{else.} \end{cases}$$

A maximal orthonormal system is called a *Hilbert space basis.* By means of Zorn's Lemma Hilbert space bases exist. One can show even more, namely that each orthonormal system is contained in a maximal one. For a Hilbert space of infinite dimension the following conditions are equivalent:

1) Every Hilbert space basis is countable.
2) There exists a countable Hilbert space basis.
3) There exists a countable dense subset.

Hilbert spaces with the properties 1)–3) are called *separable.* Usually only separable Hilbert spaces are of interest. Two Hilbert spaces H_1 and H_2 are called isomorphic (as Hilbert spaces) if there exists an isomorphism $H_1 \to H_2$ that preserves the scalar products. From linear algebra one knows that Hilbert spaces of finite dimension are isomorphic if and only if their dimensions agree.

2.3 Proposition. *Two separable Hilbert spaces of infinite dimension are isomorphic.*

More precisely: Let $e_1, e_2 \dots$ be an orthonormal basis of the Hilbert space H (of infinite dimension). There exists a unique isomorphism

$$l^2 \xrightarrow{\sim} H, \quad E_i \longmapsto e_i.$$

Here E_i (unit vector) denotes the sequence with a 1 at the i^{th} position and zeros else.

3. Banach spaces

A norm $\|\cdot\|$ on a (real) vector space V is a function $V \to \mathbb{R}$ with the properties

1) $\|a\| \geq 0 \quad$ and $(\|a\| = 0 \Longrightarrow a = 0)$.
2) $\|ta\| = |t|\,\|a\| \quad (t \in \mathbb{R},\ a \in V)$.
3) $\|a + b\| \leq \|a\| + \|b\|$.

A normed space is a space that has been equipped with a distinguished norm. A normed space is called a Banach space if every Cauchy sequence (with respect to the metric $d(a,b) = \|b - a\|$) converges. So Hilbert spaces can be considered as Banach spaces. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called equivalent if there exist constants C_1, C_2 with the property

$$\|a\|_1 \leq C_1 \|a\|_2\,, \quad \|a\|_2 \leq C_2 \|a\|_1\,.$$

Of course equivalent norms lead to the same topologies.

3.1 Proposition. *One a finite dimensional normed vector space any two norms are equivalent. Each finite dimensional normed vector space is a Banach space.*

A linear form $L : V \to \mathbb{R}$ on a normed vector space is continuous if and only of there exists a number C with the property

$$\|L(a)\| \leq C \|a\|\,.$$

The infimum of all possible C is called the norm of L. We denote it by $\|L\|$. We denote by V' the set of all continuous linear forms on V. This is a vector space. Through the just introduced $\|L\|$ this space get's also a normed space. But even more: It is easy to show that V' with this norm is a Banach space.

So far the dual V' could be zero even if V is not zero. Immediately one can see that this happens not for Hilbert spaces since there we have the linear forms $x \mapsto \langle a, x\rangle$ for a given a. But also in the case of Banach spaces continuous linear forms do exist:

3.2 Theorem of Hahn Banach for normed vector spaces.
Let V be a normed space and $W \subset V$ a subvector space, equipped with the restricted norm. Each continuous linear form on W extends to a continuous linear form on V.

Since the dual V' of a normed space is a normed space again we can consider the double dual V''. As usual there is a natural map

$$V \longrightarrow V'', \quad a \longmapsto (L \mapsto L(a)).$$

Of course one has to show that $L \mapsto L(a)$ is continuous on V' but this is easy. The Hahn Banach theorem shows that $V \to V''$ is injective and one can show even more, namely that this map is norm preserving. Hence we can think of V as a subspace of the Banach space V'' equipped with the restricted norm. We denote by $\bar{V}$ the closure of V in V'' and equip it also with the restricted norm. It is clear that $\bar{V}$ is a Banach space. The space $\bar{V}$ is called the completion of V. It is a Banach space that contains V as a dense subspace and such that V carries the restricted norm. These properties characterize $\bar{V}$:

3.3 Lemma. *Let V be a normed vector space. Assume that there are two Banach spaces $\bar{V}_1$, $\bar{V}_2$ so that both contain V as subvector space. Assume that V is dense in both and that the norm on V is the restrictions of the norms on $\bar{V}_1, \bar{V}_2$. Then there exists a unique norm preserving isomorphism $\bar{V}_1 \to \bar{V}_2$ that induces the identity on V.*

There is basic characterization of finite dimensional vector spaces:

3.4 Proposition. *A normed vector space V is of finite dimension if and only if the ball*

$$\{a \in V; \quad \|a\| \leq 1\}$$

is compact.

4. Fréchet spaces

A (real) topological vector space is a $\mathbb{R}$-vector space V that has been equipped with a topology such that the maps

$$V \times V \longrightarrow V, \quad (a, b) \longmapsto a + b, \qquad \mathbb{R} \times V \longrightarrow V, \quad (t, a) \longmapsto ta,$$

are continuous.

A *semi-norm* p on a $\mathbb{R}$ vector space V is a map $p : V \to \mathbb{R}$ with the properties

a) $p(a) \geq 0$ for all $a \in V$,
b) $p(ta) = |t|p(a)$ for all $t \in \mathbb{R}$, $a \in V$,
c) $p(a + b) \leq p(a) + p(b)$.

The ball of radius $r > 0$ with respect to p is defined as

$$U_r(a, p) := \big\{\, x \in V;\ p(a - x) < r \,\big\}.$$

Let $\mathcal{M}$ be a set of semi-norms. A subset $B \subset V$ is called a semi-ball around a with respect to $\mathcal{M}$ if there exists a finite subset $\mathcal{N} \subset \mathcal{M}$ and a $r > 0$ such that

$$B = \bigcap_{p \in \mathcal{N}} U_r(a, p).$$

A subset U of V is called open (with respect to $\mathcal{M}$) if for every $a \in U$ there exists a semi-ball B around a with $B \subset U$.

It is clear that this defines a topology on V such that all $p : V \to \mathbb{R}$ are continuous. (It is actually the weakest topology with this property.) It is also easy to to see that this topology gives V a structure as topological vector space. Moreover a sequence (a_n) in V converges to $a \in V$ if and only if $p(a_n - a) \to 0$ for all $p \in \mathcal{M}$.

4.1 Definition. *A toplogical vector space is called* ***locally convex*** *if there exists a set $\mathcal{M}$ of semi norms that induces the topology of V.*

This notion results form the fact the semi-balls are convex sets.

The set $\mathcal{M}$ is called definit, if

$$p(a) = 0 \quad \text{for all} \quad p \in \mathcal{M} \quad \Longrightarrow \quad a = 0.$$

It is easy to prove that V is a Hausdorff space for definit $\mathcal{M}$.

A sequence (a_n) in a topological vector space V is called a *Cauchy sequence* if for each neighborhood U of the origin there exists N such that $a_n - a_m \in U$ for all $n, m \geq N$. If V is locally convex and $\mathcal{M}$ a defining set of semi-norms this means that for every $\varepsilon > 0$ and every $p \in \mathcal{M}$ there exists an $N = N(p, \varepsilon)$ such that

$$p(a_n - a_m) < \varepsilon \quad \text{for} \quad n, m \geq N.$$

Of course Cauchy sequences converge.

4.2 Definition. *A topological space V is called a Fréchet space if its topology can be defined by a countable definit set of semi-norms and such that any Cauchy sequence converges.*

Notice that a Banach space can be considered as a Fréchet space. Here $\mathcal{M}$ can to taken as the set consisting of a single element (the defining norm).

4.3 Lemma. *Fréchet spaces are metrizable.*

Corollary. *A subset K of a Fréchet space is compact if any sequence admits a subsequence that has a limit in K.*

Proof. We choose some ordering of a countable defining sets of semi-norms $\mathcal{M} = \{p_1, p_2, \dots\}$. Then one defines

$$d(a, b) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(a - b)}{1 + p(a_n) + p(b_n)}.$$

It is easy to show that this is a metric which defines the original topology.

□

Basic examples of Fréchet spaces

Let X be a locally compact topological space and K an open subset. We define on $\mathcal{C}(X)$ the semi-norm

$$p_K(f) = \max_{x \in K} |f(x)|.$$

We consider the locally convex space defined by these semi-norms. We want to have that this set is of countable type. For this we assume that X has countable topology. Then X can be written as a union

$$K_1 \subset K_2 \subset \cdots$$

of an ascending chain of compact subsets. One can achieve that each member K_i is contained in the interior of its successor K_{i+1}. Then each compact subset of X is contained in some K_i. As a consequence the set of norms p_{K_i} already defines the topology. A Cauchy sequence in this space is a sequence whose restriction to each compact subset is a usual Cauchy sequence with respect to the maximum norm. Since the space of continuous functions on a compact space is a Banach space we easily can derive that $\mathcal{C}(X)$ is a Fréchet space. A sequence converges in this Fréchet space if and only if it converges uniformly on each compact subset.

A closed subspace of a Fréchet space, equipped with the induced topology, is a Fréchet space as well. This gives the possibility to define more Fréchet spaces.

Let now X be an analytic manifold (with countable basis of the topology) and let $\mathcal{O}(X)$ be the space of all holomorphic functions. From complex analysis it is known that the limit of a sequence of holomorphic functions that converges uniformly on compact subsets is holomorphic too. This shows that $\Omega(X)$ is a closed subspace of X. Hence $\Omega(X)$ is a Fréchet space.

If X is a differentiable manifold (with countable basis of the topology) then $\mathcal{E}(X) = \mathcal{C}^\infty(X)$ usually is not closed in $\mathcal{C}(X)$. Hence it is no Fréchet space with the topology induced form $\mathcal{C}(X)$. But there is an other topology that makes $\mathcal{C}^\infty(X)$ in natural way to a Fréchet space. We treat the local case first:

4.4 Lemma. *Let $\Omega \subset \mathbb{R}^n$ be an open subset. There is a unique structure as Fréchet space on $\mathcal{E}(\Omega)$ with the following property: A sequence (f_m) converges to f if and only if it converges locally uniformly and if this is true for for all derivatives of arbitrary order.*

Proof. The uniqueness is clear since the topology of a metric space is determined by the convergent sequences. So we have to construct suitable semi-norms. Let K be a compact subset and $m \geq 0$ an integer. We define a semi-norm on $\mathcal{C}^\infty(\Omega)$:

$$p_{K,m}(f) = \max_{|\alpha| \leq m,\ x \in K} |(\partial_\alpha f)(x)| \qquad (\alpha = (\alpha_1, \ldots, \alpha_m)).$$

The set of these semi-norms defines a structure as locally convex space on $\mathcal{C}^\infty(\Omega)$. It is sufficient to take the compacta K from an exhausting system as above. Hence a countable system of semi-norms is enough to define the topology. A sequence f_m converges to a function f with respect to this topology if an only if it converges uniformly on each compact subset and if the same is true for all derivatives of arbitrary order. So it is clear that $\mathcal{C}^\infty(\Omega)$ gets a Fréchet space with this structure.

We briefly explain that more generally for a differentiable manifold X one can equip $\mathcal{C}^\infty(X)$ with a natural structure of a Fréchet space. Instead of compact subsets one now considers pairs (K, φ) where K is a compact subset of X and $\varphi : U \to V$ is a differentiable chart such that $K \subset U$. Then for integral $m \geq 0$ one can define the semi-norm $p_{K,\varphi,m}$ on $\mathcal{C}^\infty(X)$. For this one transports a function $f \in \mathcal{C}^\infty(X)$ to a function f_φ on V. Then one defines

$$p_{K,\varphi,m}(f) = p_{\varphi(K),m}(f_\varphi).$$

The set of these semi-norms gives a structure as locally convex space. In the case of an open subset of $\mathbb{R}^n$ we get the old structure. It is easy to see that one gets a Fréchet space.

4.5 Proposition. *Let X be a differentiable manifold (with countable basis of the topology). Then the space*

$$\mathcal{E}(X) := \mathcal{C}^\infty(X)$$

is a Fréchet space.

Direct product of Fréchet spaces

4.6 Lemma. *Let $E_1, \dots, E_n$ be Fréchet spaces. Then the direct product*

$$E = E_1 \times \cdots \times E_n,$$

equipped with the product topology is a Fréchet space too.

The proof uses the following construction. Let p_i be semi-norms on E_i then

$$p(a_1, \dots, a_n) = \max_{1 \leq i \leq n} p_i(a_i)$$

is semi-norm. □

Factor spaces of Fréchet spaces

4.7 Lemma. *Let F be a closed subspace of a Fréchet space E. We equip the factor space E/F with the quotient topology. Then E/F is a Fréchet space too.*

The proof uses the following construction. Let p by a semi-norm on E then

$$\bar{p}(a) = \inf\{p(a + x);\ x \in F\}$$

is a semi-norm on E. □

The dual of a Fréchet space

Let V be a locally convex vector space. The dual V' of V is the set of all continuous linear forms on V. We want to equip V' with a structure as locally convex space.

4.8 Definition. *A set B of a topological vector space is called **bounded** if for each open neighborhood $0 \in U \subset F$ there exists a constant $\varepsilon > 0$ with $\varepsilon B \subset U$.*

If V is a locally convex space and $\mathcal{M}$ is a set of semi-norms that defines the topology, then this means that for each $p \in \mathcal{M}$ there exists a constant C_p such that

$$p(a) \le C_p \quad \text{for all} \quad p \in \mathcal{M}.$$

Let $B \subset V$ be a non-empty bounded set. Then each linear form $L \in V'$ is bounded on B. This is clear since one just can choose an open neighborhood U of zero such that $L(U) \subset (-1, 1)$. We define

$$p_B(L) := \sup\{|L(a)|; \quad a \in B\}.$$

It is easy to see that this is a semi-norm. The set of these semi-norms equips V' with a structure es locally convex space.

4.9 Lemma. *If V is a Fréchet space than its dual V' is also a Fréchet space.*

The Hahn-Banach theorem holds also for Fréchet spaces.

4.10 Theorem of Hahn-Banach for locally convex spaces. *Let V be a locally convex space and $W \subset V$ a subspace which is equipped with the induced topology. Each continuous linear on W extends to continuous linear form on V.*

5. Montel spaces

A basic theorem which will needed in the following is:

5.1 Open mapping theorem. *Let $f : E \to F$ a surjective continuous linear map between Fréchet spaces. Then f is open, i.e. the images of open sets are open.*

5.2 Definition. *A **Montel space** is a topological vector space such that the closure of each bounded set is compact.*

For example a Banach space is a Montel space if and only if it is of finite dimension. But there exist Montel spaces of infinity dimension:

5.3 Theorem. *Let X be a differentiable manifold (with countable basis) of topology. Then $\mathcal{E}(X)$ is a Montel space.*

For a complex analytic manifold X we used the inclusion $\mathcal{O}(X) \subset \mathcal{C}(X)$ to equip $\mathcal{O}(X)$ with a structure as Fréchet space. But we can consider it also as subspace of $\mathcal{E}(X)$. Actually the induced topology of $\mathcal{E}(X)$ is the same as that of $\mathcal{C}(X)$. This comes from the well-known fact from complex analysis that taking complex derivatives is compatible with locally uniform convergence. This also shows that $\mathcal{O}(X)$ is closed in $\mathcal{E}(X)$. So we see:

5.4 Theorem. *Let X be an analytic manifold (with countable basis) of topology. Then $\mathcal{O}(X)$ (equipped with the topology of locally uniform convergence) convergence of all derivatives) is a Montel space.*

6. Distributions, elementary calculus

let $\Omega \subset \mathbb{R}^n$ be an open subset.

6.1 Definition. *A distribution on Ω is a linear form $D : \mathcal{D}(\Omega) \to \mathbb{R}$ with the following properties. Let (h_m) be a sequence in $\mathcal{D}(\Omega)$ and $h \in \mathcal{D}(\Omega)$. Assume that there is a compact subset $K \subset \Omega$ such that all supports of the functions h_m and h are contained in K. Assume furthermore that (h_m) converges uniformly on K to f and that this is true for all derivatives of arbitrary order. Then $\lim_{m\to\infty} D(h_m) = D(h)$.*

We denote the vector space of all distributions by $\mathcal{D}'(\Omega)$. There is also the notion of a distribution with compact support.

6.2 Definition. *A distribution with compact support on* Ω *is a linear form* $D : \mathcal{E}(\Omega) \to \mathbb{R}$ *with the following properties. Let* (h_m) *be a sequence in* $\mathcal{D}(\Omega)$ *and* $h \in \mathcal{D}(\Omega)$. *Assume that* (h_m) *converges locally uniformly to* h *and that this is true for all derivatives of arbitrary order. Then* $\lim_{m\to\infty} D(h_m) = D(h)$.

We denote the vector space of all distributions with compact support by $\mathcal{E}'(\Omega)$. If D is a distribution with compact support, then the restriction $D|\mathcal{D}(\Omega)$ is a distribution. This gives an injective linear map $\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. Its is not difficult to show that this map is injective. Usually we will identify D and $D|\mathcal{D}(\Omega)$. Hence distributions with compact support are just distributions with an additional property.

Some examples of distributions

Let $f \in \mathcal{E}(\Omega)$. Then

$$D_f(h) = \langle f, h\rangle := \int\limits_\Omega f(x)h(x)dx \qquad (h \in \mathcal{D}(\Omega).$$

is a distribution. This gives an embedding

$$\mathcal{E}(\Omega) \longrightarrow \mathcal{D}'(\Omega).$$

Frequently we identify f with its associated distribution D_f and we consider distributions as generalized functions. For a distribution D we use also the notation

$$\langle D, h\rangle = D(h).$$

Similarly we can embed

$$\mathcal{D}(\Omega) \longrightarrow \mathcal{E}'(\Omega).$$

This example D_f can be generalized considerably. Why should one assume that f is differentiable? All what we need is that f is integrable over each compact subset $K \subset \Omega$. Then fh is integrable over Ω for all $h \in \mathcal{D}(\Omega)$ and it is easy to see that $D_f = \langle f, h\rangle$ is a distribution. For example each continuous function defines a distribution in this way.

Differentiation of distribution

We define the derivative $\partial D/\partial x_i$ of a distribution D by the formula

$$\frac{\partial D}{\partial x_i}(h) := -D\Big(\frac{\partial h}{\partial x_i}\Big).$$

The minus sign has the following meaning. Let $f \in \mathcal{D}(\Omega)$ and D_f the associated distribution. Then one has

$$\frac{\partial D_f}{\partial x_i} = D_{\partial f/\partial x_i}.$$

This follows easily using partial integration. Just to make the idea clear, we consider the case $\Omega = \mathbb{R}$. Then partial integration gives for $h \in \mathcal{D}(\mathbb{R})$

$$\int_{-\infty}^{\infty} f'(x)h(x)dx = -\int_{-\infty}^{\infty} f(x)h'(x)dx$$

which is the desired formula. Hence we have shown:

6.3 Remark. *The definition of the derivative $\partial D/\partial x_i$ of a distribution by means of the formula*

$$\frac{\partial D}{\partial x_i}(h) := -D\Big(\frac{\partial h}{\partial x_i}\Big)$$

generalizes the usual derivative of a function $f \in \mathcal{D}(X)$.

We consider some examples:

The Heaviside function

The Heaviside function by definition is the function

$$Y : \mathbb{R} \longrightarrow \mathbb{R}, \quad Y(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$

It is integrable over each compact set and hence defines a distribution. The derivative of the distribution is

$$Y'(h) = -\int_{-\infty}^{\infty} (Y(x)h'(x)dx = -\int_{0}^{\infty} h'(x)dx = h(0).$$

This is the so-called delta distribution

$$\delta(h) = h(0).$$

It can be considered as distribution with compact support. The delta distribution is an example of a distribution that is not induced by a function f. The reason is that this function would be zero outside 0 and one would have $\langle f, 1\rangle = \int_{-\infty}^{\infty} f(x)dx = 1$. Such a function cannot exist. (The delta "function" in physics should be considered as distribution.)

One can of course take higher derivatives of distributions and as a consequence one can apply the Laplace operator Δ to a distribution.

We give an example: Consider the function $r(x, y) = \sqrt{x^2 + y^2}$ on $\mathbb{R}^n$. We then can consider the function

$$\log \frac{1}{r}$$

on $\mathbb{R}^2$. In the origin, where it is a priori not defined we take an arbitrary value, for example zero. Using polar coordinates it is easy to see that this function is integrable over a disk $r \leq 1$. This implies that this function is integrable over each compact set. Hence it defines a distribution. We can apply the Laplace operator. The formula

$$\Delta \log \frac{1}{r} = 2\pi\delta$$

is left as an exercise.

7. Distributions and topological vector spaces

In the following $\Omega \subset \mathbb{R}^n$ is an open subset. We equipped already the space $\mathcal{E}(\Omega)$ with a structure as Fréchet space and we proved that the space $\mathcal{E}'(\Omega)$ of all continuous linear forms carries a structure as Fréchet space again. The elements of $\mathcal{E}'(\Omega)$ are the distributions with compact support. Since $\mathcal{E}'(\Omega)$ is a Fréchet space we can define the double dual $\mathcal{E}''(\Omega)$ and equip it with a structure as Fréchet space.

7.1 Proposition. *The canonical map*

$$\mathcal{E}(\Omega) \longrightarrow \mathcal{E}''(\Omega)$$

is an isomorphism of topological vector spaces.

One may ask whether $\mathcal{D}(\Omega)$ is aFréchet space too. Here is a problem. If one equips $\mathcal{D}(\Omega)$ with the induced topology of $\mathcal{E}(\Omega)$ it will not be complete. The reason that a sequence (f_m) from $\mathcal{D}(\Omega)$ may converge in $\mathcal{E}(\Omega)$ but the limit needs not to be compactly supported. The situation is better if we consider for a compact subset $K \subset \Omega$ the space

$$\mathcal{D}_K(\Omega) := \{f \in \mathcal{E}(\Omega); \quad \text{support}(f) \subset K\}.$$

This is a closed subspace of $\mathcal{E}(\Omega)$ and hence carries an induced structure as Fréchet space. We use the topologies on the $\mathcal{D}_K(\Omega)$ to define a topology on their union which is $\mathcal{D}(\Omega)$:

A subset $U \subset \mathcal{D}(\Omega)$ is called open if for every $a \in U$ there exists a *convex* subset $W \subset \mathcal{D}(\Omega)$ which has the following two properties:

1) The intersection $W \cap \mathcal{D}_K(\Omega)$ is a neighborhood of the origin in $\mathcal{D}_K(\Omega)$ for each compact subset $K \subset \Omega$.
2) $a + W \subset U$.

The condition that W is convex is essentially for the proof of:

7.2 Proposition. *With the topology described above, $\mathcal{D}(\Omega)$ get's a locally convex vector space. A sequence (f_m) in $\mathcal{D}(\Omega)$ converges to f, if there exists a K such that all f_m and f are contained in $\mathcal{D}_K(\Omega)$ and $f_m \to f$ there. Every Cauchy sequence converges. A linear form $D : \mathcal{D}(\Omega) \to \mathbb{R}$ is continuous if and only if its restriction to every $\mathcal{D}_K(\Omega)$ is continuous. Hence the space of continuous linear forms is the usual space $\mathcal{D}'(\Omega)$ of distributions.*

But $\mathcal{D}(\Omega)$ is no Fréchet space. The reason that it is not possible to find a *countable* system of semi-norms that define the topology.

Recall that $\mathcal{D}'(\Omega)$ carries a structure as locally convex space. (To each bounded set B there is an associated semi-norm.)

7.3 Proposition. *A sequence of distributions D_m converges in $\mathcal{D}'(\Omega)$ against zero if and only of for each bounded subset $B \subset \mathcal{D}(\Omega)$ the sequence $D_m(h)$ converges to 0 uniformly for $h \in B$. Every Cauchy sequence in $\mathcal{D}'(\Omega)$ converges.*

But $\mathcal{D}'(\Omega)$ is no Fréchet space. As in the case $\mathcal{D}(\Omega)$, the topology cannot be defined by a countable set of semi-norms. Since $\mathcal{D}'(\Omega)$ is topological vector space we can consider the double dual $\mathcal{D}''(\Omega)$ and equip it with a structure as locally convex space. There is a natural map $\mathcal{D}(\Omega) \to \mathcal{D}''(\Omega)$. Again we have:

7.4 Proposition. *The canonical map*

$$\mathcal{D}(\Omega) \longrightarrow \mathcal{D}''(\Omega)$$

is an isomorphism of topological vector spaces.

Some density results

Recall that $\mathcal{D}(\Omega)$ has been embedded into $\mathcal{E}'(\Omega)$.

7.5 Lemma. *The set $\mathcal{D}(\Omega)$ is a dense subset of $\mathcal{E}'(\Omega)$ with respect to the Fréchet space structure of $\mathcal{E}'(\Omega)$.*

8. The Schwartz lemma for Banach spaces

A continuous linear map $f : E \to F$ between Fréchet spaces is a *compact operator,* if there exists a non-empty open subset of E such that the closure of its image is compact. It is clear that this is the case if $f(E)$ is of finite dimension. We give an important example where this is not the case:

Let X be a compact differentiable manifold and dx a Radon measure. We assume that for a non negative function $f \in \mathcal{C}_c(X)$ that is not identicall zero we have $\int_X f(x) > 0$. Let $K \in \mathcal{C}^\infty(X \times X)$. We call K a kernel function. It defines an operator

$$A : \mathcal{E}(X) \longrightarrow \mathcal{E}(X), \quad (Af)(x) = \int\limits_X K(x,y)f(y)dy.$$

It is clear that this integral exists (X is compact) and that Af actually is differentiable (Leibniz-criterion). It is also easy to see that A is continuous operator. It is easy to see that the kernel K is determined by the operator A. We call K the kernel of A.

8.1 Lemma. *Let X be a compact differentiable manifold with a Radon measure such that the integral of a nonnegative function $f \in \mathcal{C}_c(X)$ is positive if f doesn't vanish identically. Let A be the integral operator with kernel $K \in \mathcal{C}^\infty(X \times X)$. The operator*

$$A : \mathcal{E}(X) \longrightarrow \mathcal{E}(X)$$

is compact.

Proof.

A linear map $f : V \to W$ is called *nearly surjective* if $f(V)$ is closed in W and if $W/f(V)$ has finite dimension. This is automatically the case when W is finite dimensional.

8.2 Theorem of Schwartz. *Let $f : E \to F$ be a surjective continuous linear map between Fréchet spaces and let $g : E \to F$ be a compact operator. Then $f + g$ is nearly surjective.*

If one applies Schwartz's theorem in the case $E = F$, $f = -\operatorname{id}$ and $g = \operatorname{id}$ on obtains:

8.3 Corollary. *When the identity operator* $\operatorname{id} : E \to E$ *of a Fréchet space is compact, then E is finite dimensional.*

9. The Schwartz lemma for Fréchet spaces

We need a technique to link Fréchet spaces to Banach spaces. Let p be a semi-norm on a Vector space E. Then the null space

$$N(p) := \{a \in E;\ p(a)\}$$

is a subvector space. The semi-norm p factors through a norm on the factor space $E/N(p)$. We denote this normed vector space by E_p.

9.1 Lemma. *Let E be a Fréchet space and p a continuous semi-norm on E. Then E_p is a Banach space. The natural projection $E \to E_p$ is continuous.*

Fréchet spaces as projective limits of Banach spaces

Actually E can be reconstructed from the E_p. In the literature this is written as "projective limit"

$$E = \varprojlim E_p.$$

There is no need for us to introduce the definition of a projective limit here. Instead of this we only formulate, what is behind this formula.

9.2 Lemma. *Let E be a Fréchet space and $\mathcal{M}$ be a defining system of semi-norms. We assume that for each $p_1, p_2 \in \mathcal{M}$ there exists a semi-norm $p \in \mathcal{M}$ such that $p \geq p_1, p_2$. Then each open subset $U \subset E$ is the union of finite intersections of inverse images of open sets in E_p.*

Applications

We want to give a direct proof of the following theorem:

9.3 Theorem. *Let X be a compact complex manifold. Then the space $\Omega^p(X)$ of holomorphic differential forms of degree p is finite dimensional for all p.*

it is possible to equip each complex manifold with a hermitian metric. Then we can apply our deep theorems (resting on the elliptic operator theory) to get this result. Instead of this we give here a more direct proof which has the advantage that it admits generalizations on complex spaces.

Proof of 9.3.

10. Elliptic operators on compact manifolds

Let X be a differentiable manifold. Since the notion of pseudodifferential operator on open subsets of $\mathbb{R}^n$ is invariant under diffeomorphism and since it is a local notation, it is possible to define the notion of a pseudodifferential operator

$$A : \mathcal{D}(X) \longrightarrow \mathcal{E}(X).$$

It is also possible to define what it means that A is an elliptic operator.

We assume that a Radon-measure dx is given on X such that the integral of function $f \in \mathcal{C}_c(X)$, $f \geq 0$, is not zero if f is not identically zero. This measure then induces an embedding

$$\mathcal{D}(X) \longrightarrow \mathcal{E}'(X), \quad f \longmapsto \Big(g \mapsto \int_X f(x)g(x)dx\Big).$$

For $K \in \mathcal{C}^\infty(X \times X)$ we can define the associated integral operator

$$A : \mathcal{D}(X) \longrightarrow \mathcal{E}(X), \quad Af(x) = \int_X K(x,y)f(y)dy.$$

As in the local case one can prove the existence of a parametrix B for a given proper elliptic operator A. This is an elliptic operator such that $A \circ B - \mathrm{id}$ and $B \circ A - \mathrm{id}$ are integral operators as explained above.

Now we assume that X is compact. Then $\mathcal{D}(X) = \mathcal{E}(X)$ and each pseudifferential operator is proper. Since a pseudifferential operator is continuous in the topology of $\mathcal{E}(X)$, the dual operator

$$A' : \mathcal{E}'(X) \to \mathcal{E}'(X)$$

is defined. Recall that $\mathcal{E}(X)$ has been embedded into $\mathcal{E}'(X)$. As in the local case we have that A' maps $\mathcal{E}(X)$ into $\mathcal{E}(X)$. Restricting A' we get an operator

$$A^* : \mathcal{E}(X) \longrightarrow \mathcal{E}(X).$$

This is also en elliptic operator. It is the formal adjoint of A which means that it has the property

$$\langle Af, g\rangle = \langle f, A^* g\rangle,$$

where $\langle\cdot\rangle$ denotes the scalar product

$$\langle f, g\rangle = \int_X f(x)g(x)dx.$$

The basic theorem is:

10.1 Theorem. *Let $A : \mathcal{E}(X) \to \mathcal{E}(X)$ be an elliptic operator on a compact differentiable manifold. Then the kernel of A is finite dimensional and one has*

$$\mathcal{E}(X) = \mathrm{kernel}(A) \oplus \mathrm{image}(A^*).$$

Proof. We choose a parametrix B. The kernel of A is contained in the kernel of $B \circ A$ and this space is finite dimensional by the theorem of Schwartz. One has to use that $B \circ A$ is the sum of a surjective operator (namely id) and a compact operator (actually an integral operator on the compact manifold X).

It is clear that the intersection of kernel(A) and image(A^*) is zero. Next we show that image(A^*) is closed in $\mathcal{E}(X)$. For this we use that image(A^*) $\subset$ image($A^* \circ B^*$). We have that $A^* \circ B^* = (B \circ A)^*$. We know that $B \circ A - \mathrm{id}$ is an integral operator. The same is true then for $(B \circ A - \mathrm{id})^*$. We obtain that image($A^* \circ B^*$) is a closed subspace of finite codimension in $\mathcal{E}(X)$. The Fréchet space $\mathcal{E}(X)/\text{image}(A^* \circ B^*)$ is finite dimensional. Hence each subspace of it is closed. This implies that image(A^*) is closed and since kernel(A) is closed we get that kernel(A) $\oplus$ image(A^*) is closed in $\mathcal{E}(X)$.

Now we can consider $\mathcal{E}(X)/(\text{kernel}(A) \oplus \text{image}(A^*))$ as a Fréchet space. Our claim is that it is zero. We argue by contradiction. Then there exists a (continuous) linear form that is not identically zero. In other words there exists an element $D \in \mathcal{E}'(X)$ that is not identically zero but vanishes on kernel(A) and on image(A^*). Especially we have $D(A^* f) = 0$ for all $f \in \mathcal{E}(X)$. By definition of the dual operator we have $D(A^* f) = A'(D)(f)$. So we see $A'(D) = 0$. The smoothing theorem gives that D is already contained in $\mathcal{E}(X)$. (The smoothing theorem is an immediate consequence of the existence of a parametrix.) We write $D = f$ and we have $\langle f, A^* g\rangle = 0$ for all g. This gives $\langle Af, g\rangle = 0$ for all g and hence $Af = 0$. Since D vanishes on the kernel of f we get $\langle f, f\rangle = 0$ and hence $f = 0$. This contradicts to the assumption that D is different form zero. □