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Complex spaces, Grauert's finiteness theorem

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Chapter I. Complex Spaces

1. Complexes

Rings are assumed to be commutative and associative and with unit. Homomorphisms of rings are assumed to map the unit element into the unit element. By an algebra over a ring A we understand a homomorphism of rings $f : A \to B$. A homomorphism between A-algebras $A \to B$ and $A \to C$ is just a commutative diagram

$$\begin{tikzcd} A \arrow[d] \arrow[dr] & \\ B \arrow[r] & C \end{tikzcd}$$

of rings.

An A-algebra $f : A \to B$ carries a structure as A-module,

$$ab := f(a)b.$$

This is a functor from the category of A-algebras into the category of A-modules. More general, every B-module carries a structure as A-module. This is a functor from the category of B-modules into the category of A-modules.

Chain complexes

let A be an associative and commutative ring with unit. By a (chain-) complex $M_\bullet$ we understand a sequence of A-modules

$$\cdots \longrightarrow M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$$

such that n runs over all integers and where $d_{n-1} \circ d_n = 0$ for all n. The homology groups of the complex are

$$H_n(M_\bullet) = \frac{\text{kernel}(M_n \longrightarrow M_{n-1})}{\text{image}(M_{n+1} \longrightarrow M_n)}.$$

A homomorphism of complexes $M_\bullet \to N_\bullet$ is a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & M_{n+1} \arrow[r] \arrow[d] & M_n \arrow[r] \arrow[d] & M_{n-1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & N_{n+1} \arrow[r] & N_n \arrow[r] & N_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

So we can talk about the category of (chain-) complexes. This is an additive category (morphisms can be added in a natural way) and even an abelian category. The notions "subcomplex", image of a homomorphism of complexes is defined in the obvious way. A sequence $M_\bullet \to N_\bullet \to P_\bullet$ is called exact if the image of $M^\bullet \to N^\bullet$ equals the kernel of $N^\bullet \to P^\bullet$, This means that $M_n \to N_n \to P_n$ is exact for all n.

A homomorphism of complexes induces natural maps of the homology groups $H_n(M_\bullet) \to H_n(N_\bullet)$.

A sequence of complexes $E^\bullet \to F^\bullet \to G^\bullet$ is called exact if $E^n \to F^n \to G^n$ is exact for every n.

1.1 Definition. *A homomorphism of chain complexes $f : E_\bullet \to F_\bullet$ is called a **quasi-isomorphism** if the induced homomorphisms $H_n(E_\bullet) \to H_n(F_\bullet)$ are isomorphisms.* Dqi

A fundamental lemma of homological algebra states.

1.2 Lemma. *Let $0 \to E_\bullet \to F_\bullet \to G_\bullet \to 0$ be an exact sequence of complexes. Then there can be constructed combining homomorphisms $\partial : H^n(G_\bullet) \to H^{n-1}(E_\bullet)$ such that the long sequence* LES

$$\cdots \longrightarrow H_n(E_\bullet) \longrightarrow H_n(F_\bullet) \longrightarrow H_n(G_\bullet) \xrightarrow{\partial} H_{n-1}(E_\bullet) \longrightarrow \cdots$$

is exact.

In many cases one needs only the existence of ∂ but sometimes it is necessary to know how it is constructed. We sketch the construction. One starts with $a \in H_n(G^\bullet)$. It is represented by $b \in G_n$. Its image in G_{n-1} is zero. Let $b \in F_n$ be an inverse image of a. Its image in F_{n-1} is in the kernel of $F_{n-1} \to G_{n-1}$. Hence it is the image of a $c \in E_{n-1}$. One shows $d(c) = 0$ so c represents a homology class in $H^{n-1}(E^\bullet)$. □

1.3 Definition. *A homomorphism of chain complexes $f_\bullet : M_\bullet \to N_\bullet$ is called **null-homotopic** if there exists a sequence $h_n : M_n \to N_{n+1}$ such that* Dheq

$$f_n = d \circ h_n + h_{n-1} \circ d.$$

This condition can be visualized through a diagram

$$\begin{tikzcd}
M_{n+1} \arrow[r, "f_n"] \arrow[d, "d"'] & N_{n+1} \arrow[d, "d"] \\
M_n \arrow[r, "f_{n-1}"'] \arrow[ur, "h_n"] & N_n
\end{tikzcd}$$

1.4 Remark. *If $f_\bullet : M_\bullet \to N_\bullet$ is null-homotopic, then it induces zero maps in the homology.* Rizh

The proof is easy. □

1.5 Definition. *A homomorphism $f_\bullet : M_\bullet \to N_\bullet$ is called a **homotopy equivalence** if there exists a homomorphism $g_\bullet : N_\bullet \to M_\bullet$ such that $g_\bullet \circ f_\bullet$ and $f_\bullet \circ g_\bullet$ are homotopy equivalent to the identity maps.* Dhen

On can consider g as a weak substitute of the inverse of f. So homotopy equivalences admit inverses in some sense. But quasi-isomorphisms usually can not be inverted. (We don't want to introduce the concept of the derived category, where quasi-isomorphisms can be inverted.)

1.6 Definition. *Two homomorphisms of complexes* Dhei

$$f_\bullet, g_\bullet : M_\bullet \longrightarrow N_\bullet$$

*are called **homotopic** if their difference is null-homotopic.*

1.7 Remark. *Two homotopic homomorphisms of complexes induce the same map in the homology.* Rts

1.8 Remark. *Each homotopy equivalence is a quasi-isomorphism.* Rhei

1.9 Definition. *Let $f_\bullet : M_\bullet \to N_\bullet$ be a homomorphism of chain complexes. The **mapping cone** $C(f)_\bullet$ is the following complex.* Dmc

a) $C(f)_k = M_{k-1} \oplus N_k$.

b) *The differential d is*

$$\begin{pmatrix} -d & 0 \\ -f & d \end{pmatrix} : M_{k-1} \oplus N_k \longrightarrow M_{k-2} \oplus N_{k-1},$$
$$(a, b) \longmapsto (-d(a), d(b) - f(a)).$$

We explain the meaning of the mapping cone. For this we introduce a notation. Let $M_\bullet$ be a complex of modules and let m be an integer. We define the shifted complex $M[m]_\bullet$ through $M[m]_n := M_{nm+n}$ with the obvious differentials. For a complex homomorphism $M_\bullet \to N_\bullet$ we can define complex homomorphisms

$$N_\bullet \longrightarrow C(f)_\bullet \quad \text{and} \quad C(f)_\bullet \longrightarrow M[-1]_\bullet.$$

The first one is given by $b \mapsto (0, b)$ the second one by $(a, b) \mapsto a$. One can check that this gives a short exact sequence of complexes

$$0 \longrightarrow N_\bullet \longrightarrow C(f)_\bullet \longrightarrow M[-1]_\bullet \longrightarrow 0$$

(Recall that a sequence $E_\bullet \to F_\bullet \to G_\bullet$ is called exact if $E_n \to F_n \to G_n$ is exact for all n.) The long exact sequence that is associated to this short exact sequence is

$$\cdots \longrightarrow H_n(N_\bullet) \longrightarrow H_n(C(f)_\bullet) \longrightarrow H_{n-1}(M_\bullet) \xrightarrow{\partial} H_{n-1}(N_\bullet) \longrightarrow \cdots.$$

A concrete calculation shows that the combining homomorphism ∂ equals the map induced by f.

We consider exact sequences of A-modules

$$\cdots \longrightarrow L_n \xrightarrow{d_n} L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

They are called resolutions of M from he left. Sometimes we interpret them in the following way.

We make the sequence (L_n, d_n) to a complex $L_\bullet$ through $L_n = 0$ for $n < 0$,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & L_1 & \longrightarrow & L_0 & \longrightarrow & 0 \longrightarrow & \cdots \\ & & & & \uparrow & & & \\ & & & & \text{zero-position} & & & \end{array}$$

The homology groups are

$$H_0(L_\bullet) = 0 \text{ for } n \neq 0 \quad \text{and } H_0(L_\bullet) = M.$$

One introduces also the complex $M_\bullet$,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \uparrow & & & & \\ & & & & \text{zero-position} & & & & \end{array}$$

where $M_0 = M$ and all other M_n are zero. Then one can express the sequence above as a complex homomorphism $L_\bullet \longrightarrow M_\bullet$. This is a quasi-isomorphism . For sake of simplicity we identify M with the associated complex $M_\bullet$ and then can write also $L_\bullet \to M$ for this complex homomorphism

Let M be an A-module. A resolution of M

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is called free if the modules F_n all are free (not necessarily finitely generated). As above we consider the F_n as a complex (F_n, d_n), $n \in \mathbb{Z}$ ($F_n = 0$ for $n < 0$) and we write $F_\bullet \to M$ for the resolution.

1.10 Definition. *Let $F_\bullet \to M$ and $G_\bullet \to N$ be two free resolutions and let $M \to N$ be a linear map. A lifting of this map is a homomorphism $F_\bullet \to G_\bullet$ such that the diagram* Dlm

$$\begin{array}{ccc} F_\bullet & \longrightarrow & G_\bullet \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

commutes.

This diagram stands of course for

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

1.11 Remark. *Let $F_\bullet \to M$ and $G_\bullet \to N$ be two free resolutions and let $M \to N$ be a linear map. There exists a lifting $F_\bullet \to G_\bullet$ and such a lifting is unique up to homotopy.* Rlex

Let M be another A-module. Then we can consider the complex $M \otimes_A F_\bullet$,

$$\cdots \longrightarrow M \otimes_A F_{n+1} \longrightarrow M \otimes_A F_n \longrightarrow \cdots.$$

The homology groups of this complex are the Tor groups

$$\mathrm{Tor}_n(M, N) = H_n(M \otimes_A F_\bullet).$$

Clearly $\mathrm{Tor}_0(N, N) = M \otimes_A N$.

Cochain complexes

Besides chain complexes $M_\bullet$ we can also consider cochain complexes $M^\bullet$. These are sequences

$$\cdots \longrightarrow M^n \xrightarrow{d_n} M^{n+1} \longrightarrow \cdots$$

such that n runs over all integers and where $d_{n+1} \circ d_n = 0$ for all n. There is no essential difference between chain- and cochain complexes. If $M_\bullet$ is a chain complex than we can define the cochain complex $M^n := M_{-n}$ and conversely. We will call both kinds simply complexes. The context, in particular the notations $M_\bullet$, $M^\bullet$ will show which kind of complex we consider.

For sake of completeness we repeat shortly some basic notions of chain complexes for cochain complexes.

The cohomology groups of a cochain complex are

$$H^n(M^\bullet) = \frac{\mathrm{kernel}(M^n \longrightarrow M^{n+1})}{\mathrm{image}(M^{n-1} \longrightarrow M^n)}.$$

The notion of homomorphism is the same as for chain complexes.

The definitions of "quasi-isomorphism", "homotopy equivalence of complexes" and "homotopic mappings of complexes" is literally the same and the Remarks 1.7 and 1.8 hold.

The notion of null homotopy is visualized by the diagram

$$\begin{tikzcd} M^{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d"'] & N^{n-1} \arrow[d, "d"] \\ M^n \arrow[r, "f_n"'] \arrow[ur, "h_n"] & N^n \end{tikzcd}$$

Let $f : M^\bullet \to N^\bullet$ be a homomorphism of cochain complexes. Then the mapping cone is defined through

$$C(f)^k = M^{k+1} \oplus N^k.$$

The differential $d : C(f)^k \to C(f)^{k+1}$ is given by the same formula as in the case of chain complexes,

$$d(a,b) = (-d(a), d(b) - f(a).$$

In this case the short exact sequence looks like

$$0 \longrightarrow F^\bullet \longrightarrow C(f)^\bullet \longrightarrow E[1]^\bullet \longrightarrow 0.$$

2. Sheaves

2.1 Definition. *Let X be a topological space. A presheaf F of sets on X is an assignment that associates to each open subset $U \subset X$ a set $F(U)$ and to each pair of open sets $V \subset U$ a map $r_V^U : F(U) \to F(V)$ such that $r_U^U = \mathrm{id}$ for every open U and such that for every three open subsets $W \subset V \subset U$ the diagram* Dpr

$$\begin{tikzcd} F(U) \arrow[r, "r_V^U"] \arrow[d, "r_W^U"'] & F(V) \arrow[dl, "r_W^V"] \\ F(W) & \end{tikzcd}$$

commutes.

The elements of $F(U)$ often are called *sections* of F over U and the elements of $F(X)$ are called *global sections.* We will call r_V^U the restriction maps and we will use frequently the notation

$$s|V = r_V^U(s).$$

In many cases the sections of a presheaf are functions or something similar. In these cases the restriction maps are restrictions in the usual sense.

2.2 Definition. *Let F, G be two presheaves of sets. A morphism (also called simply a map) $f : F \to G$ is a collection of maps $f_U : F(U) \to G(U)$ such that for every pair $V \subset U$ of open subsets the diagram* Dms

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d] & G(U) \arrow[d] \\ F(V) \arrow[r] & G(V) \end{tikzcd}$$

commutes.

We can talk about the category of presheaves of sets.

2.3 Definition. *A presheaf F of sets is called a sheaf if the following conditions are satisfied.* Dss

1) *$F(\emptyset)$ is a set that consists of one element.*
2) *Assume that $U = \bigcup U_i$ is an open covering of an open subset $U \subset X$. Assume that $s, t \in F(U)$ are two sections with the property $s|U_i = t|U_i$ for all i, then $s = t$.*
3) *Assume that $U = \bigcup U_i$ is an open covering of an open subset $U \subset X$. Let $s_i \in F(U_i)$ be sections with the property $s_i|(U_i \cap U_j) = s_j(U_i \cap U_j)$ for all i, j. Then there exists $s \in F(U)$ with the property $s|U_i = s_i$ for all i.*

By definition, the category of sheaves of sets is the full subcategory of the category of sheaves whose objects are sheaves. There is a trivial functor from the category of sheaves of sets into the category of presheaves of sets.

Stalks

Let F be a presheaf of sets on a topological space X and let $a \in X$. We want to consider the direct limit

$$F_a := \varinjlim_{a \in U \subset X \text{ open}} F(U).$$

Its elements are classes [U,s] of pairs (U, s) where U is an open neighborhood of a and where $s \in U$. Two such pairs are called equivalent, $(U, s) \sim (V, t)$, if there exists an open W, $a \in W \subset U \cap V$ with the property $s|W = t|W$. The elements of F_a are called *germs* and F_a is called the *stalk* of F at a. A map of presheaves $f : F \to G$ induces for each $a \in X$ a map $f_a : F_a \to G_a$. This is for each $a \in X$ a functor form the category of presheaves of sets into the category of sets.

Let F be a presheaf. Assume that for each open U a subset $G(U) \subset F(U)$ is given such that the restriction maps $F(U) \to F(V)$ map $G(U)$ into $G(V)$. Then the assignment $U \longmapsto G(U)$ (with the obvious restriction maps) is a presheaf. We call it a subpresheaf. There is a natural homomorphism $G \to F$.

It is easy to check that the canonical maps $G_a \to F_a$ are injective. Usually we will consider G_a as a subset of F_a.

Let $f : F \to G$ be a map of presheaves. Then there is an obvious subpresheaf $f_{\text{pre}}(F)$ ind G such that $f(F)(U) = f_U(F(U))$. We call it the *presheaf image* of f. The presheaf image needs not to be a sheaf even if F, G both are sheaves.

The generated sheaf

2.4 Definition. *Let F be presheaf of sets. A generated sheaf $(\hat{F}, \kappa)$ is a sheaf of sets $\hat{F}$ together with a map $\kappa : F \to \hat{F}$ such that the following property holds. Let $f : F \to G$ be a map of F into a sheaf G. Then there exists a unique map of sheaves $\hat{F} \to G$ such the diagram* Dgs

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \hat{F} \arrow[dl] \\ G & \end{tikzcd}$$

commutes.

It is clear that the pair $(\hat{F}, \kappa)$ is uniquely determined up to canonical isomorphism. Hence we can talk about *the* generated sheaf.

A concrete construction of $\hat{F}$ runs a follows. First one treats a special case. Let G be a sheaf and F be a subpresheaf. Then one can define a subpresheaf $\tilde{F} \subset G$ in the following way.

$$\tilde{F}(U) = \{s \in G(U); \quad s_a \in F_a\}.$$

One can also say that $\tilde{F}(U)$ consists of all $s \in G(U)$ such there exists an open covering $U = \bigcup U_i$ such that $s|U_i \in F(U_i)$.

We have $F \subset \tilde{F} \subset G$. Let $\kappa : F \to \tilde{F}$ be the canonical map. Then $(\tilde{F}, \kappa)$ is a generated sheaf. So we can write

$$\hat{F} = \tilde{F}.$$

Now we treat the general case. There is an obvious presheaf $F^{(0)}$ with the property

$$F^{(0)}(U) = \prod_{a \in U} F_a.$$

It is obvious that this is a sheaf. One calls it the *Godement sheaf* related to F. There is a natural map $f : F \to F^{(0)}$. Then one defines

$$\hat{F} := \widetilde{f_{\text{pre}}(F)}.$$

There is a natural map $\kappa : F \to \hat{F}$. It is not difficult to check the universal property.

It is also clear that the assignment $F \mapsto \hat{F}$ is a functor form the category of presheaves of sets into the category of sheaves of sets.

Using this explicit construction one can show the following lemma.

2.5 Lemma. *Let F be a presheaf of sets. The canonical map $\kappa : F \to \hat{F}$ induces bijections* Lkb

$$F_a \longrightarrow \hat{F}_a.$$

Let X be a topological space. We can consider (pre-)sheaves of sets or of groups or of rings. Each of them defines a category. If F is a presheaf of groups (rings) then the stalks F_a are groups (rings) in the obvious way. A homomorphism $F \to G$ of presheaves of groups (rings) induces a homomorphism $F_a \to G_a$ of groups (rings). Let F be a sheaf of groups (rings). Then the generated sheaf carries a structure as sheaf of groups (rings).

We defined already the notion of the presheaf image f_{pre} of a map $f : F \to G$ of presheaves. In the case that F, G are sheaves. The presheaf image needs not to be a sheaf. Hence we have to define the sheaf image $f_{\text{sheaf}}(F)$ in a different way

$$f_{\text{sheaf}}(F) = \widetilde{f_{\text{pre}}(F)}.$$

There is a natural map $f_{\text{sheaf}}(F) \to G$ which is an isomorphism onto a subsheaf of G. Hence we can identify $f_{\text{sheaf}}(F)$ with a subsheaf of G.

2.6 Definition. *A map of presheaves $f : F \to G$ is called presheaf surjective if $f_{\text{pre}}(F) = G$. Assume that F, G are sheaves. Then f is called sheaf surjective if $f_{\text{sheaf}}(F) = G$.* Dss

Remarkably this difficulty does not occur for the notion of injectivity.

2.7 Definition. *A map of presheaves (sheaves) $f : F \to G$ is called injective if $F(U) \to G(U)$ is injective for all open U.* Dsis

Let $f : F \to G$ be a map of sheaves. We assume that it is injective in the sense of presehaves. Then the presheaf image is already a sheaf.

We turn now to sheaves of abelian groups. The kernel of a homomorphism $f : F \to G$ of presheaves is the subpresheaf of f defined through

$$\text{Kernel}(f)(U) = \text{Kernel}(f_U).$$

Similarly to the notion of injectivity there is no difference between presheaf- and sheaf kernel. The reason is a s follows. Let $f : F \to G$ be a homomorphism of sheaves. Then the kernel is already a sheaf.

2.8 Definition. *A sequence of presheaves of abelian groups $F \xrightarrow{f} G \xrightarrow{g} H$ is called presheaf exact if $f_{\text{pre}}(f) = \text{Kernel}(g)$ or, which means the same, $F(U) \to G(U) \to H(U)$ is exact for all open U.* Dspe

2.9 Definition. *A sequence of sheaves of abelian groups $F \xrightarrow{f} G \xrightarrow{g} H$ is* Dspe
called sheaf exact if $f_{\text{sheaf}}(f) = \text{Kernel}(g)$.

2.10 Proposition. *A sequence of sheaves $F \to G \to H$ is sheaf exact if and* Pssse
only if the sequences $F_a \to G_a \to H_a$ are exact for all points a.

We omit the proof. □

Let $F \subset G$ be a subpresheaf of a presheaf G. Then the presheaf quotient is defined through

$$(F/_{\text{pre}}G)(U) = F(U)/G(U)$$

with obvious restriction maps. The sequence $0 \to F \to G \to F/_{\text{pre}} \longrightarrow 0G$ is presheaf exact.

If $F \subset G$ is a subsheaf of a sheaf then the factor sheaf is defined through

$$F/_{\text{sheaf}}G = \widehat{F/_{\text{pre}}}G.$$

The sequence

$$0 \to F \to G \to F/_{\text{sheaf}}G \to 0$$

is sheaf exact.

Finally we mention.

2.11 Remark. *The functor "generated sheaf" from the category of presheaves* Rfsa
of abelian groups into the category of sheaves of abelian groups is exact.

We can talk about (chain- or cochain-) complexes of sheaves. Let

$$\cdots \longrightarrow F_{n-1} \longrightarrow F_n \longrightarrow \cdots$$

be a cochain complex. Then we can define its cohomology sheaves $H^n(F^\bullet)$. We also can talk about homomorphisms of complexes of shaves in the obvious way and we we can say what it means that such a homomorphism is a quasi-isomorphism.

Direct image

Let $f : X \to Y$ be a continuous map of topological spaces and let F be presheaf of abelian groups on X. Then there is an obvious presheaf $f_*(F)$ on Y with the property

$$f_*(F)(V) = F(f^{-1}(V)).$$

It is called the direct image of F. This is a functor of the category of sheaves of abelian groups on X to those on Y. In the case that F is a sheaf, we have that $f_*(F)$ is a sheaf. So we get a functor from the category of sheaves of abelian groups into those on Y. There is also a functor that associates to a sheaf on Y a sheaf on X. We only need a special case of this which we explain now.

There is an obvious restriction of sheaves for open subsets $U \subset X$. A (pre)sheaf F on X can be restricted to a (pre)sheaf $F|U$ in a trivial way,

$$(F|U)(V) = F(V) \qquad (V \subset U \text{ open}).$$

But there is also a restriction in the following situation. Let $A \subset X$ be a closed subset of a topological space and let F be a sheaf of abelian groups on X. We make the strong assumption that

$$F|(X - A) = 0.$$

Let $U_1 \subset U_2 \subset X$ be open subsets with the property $U_1 \cap A = U_2 \cap A$. Then the restriction $F(U_1) \to F(U_2)$ is an isomorphism. This is easy to show by means of the open coverings $X = U_i \cup (X - A)$. Hence we can try the following definition. For each open subset $V \subset X$ one chooses an open $U \subset X$ with the property $U \cap A = V$. Then we define

$$(F|A)(V) = F(U).$$

To avoid conflict with the axiom of choice, it looks natural to modify this definition

$$(F|A)(V) := \varinjlim_{U \subset X \text{ open},\ U \cap A = V} F(U).$$

Then the natural map to $F(U)$ is an isomorphism for each open $U \subset X$, $U \cap A = V$.

Let U be closed and open and assume $F|(X - U) = 0$, then the two restrictions agree

2.12 Lemma. *Let A be a closed subspace of a topological space X and denote by $j : A \to X$ the canonical injection. Let F be a sheaf of abelian groups on A. There is a canonical isomorphism* Laxi

$$F_a \xrightarrow{\sim} j_*(F)_a \quad \textit{for} \quad a \in A.$$

2.13 Lemma. *Let A be a closed subspace of a topological space X and denote by $j : A \to X$ the canonical injection.* Ljsr

1) *Let F be a sheaf of abelian groups on A. There is a natural isomorphism*

$$F \xrightarrow{\sim} (j_*F)|A.$$

2) *Let G be a sheaf of abelian groups on X whose restriction to $X - A$ vanishes. Then there is a natural isomorphism*

$$G \xrightarrow{\sim} j_*(G|A).$$

We conclude this section with a remark that is occasionally useful.

2.14 Remark. *Let F, G be two sheaves on X and let $\mathcal{B}$ be a basis of the topology. Assume that for each $U \in \mathcal{B}$ a map $F(U) \to G(U)$ is given. Assume also that this system of maps is compatible with restrictions. Then this system extends in a unique way to a map of sheaves $F \to G$.* Rbdt

The proof is easy and can be omitted. □

3. Finitely generated sheaves

Let $\mathcal{O}$ be a sheaf of rings. There is the notion of an $\mathcal{O}$-module $\mathcal{M}$. This is a sheaf of abelian groups together with a homomorphism of sheaves of abelian groups

$$\mathcal{O} \times \mathcal{M} \longrightarrow \mathcal{M}$$

such that the induced maps

$$\mathcal{O}(U) \times \mathcal{M}(U) \longrightarrow \mathcal{M}(U)$$

equip $\mathcal{M}(U)$ with a structure as $\mathcal{O}(U)$-module. Here we use the obvious definition for the direct product of (pre-)sheaves.

$$(\mathcal{M} \times \mathcal{N})(U) = \mathcal{M}(U) \times \mathcal{N}(U).$$

This definition can be extended to more than one factor and one can define

$$\mathcal{M}^n = \mathcal{M} \times \cdots \times \mathcal{M}.$$

There is an obvious notion of an $\mathcal{O}$-linear map $\mathcal{M} \to \mathcal{N}$ of $\mathcal{O}$-modules. So we can talk about the category of $\mathcal{O}$-modules.

This category has the same exactness property as the category of abelian groups. One can define the kernel and the sheaf image in this category and one can define direct products. We also mention that the stalk $\mathcal{M}_a$ of an $\mathcal{O}$-module carries a natural structure as $\mathcal{O}_a$-module.

In the following we will understand by an exact sequence of $\mathcal{O}$-moduls a sheaf exact sequence and we use the notations

$$f(\mathcal{M}) := f_{\text{sheaf}}(\mathcal{M}) \quad \text{and} \quad \mathcal{M}/\mathcal{N} = \mathcal{M}/_{\text{sheaf}}\mathcal{N}.$$

(These are $\mathcal{O}$-modules.)

Let $\mathcal{M}$ be an $\mathcal{O}$-module and let $\mathcal{O}^m \to \mathcal{M}$ be an $\mathcal{O}$-linear map. There is an induced map $\mathcal{O}(X)^m \to \mathcal{M}(X)$. Hence there are m distinguished global sections $s_1, \dots, s_m \in \mathcal{M}(X)$ (the images of the elements of the standard basis $e_1, \dots, e_m$ of $\mathcal{O}(X)^m$). These global sections determine the map, since for any open $U \subset X$ an arbitrary section of $\mathcal{O}^m$ can be written in the form $s = f_1 e_1|U + \cdots + f_m e_m|U$. The image of this section is $f_1 s_1|U + \cdots + f_m s_m|U$. Conversely we obtain an $\mathcal{O}$-linear map through this formula for any choice of global sections $s_1 \dots, s_m$. This shows:

3.1 Lemma. *There is a natural one to one correspondence between $\mathcal{O}$-linear maps $\mathcal{O}^m \to \mathcal{M}$ and m-tuples of global sections of $\mathcal{M}$.* FrtoM

An $\mathcal{O}$-module is called finitely generated if there is a surjective map of $\mathcal{O}$-modules $\mathcal{O}^m \to \mathcal{M}$. Surjectivity of course is understood in the sense of sheaves. So this means that $\mathcal{O}_a^m \to \mathcal{M}_a$ is surjective for each point $a \in X$.

The support of a sheaf F of abelian groups, rings, algebras is defined as

$$\mathrm{supp} F := \{a \in F;\ F_a \neq 0\}.$$

3.2 Lemma. *Let $\mathcal{M}$ be a finitely generated $\mathcal{O}$-module. The support of $\mathcal{M}$ is a closed subset.* SuppCl

Proof. We show that the complement of the support is open. Let a be a point such that $\mathcal{M}_a = 0$. Consider generators $s_1, \dots, s_m$ of $\mathcal{M}$. The germs $(s_i)_a$ are zero. Hence there exists an open neighborhood U such that all $s_i|U = 0$. This shows $\mathcal{M}_b = 0$ for all $b \in U$. □

3.3 Lemma. *Let $\mathcal{M}, \mathcal{N}$ be two finitely generated submodules of an $\mathcal{O}$-module $\mathcal{P}$. Let a be a point such that $\mathcal{M}_a \subset \mathcal{N}_a$. Then there exists an open neighborhood $a \in U$ such that $\mathcal{M}|U \subset \mathcal{N}|U$.* SubCon

Proof. Take generators $s_1, \dots, s_m$ of $\mathcal{M}$ and $t_1, \dots, t_n$ of $\mathcal{N}$. Express the germs $(t_i)_a$ by the $(s_j)_a$. Since there are only finitely coefficients involved, these equations extend to a small open neighborhood of a. □

A similar argument gives:

3.4 Lemma. *Let $\mathcal{M} \to \mathcal{N}$ be an $\mathcal{O}$-linear map of finitely generated $\mathcal{O}$-modules. Let a be a point such that $\mathcal{M}_a \to \mathcal{N}_a$ is surjective. Then there exists an open neighborhood U such that $\mathcal{M}|U \to \mathcal{N}|U$ is surjective.* SurUmg

Lifting of maps

A very simple fact of commutative algebra says. Let $M \to N$ be a surjective R-linear map of R-modules and let $R^n \to N$ be a linear map too. Then there exists a lift $R^n \to M$. Denote the images of the standard basis $e_1, \dots, e_n$ in N by $b_1, \dots, b_n$ and take pre-images a_i in M. Then map e_i to a_i.

To get an analogue for sheaves, we consider a surjective $\mathcal{O}$-linear map $\mathcal{M} \to \mathcal{N}$ of $\mathcal{O}$-modules and an $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{N}$. Now we get a problem since the map $\mathcal{M}(X) \to \mathcal{N}(X)$ needs not to be surjective. So we can not repeat the above argument. We only can say:

3.5 Lemma. *Let $\mathcal{M} \to \mathcal{N}$ be a surjective $\mathcal{O}$-linear map and $\mathcal{O}^n \to \mathcal{N}$ also an $\mathcal{O}$-linear map. For each point a there exists an open neighborhood U and an $\mathcal{O}|U$-linear map such the diagram* LiftLoc

$$\begin{tikzcd} \mathcal{O}|U^n \arrow[d] \arrow[dr] & \\ \mathcal{M}|U \arrow[r] & \mathcal{N}|U \end{tikzcd}$$

commutes.

4. Coherent sheaves

Let us recall a basic property of noetherian rings R. Let M be a finitely generated module, i.e. there exists a surjective R-linear map $R^n \to M$. Then the kernel K of this map is finitely generated as well. Hence there exists an exact sequence $R^n \xrightarrow{\varphi} R^m \to M$. The map φ determines $M \cong R^n/\mathrm{Im}(\varphi)$. The map φ just given by a matrix with m rows and n columns. This is the way how computer algebra can manage computations for finitely generated modules over noetherian rings as polynomial rings. Serre found a weak substitute for $\mathcal{O}$-modules.

4.1 Definition. *A sheaf of rings $\mathcal{O}$ is called **coherent** if for any open subset $U \subset X$ and any $\mathcal{O}|U$-linear map $\mathcal{O}^n|U \to \mathcal{O}^m|U$ the kernel is locally finitely generated.* DCoh

Recall that an $\mathcal{O}$-module $\mathcal{M}$ is called locally finitely generated if there exists an open covering $X = \bigcup_i U_i$ such that $\mathcal{M}|U_i$ is a finitely generated as $\mathcal{O}_X|U_i$-module for all i.

4.2 Definition. *Let $\mathcal{O}$ be a coherent sheaf of rings. An $\mathcal{O}$-module $\mathcal{M}$ is called coherent if for every point there exists an open neighborhood U and an exact sequence* CohMod

$$\mathcal{O}|U^n \longrightarrow \mathcal{O}|U^m \longrightarrow \mathcal{M}|U \longrightarrow 0.$$

Of course $\mathcal{O}$ considered as $\mathcal{O}$-module is coherent. Just consider $0 \to \mathcal{O} \to \mathcal{O} \to 0$.

An $\mathcal{O}$-module is called a (finitely generated) free sheaf if it is isomorphic to $\mathcal{O}^m$ for suitable m. It is called locally free if every point admits an open neighborhood such that the restriction to it is free. A locally free sheaf is also called a vector bundle. For trivial reasons a (finitely generated) free sheaf over a coherent sheaf of rings is coherent. Since coherence is a local property, every vector bundle is coherent. The property "coherent" is stable under standard constructions. The proves are not difficult. We will keep them short:

First we treat some special cases for free $\mathcal{O}$-modules. A first trivial observation is that the image of an $\mathcal{O}$-linear map $\mathcal{O}^p \to \mathcal{O}^q$ is coherent. The next observation is that the intersection $\mathcal{M} \cap \mathcal{N}$ of two coherent subsheaves $\mathcal{M}, \mathcal{N}$ of $\mathcal{O}^n$ is coherent. (The intersection $\mathcal{M} \cap \mathcal{N}$ is defined in the naive sense as presheaf and turns to be out a sheaf, more precisely an $\mathcal{O}$-module.) The idea is to write the intersection as a kernel. We explain the principle for individual modules $M, N \subset R^n$ of finite type over a ring R: Let $F : R^p \to R^m$, $G : R^q \to R^m$ be linear maps and let M, N be their images. We denote by K the kernel of the linear map

$$R^{p+q} \longrightarrow R^m, \quad (m,n) \longmapsto F(m) - G(n).$$

The image of K under the map

$$R^{p+q} \longrightarrow R^m, \quad (m,n) \longmapsto F(m).$$

is precisely the intersection $M \cap N$.

The last observation is the following. Let $\mathcal{O}^p \to \mathcal{O}^q$ be $\mathcal{O}$-linear and let $\mathcal{M} \subset \mathcal{O}^q$ be coherent. We claim that its inverse image in $\mathcal{O}^p$ is coherent. We explain again the algebra behind this result. Let $F : R^m \to R^l$ be a R-linear map and $N \subset R^l$ be an R-module of finite type. We assume that $F(R^m) \cap N$ is finitely generated. Then there exists a finitely generated submodule $P \subset R^m$ such that $F(P) = F(R^m) \cap N$. We also assume that the kernel K of F is finitely generated. It is easily proved that $F^{-1}(N) = P + K$ and we obtain that the inverse image is finitely generated.

These observations carry over to arbitrary coherent $\mathcal{O}$-modules.

4.3 Lemma. *Let $\mathcal{M} \to \mathcal{N}$ be an $\mathcal{O}$-linear map of coherent sheaves. The image sheaf is coherent.* ImCohCoh

Corollary. *A locally finitely generated sub-sheaf of a coherent sheaf is coherent.*

Proof. It is sufficient to show that the image of a map $\mathcal{O}^m \to \mathcal{M}$ is coherent. By definition of coherence it is sufficient to show that the kernel $\mathcal{K}$ is locally finitely generated. We can assume that there exists an exact sequence

$$\mathcal{O}^p \longrightarrow \mathcal{O}^q \longrightarrow \mathcal{M} \longrightarrow 0.$$

Since $\mathcal{O}^q \to \mathcal{M}$ is surjective we can assume (use Lemma 3.5) that there exists a lift $\mathcal{O}^m \to \mathcal{O}^q$ such that the diagram

$$\begin{tikzcd}
\mathcal{O}^m \arrow[r] \arrow[dr] & \mathcal{M} \\
& \mathcal{O}^q \arrow[u] \\
& \mathcal{O}^p \arrow[u]
\end{tikzcd}$$

commutes. Take the image of $\mathcal{O}^p \to \mathcal{O}^q$ and then its pre-image in $\mathcal{O}^m$ It is easy to check that this is the kernel $\mathcal{K}$. □

4.4 Lemma. *The kernel of a map $\mathcal{M} \to \mathcal{N}$ of coherent sheaves is coherent.* KeCohCoh

Proof. Because of Lemma 4.3 we can assume that $\mathcal{M} \to \mathcal{N}$ is surjective. We choose presentations

$$\mathcal{O}^a \longrightarrow \mathcal{O}^b \longrightarrow \mathcal{M}, \quad \mathcal{O}^c \longrightarrow \mathcal{O}^d \longrightarrow \mathcal{N}.$$

We can assume that there is commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{N} \arrow[r] & 0 \\
& & \mathcal{O}^b \arrow[u] \arrow[r, "\varphi"] & \mathcal{O}^d \arrow[u] & \\
& & \mathcal{O}^a \arrow[u] \arrow[r] & \mathcal{O}^c \arrow[u, "\psi"'] &
\end{tikzcd}$$

The existence of φ follows from Lemma 3.5 (after replacing X by a small open neighborhood of a given point). The existence of $\mathcal{O}^a \to \mathcal{O}^c$ is trivial. Then we get a natural surjection $\varphi^{-1}(\psi(\mathcal{O}^c)) \to \mathcal{K}$. □

4.5 Lemma. *The cokernel $\mathcal{N}/\varphi(\mathcal{N})$ of a map $\varphi : \mathcal{M} \to \mathcal{N}$ of coherent sheaves is coherent.* KoCohCoh

Proof. We can assume that $\mathcal{N}$ is a sub-sheaf of $\mathcal{M}$ and that φ is the canonical injection. We can assume that a commutative diagram with exact columns exists:

$$\begin{tikzcd} & 0 & 0 & & \\ 0 \arrow[r] & \mathcal{N} \arrow[r] \arrow[u] & \mathcal{M} \arrow[r] \arrow[u] & \mathcal{M}/\mathcal{N} \arrow[r] & 0 \\ & \mathcal{O}^b \arrow[r] \arrow[u] & \mathcal{O}^d \arrow[u] & & \\ & \mathcal{O}^a \arrow[r] \arrow[u] & \mathcal{O}^c \arrow[u] & & \end{tikzcd}$$

It is easy to construct from this diagram an exact sequence

$$\mathcal{O}^b \oplus \mathcal{O}^c \longrightarrow \mathcal{O}^d \longrightarrow \mathcal{M}/\mathcal{N} \longrightarrow 0. \qquad \square.$$

4.6 The two of three lemma. *Let $\mathcal{O}$ be a coherent sheaf of rings and* TwoThree

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

an exact sequence of $\mathcal{O}$-modules. Assume that two of them are coherent. Then the third is coherent too.

Proof. All what remains to show is that $\mathcal{M}_2$ is coherent if $\mathcal{M}_1, \mathcal{M}_2$ are. We can assume that there is a commutative diagram

$$\begin{tikzcd} & & & 0 & \\ 0 \arrow[r] & \mathcal{M}_1 \arrow[r] & \mathcal{M}_2 \arrow[r] & \mathcal{M}_3 \arrow[r] \arrow[u] & 0 \\ & & & \mathcal{O}^q \arrow[u] \arrow[ul, "\alpha"] & \\ & & & \mathcal{O}^p \arrow[u] & \end{tikzcd}$$

We use this to produce a map

$$\mathcal{M}_1 \oplus \mathcal{O}^q \longrightarrow \mathcal{M}_2, \quad (x, y) \longmapsto x - \alpha(y).$$

It is easy to check that this map is surjective. The kernel is defined by $x = \alpha(y)$. Hence it can be identified with the part of $\mathcal{O}^q$ that is mapped into $\mathcal{M}_1$ under α. But this precisely the kernel of $\mathcal{O}^q \to \mathcal{M}_3$ hence the image of $\mathcal{O}^p$. We get an exact sequence

$$\mathcal{O}^p \longrightarrow \mathcal{M}_1 \oplus \mathcal{O}^q \longrightarrow \mathcal{M}_2 \longrightarrow 0.$$

This shows that $\mathcal{M}_2$ is coherent (use Lemma 4.5).). $\square$

4.7 Lemma. *The intersection of two coherent subsheaves of a coherent sheaf is coherent.* SubSc

Proof. One uses the fact that intersections can be constructed as kernels. Let $\mathcal{M}, \mathcal{N} \subset \mathcal{X}$ be two submodules of an $\mathcal{O}$-module $\mathcal{X}$. Then $\mathcal{M} \cap \mathcal{N}$ is isomorphic to the kernel of $\mathcal{M} \times \mathcal{N} \to \mathcal{X}$, $(a, b) \mapsto a - b$. □

4.8 Remark. *Let $\mathcal{M}$ be a coherent $\mathcal{O}$-module. Then the support of $\mathcal{M}$ is a closed subset.* SupC

Proof. We show that the set of all a such that $\mathcal{M}_a = 0$ is open. We can assume that $\mathcal{M}$ is finitely generated by sections $s_1, \dots, s_n$. If there germs at a are zero then $s_1, \dots, s_n$ are zero in a full neighbourhood of a. □

We collect some of the permanence properties of coherent sheaves.

4.9 Proposition. PointEx

1) *Let $\mathcal{M}, \mathcal{N}$ be two coherent sub-sheaves of a coherent sheaf. Assume $\mathcal{M}_a \subset \mathcal{N}_a$ for some point a. Then there exists an open neighborhood U such that $\mathcal{M}|U \subset \mathcal{N}|U$.*
2) *Let $\mathcal{M}, \mathcal{N}$ be two coherent subsheaves of a coherent sheaf. Assume $\mathcal{M}_a = \mathcal{N}_a$ for some point a. Then there exists an open neighborhood U such that $\mathcal{M}|U = \mathcal{N}|U$.*
3) *Let $f, g : \mathcal{M} \to \mathcal{N}$ be two $\mathcal{O}$-linear maps between coherent sheaves such that $f_a = g_a$ for some point a. Then there exists an open neighborhood U such that $f|U = g|U$.*
4) *Let $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ be $\mathcal{O}$-linear maps of coherent sheaves and a a point. The following two conditions are equivalent:*
   a) *The sequence $\mathcal{M}_a \to \mathcal{N}_a \to \mathcal{P}_a$ is exact.*
   b) *There is an open neighborhood U such that the sequence $\mathcal{M}|U \to \mathcal{N}|U \to \mathcal{P}|U$ is exact.*

Proof.

1) Use that $\mathcal{M}_a \subset \mathcal{N}_a$ is equivalent to $\mathcal{N}_a = \mathcal{M}_a \cap \mathcal{N}_a$ $(= (\mathcal{M} \cap \mathcal{N})_a$.

2) follows from 1).

3) Consider the kernel of $f - g$.

4) Consider the image $\mathcal{A}$ of $\mathcal{M} \to \mathcal{N}$ and the kernel $\mathcal{B}$ of $\mathcal{N} \to \mathcal{P}$. Both are coherent. We can assume that they are finitely generated. From assumption we know $\mathcal{A}_a = \mathcal{B}_a$. □

4.10 Proposition. *Let $\mathcal{M}, \mathcal{N}$ coherent $\mathcal{O}$-modules and $\mathcal{M}_a \to \mathcal{N}_a$ an $\mathcal{O}_a$-linear map. There exists an open neighborhood U and an extension $\mathcal{M}|U \to \mathcal{N}|U$ as $\mathcal{O}|U$-linear map.* ExtPtoU

Additional remark. *By Proposition 4.9 this extension is unique in the obvious local sense.*

Proof. We can assume that there is a surjective $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{M}$. We consider the composed map $\mathcal{O}_a^n \to \mathcal{M}_a \to \mathcal{N}_a$. It is no problem to extend to $\mathcal{O}_a^n \to \mathcal{N}_a$ to an open neighborhood $\mathcal{O}|U^n \to \mathcal{N}|U$. We can assume that U is the whole space. The kernel of $\mathcal{O}_a^n \to \mathcal{M}_a$ is contained in the kernel of $\mathcal{O}_a^n \to \mathcal{N}_a$. Since the kernels are coherent this extends to a full open neighborhood U. Hence we get a factorization $\mathcal{M}|U \to \mathcal{N}|U$. □

4.11 Lemma. *Let $\mathcal{O}_X$ be a coherent sheaf of rings on a topological space X. Let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent sheaf of ideals. Let Y be the support of $\mathcal{O}_X/\mathcal{J}$. Then the restriction of $\mathcal{O}_X/\mathcal{J}$ to Y is a coherent sheaf of rings $\mathcal{O}_Y$. The category of coherent $\mathcal{Y}$-modules is equivalent to the category of coherent $\mathcal{O}_X$ modules which are annihilated by $\mathcal{J}$.* CohSub

Proof. This is an application of Lemma 2.12. It is easy to see that $\mathcal{O}_X/J|Y$ is a sheaf Let $\mathcal{M}$ be an $\mathcal{O}_X$-module which is annihilated by $\mathcal{J}$. (This means $J(U)\mathcal{M}(U) = 0$ for all open U). The support of $\mathcal{M}$ is contained in Y. Then $\mathcal{M}|Y$ is defined and carries a natural structure as $\mathcal{O}_Y$-module. The rest is clear. □

□

5. Complex spaces

In the following by a ringed space $(X, \mathcal{O}_X)$ we always understand a topological space that has been equipped with a sheaf of $\mathbb{C}$-algebras. By definition, a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair, consisting of a continuous map $f : X \to Y$ and a homomorphism

$$\varphi : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X)$$

of sheaves of $\mathbb{C}$-algebras. In practice this means that we have homomorphisms

$$\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V)).$$

It is clear how to define the composition of two morphisms and there is the identity morphism. (This means that we defined a category). In particular, we have the notion of an isomorphism of ringed spaces. We write the morphism as (f, φ), or simply by f it is clear which φ is considered. But one should have in mind that φ is usually not determined by f.

We equip $\mathbb{C}^n$ with the sheaf of all holomorphic functions (on open subsets). We denote this sheaf by $\mathcal{O}_{\mathbb{C}^n}$. The restricted sheaf to an open subset we denote by $\mathcal{O}_U = \mathcal{O}_{\mathbb{C}^n}|U$. Let $(f_1, \ldots, f_m)$ a finite set of holomorphic functions on U. Then we can consider the ideal sheaf $\mathcal{I}$ generated by the f_i . The factor $\mathcal{O}_U/\mathcal{J}$ is a sheaf of $\mathbb{C}$-algebras. The support of this sheaf is

$$Y = \{z \in \mathbb{C}^n; \quad f_1(z) = \cdots = f_m(z) = 0\}.$$

which is a closed subset. Then we can consider the ringed space

$$(Y, \mathcal{O}_Y) \quad \text{where} \quad \mathcal{O}_Y = (\mathcal{O}_X/\mathcal{I})|Y.$$

Of course $\mathcal{O}_Y$ depends on the choice of $f_1, \dots, f_m$. Such a ringed space is called a *model space.*

5.1 Definition. *A complex space (in the sense of Grothendieck) is a ringed space that is locally isomorphic to a model space. A holomorphic map $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism in the sense of ringed spaces.* Dcsp

Notice that a holomorphic map consists of two parts, a continuous map $f : X \to Y$ and a homomorphism of sheaves $\varphi : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

5.2 Oka's coherence theorem. *The structure sheaf of a complex space is coherent.* OC

We consider the stalk $\mathcal{O}_{X,a}$ of a complex space. In the case $\mathbb{C}^n$ (equipped with the sheaf of holomorphic functions this algebra is isomorphic as $\mathbb{C}$-algebra to the ring of convergent power series.

$$\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \dots, z_n\}.$$

An analytic algebra A is a $\mathbb{C}$-algebra A that is different from 0 and is isomorphic to a factor algebra of $\mathcal{O}_n$, n suitable,

$$A = \mathcal{O}_n/\mathfrak{a}, \quad \mathfrak{a} \neq A.$$

Analytic algebras are local algebras. If $\mathcal{O}_n \to A$ is a surjective algebra homomorphism then the image of the maximal ideal of $\mathcal{O}_n$ is the maximal ideal of A. In particular we get an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow A \longrightarrow A/\mathfrak{m} \longrightarrow 0$$

The composition $\mathbb{C} \to A/\mathfrak{m}$ is an isomorphism. We will use it to identify $A/\mathfrak{m} = \mathbb{C}$. We finally mention that algebra homomorphisms $A \to B$ are automatically local, i.e. the image of the maximal ideal of A is contained in the maximal ideal of B. So we get a natural homomorphism

$$A/\mathfrak{m}(A) \longrightarrow B/\mathfrak{m}(B)$$

which is the identity if we identify both sides with $\mathbb{C}$.

Let $f \in \mathcal{O}_X(X)$ be a global section of the structure sheaf of a complex space and let $x \in X$ be a point. We can consider the germ f_x and take its coset mod $\mathfrak{m}(\mathcal{O}_{X,x})$. This is a number which we denote by $f(x)$. In this way we get a function

$$\tilde{f} : X \longrightarrow \mathbb{C}, \quad \tilde{f}(x) := f(x).$$

A look at the definition of model spaces shows that $\tilde{f}$ is continuous. Hence we have constructed an algebra homomorphism

$$\mathcal{O}_X(X) \longrightarrow \mathcal{C}_X(X).$$

The same can be done for open subsets. We can read this as map of sheaves of $\mathbb{C}$-algebras

$$\mathcal{O}_X \to \mathcal{C}_X.$$

We denote the kernel of this map by $\mathcal{N}_X$. Clearly $\mathcal{N}_X(U)$ contains all nilpotent elements of $\mathcal{O}_X(U)$.

Let $(X, \mathcal{O}_X)$ be an arbitrary ringed space. The nilradical $\mathcal{N}$ is the subsheaf of $\mathcal{O}_X$ that is defined through

$$U \longmapsto \{g \in \mathcal{O}_X(U), \quad \text{locally nilpotent}\}.$$

There is the subsheaf generated by the presheaf

$$U \longmapsto \{g \in \mathcal{O}_X(U), \quad \text{nilpotent}\}.$$

It also can be defined through

$$\mathcal{N}_X(U) = \{f \in \mathcal{O}_X(U); \quad f_a \text{ nilpotent in } \ \mathcal{O}_{X,a} \text{ for all } a \in U\}.$$

Basic results of local complex analysis show.

5.3 Hilbert-Rückert nullstellensatz. *Let $(X, \mathcal{O}_X)$ be a complex space.* HR
Then the kernel of the natural map $\mathcal{O}_X \to \mathcal{C}_X$ is the nilradical $\mathcal{N}_X$.

5.4 Cartan's coherence theorem. *Let $(X, \mathcal{O}_X)$ be a complex space. The* CC
nilradical is coherent.

(This is equivalent to the fact that the nilradical is locally finitely generated.)

Holomorphic functions on complex spaces

By a holomorphic function on a complex space $(X, \mathcal{O}_X)$ we understand a holomorphic map

$$(f, \varphi) : (X, \mathcal{O}_X) \longrightarrow (\mathbb{C}, \mathcal{O}_\mathbb{C}).$$

So $\varphi : \mathcal{O}_\mathbb{C} \to f_*\mathcal{O}_X$. Such a morphism is determined by the image of the global section $1 \in \mathcal{O}_\mathbb{C}$. This is an element of $\mathcal{O}_X(X)$. This gives the following result.

5.5 Remark. *The holomorphic mappings $(X, \mathcal{O}_X) \to (\mathbb{C}, \mathcal{O}_\mathbb{C})$ are in one-to-* Rhm
one correspondence to the global sections in $\mathcal{O}_X(X)$.

Open subspaces

Let $(X, \mathcal{O}_X)$ be a complex space and let $U \subset X$ be an open subset. Then $(U, \mathcal{O}_X|U)$ is a complex space too, The natural inclusion $i : U \to X$ together with the natural map $\varphi : \mathcal{O}_U \to i_*\mathcal{O}_X)$ gives a holomorphic map $(U, \mathcal{O}_X|U) \to (X, \mathcal{O}_X)$. The following universal property is satisfied. Let $(g, \psi) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a holomorphic map of a third complex space Z into X such that $f(X) \subset U$, then (p, ψ) factors through a unique holomorphic map $(g_0, \psi_0) : (Z, \mathcal{O}_Z) \to (U, \mathcal{O}_X|U)$.

A holomorphic map $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called an *open embedding* if there is an open subset $U \subset Y$ such that (f, φ) factors through an isomorphism $(X, \mathcal{O}_X) \xrightarrow{\sim} (U, \mathcal{O}_X|U)$. The composition of two open embeddings is an open embedding.

Closed subspaces

Let $(X, \mathcal{O}_X)$ be a complex space and let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent ideal sheaf. (It is enough to know that $\mathcal{J}$ is locally finitely generated.) We then can consider the sheaf $\mathcal{O}_X/J$. The support of this sheaf is a closed subset $Y \subset X$. We then can consider the restriction

$$\mathcal{O}_Y := (\mathcal{O}_X/\mathcal{J})|Y.$$

Then $(Y, \mathcal{O}_Y)$ is a complex space. We call this the (closed) complex subspace of $(X, \mathcal{O}_X)$ related to the ideal sheaf $\mathcal{J}$. There is a natural holomorphic map $i : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$. A holomorphic map $j : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is called a closed embedding (of complex spaces) if there exists a coherent ideal sheaf $J \subset \mathcal{O}_X)$ such that j factors through a biholomorphic map

$$(Z, \mathcal{O}_Z) \xrightarrow{\sim} (Y, \mathcal{O}_Y) \quad \text{where} \quad Y = \operatorname{supp}(\mathcal{O}_X/J),\ \mathcal{O}_Y = (\mathcal{O}_X/J)|Y.$$

It is easy to show that the composition of two closed embeddings is a closed embedding.

Finally we call a holomorphic map $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ a locally closed embedding if it is the composition of a closed embedding $f : (Y, \mathcal{O}_Y) \to (U, \mathcal{O}_U)$ and an open embedding $(U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$.

Direct product of complex spaces

Let X, Y be two objects in a category. The direct product of X, Y is a triple $(X \times Y, p, q)$ consisting of an object $X \times Y$ and two morphisms $p : X \times Y \to X$, $q : X \times Y \to Y$ such that the natural map

$$\operatorname{Mor}(X \times Y, Z) \longrightarrow \operatorname{Mor}(X \times Z) \times \operatorname{Mor}(X, Z)$$

is bijective. It is well-known and easy to show that the direct product is unique up to canonical isomorphism in the obvious sense. On says that a category admits direct products if the direct product for two arbitrary objects exists.

5.6 Proposition. *In the category of complex spaces direct products exist.* Pdpcs

We will not give the proof in all details. But we will describe several tools which lead to a proof.

1) The first is a gluing principle for sheafs. Assume that $X = \bigcup U_i$ is an open covering of a topological space. Assume also that for each i there is given a sheaf F_i on U_i and for each pair (i,k) of indices there is given an isomorphism $h_{ij} : F_i|(U_i \cap U_j) \to F_j|(U_i \cap U_j)$ with the conditions
$$h_{ik} = h_{ij} \circ h_{jk} \qquad \text{on } U_i \cap U_j \cap U_j.$$
Then there exists a sheaf F and a system of isomorphisms $h_i : F|U_i \to F_i$ with the properties
$$h_{ik} = h_i h_k^{-1} \qquad \text{on } U_i \cap U_k.$$
2) Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be complex spaces such that their direct product $(X \times Y, \mathcal{O}_{X\times Y})$ exists. Assume that $U \subset X$, $V \subset Y$ are open subsets. Then the direct product of $(U, \mathcal{O}_X|U)$ and $(V, \mathcal{O}_Y|V)$ exists and can be identified with
$$(U \times V, \mathcal{O}_{X\times Y}|U \times V).$$
3) Let X, Y be two model spaces which are closed in open subsets $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ and which are defined through holomorphic functions $f_1, \dots, f_\nu$ on U and $g_1, \dots, g_\mu$ on V. Then the direct product of the complex spaces X, Y exists. It can be identified with the model space in $U \times V$ defined through the holomorphic functions $f_i(x)g_k(y)$.

Complex spaces in the sense of Serre

A complex space is called a complex space in the sense of Serre, if the natural map $\mathcal{O}_X \to \mathcal{C}_X$ is injective. Due to the nullstellensatz this is equivalent to the fact that the rings $\mathcal{O}_{X,a}$ are nilpotent-free. Then we can consider the elements of $\mathcal{O}_X(U)$ as usual functions in U. The category of complex spaces in the sense of Serre is the full subcategory of the category of complex spaces in the sense of Grothendieck. If $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a holomorphic map between complex spaces in the sense of Serre, then φ is determined by f. There is a natural functor $X \mapsto X_{\text{red}}$ of the category of complex spaces of Grothendieck to that of Serre. Just associate to $(X, \mathcal{O}_X)$ the ringed space $(X, \mathcal{O}_X/\mathcal{N})$ where $\mathcal{N}$ is the nil-radical. Due to Cartan's coherence theorem this is a complex space.

Example of a non-reduced complex space

Consider the complec plane $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$. Consider the ideal sheaf that is generated by z^2. This is coherent. Its support is one point (the origin) and the restricted sheaf is given by $\mathbb{C}\{z\}/z^2 \cong \mathbb{C}^2$. This ring is not nilpotent free. So the ringed space $(\text{pt}, \mathbb{C}\{z\}/z^2 \cong \mathbb{C}^2)$ is a non-reduced complex space.

6. Cohomomology of sheaves

Let X be a topological space and F a presheaf of abelian groups. We denote the stalk of F at a point a by F_a. For an open subset $U \subset X$ we defined already

$$F^{(0)}(U) = \prod_{a \in U} F_a.$$

The assignment $U \mapsto F^{(0)}(U)$, with obvious restriction maps, is a presheaf of abelian groups. It is in fact a sheaf. There is a natural homomorphism $F \to F^{(0)}$ and the assignment $F \mapsto F^{(0)}$ is a functor, in particular a homomorphism $F \to G$ induces a natural homomorphism $F^{(0)}(U) \to G^{(0)}(U)$.

Now we restrict to sheaves of abelian groups. The functor $F \mapsto F^{(0)}$ is exact. This means that for a exact sequence $F \to G \to H$ of sheaves the associated sequence $F^{(0)}(U) \to F^{(0)}(U) \to F^{(0)}(U)$ is exact.

The natural homomorphism $F \to F^{(0)}$ is injective. So the sequence

$$0 \longrightarrow F \longrightarrow F^{(0)}$$

is exact. We will extend this sequence as follows. Consider the factor sheaf $F/F^{(0)}$ and define

$$F^{(1)} = (F^{(0)}/F)^{(0)}.$$

Inductively we can define

$$F^{(n+1)} = (F^{(n)}/F^{(n-1)})^{(n-1)}.$$

This construction gives an exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)} \longrightarrow F^{(1)} \longrightarrow \cdots$$

Sometimes it is good to look in a different way at this sequence. We also have to consider the reduced sequence (cancellation of F)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F^{(0)} & \longrightarrow & F^{(1)} & \longrightarrow & \cdots \\ & & \uparrow & & & & \\ & & \text{zero-position} & & & & \end{array}$$

This is a complex of sheaves which is exact outside the zero-position. We apply the functor “global sections” to obtain a complex of abelian groups

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F^{(0)}(X) & \longrightarrow & F^{(1)}(X) & \longrightarrow & \cdots \\ & & \uparrow & & & & \\ & & \text{zero-position} & & & & \end{array}$$

We denote the cohomology groups of this complex by $H^n(X,F)$. This is zero if $n<0$. In the case $n=0$ it is the kernel of $F^{(0)}\to F^{(1)}$ which is obviously canonically isomorphic to $F(X)$,

$$H^0(X,F)=F(X).$$

There is a generalization of the cohomology of sheaves. We consider a continuous map of topological spaces $f:X\to Y$. We recall the functor "direct image". If F is a sheaf of abelian groups of Y, then f_*F is a sheaf of abelian groups on Y. It is defined through $(f_*F)(V)=F(f^{-1}(V))$. The assignment $V\mapsto H^n(F|U,U)$ defines a presheaf. Its associated sheaf is denoted by R^nf_*F. Clearly R^nf_* is zero for $n<0$ and

$$R^0f_*(F)=f_*F.$$

The sheaves R^nf_*F are called the *higher direct images* of f.

Long exact sequences

6.1 Remark. *The functor "global sections" $F\mapsto F(X)$ for sheaves of abelian groups on a topological space is left exact. This means the following. Let $0\to F\to G\to H\to 0$ be an exact sequence of sheaves, then* Rle

$$0\longrightarrow F(X)\longrightarrow G(X)\longrightarrow H(X)$$

is exact.

Let $f:X\to Y$ be a continuous map. Then the functor "direct image" $F\mapsto f_*F$ is also left exact.

Let $0\to F\to G\to H\to 0$ be an exact sequence of sheaves of abelian groups on X. One can show that this induces a short exact sequence of complexes

$$0\longrightarrow F^{(\bullet)}(X)\longrightarrow G^{(\bullet)}(X)\longrightarrow H^{(\bullet)}(X)\longrightarrow 0.$$

The associated long exact cohomology sequence gives the long exact cohomology sequence of sheaf cohomology

$$0\longrightarrow F(X)\longrightarrow G(X)\longrightarrow H(X)\longrightarrow H^1(X,F)\longrightarrow\cdots.$$

A relative version of this is the long exact sequence for the higher direct images: Let $f:X\to Y$ a continuous map of topological spaces and let $0\to F\to G\to H\to 0$ be an exact sequence of sheaves of abelian groups on X. Then one gets a long exact sequence of sheaves

$$0\longrightarrow f_*(F)\longrightarrow f_*(G)\longrightarrow f_*(H)\longrightarrow R^1f_*(F)\longrightarrow\cdots.$$

7. Cech cohomology

We have to work with open coverings $\mathfrak{U} = (U_i)_{i\in I}$ of the given topological space X. For indices $i_0, \dots, i_p$ we use the notation

$$U_{i_0,\dots,i_p} = U_{i_0} \cap \ldots \cap U_{i_p}.$$

Let F be a sheaf on X. A p-cochain of F with respect to the covering $\mathfrak{U}$ is a family of sections is an element of

$$\prod_{(i_0,\dots,i_p)\in I^{p+1}} F(U_{i_0,\dots,i_p}).$$

This means that to any $(p+1)$-tuple of indices $i_0, \dots, i_p$ there is associated a section $s(i_0, \dots, i_p) \in F(U_{i_0,\dots,i_p})$. We denote the group of all cochains by $C^p(\mathfrak{U}, F)$. The derivative ds of a p-cochain the $(p+1)$-cochain defined by

$$ds(s_0, \dots s_{p+1}) = \sum_{j=0}^{p+1} (-1)^j s(i_0, \dots, \hat{i}_j, \dots, i_{p+1}) | U_{i_0,\dots,i_{p+1}}.$$

The rule $d^2 = 0$ is obvious, hence we obtain a complex, the so called Cech complex,

$$\cdots \longrightarrow C^{p-1}(\mathfrak{U}, F) \longrightarrow C^p(\mathfrak{U}, F) \longrightarrow C^{p+1}(\mathfrak{U}, F) \longrightarrow \cdots$$

Here for negative p we set $C^p(\mathfrak{U}, F) = 0$. The cohomology groups of this complex are the Čech cohomology groups $\check{\mathrm{H}}^p(\mathfrak{U}, F)$.

7.1 Lemma. *There is a natural isomorphism* HnullC

$$\check{H}^0(\mathfrak{U}, F) = H^0(X, F) \qquad (= F(X)).$$

Proof. A zero-cochain s is just a family $s_i \in F(U_i)$. The condition $ds = 0$ means $s_i | U_i \cap U_j = s_j | U_i \cap U_j$. By the sheaf axioms they glue to a global section. □

A sheaf F is called flabby if the restriction maps $F(U) \to F(V)$ are surjective. The Godement sheaves $F^{(0)}$ are examples of flabby sheaves.

7.2 Proposition. *Let F be a flabby sheaf. Then for every open covering* PCwv

$$\check{H}^p(\mathfrak{U}, F) = 0 \quad \textit{for} \quad p > 0.$$

7.3 Theorem of Leray. *Let F be a sheaf on X and $\mathfrak{U} = (U_i)$ an open covering of X. Assume that $H^p(U, F|U) = 0$ for all $p > 0$ and for arbitrary intersection of finitely many U_i. Then there is a natural isomorphism* ToL

$$H^p(X, F) \cong \check{H}^p(\mathfrak{U}, F)$$

for all p.

Proof. We consider the Godement resolution $0 \to F \to F_0 \to F_1 \to \cdots$. There is a natural diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F(X) & \to & F_0(X) & \to & F_1(X) & \to & F_2(X) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^0(\mathfrak{U}, F) & \to & C^0(\mathfrak{U}, F_0) & \to & C^0(\mathfrak{U}, F_1) & \to & C^0(\mathfrak{U}, F_2) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^1(\mathfrak{U}, F) & \to & C^1(\mathfrak{U}, F_0) & \to & C^1(\mathfrak{U}, F_1) & \to & C^1(\mathfrak{U}, F_2) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^2(\mathfrak{U}, F) & \to & C^2(\mathfrak{U}, F_0) & \to & C^2(\mathfrak{U}, F_1) & \to & C^2(\mathfrak{U}, F_2) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

All rows but the first one are exact. Similarly all columns but first one are exact. Now a homological lemma gives the desired result. □

The oriented Čech complex

A Čech cocycle s is called *alternating* if for every permutation σ of $0, \dots p$ one has

$$s(\sigma(0), \dots, \sigma(p)) = \operatorname{sgn}(\sigma) s(0, \dots p)$$

The subspace of all alternating cocycles with values in a sheaf F is denoted by

$$C^p_{\text{alt}}(\mathfrak{U}, F) \subset C^p(\mathfrak{U}, F).$$

This is a sub-complex, so, in particular, we have a homomorphism of complexes. This is a homotopy equivalence. Hence it is a quasi-isomorphism of complexes.

Chapter II. Stein spaces

1. The notion of a Stein space

From now on we assume that all complex spaces are Hausdorff and with countable basis of the topology. If the reader wants, he can assume that the notion of a complex space is understood in the sense of Serre.

Probably the reader knows that on a connected compact complex manifold any holomorphic function is constant. Assume that the dimension is > 1. If one removes from this manifold a single point the situation does not remedy, since in more than one variable there do not exist isolated singularities. Hence there exist also non-compact manifolds that admit no non-constant analytic function. Stein spaces are opposite to this situation. They are spaces that admit many holomorphic functions. We are going to explain in which sense this has to be understood.

Let K be a non-empty compact subset of a topological space X. We use the notation

$$||f||_K := \max\{|f(x)|;\ x \in K\}$$

for a continuous function f on X.

1.1 Definition. *Let K be a non-empty compact subset of a complex space. The* ***holomorphic convex hull*** *$\hat{K}$ of K is the set of all $x \in X$ such that $|f(x)| \leq ||f||_K$ for all $f \in \mathcal{O}_X(X)$.* HolCon

1.2 Definition. *A complex space is called* ***holomorphically convex*** *if the holomorphic convex hull of any compact subset is compact.* HolConS

Assume that X is a complex space with the following property: for every infinite closed discrete subset $S \subset X$ there exists a holomorphic function $f : X \to \mathbb{C}$ that is unbounded on S. Then X is holomorphically convex. This can be seen by an indirect argument. Let K be a compact subset such that $\hat{K}$ is not compact. Then there exists a sequence in $\hat{K}$ with no convergent subsequence. This gives an infinite subset $S \subset \hat{K}$ that is closed in X and discrete. Then there exists a global holomorphic function which is unbounded on $\hat{K}$. This is not possible.

From this observation we can deduce that open subsets U of the plane $\mathbb{C}$ are holomorphically convex. To show this we consider an infinity closed discrete subset S. If S is unbounded then we take $f(z) = z$. In the case that S is bounded their must be an accumulation point a of S which lies on the boundary of U. Then take $f(z) = 1/(z - a)$.

In more then one variable the situation is completely different. A polydisk (around 0)

$$U = U_r = U_{r_1} \times \cdots \times U_{r_n}$$

is a cartesian product of discs in $\mathbb{C}$ around zero. On calls $r = (r_1, \dots, r_n)$ the multi-radius of U Let $U = U_r$ be a polydisk. We claim that $U - \{0\}$ is not holomorphically convex. For this we consider the subset K consisting of all z with $|z_i| = r_i/2$. We know that every holomorphic function f on $U - \{0\}$ extends holomorphically to U. From the maximum principle one deduces $\hat{K} = \{z \in U;\ |z_i| \le r_i/2\}$. This set is not compact.

1.3 Definition. *A complex space X is called a Stein space if the following conditions are satisfied:* SteinS

1) *It is holomorphically convex.*
2) *(Point separation) For two different points $x, y \in X$ there exists a global $f \in \mathcal{O}_X(X)$ with $f(x) = 0$, $f(y) = 1$.*
3) *(Infinitesimal point separation) For any point $a \in X$ there exist global $f_1, \dots, f_m \in \mathcal{O}_X(X)$ whose germs generate the maximal ideal of $\mathcal{O}_{X,x}$.*

It is clear that open subsets of the complex plane are Stein spaces. More generally it is clear that a cartesian product $D = D_1 \times \cdots \times D_n$ of open subsets $D_i \subset \mathbb{C}$ is Stein. In particular, polydisks are Stein. It is already a deep result that all non-compact connected Riemann surfaces are Stein spaces. We will not proof this result here. A proof can be found in [Fo]. As we have seen it is false that open subsets of $\mathbb{C}^n$ are always Stein in the case $n > 1$.

1.4 Remark. *Let X be a Stein space. Then every closed analytic subspace is a Stein space too.* SubStein

2. Theorem A and B for Stein spaces

The basic theorems about Stein spaces are

2.1 Theorem A for Stein spaces. *Let X be a Stein space and $\mathcal{M}$ a coherent sheaf. For each $a \in X$ the stalk $\mathcal{M}_a$ can be generated by (the germs of) finitely many global sections.* TheoA

2.2 Theorem B for Stein spaces. *Let X be a Stein space and $\mathcal{M}$ a coherent sheaf. Then* TheoA

$$H^q(X, \mathcal{M}) = 0 \quad \textit{for} \quad q > 0.$$

The formulation seems to indicate that we have two independent theorems. Actually theorem A is an easy consequence of theorem B. To prove this we consider the vanishing ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$ of the point a and then for an arbitrary natural number Then we use the exact sequence

$$0 \longrightarrow \mathcal{J}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{J} \longrightarrow 0.$$

From theorem B we get that $\mathcal{M}(X) \to (\mathcal{M}/\mathcal{J})(X)$ is surjective. Notice that $(\mathcal{M}/\mathcal{J})(X) = \mathcal{M}_a/\mathfrak{m}\mathcal{M}_a$. Here $\mathfrak{m}$ means the maximal ideal of $\mathcal{O}_{X,a}$. We denote by M the submodule of $\mathcal{M}_a$ that is generated by the image of $\mathcal{M}(X)$ and by $N = \mathcal{M}_a/M$ the factor module. The above argument shows $\mathcal{M}_a = M + \mathfrak{m}\mathcal{M}_a$ or $\mathfrak{m}N = N$. The proof now follows from the lemma of Nakayama of commutative algebra.

We formulate basic properties of Stein spaces which are related to Theorem A and B. First we introduce a notation. Let X be a topological spaces and U a subset. We will write

$$U \subset\subset X$$

if U is open in X and if the closure of U in X is compact.

2.3 Proposition. *Let X be a Stein space and K a compact subset. Then there exists an open Stein subspace U of X with the property* PKop

$$K \subset U \subset\subset X.$$

Let

$$X = \bigcup_{i \in I} U_i = \bigcup_{j \in J} V_j$$

be two coverings of a set. One calls the second covering a *refinement* of the first one if for every $j \in J$ there exists $i \in I$ with the property $V_j \subset U_i$. If one chooses for each j an i one gets a so-called refinement map $J \to I$ (which is not uniquely determined).

2.4 Proposition. *Every open covering of a complex space admits a refinement $X = \bigcup_{i\in I} U_i$ such that I is finite or countable and such that all U_i are Stein.* PSc

2.5 Proposition. *Let $X = \bigcup_{i\in I} U_i$ be a Stein covering of X. Then there exists for each i* PVs

$$V_i \subset\subset U_i$$

such that $X = \bigcup_{i\in I} V_i$ is still a Stein covering of X.

2.6 Proposition. *Let X be a complex space and let U, V be two open Stein subspaces. Then $U \cap V$ is Stein too.* Puv

Chapter III. Nuclear spaces

1. Locally convex spaces

A *seminorm* p on a complex vector space E is a map $p : E \to \mathbb{R}$ with the properties

a) $p(a) \geq 0$ for all $a \in E$,
b) $p(ta) = |t|p(a)$ for all $t \in \mathbb{C}$, $a \in E$,
c) $p(a + b) \leq p(a) + p(b)$.

A basic result for seminorms is the Theorem of Hahn-Banach.

1.1 Theorem of Hahn-Banach. *Let p be a seminorm on a complex vector space E. Let $F \subset E$ be a $\mathbb{C}$-linear subspace and let $L : F \to \mathbb{C}$ be a linear form with the property $|L(a)| \leq p(a)$ for all $a \in F$. Then there exists a linear extension $L : E \to \mathbb{C}$ of L with the property $|L(a)| \leq p(a)$ for all $a \in E$.* THB

Corollary. *Let p be a seminorm on a complex vector space E. Let $a \in E$. Then there exists a linear form L on E with the property*

$$L(a) = p(a), \qquad |L(x)| \leq p(x) \text{ for all } x \in E.$$

The ball of radius $r > 0$ around a and with respect to p is defined as

$$U_r(a, p) := \{\, x \in E;\ p(a - x) < r \,\}.$$

Let $\mathcal{P}$ be a non-empty set of seminorms. We assume that $\mathcal{P}$ is filtrating. This means that for any two $p_1, p_2 \in \mathcal{P}$ there exists a $p \in \mathcal{P}$ with $p \geq p_1$, $p \geq p_2$. We will consider only filtrating sets of seminorms. This is not a big restriction, since for any set of seminorms, one can consider the set of all seminorms $\mathcal{Q}$ which consist of all $\max p$ where p runs through a finite subset of $\mathcal{P}$. Then $\mathcal{Q}$ is a filtrating set that contains $\mathcal{P}$.

A subset U of E is called open (with respect to $\mathcal{P}$) if for every point $a \in E$ there exists a ball B around a with respect to a seminorm $p \in \mathcal{P}$ which is contained U. It is easy to show that this defines a topology which equips E with a structure as topological vector space. This means that the maps

$$E \times E \longrightarrow E, \quad \mathbb{C} \times E \longrightarrow E$$

(addition and scalar multiplication) are continuous. The balls are open subsets. We call them also open balls to differ them from the closed balls

$$\bar{U}_r(a, p) := \{\, x \in E;\ p(a - x) \leq r \,\}.$$

These are closed subsets.

It is clear that all $p \in \mathcal{P}$ are continuous. (It is actually the weakest topology with this property.)

We call $\mathcal{P}$ a defining set for this topology of E. Let $\mathcal{P}_{\max}$ be the set of all continuous seminorms on E. It is clear that $\mathcal{P}_{\max}$ is also filtrating and defines the same topology and even more: Two sets $\mathcal{P}$, $\mathcal{Q}$ of seminorms define the same topology if and only if $\mathcal{P}_{\max} = \mathcal{Q}_{\max}$.

A subset M of a vector space is called *convex* if for any two points $a, b \in M$ the straight line joining them is contained in M. Let $(E, \mathcal{P})$ be a vector space equipped with some filtrating set of seminorms. Then there exists a basis of neighborhoods of the origin that consists of convex sets. This is trivial since the balls obviously are convex.

1.2 Definition. *A topological vector space is called* ***locally convex*** *if there exists a filtrating set of seminorms wich defines its topology.* Dlc

(One can show that this is equivalent to the fact that E admits a basis of convex neighbourhoods of the origin. But we don't need it.)

In an arbitrary topological vector space the notion of convergent sequences and Cauchy sequences is defined.

A sequence (a_n) in a topological vector space E converges to $a \in E$ if and only if for every neighborhood U of the origin $a_n - a \in U$ holds up to finitely many n.

A sequence (a_n) in a topological vector space E is a Cauchy sequence if and only if and only if for every neighborhood U of the origin there exists a natural number N such that $a_n - a_m \in U$ holds for all $n, m \geq N$.

In the case of a locally convex space these notions can be defined as follows. Let $\mathcal{P}$ be a defining system of seminorms.

A sequence (a_n) in E converges to $a \in E$ if and only if $p(a_n - a) \to 0$ for all $p \in \mathcal{P}$.

And a sequence is a Cauchy sequence if for every $\varepsilon > 0$ and every $p \in \mathcal{P}$ there exists a natural number N such that

$$p(a_n - a_m) < \varepsilon \quad \text{for} \quad n, m \geq N.$$

Of course both statements carry over from $\mathcal{P}$ to $\mathcal{P}_{\max}$. Clearly each convergent series is a Cauchy sequence.

1.3 Definition. *A topological vector space is called* ***sequence complete*** *if every Cauchy sequence converges.* Dsc

In the general context sequence completeness is not the correct definition for completeness. One has to replace sequences by nets or filters. For our purpose this is not necessary because the spaces which occur in our context have the property that there exists a countable basis of neighbourhoods of the origin. In this case the two notions of completeness agree, so we have not to deal with filters or nets.

2. Fréchet spaces

A set $\mathcal{P}$ of seminorms on a vector space is called definite if

$$p(a) = 0 \quad \text{for all} \quad p \in \mathcal{P} \quad \Longrightarrow \quad a = 0.$$

It is easy to prove that $\mathcal{P}$ is definite if and only if the associated locally convex space is a Hausdorff space.

2.1 Definition. *A Fréchet space is a locally convex vector space with the following properties.* DFs

1) *It is Hausdorff.*
2) *There exists a countable defining set of seminorms.*
3) *It is sequence complete.*

Obviously 2) implies that there exists a countable basis of neighborhoods of the origin and the converse is also true.

Let p_n be a defining system p_n of seminorms. We can replace it by the new system

$$\max\{p_1, \dots, p_n\}.$$

This gives the following remark.

2.2 Remark. *On a Fréchet space E there exists a countable defining system (p_n) of seminorms with the property* Rcss

$$p_1 \leq p_2 \leq \cdots.$$

Permanence properties of Fréchet spaces

A closed subspace $F \subset E$ of a Fréchet space, equipped with the induced topology, is a Fréchet space too. A defining system of seminorms is obtained if one restricts the seminorms p of a defining system on E to F.

Let $F \subset E$ a closed subspace of a Fréchet space. Then the quotient space E/F equipped with the quotient topology, is a Fréchet space. A defining system of seminorms is obtained as follows. Denote the quotient map by $f : E \to E/F$. Let p be a continuous seminorm on E (from a defining system is enough). Then

$$\tilde{p}(y) = \inf_{f(x)=y} p(x) \qquad (x \in E),$$

is a seminorm on E/F.

Let $(E_s)_s \in I$ be a finite or countable family of Fréchet spaces. Then their direct product

$$E = \prod_{s \in S} E_s$$

equipped with the product topology, is a Fréchet space. In terms of seminorms this can be described as follows. Take a finite subset $T \subset S$ and for each $t \in T$ take a continuous seminorm p_i, $i \in J$ on E_i (from a defining system is enough). Then one can define a seminorm on the product

$$p((x_i)) = \max_{j \in J} p_j(x_j).$$

Let E be a locally convex Hausdorff space such that there exists a countable defining set of seminorms. A *completion* is a Fréchet space $\hat{E}$ with the following properties.

1) E is a vector subspace of $\hat{E}$.
2) It carries the induced topology.
3) It is dense in $\hat{E}$.

A standard construction shows that a completion always exists. It is unique up to canonical isomorphism, since it satisfies a universal property.

Let $E \to F$ be a continuous linear map into an arbitrary Fréchet space. Then there exists a unique continuous linear extension $\hat{E} \to F$.

As a consequence, every continuous linear map $E \to F$ between locally convex spaces with countable bases of the neighborhoods of 0 extends to the completions $\hat{E} \to \hat{F}$.

We recall some basic facts about Fréchet spaces.

2.3 Open mapping Theorem. *Any linear continuous surjective map between Fréchet spaces is open. In particular, it is topological if in addition it is bijective.* TOmT

Banach spaces

A seminorm p on a vector space E is called a norm if it is definite, $p(a) = 0 \Rightarrow a = 0$. Usually on writes $\|a\| := p(a)$. A normed space is a pair $(E, \|\cdot, \cdot\|)$ consisting of a vector space and a distinguished norm. A normed space has an underlying structure of a Hausdorff locally convex vector space. The origin admits a countable basis of neighborhoods. Hence we can consider the completion $\hat{E}$. The norm of E extends to a norm on $\hat{E}$. So the completion can be considered as normed space as well. A *Banach space* is a complete normed space. We give two examples of Banach spaces.

ℓ^1 is the space of all sequences $(a_n)_{n\geq 1}$ of complex numbers such that $\sum |a_n|$ converges. Such sequences are also called absolutly summable. This is a Banach space with respect to the norm

$$\|(a_n)\|_1 = \sum_{n=1}^{\infty} |a_n|.$$

A second example is ℓ^∞. It is the space of all bounded sequences of complex numbers. This is Banach space with the norm

$$\|(a_n)\| = \sup_n |a_n|.$$

These spaces admit an obvious generalization. We start with a Banach space B. Then we define $\ell^1(E)$ to be the space of all sequences (a_n) in E such that $\sum |a_n|$ converges. This is a Banach space with the norm $\|(a_n)\| = \sup_n |a_n|$. Similarly we define to be $\ell^\infty(E)$ the space of all bounded sequences in E. We equip this with the obvious norm to obtain a Banach space again.

In the next section we will generalize $\ell^1(E)$ to Fréchet spaces.

3. Summability in Fréchet spaces

We start with a locally convex space E. We assume that there exists a countable system of neighborhoods of the origin. Then we want to consider sequences (a_n) where $a_n \in E$. There are several generalizations of the notion “summable”.

3.1 Definition. *A sequence (a_n) in a locally convex space is called **weakly*** Dwss
***summable** if for each continuous linear form $L \in E'$ the sequence $(L(a_n))$ of numbers is absolutely summable. We denote the space of all weakly summable sequences by $\ell^1[E]$.*

*It is called **summable** if the following condition holds.*

For each continuous seminorm p (of a defining system is enough) and for each $\varepsilon > 0$ there exists a finite set $J \subset \mathbb{N}$ such that for any two finite sets J_1, J_2, $J \subset J_1, J_2 \subset \mathbb{N}$, one has

$$p\Big(\sum_{n\in J_1} a_n - \sum_{n\in J_2} a_n\Big) < \varepsilon.$$

We denote the space of all absolutely summable sequences by $\ell^1(E)$.

It is called ***absolutely summable*** *if for each continuous seminorm p (from a defining system is enough) the sequence of numbers $(p(a_n))$ is summable. We denote the space of all absolutely summable sequences by $\ell^1\{E\}$.*

It is known that for a Banach space the notions summable and absolutely summable are the same.

3.2 Lemma. *One has* Lta

$$\ell^1\{E\} \subset \ell^1(E) \subset \ell^1[E].$$

In the case $E = \mathbb{C}$ the three spaces agree.

Proof. Let $(a_n) \in \ell^1\{E\}$. Fix a p and a ϵ. Then there exists a N such that

$$\sum_{n>N} p(a_n) < \varepsilon.$$

Hence

$$p\Big(\sum_{n\in J_1} a_n - \sum_{n\in J_2} a_n\Big) < 2\sum_{n>N} p(a_n) < 2\varepsilon.$$

This implies $(a_n) \in \ell^1(E)$.

Now we assume $(a_n) \in \ell^1(E)$. Let $L \in E'$. There exists a continuous seminorm p such that $|L| \le p$. Then $(p(a_n))$ is a Cauchy sequence. Hence $(L(a_n)$ is a Cauchy sequence. Hence it converges. This remains true for any reordering of (a_n). This means that the series converges unconditional. It is known from basic calculus that this implies absolute convergence. $\square$

We want to equip each of the three spaces with a structure as locally convex space. We start with $\ell^1[E]$. For each continuous seminorm p on E (of a defining system is enough) we define a seminorm ε_p on $\ell^1[E]$.

$$\varepsilon_p((a_n)) = \sup \sum_n |L(a_n)|, \quad |L| \le p.$$

Here the supremum is taken over all $L \in E'$ such that $|L(x)| \le p(x)$ for $x \in E$. Of course one has to show that this supremum is finite. The family of all ε_p, a defining system of p is enough, defines a structure as locally convex space on $\ell^1[E]$. It is clear that $\ell^1[E]$ with this structure as locally convex space has a countable basis of neighborhoods of the origin. We call this topology on $\ell^1[E]$ the ε-topology. The restriction of this topology to $\ell^1(E)$ is also called ε-topology.

There is also a natural topology on $\ell^1\{E\}$. For each continuous seminorm p (from a defining system is enough) we define

$$\pi_p((a_n)) = \sum_n p(a_n).$$

We call the topology defined through this system the π-topology on $\ell^1\{E\}$.

3.3 Remark. *The natural inclusion* Rnic

$$(\ell^1\{E\}, \pi\text{-}topology) \longrightarrow (\ell^1(E), \varepsilon\text{-}topology)$$

is continuous.

Proof. This follows from the trivial inequality $\varepsilon((a_n)) \le \pi((a_n)$ for $(a_n) \in \ell^1\{E\}$. □

3.4 Lemma. *Let E be a Fréchet space. Then the spaces $\ell^1[E]$, $\ell^1(E)$, equipped with the ε-topology or $\ell^1\{E\}$ equipped with the π-topology are Fréchet spaces* Lepf

4. Tensor products of locally convex spaces

We need the tensor product of vector spaces. Let E, F be two vector spaces. Recall that the algebraic tensor product $(E \otimes_{\mathbb{C}} F, B)$ is a pair, consisting of a vector space $E \otimes_{\mathbb{C}} F$ and a bilinear map

$$B : E \times F \longrightarrow E \otimes_{\mathbb{C}} F,$$

such that the following universal property holds. Let $\beta : E \times F \longrightarrow X$ be a bilinear map into an arbitrary vector space X, then there exists a unique linear map $L : E \otimes_{\mathbb{C}} F \to X$ such the diagram commutes. Usually one writes

$$a \otimes b = \beta(a, b).$$

One can show that each element of $E \otimes_{\mathbb{C}} F$ can be written as finite sum of "pure" elements $a \otimes b$. But this presentation needs not to be unique.

Using bases it is easy to show that a tensor product exists and is unique up to canonical isomorphism in an obvious sense. The tensor product is exact in the following sense. Let $E_1 \to E_2 \to E_3$ be an exact sequence of vector spaces. Then for each vector space F the sequence $E_1 \otimes_{\mathbb{C}} F \to E_2 \otimes_{\mathbb{C}} F \to E_3 \otimes_{\mathbb{C}} F$ remains exact.

The tensor product can be generalized to more the two factors, $E_1 \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} E_n$ together with a multilinear map

$$E_1 \times \ldots \times E_n \longrightarrow E_1 \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} E_n, \quad (e_1, \ldots, e_n) \longmapsto e_1 \otimes \ldots \otimes e_n,$$

that satisfies an obvious universal property. The natural map

$$E_1 \otimes_{\mathbb{C}} E_2 \otimes_{\mathbb{C}} E_3 \longrightarrow (E_1 \otimes_{\mathbb{C}} E_2) \otimes_{\mathbb{C}} E_3$$

is an isomorphism. In this way one obtains the associativity law

$$(E_1 \otimes_{\mathbb{C}} E_2) \otimes_{\mathbb{C}} E_3 = E_1 \otimes_{\mathbb{C}} (E_2 \otimes_{\mathbb{C}} E_3).$$

Let now E, F be two topological vector spaces. It is possible to equip the tensor product $E \otimes_{\mathbb{C}} F$ with a topology such that for every continuous bilinear map $E \times_{\mathbb{C}} F \to X$ into a topological vector space the induced map $E \otimes_{\mathbb{C}} F \to X$ is continuous. Unfortunately this topology has the bad property that $E \otimes F$ needs not to be locally convex even if E, F are. Therefore we modify this construction for locally convex spaces.

The idea is to combine two seminorms p, q on E, F to a seminorm $p \otimes q$ on $E \otimes_{\mathbb{C}} F$. It seems natural to demand

$$(p \otimes q)(a \otimes b) = p(a)q(b).$$

But there are problems. One problem is that not every element of $E \otimes_{\mathbb{C}} F$ can be written in the form $a \otimes b$. We only know that each element of the tensor product can be written as finite sum of elements of this form.

We will describe now two seminorms $p \otimes_\pi q$, $p \otimes_\varepsilon q$ which both have the property $(p \otimes q)(a \otimes b) = p(a)q(b)$. But in general they are different.

4.1 Definition. *Let p, q be seminorms on vector spaces E, F. Their π-tensor product is* Dpit

$$(p \otimes_\pi q)(a) = \inf \sum_{i=1}^{n} p(x_i)q(y_i)$$

where $(x_1, \ldots, x_n)$, $(y_1, \ldots y_n)$ runs through all presentations

$$a = x_1 \otimes y_1 + \cdots + x_n \otimes y_n.$$

It is rather clear that this is a seminorm. Let E, F be two locally convex spaces with defining systems $\mathcal{P}$, $\mathcal{Q}$, then

$$\mathcal{P} \otimes_\pi \mathcal{Q} = \{p \otimes_\pi q; \quad p \in \mathcal{P},\ q \in \mathcal{Q}\}$$

defines the desired topology on the tensor product. This topology is independent of the choice of the defining systems. We denote the corresponding locally convex space by $E \otimes_\pi F$.

4.2 Lemma. *Let E, F be two locally convex vector spaces. Then the space $E \otimes_\pi F$ has the following universal property. For each locally convex vector space X and each continuous bilinear map $E \times F \longrightarrow X$ the induced linear map $E \otimes_{\mathbb{C}} F \to X$ is continuous.* Ltt

4.3 Lemma. *Let E, F be two Hausdorff locally convex spaces. Then their π-tensor product is Hausdorff too.* Lthh

Proof. Let $a \in E \otimes_\pi F$ be an element of the nullspace. We have to show that a is zero. We argue indirectly and assume $a \neq 0$. It is sufficient to construct a continuous linear form on $E \otimes_\pi F$ such that $L(a) \neq 0$. For this we write a in the form

$$a = x_1 \otimes y_1 + \cdots x_n y_n.$$

We can assume that the vectors $x_1, \dots, x_n$ are linearly independent and the same for $y_1, \dots, y_n$. We can find linear forms $L_1 \in E'$, $L_2 \in F'$ such that

$$L_1(x_1) = 1, \quad L_2(y_2) = 1 \text{ and } L_1(x_i) = L_2(y_i) = 0 \quad \text{for } i > 1.$$

Then we can consider the (continuous) bilinear form

$$E \times F \longrightarrow \mathbb{C}, \quad (x, y) \longmapsto L_1(x) L_2(y).$$

It factors through a linear form $L : E \otimes_\pi F \to \mathbb{C}$. It has the desired property. □

Of course $\mathcal{P} \otimes_\pi \mathcal{Q}$ is countable if $\mathcal{P}$, $\mathcal{Q}$ are. Hence we can complete $E \otimes_\pi F$ if E, F are Fréchet spaces and we obtain a Fréchet space

$$E \hat{\otimes}_\pi F = \text{completion of } E \otimes_\pi F.$$

4.4 Lemma. *Let E, F be two Fréchet paces. Then the space $E \hat{\otimes}_\pi F$ has the following universal property. For each Fréchet space X and each continuous bilinear map $E \times F \longrightarrow X$ there exists a unique continuous linear factorization $E \hat{\otimes}_{\mathbb{C}} F \to X$.* Lttz

The proof is clear. □

4.5 Proposition. *The π-tensor product of locally convex vector spaces is commutative and associative. This means that the natural (algebraic) isomorphism* Ptass

$$E \otimes_\pi F = F \otimes_\pi E, \quad (E \otimes_\pi F) \otimes_\pi G = E \otimes_\pi (F \otimes_\pi G)$$

is an isomorphism of locally convex spaces.

Corollary. *Let E, F, G be Fréchet spaces, then*

$$E \hat{\otimes}_\pi F = F \hat{\otimes}_\pi E, \quad (E \hat{\otimes}_\pi F) \hat{\otimes}_\pi G = E \hat{\otimes}_\pi (F \hat{\otimes}_\pi G).$$

There are several other possible topologies on the tensor product. The topology we have defined is called the *projective* topology or the π-topology.

Now we define another topology on the tensor product $E \otimes_{\mathbb{C}} F$ of two locally convex spaces which will be denoted as ε-topology, written as $E \otimes_\varepsilon F$.

For the construction we define also a tensor product $p \otimes_\varepsilon q$ for seminorms p, q on E, F. The definition is related to the following lemma.

4.6 Lemma. *Let E be a vector space and let p be a seminorm on E. Then* Lxxx

$$p(a) = \sup\{|L(a)|; \quad |L| \le p\}.$$

This follows immediately from the Hahn Banach theorem. □

Let E, F be two vector spaces and $L : E \to \mathbb{C}$, $M : E \to \mathbb{C}$ be two linear forms. Then one can consider the bilinear form

$$B : E \times F \longrightarrow \mathbb{C}, \quad B(x, y) = L(x)M(y).$$

By the universal property of the tensor product it factors through a linear map

$$E \otimes_{\mathbb{C}} F \longrightarrow \mathbb{C}, \quad x \otimes y \longmapsto L(x)M(y).$$

As a consequence the sum

$$\sum_{i=1}^{n} L(x_i)M(y_i)$$

depends only on

$$a = \sum_{i=1} x_i \otimes y_i$$

(and not on the special decomposition).

4.7 Definition. *Let p, q be seminorms on vector spaces E, F. Their ε-tensor product is* Detp

$$(p \otimes_\varepsilon q)(a) = \sup\left\{ \left| \sum_i L(x_i)M(y_i) \right| \right\}$$

where L, M run through all elements of E', F' with the property $|L| \le p$, $|M| \le q$ and where $a = \sum_i x_i \otimes y_i$ is an arbitrarily chosen decomposition of a.

It is clear that $p \otimes_\varepsilon q$ is a seminorm. Let E, F be two locally convex spaces with defining systems $\mathcal{P}$, $\mathcal{Q}$, then

$$\mathcal{P} \otimes_\varepsilon \mathcal{Q} = \{p \otimes_\pi q;\ p \in \mathcal{P},\ q \in \mathcal{Q}\}$$

defines the desired topology on the tensor product. This topology is independent of the choice of the systems. We denote the corresponding locally convex space by $E \otimes_\varepsilon F$.

4.8 Lemma. *The π and the ϵ-tensor product of two seminorms have the properties* Lpiep

$$p \otimes_\varepsilon q \le p \otimes_\pi q$$

and

$$(p \otimes_\pi q)(x \otimes y) = (p \otimes_\varepsilon q)(x \otimes y) = p(x)q(y).$$

Proof. The stated inequality is trivial, since

$$\Big| \sum_i L(x_i)M(y_i) \Big| \le \sum p(x_i)q(x_i).$$

The inequality $(p \otimes_\pi q)(x, y) \le p(x)p(y)$ is also trivial. So it remains to prove

$$(p \otimes_\varepsilon q)(x \otimes y) \ge p(x)p(y).$$

From the definition of the ε-tensor product we see

$$(p \otimes_\varepsilon q)(x \otimes y) \ge |L(x)M(y)|$$

for all linear forms L, M with $|L| \le p$, $|M| \le q$. Due to the Hahn Banach theorem we can find L, M such that $|L(x)| = p(x)$ and similar for M. □

For (a_n) in ℓ^1 and $x \in E$ we obviously have $(a_n x) \in \ell^1\{E\}$. So we get a bilinear map

$$\ell^1 \times E \longrightarrow \ell^1\{E\}.$$

This map induces a linear map

$$\ell^1 \otimes_{\mathbb{C}} E \longrightarrow \ell^1\{E\}.$$

4.9 Proposition. *The natural map* Pnmrl

$$\ell^1 \hat{\otimes}_\pi E \longrightarrow \ell^1\{E\}$$

is an isomorphism of topological vector spaces, where the right hand side is equipped (as the left hand side) with the π-topology.

Proof. Here ℓ^1 is a locally convex space with a single norm $\|(\alpha)_n\| = \sum_n |\alpha_n|$ as defining family. Hence a defining family on $\ell^1 \otimes_\pi E$ is given by $\|\cdot\| \otimes_\pi p$ where p runs through a defining family of E. Recall also that a defining family on $\ell^1\{E\}$ is given by

$$\pi_p((a_n)) = \sum_n p(a_n).$$

Claim. If one pulls back the seminorm π_p to $\ell^1 \otimes_\pi E$, one gets the seminorm $\ell^1 \otimes_\pi E$ (for the same p).

Proof of the claim. Let $a \in E \otimes_\pi F$. We write it in the form $a = x^{(1)} \otimes y_1 + \cdots + x^{(m)} \otimes y_m$. Here the $x^{(i)}$ are sequences in ℓ^1,

$$x^{(i)} = x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, \ldots$$

and $y_i \in E$. Then

$$(\|\cdot\| \otimes_\pi p)(a) = \inf \sum_{n=1}^{\infty} \sum_{i=1}^{m} \left|x_n^{(i)}\right| p(y_i)$$

where the infimum runs over all these presentations. Now we consider the image of a in $\ell^1\{E\}$. It is the sequence (S_n) with

$$S_n = \sum_{i=1}^{m} (x_n^{(i)}) y_i \quad \text{(sequence label } n\text{)}$$

We get

$$\pi_p((S_n)) = \sum_{n=1}^{\infty} \left| \sum_{i=1}^{m} x_n^{(i)} p(y_i) \right|.$$

We get

$$(\|\cdot\| \otimes_\pi p)(a) \geq \pi_p((S_n)).$$

It follows that the map $\ell^1 \otimes_\pi E \to \ell^1\{E\}$ is continuous. Now we construct a new presentation for $a = \sum_i x^{(i)} \otimes y_i$. We will use that the natural map $\ell^1 \times E \to \ell^1 \otimes_\pi E$ is bilinear and continuous. We want to use the equality (in the Banach space ℓ^1).

$$x^{(i)} = \sum_{n=1}^{\infty} x_n^{(i)} e^{(n)} = \lim_{N\to\infty} \sum_{n=1}^{N} x_n^{(i)} e^{(n)}.$$

Here $e^{(n)}$ is the sequence that has 1 at the nth place and 0 else. Now we obtain

$$a = \sum_i x^{(i)} \otimes y_i = \lim_{N\to\infty} \sum_i \sum_{n=1}^{N} x_n^{(i)} e^{(n)} \otimes y_i = \lim_{N\to\infty} \sum_{n=1}^{N} e^{(n)} \otimes \sum_i x_n^{(i)} y_i.$$

Using this presentation of a we derive the inequaltity

$$(\|\cdot\| \otimes_\pi p)(a) \leq \pi_p((S_n)).$$

So the equality must hold. This proves the claim.

Proof of Proposition 4.9 continued. The equality of the two seminorms shows that the map $\ell^1 \otimes_\pi E \to \ell^1\{E\}$ is an embedding of topological spaces. So $\ell^1 \otimes_\pi E$ can be considered as subspace of $\ell^1\{E\}$ and its π-topology is the induced topology. It is easy to show that this subspace consists of all sequences (a_n) such that the $a_{(n)}$ generate a finite dimensional vector space. It is also clear that this space is dense in $\ell^1\{E\}$. This shows that the induced map $\ell^1 \hat{\otimes}_\pi E \to \ell^1\{E\}$ is a topological isomorphism. □

4.10 Proposition. *The natural map* Pnmrlz

$$\ell^1 \hat{\otimes}_\epsilon E \longrightarrow \ell^1(E)$$

is an isomorphism of topological vector spaces, where the right hand side is equipped (as the left hand side) with the ϵ-topology.

The proof is similar to that of Proposition 4.10. We omit it. □

5. Trace class operators in Hilbert spaces

All Hilbert spaces are assumed to be separable. This means that they admit a finite or countable orthonormal basis. Let H be a separable Hilbert space and let $f : H \to H$ be a bounded linear operator. If H is finite dimensional, one can define the trace $\mathrm{tr}(f)$ through

$$\mathrm{tr}(f) = \sum \langle f(e_n), e_n \rangle$$

where e_n denotes an orthonormal basis. It is known from linear algebra that the trace is independent of the choice of the basis.

We want to carry over the notion of the trace for a certain class of *trace class operator*s. The idea is to consider an orthonormal basis $e_1, e_2, \ldots$ and then to define

$$\mathrm{tr}(f) = \sum_{n=1}^{\infty} \langle f e_n, e_n \rangle.$$

Of course there is a problem with convergence and questions like independence of the choice of the basis have to be settled. We introduce now the so-called *trace class operators*, also called *nuclear operators*, which allow an easy definition of the trace.

5.1 Definition. *A bounded linear operator $f : H_1 \to H_2$ of Hilbert spaces is called* ***of trace class*** *or* ***nuclear*** *if there exist sequences $(x_n \in H_1)$, $(y_n \in H_2)$ such that* Dsk

1) $\displaystyle\sum_{n=1}^{\infty} \|x_n\| < \infty, \quad \|y_n\| = 1$ *for all n.*

2) $\displaystyle f(x) = \sum_{n=1}^{\infty} \langle x, x_n \rangle y_n.$

Notice that 2) converges absolutely as consequence of 1).

Let (e_m) be an orthonormal basis on a Hilbert space H. Then the following formula holds for $x, y \in H$,

$$\sum_m \langle x, e_m \rangle \langle e_m, y \rangle = \langle x, y \rangle,$$

where the left hand side converges absolutely. This formula holds for basis elements x, y for trivial reason and then carries over to the general case.

We derive a consequence. Let $f : H \to H$ be a nuclear operator of a Hilbert space into itself. Let (x_n), (y_n) be defining sequences for f and let (e_m) be an orthonormal basis. Then the following formula

$$\sum_m \langle f_m(e_m), e_m \rangle = \sum_n \langle x_n, y_n \rangle$$

holds where both sides converge absolutely. We obtain the following important result.

5.2 Remark. *For a nuclear operator on a Hilbert space $f : H \to H$, the trace $\operatorname{tr}(H)$ can be defined through the formula* Rtr

$$\operatorname{tr}(f) = \sum_m \langle f(e_m), e_m \rangle$$

where (e_m) is an arbitrary orthonormal basis.

The condition "nuclear" is sometimes difficult to check. There is a related condition which is easier.

5.3 Definition. *A bounded linear operator $f : H_1 \to H_2$ of Hilbert spaces is called a **Hilbert-Schmidt operator** if there exists an orthonormal basis $e_1, e_2, \dots$ such that* DbHS

$$\sum_n \|f(e_n)\|^2 < \infty.$$

It is easy to show that then the sum converges for all orthonormal bases and that this sum is independent of the choice of this basis.

5.4 Proposition. PHnc

1) *Nuclear operators between Hilbert spaces are Hilbert Schmidt operators.*
2) *The composition of two Hilbert Schmidt operators $H_1 \to H_2 \to H_3$ is nuclear.*
3) *Hilbert Schmidt operators (hence nuclear ones) are compact operators.*

We omit the proof. □

Sometimes the following reformulation of nuclearity is useful. Let

$$f(x) = \sum_{n=1}^{\infty} \langle x, x_n \rangle y_n, \quad \sum_{n=1}^{\infty} \|x_n\| < \infty, \quad \|y_n\| = 1 \text{ for all } n.$$

be a nuclear operator. We can assume that all x_n are different form 0. We consider the continuous linear forms $L_n(x) = \langle x, x_n \rangle$. We know (Theorem of

Riesz) that every continuous linear form $L : H \to \mathbb{C}$ is of the form $L(x) = \langle x, a\rangle$ with a unique $a \in H$. The space of all continuous forms H' carries an obvious structure as Banach space. With this structure we have $\|L\| = \|a\|$. Now we rewrite the formula of the presentation:

$$f(x) = \sum_n L_n(x)y_n, \quad \sum_n \|L_n\| < \infty, \quad \|y_n\| = 1.$$

We rewrite this in the form

$$f(x) = \sum_n \lambda_n \tilde{L}_n(x)y_n$$

where

$$\lambda_n = \|L_n\|, \quad \tilde{L}_n = \frac{L_n}{\|L_n\|}.$$

Then $\sum \lambda_n$ converges absolutely. These presentations have the following properties.

a) λ_n converges absolutely.
b) The sequence y_n is bounded.
c) The sequence $\tilde{L}_n(x)$ is bounded for each x. Now it is easy to prove the following result.

5.5 Lemma. *A bounded linear operator $f : H_1 \longrightarrow H_2$ is nuclear if and only it has a presentation of the form* Lcbo

$$f(x) = \sum_n \lambda_n L_n(x)y_n$$

where λ_n is a sequence of complex numbers such that $\sum_n |\lambda_n|$ converges, L_n is a bounded sequence of continuous linear forms and (y_n) is bounded in H.

This formulation carries over form Hilbert to Banach spaces. But more is true, it is the basis of the definition of nuclearity for Fréchet spaces.

6. Nuclear operators for locally convex spaces

The notion “nuclear operator” can be generalized from Hilbert spaces to Fréchet spaces. This is due to Alexander Grothendieck.

We need the notion of a bounded set in a locally convex space.

6.1 Definition. *A subset $M \subset E$ of a locally convex space is* ***bounded*** *if it is bounded with respect to each continuous seminorm of E.* Dlcb

It is easy to show that it is enough to have boundedness for a defining system of seminorms. If M is a bounded set and U any neighborhood of the origin then one has

$$M \subset \bigcup_{t>0} tU.$$

It is also easy to show the converse. Hence a subset M of E is bounded if and only $M \subset \bigcup_{t>0} tU$ for any neighborhood of the origin. (One can use this to extend the notion of a bounded set to arbitrary topological vector spaces.)

6.2 Definition. *A subset $A \subset E$ of a vector space is called* ***absolutely convex*** *if for $a, b \in E$ and complex numbers α, β with the property $|\alpha| + |\beta| \leq 1$ one has $\alpha a + \beta b \in A$.* Dac

An easy result whose proof we omit says.

6.3 Lemma. *Each bounded subset of a locally convex space is contained in a closed, absolutely convex, bounded subset.* Lbca

Subsets as in the lemma can be used to define certain normed spaces.

6.4 Remark. *Let A closed, absolutely convex, bounded subset of a Fréchet space E. We denote by $E(A)$ the subspace generated by A. This is a Banach space if one equips it with the norm* Rean

$$\|x\|_A := \inf\{t > 0; \quad x \in tA\}.$$

The proof is easy and can be omitted. □

There is second way to associate Banach spaces to Fréchet spaces. Let E be a locally convex space and p a continuous seminorm. It is easy to show that the nullspace $N_p = \{x \in E;\ p(x) = 0\}$ is a vector subspace. The seminorm p factors through $E \to E/N_p$ and defines a norm on E/N_p. We denote by E_p the completion of this normed space. We will denote the norm on E_p that is induced by p by $\|\cdot\|_p$. Let $q \geq p$ two continuous seminorms. Then there is a natural continuous linear map $E_q \to E_p$ of Banach spaces.

We also have to work with the dual space E'. This is the space of all continuous linear forms $E \to \mathbb{C}$. It can be defined for every topological vector space E. We mention that a linear form $L : E \to \mathbb{C}$ on a locally convex space is continuous if there exists a continuous seminorm p with the property

$$|L(a)| \leq p(a).$$

We call a sequence (L_n) of continuous linear forms on a locally convex space *equicontinuous* if there exists a continuous seminorm such that

$$|L_n(x)| \le p(x) \quad \text{for all} \quad n, x.$$

Let E be a Banach space with norm $\|\cdot\|$, then E' is a Banach space too with the norm

$$\|L\| = \sup\{|L(x)|;\ \|x\| = 1\}.$$

A sequence of continuous linear forms on a Banach space E is equicontinuous if and only if it is bounded with respect to the norm on E'.

6.5 Definition. *A continuous linear operator $f : E \to F$ between Fréchet* Dclfn
spaces is called ***nuclear*** *if it has a presentation*

$$f(x) = \sum_n \lambda_n L_n(x) y_n$$

where
1) (λ_n) *is a sequence of complex numbers such that* $\sum |\lambda_n| < \infty$.
2) (y_n) *is a bounded sequence in* F.
3) L_n *is an equicontinuous sequence of linear forms.*

It is clear that the series in the Definition converges absolutely. Lemma 5.5 shows that in the case of Hilbert spaces this agrees with Definition 5.3. There is a more general result for Banach spaces.

6.6 Lemma. *A bounded linear operator $f : E_1 \to E_2$ of Banach spaces spaces* Dskz
is nuclear if there exist sequences $(b_n \in E_2)$, $(L_n \in E_1')$ such that

1) $\sum_{n=1}^{\infty} \|L_n\| < \infty, \quad \|y_n\| = 1$ *for all* n,
2) $f(x) = \sum_{n=1}^{\infty} L_n(x) y_n$.

A simple result about nuclear maps is the "ideal property".

6.7 Remark. *Let $f : E \to F$ and $g : F \to G$ be continuous linear maps of* Rnn
Fréchet spaces. Assume that f or g is nuclear. Then $g \circ f$ is nuclear.

The proof is trivial. □

6.8 Proposition. *Assume that $f : E \to F$ be a nuclear map of Fréchet* Pfee
spaces. Then there exists a commutative diagram of continuous linear maps

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & F \\ E_1 \arrow[r, "f_1"'] & F_1 \arrow[u] \end{tikzcd}$$

such that E_1, F_1 are Banach spaces an that f_1 is nuclear.

Proof. Let f be nuclear as in Definition IV.3.1. Since the L_n are equicontinuous there exists a continuous seminorm p on E such that $|L|_n \le p$ for all n. We can consider the Banach space $E_1 := E_p$. The natural map $E \to E_1$ clearly is continuous. The map $E \to F$ factors as

$$\begin{tikzcd} E \arrow[r] \arrow[d] & F \\ E_1 \arrow[ur] & \end{tikzcd}$$

Let f be nuclear as in Definition IV.3.1. Choosing a natural number N we can separate f into two parts,

$$f = f' + f'' = \sum_{n=1}^{N} L_n(x) y_n + \sum_{n=N+1}^{\infty} L_n(x) y_n.$$

Then the image of f' is finite dimensional. If we take N big enough we get $\|f\|'' < 1$. It follows

$$(\mathrm{id} + f)E + f''E = (\mathrm{id} + f' + f'')E + f''E = (\mathrm{id} + f')E + f''E.$$

But $\mathrm{id} + f'$ is invertible (geometric series). Hence the right hand side is E. This gives us the following result.

6.9 Proposition. *Let $f : E \to E$ be a nuclear map of Banach spaces, then the cokernel of $\mathrm{id} + f$ is finite dimensional.* Pkc

7. Nuclear spaces

7.1 Definition. *A Fréchet space E is called **nuclear** if for every continuous seminorm p there exists a seminorm $q \ge p$ such that $E_q \to E_p$ is nuclear.* DnS

This means concretely the following. There exists a sequence L_n of continuous seminorms on E_q and a sequence (b_n) in E_p with the following properties.

$$\sum \|L_n\|_q < \infty, \;\; \|b_n\|_p = 1,$$
$$(\text{image in } E_p \text{ of}) \; a = \sum_{n=1}^{\infty} L_n(a) b_n \quad (a \in E_q).$$

The elements b_n actually can be taken from E/N_q. To prove this we mention that each $b \in E_q$ can be written as absolutely convergent series of elements in

E/N_p. Just write $b = \lim B_m$, $B_m \in E$. One can achieve that $\|b - B_m\| \leq 2^m$. Then one gets

$$b = \sum_{m=1}^{\infty} b_m = B_1 + (B_2 - B_1) + (B_3 - B_2) + \cdots$$

and this series is absolutely convergent. Wy apply this to b_n and obtain

$$b_n = \sum_{m=1}^{\infty} b_{mn}, \quad b_{mn} \in E/N_p.$$

Now we get

$$\sum_{n=1}^{\infty} L_n(a)b_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} L_n(a)b_{mn} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (L_n(a)\,\|b_{mn}\|)\frac{b_{mn}}{\|b_{mn}\|}.$$

Now we use a bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(m,n) \to k$. Then we set

$$\tilde{L}_k(a) = L_n(a)\,\|b_{mn}\|\,.$$

We obtain

$$\sum_{n=1}^{\infty} L_n(a)b_n = \sum_k \tilde{L}_k(a)b_k.$$

In the following we identify continuous linear forms on the Banach space E_q with continuous linear forms on the normed space E/N_q. The latter can be identified with linear forms on E which are vanish along N_p and are continuous with respect to q.

7.2 Lemma. *A Fréchet space E is nuclear if for every continuous seminorm p there exists a continuous seminorm $q \geq p$ such that there exists a sequence L_n of continuous seminorms on E_q and a sequence (b_n) in E with the following properties.* LFc

$$\sum \|L_n\|_q < \infty,\ \ \|b_n\|_p = 1,$$
$$\lim_{N\to\infty} p\Big(a - \sum_{n=1}^{\infty} L_n(a)b_n\Big) \quad (\textit{for } a \in E).$$

It is sufficient to show this identity for all a from a dense subspace of E.

7.3 Proposition. *Let E be a nuclear space. Then there is a natural isomorphism of topological vector spaces* Pnipe

$$\ell^1\hat{\otimes}_\pi E = \ell^1\hat{\otimes}_\epsilon E.$$

Proof. We know already $\varepsilon_p \le \pi_p$. Hence it suffices to prove the following. Let p be an arbitrary continuous seminorm on E. Then there exists a continuous seminorm q such that

$$\pi_p((a_n)) \le \varepsilon_q((a_n)).$$

We start with a q such that $E_p \longrightarrow E_q$ is nuclear. By definition of nuclearity there exists a sequence of linear forms continuous linear forms L_m on E_p and a sequence $b_m \in E/N_p$ such that

$$\sum_m L_m(a)b_m = (\text{image in } E_q \text{ of})\ a.$$

We apply this to each member of the sequence (a_n),

$$\sum_m L_m(a_n)b_m = a_n.$$

This implies

$$\pi_q((a_n)) \le \sum_{m,n} |L_m(a_n)|.$$

The definition of ε_p gives

$$\sum_n |L(a_n)| \le \varepsilon_p((a_n)) \quad \text{for} \quad |L| \le p.$$

A simple consequence is

$$\sum_n |L(a_n)| \le \varepsilon((a_n)) \left\|L\right\|_p \quad \text{for all } L.$$

Here $\left\|L\right\|_p$ denotes the norm of the linear form $L : E_p \to \mathbb{C}$. From the assumption of the nuclearity we know that

$$C := \sum \left\|L\right\|_p < \infty.$$

Now we get

$$\pi_q((a_n)) \le C\varepsilon_p((a_n)).$$

8. Tensor product of nuclear spaces

8.1 Theorem. *Let E, F be two Fréchet spaces, one of them nuclear. Then the ε- and the π-tensor product agree.* Tvpa

If E, F are two spaces, we will use the notation

$$E\hat{\otimes}_{\mathbb{C}}F = E\hat{\otimes}_{\varepsilon}F = E\hat{\otimes}_{\pi}F.$$

Proof of the theorem. Let E be nuclear. We will show that for each continuous seminorm P on $E \otimes_\pi F$ there exists a continuous seminorm Q on $E \otimes_\varepsilon F$ such that $P(z) \le CQ(z)$ with a constant C that is independent of $z \in E \otimes_{\mathbb{C}} F$. (Then the ε-topology is finer than the π-topology. Recall that the reverse is always true.) We can assume that $P = p \otimes_\pi q$, $p \in E'$, $q \in F'$. Then we will construct Q in the form $Q = r \otimes_\varepsilon q$. The condition for $r \in E'$ is that $r \ge q$ and that $E_r \to E_q$ is nuclear. We apply Lemma 7.2 to an element a that is in the image of $E \to E_r$. We get that there exist sequences $L_n \in E_r'$ and $b_n \in E$ with the properties

$$\sum_n \|L_n\|_p < \infty, \quad \|b_n\|_q = 1$$

and such that for $a \in E$ we have

$$\left\| a - \sum_{n=1}^N L_n(a) b_n \right\| \longrightarrow 0 \text{ for } N \longrightarrow \infty.$$

So we have to prove

$$(p \otimes_\pi r)(z) \le C(q \otimes_\varepsilon r)(z) \qquad (z \in E \otimes_{\mathbb{C}} F).$$

For the constant we will take

$$C = \sum_n \|L_n\|_p .$$

The proof will use the following inequality. Let z be an element of $E \otimes_{\mathbb{C}} F$. We can write it in the form

$$z = \sum_{\text{finite}} x_i \otimes y_i, \quad x_i \in E,\ y_i \in F.$$

Let now $L \in E_r'$ (for example one of the L_n. We will write

$$L(x) = L(\text{image in } E_r \text{ of})\ x \quad \text{for } x \in E$$

and similar for other linear forms. Then

$$\begin{aligned}\Big\|\sum_i L(x_i)y_i\Big\|_r &= \sup\Big\{\Big|M\Big(\sum_i L(x_i)y_i\Big)\Big|, \quad M \in F_q',\ |M| \le q\Big\}\\ &= \sup\Big\{\Big|\Big(\sum_i L(x_i)M(y_i)\Big)\Big|, \quad M \in F_q',\ |M| \le q\Big\}\\ &\le \|L\|_r\,(r \otimes_\varepsilon q)(z).\end{aligned}$$

By the Hahn Banach theorem there exists a linear form N on $E \otimes_{\mathbb{C}} F$ with the property

$$N(z) = (p \otimes_\pi q)(z) \quad \text{and} \quad |N(z')| \le (p \otimes_\pi q)(z') \text{ for all } z' \in E \otimes_{\mathbb{C}} F.$$

We apply this to an element z' of the form $z' = a \otimes b$, $a \in E$, $b \in F$ to obtain

$$|N(a \otimes b)| \le \|a\|_p\,\|b\|_q\,.$$

$$N(a \otimes b) = \sum_{n=1}^{\infty} L_n(a)N(b_n \otimes b).$$

This implies

$$\begin{aligned}(p \otimes_\pi q)(z) = N(z) &= \sum_{n=1}^{\infty} N\Big(\sum_i L_n(x_i)(b_n \otimes y_i)\Big)\\ &= \sum_{n=1}^{\infty} N\Big(b_n \otimes \Big(\sum_i L_n(x_i)y_i\Big)\Big)\\ &\le \sum_{n=1}^{\infty} \|b_n\|_p \Big\|\sum_i L_n(x_i)y_i\Big\|_q\\ &= \sum_{n=1}^{\infty} \Big\|\sum_i L_n(x_i)y_i\Big\|_q\\ &\le \sum_{n=1}^{\infty} \|L_n\|_r\, r \otimes_\varepsilon q)(z)\\ &= C(r \otimes_\varepsilon q)(z).\end{aligned}$$

This proves Theorem 8.1. □

8.2 Theorem. *Let E_1, E_2 be nuclear spaces, then $E_1 \hat{\otimes}_{\mathbb{C}} E_2$ is nuclear as well.* Tnasw

Proof. We will show this for the ε-tensor product. We will show that for any continuous seminorm P on $E_1 \otimes_{\mathbb{C}} E_2$ there exists a continuous seminorm $Q \geq P$ such that Lemma 7.2 applies to $E_1 \hat{\otimes}_\varepsilon E_2$ (instead of E). We can assume that $P = p_1 \otimes_\varepsilon p_2$. The we choose $q_1 \geq p_1$ and $q_2 \geq p_2$ such that there exist for $i = 1, 2$ sequences of linear forms L_{in} on E_{q_i} and sequences of elements $b_{in} \in E$ such that

$$\begin{aligned}
&\sum \|L_{in}\|_{q_i} < \infty, \ \|b_{in}\|_{p_i} = 1, \\
&\lim_{N\to\infty} p_i\left(a_i - \sum_{n=1}^N L_{in}(a_i)b_{in}\right) = 0 \quad (\text{for } a_i \in E_i).
\end{aligned}$$

We abbreviate for $i = 1, 2$

$$S_i = \sum_{n=1}^N L_{in}(a_i)b_{in}.$$

We consider

$$\begin{aligned}
(p_1 \otimes_\varepsilon p_2)(a_1 \otimes a_2 - S_1 \otimes S_2) &= (p_1 \otimes_\varepsilon p_2)(a_1 \otimes (a_2 - S_2) + (a_1 - S_1) \otimes S_2) \\
&\leq p_1(a_1)p_2(a_2 - S_2) + p_1(a_1 - S_1)p_2(S_2).
\end{aligned}$$

This tends to zero if $N \to \infty$. Now we consider

$$S_1 \otimes S_2 = \sum_{1\leq m,n\leq N} L_{1m}(a_1)L_{2n}(a_2)b_{1m} \otimes b_{2n}.$$

Now we choose a bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(m, n) \mapsto k$. If we set

$$L_k = L_{1m} \otimes L_{2n}, \quad b_k = b_{1m} \otimes b_{2n}$$

we get

$$S_1 \otimes S_2 - \sum_{1\leq k\leq N} L_k(a_1 \otimes a_2)b_k \longrightarrow 0 \quad \text{for } N \longrightarrow \infty.$$

9. Exactness of the completed tensor product

We recall that a sequence of topological vector spaces $E_1 \to E_2 \to E_3$ is called exact if it is exact in the algebraic sense, i.e. the image of $E_1 \to E_2$ equals the kernel of $E_2 \to E_3$. There is a problem with completion. Completion usually does not preserve exactness. So the following theorem is remarkable.

9.1 Theorem. *Let $0 \to E_1 \to E_2 \to E_3 \to 0$ be an exact sequence of nuclear spaces. and let F be another nuclear space. Then the sequence* Texact

$$0 \to E_1 \hat{\otimes}_{\mathbb{C}} F \to E_2 \hat{\otimes}_{\mathbb{C}} F \to E_3 \hat{\otimes}_{\mathbb{C}} F \to 0$$

is exact.

The goal of the section is to give the proof of this theorem. We will use the following notation. An *embedding* $f : E \to F$ of Fréchet spaces is a linear continuous map such that $E \to f(E)$ is a topological isomorphism if $f(E)$ is equipped with the induced topology. In particular $f(E)$ is a Fréchet space and hence closed in F.

9.2 Lemma. *Let $E_1 \to E_2$ be an embedding of Fréchet spaces and let F be another Fréchet space. Then $E_1 \hat{\otimes}_\varepsilon F \to E_1 \hat{\otimes}_\varepsilon F$ is an embedding of Fréchet spaces too.* Leeps

Proof. It is clear that this map is injective. Hence it remains to show that the topology on $E_1 \otimes_\varepsilon F$ is induced from $E_2 \hat{\otimes}_\varepsilon F$. It is enough to prove that the topology $E_1 \otimes_\varepsilon F$ is induced from $E_2 \otimes_\varepsilon F$. The topology on $E_2 \hat{\otimes}_\varepsilon F$ is given by seminorms $p_2 \otimes_\varepsilon q$ where p_2 is a continuous seminorm on E_2 and q on F.

$$(p_2 \otimes_\varepsilon q)\Big(\sum_i a_i \otimes b_i\Big) = \sup\Big\{\Big|\sum_i L_2(a_i)M(a_i)\Big|\Big\}$$

where L_2 runs all linear forms on E_2 with $|L_2| \le p_2$ and M runs to all linear forms on F with $|M| \le q$. Let $p_1 = p_2|E_1$. Then we have similarly that

$$(p_1 \otimes_\varepsilon q)\Big(\sum_i a_i \otimes b_i\Big) = \sup\Big\{\Big|\sum_i L_1(a_i)M(a_i)\Big|\Big\}$$

where L_1 runs all linear forms on E_1 with $|L_1| \le p_1$ and M runs to all linear forms on F with $|M| \le q$. By the Hahn Banach theorem each L_1 is the restriction of an L_2. This means

$$(p_2 \otimes_\varepsilon q)|E_1 \otimes_\varepsilon F = p_1 \otimes_\varepsilon q.$$

This proves Lemma 9.2. □

Let E be a Fréchet space and let (a_n) a sequence in E such that $\sum_n p(a_n)$ converges for each continuous seminorm (for each defining system is enough). Then

$$a = \sum_n a_n := \lim_{N \to \infty} a_n$$

converges. We claim that every element of E can be written in this form.

9.3 Lemma. *Let E be a Fréchet space and let $E_0 \subset E$ be a dense subspace. Then every element $a \in E$ can be written in the form* Lfsur

$$a = \sum_{n=1}^{\infty} a_n, \quad a_n \in E_0,$$

such that $\sum p(a_n)$ converges for each p.

Proof. Let

$$p_1 \le p_2 \le \cdots$$

be a defining system of seminorms. Let $a = \lim \alpha_n$, $\alpha_n \in E_0$. The idea is to write

$$a = \alpha_1 + (\alpha_2 - \alpha_1) + (\alpha_3 - \alpha_2) + \cdots.$$

We want to do this in such a way that

$$p_n(\alpha_{n+1} - \alpha_n) < 2^{-n}.$$

It is clear that this can be obtained if one replaces (α_n) by a suitable subsequence. □

9.4 Proposition. *Let E and F be two Fréchet spaces with defining systems $p_1 \le p_2 \le \cdots$ and $q_1 \le q_2 \le \cdots$ Then every element $c \in E\hat{\otimes}_\pi F$ can be written in the form* Pcab

$$c = \sum_{n=1}^{\infty} a_n \otimes b_n$$

such that

$$\sum p(a_n)q(b_n)$$

converges for each continuous seminorm p on E and q on F.

Proof. This follows easily from Lemma 9.3.

9.5 Proposition. *Let $E_1 \to E_2$ and $F_1 \to F_2$ be surjective continuous linear maps of Fréchet spaces Then $E_1\hat{\otimes}_\pi F_1 \to E_2\hat{\otimes}_\pi F_2$ is surjective too.* Pepi

Proof. Let

$$p_1 \le p_2 \le \cdots$$

be a defining system of seminorms on E_1. We take there quotient norms on F_1

$$\bar{p}_1 \le \bar{p}_2 \le \cdots.$$

Similarly $q_1 \le q_2 \cdots$ and $\bar{q}_1 \le \bar{q}_2 \le \cdots$ of E_2 Let $c \in F_1\hat{\otimes}_\pi F_2$. We write it in the form

$$c = \sum_{n=1}^{\infty} a_n \otimes b_n$$

such that

$$\sum \bar{p}_k(a_n)\bar{q}_k(b_n) < \infty \quad \text{for all} \quad k$$

Due to the definition of the quotient norm we find inverse images $\alpha_n \in E_1$ of a_n and similarly $\beta_n \in F_n$ such that

$$p_k(\alpha_n)q_k(\beta_n) \leq \bar{p}_k(a_n)\bar{q}_k(b_n) + 2^{-n}.$$

But then

$$\gamma := \sum_{n=1}^{\infty} \alpha_n \otimes \beta_n$$

converges and gives a preimage of c. □

Chapter IV. Nuclear modules over nuclear algebras

1. Nuclear algebras and nuclear modules

1.1 Definition. *A nuclear algebra A is a nuclear space that in addition carries a structure as associative and commutative $\mathbb{C}$-algebra with unit such that the multiplication $A \times A \to A$ is continuous.* DnFa

In the following A denotes such a nuclear algebra. Next we introduce the notion of a nuclear module over A.

1.2 Definition. *A nuclear module over A is a module E over A in the sense of commutative algebra that carries an additional structure as nuclear space such that $A \times E \longrightarrow E$ is continuous.* DnFm

The collection of all nuclear modules is a category where the morphisms are continuous A-linear maps.

A sequence $E \to F \to G$ of Fréchet modules is called exact if it is exact in the algebraic sense, i.e. the image of $E \to F$ equals the kernel of $F \to G$. The open mapping theorem shows that $E \to \mathrm{kernel}(F \to G)$ is a continuous open map.

This is not an abelian category. The image of a morphism is not necessarily closed and hence needs not to be a nuclear module. In the case $A = \mathbb{C}$ we get back the category of nuclear spaces.

Let A be a nuclear algebra and let V be a nuclear vector space. Then $A \hat{\otimes}_{\mathbb{C}} V$ is a nuclear space which carries an obvious structure as nuclear module over A. Such a nuclear module is called nuclear free.

1.3 Definition. *A nuclear module E over A is called **nuclear free** if there exists a nuclear space V such that* Dl1

$$E \cong A \hat{\otimes}_{\mathbb{C}} V.$$

We want to define a tensor product for nuclear algebras. For this we mention a simple algebraic fact. Let $A \to B$ be a unital homomorphism of commutative and associative rings and let M, N be two B-modules. Then one can consider the map

$$d : M \otimes_A B \otimes_A N \longrightarrow M \otimes_A N, \quad d(m \otimes b \otimes n) = (am) \otimes n - m \otimes (an).$$

It is easy to verify that

$$M \times N \longrightarrow M \otimes_A N / d(M \otimes_A B \otimes_A N \longrightarrow M \otimes_A N)$$

satisfies the universal property of $M \otimes_B N$. Hence we can write

$$M \otimes_B N = M \otimes_A N / d(M \otimes_A B \otimes_A N \longrightarrow M \otimes_A N).$$

We use this observation to define the complete tensor profuct $E \hat{\otimes}_A F$ for nuclear A-modules Now we start with

$$d : E \hat{\otimes}_{\mathbb{C}} A \hat{\otimes}_{\mathbb{C}} F \longrightarrow E \hat{\otimes}_{\mathbb{C}} F$$

and take the cokernel of this map

$$E \hat{\otimes}_A F := E \hat{\otimes}_{\mathbb{C}} F / d(E \hat{\otimes}_{\mathbb{C}} A \hat{\otimes}_{\mathbb{C}} F).$$

Notice that these are locally convex spaces with countable basis of neighborhoods of the origin. But they need not to be separated. But their quotient by the nullspace carries a natural structure as nuclear space. We denote this by

$$(E \hat{\otimes}_A F)_{\mathrm{sep}}.$$

This has a universal property in the category of Fréchet algebras.
Every continuous A-bilinear map $E \times F \to G$ into a Fréchet A-module G admits a unique continuous A-linear factorisation $(E \hat{\otimes}_A F)_{\mathrm{sep}} \to G$.

1.4 Definition. *Let M be a nuclear module. A* ***nuclear free resolution*** *is a sequence* Dnfr

$$\cdots \longrightarrow L_n \xrightarrow{d_n} L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

with the following properties:

1) *L_n are nuclear free Fréchet modules.*
2) *The d_n are linear and continuous.*
3) *The sequence is exact (in the usual algebraic sense).*

1.5 Definition. *A nuclear free resolution as in Definition 1.4 is called* ***direct*** *if for each n the image $d(L_n)$ is closed and if there exists a closed vector subspace $Y_n \subset L_{n-1}$ such that* Dnfd

$$L_{n-1} = d(L_n) \oplus Y_n,$$

(direct sum in the sense of topological vector spaces, i.e. L_{n-1} carries the product topology)

1.6 Lemma. *Two direct nuclear free resolutions are homotopic.* Dnfh

1.7 Lemma. *Each nuclear module admits a direct nuclear free resolution.* Lnfr

Proof. Let F be a nuclear space. We set $S_0 = A\hat{\otimes}_{\mathbb{C}}F$ which is a nuclear module through the left standing A. Then we set $S_n = A\hat{\otimes}_{\mathbb{C}}S_{n-1}$ which also gets its A-modules structure through the left standing A. So we have

$$S_n = A\hat{\otimes}_{\mathbb{C}}A\hat{\otimes}_{\mathbb{C}}\cdots\hat{\otimes}_{\mathbb{C}}A\hat{\otimes}_{\mathbb{C}}F \quad (n+1 \text{ factors } A).$$

We define the complex mappings. The map $d_0 : L_0 \to M$ is the obvious map. In general

$$d_n(a_0\otimes\cdots a_m\otimes x) = \sum_{i=0}^{n-1}(-1)a_0\otimes\cdots\otimes a_i a_{i+1}\otimes a_{i+2}\otimes\cdots\otimes x + (-1)^n a_0\otimes\cdots\otimes a_n x.$$

We denote this resolution the canonical direct free resolution.

It remains to show that the complex

$$\cdots \longrightarrow S_1 \longrightarrow S_0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

is exact. For this we show that the identity map of this complex is null-homotopic. This shows the homotopy

$$h_n : S_n \longrightarrow S_{n+1}, \quad n \geq 0,$$
$$a_0 \otimes \cdots \otimes a_n \otimes x \longmapsto 1 \otimes a_0 \otimes \cdots \otimes a_n \otimes x.$$

$$h_{-1} : M \longrightarrow S_0, \quad x \longmapsto 1 \otimes x$$

(and $h_n = 0$ for $n < -1$). □

Let F be a nuclear module over A. We consider the canonical direct free resolution $S_\bullet = S_\bullet(F)$. Let E be another nuclear module over A. The sequence

$$\cdots E\hat{\otimes}_A S_1(F) \longrightarrow E\hat{\otimes}_A S_0(F) \longrightarrow 0$$

needs not to be exact.

1.8 Definition. *The Tor groups of two nuclear modules are* Dtg

$$\widehat{\mathrm{Tor}}_n^A(E,F) = \frac{\mathrm{kernel}(S_n\hat{\otimes}_A F \longrightarrow S_{n-1}\hat{\otimes}_A F)}{\mathrm{image}(S_{n+1}\hat{\otimes}_A F \longrightarrow S_n\hat{\otimes}_A F)}.$$

In the case $n = 0$ this is, by definition of the tensor product,

$$\widehat{\mathrm{Tor}}_0^A(E,F) = E\hat{\otimes}_A F.$$

We notice

$$\begin{aligned} E\hat{\otimes}_A S_n(F) &= E\hat{\otimes}_A A\hat{\otimes}_{\mathbb{C}} \cdots \hat{\otimes}_{\mathbb{C}} A\hat{\otimes}_{\mathbb{C}} F \quad (n+1 \text{ factors } A), \\ &= E\hat{\otimes}_{\mathbb{C}} A\hat{\otimes}_{\mathbb{C}} \cdots \hat{\otimes}_{\mathbb{C}} A\hat{\otimes}_{\mathbb{C}} F \quad (n \text{ factors } A). \end{aligned}$$

From the commutativity of the tensor product we get a canonical isomorphism

$$E\hat{\otimes}_A S_n(F) = F\hat{\otimes}_A S_n(E).$$

Up to a sign this is compatible with the derivations. In this way one obtains the following result.

1.9 Remark. *There is a canonical isomorphism* RcEF

$$\widehat{\mathrm{Tor}}_n^A(E,F) = \widehat{\mathrm{Tor}}_n^A(F,E).$$

1.10 Definition. *Two nuclear A-modules E, F are called **transversal** if $E\hat{\otimes}_A F$ is separated and if $\widehat{\mathrm{Tor}}_n^A(E,F) = 0$ for $n > 0$.* Dtntr

1.11 Proposition. *Let E, F be two nuclear A-modules, one of them nuclear free. Then they are transversal.* Lnnf

Proof. We can assume that $F = A\hat{\otimes}_{\mathbb{C}} V$ is nuclear free. Then

$$E\hat{\otimes}_A F = E\hat{\otimes}_{\mathbb{C}} V$$

is nuclear free too.

It remains to show the vanishing of the higher Tors. Because of the homotopy invariance we can compute the Tors by means of an arbitrary direct nuclear resolution instead of the standard resolution. We can take for F the resolution

$$\cdots \longrightarrow 0 \longrightarrow F \longrightarrow F \longrightarrow 0.$$

For this resolution the calculation of the Tors is trivial. □

Let $0 \to F_1 \to F_2 \to F_2 \to 0$ be an exact sequence of nuclear modules and let E be a nuclear module. Then

$$E_n = M\hat{\otimes}_{\mathbb{C}} A\hat{\otimes}_{\mathbb{C}} \cdots \hat{\otimes}_{\mathbb{C}} A \quad (n \text{ factors } A).$$

is a nuclear space too and the sequence

$$0 \longrightarrow E_n\hat{\otimes}_{\mathbb{C}} F_1 \longrightarrow E_n\hat{\otimes}_{\mathbb{C}} F_2 \longrightarrow E_n\hat{\otimes}_{\mathbb{C}} F_3 \longrightarrow 0$$

remains exact. This sequence can be identified with the sequence

$$0 \longrightarrow E\hat{\otimes}_A S_n(F_1) \longrightarrow E\hat{\otimes}_A S_n(F_2) \longrightarrow E\hat{\otimes}_A S_n(F_3) \longrightarrow 0.$$

We can look at this as an exact sequence of complexes

$$0 \longrightarrow E\hat{\otimes}_A S_\bullet(F_1) \longrightarrow E\hat{\otimes}_A S_\bullet(F_2) \longrightarrow E\hat{\otimes}_A S_\bullet(F_3) \longrightarrow 0.$$

By a fundamental lemma of homological algebra this induces a long Tor sequence

$$\begin{aligned} \cdots \longrightarrow \widehat{\mathrm{Tor}}_n^A(E, F_1) \longrightarrow \widehat{\mathrm{Tor}}_n^A(E, F_2) \longrightarrow \widehat{\mathrm{Tor}}_n^A(E, F_3) \\ \longrightarrow \widehat{\mathrm{Tor}}_{n-1}^A(E, F_1) \longrightarrow \cdots \end{aligned}$$

Now we can apply Proposition 1.11.

1.12 Proposition. *Let E, F be two nuclear modules and let $L_\bullet$ a free resolution of F. Then the Tor groups $\widehat{\mathrm{Tor}}_n^A(E, F)$ equal the cohomology of $E\hat{\otimes}_A L_\bullet$.* PTgc

Another application of the long exact Tor sequence is the following result.

1.13 Lemma. *Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence of nuclear modules and E be also a nuclear module. If E is transversal to F_1 und F_2 then it is transversal to E_3.* Lntt

1.14 Lemma. *Let $A_1 \to A_2 \to A_3$ be homomorphisms of nuclear algebras and let E be a nuclear module over A. Assume that E is transversal to A_1 and to A_2. Then $E\hat{\otimes}_{A_1} A_2$ is transversal to A_3 over A_2.* Lett

2. Complex spaces and nuclear spaces

The polydisk in $\mathbb{C}^n$ of the multi-radius $r = (r_1, \dots, r_n)$, $r_i > 0$ is the set

$$P = \{z \in \mathbb{C}^n; \quad |z_i| < r_i\}.$$

We consider the space of holomorphic functions $\mathcal{O}(P)$. For each compact subset $K \subset P$ we can consider the seminorm on $\mathcal{O}(P)$,

$$p_K(a) = \max_{a \in P} |f(a)|.$$

The set of all of these seminorms equips $\mathcal{O}(P)$ with a structure as locally convex space. Due to the well known theorem of Weierstrass this space is sequence complete. If one takes only compact subsets of the form

$$K_n = \{z \in P; \quad |z_i| \le r_i(1 - 1/n)\} \qquad (n \ge 2)$$

one gets a defining system of seminorms. Hence $\mathcal{O}(P)$ has been equipped with a structure as Fréchet space.

We also have to consider the seminorms

$$q_\nu(f) = \int_{K_n} |f(z)|^2 dv_z$$

where dv_z denotes the standard measure on $\mathbb{C}^n = \mathbb{R}^{2n}$.

2.1 Lemma. *The seminorms q_ν give a defining system of the Fréchet space $\mathcal{O}(P)$. The spaces $(\mathcal{P}(P), q_\nu)$ are Hilbert spaces and the operators* LOnsp

$$(\mathcal{O}(P), q_\mu) \longrightarrow (\mathcal{O}(P), q_\nu), \quad \mu < \nu$$

are Hilbert Schmidt operators.

Corollary. *$\mathcal{O}(P)$ is a nuclear space.*

Next we consider complex spaces X that can be embedded into a polydisk as closed complex subspace, $X \hookrightarrow P$. This means that X maps biholomorphically onto a closed complex subspace of P. From Cartan's theorem B we know that the restriction map $\mathcal{O}(P) \to \mathcal{O}(X)$ is surjective. Clearly the kernel of this map is closed. Hence we can identify $\mathcal{O}(X)$ with a factor space of $\mathcal{O}(P)$ by a closed subspace. In this way we can equip $\mathcal{O}(X)$ with a structure as nuclear space.

2.2 Lemma. *Let X be a complex space that can be embedded as closed subspace into a polydisk. The structure of $\mathcal{O}(X)$ as a nuclear space is independent of the choice of the embedding.* Lsic

Let $(X, \mathcal{O}_X)$ be an arbitrary complex space. Recall that we assume that X is Hausdorff and has a countable basis of the topology. Hence X is "countable at infinity" which means that X can be written as countable union of compacta. Every point in X admits am open neighborhood U such that U can be embedded into a polydisk. As a consequence there exists a countable covering $X = U_1 \cup U_2 \cup \dots$ of open subsets with this property. So the $\mathcal{O}(U_i)$ are nuclear spaces. But then their direct product is also a nuclear space. It is clear that the image of

$$\mathcal{O}(X) \longrightarrow \prod_i \mathcal{O}(U_i)$$

is closed. So $\mathcal{O}(X)$ inherits a structure as nuclear space. This structure is independent of the choice of the covering.

Now we consider a general coherent sheaf $\mathcal{M}$ on a complex space. We will equip $\mathcal{M}(X)$ with a structure as nuclear space. The procedure is as follows. Consider an countable open covering $X = \bigcup U_i$ of open subsets such that each U_i admits a closed embedding into some polydisk. Assume also that each $\mathcal{M}|U_i$ is finitely generated,

$$\mathcal{O}_X^m \longrightarrow \mathcal{M}|U_i \longrightarrow 0.$$

From Theorem B follows that $\mathcal{O}(U_i)^m \to \mathcal{M}(U_i)$ is surjective. One can show that the kernel of this map is closed. Then one can equip $\mathcal{M}(U_i)$ with the quotient structure.

2.3 Theorem. *There can be defined for complex space $(X, \mathcal{O}_X)$ and each coherent $\mathcal{O}_X$-module $\mathcal{M}$ a structure as nuclear space on $\mathcal{M}(X)$ such that the following conditions are satisfied:* DNfs

1) *For a polydisk and $\mathcal{M} = \mathcal{O}_X$ we get the structure defined above.*
2) *Let $X = \bigcup U_i$ be a finite or countable covering by open subsets. Then*

$$\mathcal{M}(X) \longrightarrow \prod_i \mathcal{M}(U_i)$$

is a closed embedding of nuclear spaces (i.e. $\mathcal{M}(X)$ carries the induced topology of the product topology).

3) *Let $Y \hookrightarrow X$ be a closed embedding of complex spaces and let $\mathcal{M}$ be a coherent sheaf on X. Then the topologies on $\mathcal{M}(Y)$ and on $(i_*\mathcal{M})(X)$ are the same.*
4) *Let X be a Stein space and let $\mathcal{M} \to \mathcal{N}$ be a surjective map of $\mathcal{O}_X$-modules. Then $\mathcal{N}(X)$ carries the quotient structure of $\mathcal{M}(X)$. (Notice that $\mathcal{M}(X) \to \mathcal{N}(X)$ is surjective by Cartan's theorem B.)*

2.4 Proposition. *Let $(S, \mathcal{O}_S)$ be a complex space and $U \subset \mathbb{C}^n$ a polydisk. The natural map* Pnis

$$\mathcal{O}_S(S)\hat{\otimes}_{\mathbb{C}}\mathcal{O}_U(U) \longrightarrow \mathcal{O}_{S\times U}(S \times U)$$

is an isomorphism of topological vector spaces.

Proof. The natural bilinear map

$$\mathcal{O}_S(S) \times \mathcal{O}_U(U) \longrightarrow \mathcal{O}_{S\times U}(S \times U), \quad (f(x), g(y)) \longmapsto f(x)g(y),$$

is continuous. We prove the universal property for the completed π-tensor product. So let $B : \mathcal{O}_S(S) \times \mathcal{O}_U(U) \to W$ be a continuous bilinear map into a Fréchet space. We have to construct a factorization $\mathcal{O}_{S\times U}(S \times U) \to W$. For this it is convenient to write an element $f \in \mathcal{O}_{S\times U}(S \times U)$ as power series in the variable $z \in U$,

$$f(s, z) = \sum a_{\nu_1,\dots,\nu_n} z_1^{\nu_1} \cdots z_n^{\nu_n}.$$

For sake of simplicity we assume (without loss of generality) that the multiradius of U is $(1, \dots, 1)$. Then we can attach to each continuous seminorm p on $\mathcal{O}_S(S)$ and each $0 < t < 1$ a seminorm p_t on $\mathcal{O}_{S\times U}(S \times U)$,

$$p_t(f) = \sum p(a_{\nu_1,\dots,\nu_n}) t^{\nu_1+\cdots\nu_n}.$$

It is easy to show that this is a defining system of seminorms for the topology that we already have introduced. Now it is clear the the map that sends f to

$$\lim_{N\to\infty} \sum_{\nu_1+\cdots+\nu_n \le N} B(a_{\nu_1,\dots,\nu_n}, z_1^{\nu_1} \cdots z_n^{\nu_n})$$

is continuous. □

2.5 Lemma. *Let $U_0 \subset \mathbb{C}^n$ be a Stein open subset of $\mathbb{C}^n$ and let S_0 be a Stein space. Let $U' \subset U \subset\subset U_0$ be open Stein subsets and $S' \subset S \subset\subset S_0$ open Stein subspaces.* Lsss

$$\begin{array}{ccccc} U' & \subset & U & \subset\subset & U_0 \\ S' & \subset & S & \subset\subset & S_0 \end{array}$$

Assume that $\mathcal{M}$ is a coherent sheaf on $S_0 \times U_0$. Then the $\mathcal{O}(S)$-modules $\mathcal{O}(S')$ and $\mathcal{M}(S \times U)$ are transversal.

Proof. We know $\mathcal{O}(S \times U) = \mathcal{O}(S)\hat{\otimes}_{\mathbb{C}}\mathcal{O}(U)$. Hence $\mathcal{O}(S \times U)$ is a nuclear free $\mathcal{O}(S)$-module. Due to Theorem A there exists a free resolution of the sheaf $\mathcal{M}$ on $S \times U$,

$$\mathcal{F}_\bullet \longrightarrow \mathcal{M} \longrightarrow 0.$$

From theorem B follows that

$$\mathcal{F}(S\times U)_{\bullet} \longrightarrow \mathcal{M}(S\times U)$$

is a nuclear free resolution of $\mathcal{O}(S)$-modules. We have

$$\mathcal{O}(S')\hat{\otimes}_{\mathcal{O}(S)}\mathcal{O}(S\times U) = \mathcal{O}(S')\hat{\otimes}_{\mathcal{O}(S)}\mathcal{O}(S)\hat{\otimes}_{\mathbb{C}}\mathcal{O}(U) = \mathcal{O}(S'\times U).$$

It follows

$$\mathcal{O}(S')\hat{\otimes}_{\mathcal{O}(S)}\mathcal{F}_{\bullet}(S\times U) = \mathcal{F}_{\bullet}(S'\times U).$$

But $\mathcal{F}_{\bullet}(S'\times U)$ is an resolution of $\mathcal{M}(S'\times U)$. So we get

$$\mathcal{O}(S')\hat{\otimes}_{\mathcal{O}(S)}\mathcal{M}(S\times U) = \mathcal{M}(S'\times U) \quad \text{and} \quad \widehat{\mathrm{Tor}}_q(\mathcal{O}(S'),\mathcal{M}(S\times U)) = 0.$$

This finishes the proof of the lemma. □

2.6 Proposition. *Let $E^{\bullet}$ be (cochain-) complex of nuclear A-modules which is bounded from the right ($E^n = 0$ for big enough n). Let F be a nuclear A-module, transversal to all E^n. Assume that k is an integer with the property $H^n(E^{\bullet}) = 0$ for $n \ge k$. Then the analogue is true for the complex $F\hat{\otimes}_A E^{\bullet}$.* Pazaz

2.7 Corollary. *Let $E^{\bullet}$, $F^{\bullet}$ be two complexes of nuclear A-modules which both are bounded from the right. Let $f: E^{\bullet}\to F^{\bullet}$ be a quasi-isomorphism of nuclear complexes and let M be a nuclear A-module, transversal to all E^n, F^n. Then* Cqq

$$\mathrm{id}\otimes f : M\hat{\otimes}E^{\bullet} \longrightarrow M\hat{\otimes}F^{\bullet}$$

is also a quasi-isomorphism.

3. Nuclear mappings and subnuclear mappings between Fréchet modules

3.1 Definition. *Let A be a Fréchet algebra. A continuous A-linear operator $f: E\to F$ between A-Fréchet modules is called **A-nuclear** if it has a presentation* Dclfn

$$f(x) = \sum_n \lambda_n L_n(x) y_n$$

where
1) (λ_n) is a sequence of complex numbers such that $\sum|\lambda_n| < \infty$.
2) (y_n) is a bounded sequence in F.
3) L_n is an equicontinuous sequence of continuous A-linear maps $E\to A$.

*The operator f is called **subnuclear** if there exists an A-Fréchet module G and a commutative diagram such that $G\to E$ is surjective and such that $G\to F$ is nuclear.*

As in the case $A=\mathbb{C}$ (Remark 3.2) we have

3.2 Remark. *Let $f : E \to F$ and $g : F \to G$ be continuous A-linear maps of Fréchet modules Assume that f or g is nuclear. Then $g \circ f$ is nuclear.* Rnn

The proof is also trivial. $\square$

Let E be a Fréchet space. Then $A\hat{\otimes}_{\mathbb{C}}E$ is a Fréchet A-module in the obvious way.

3.3 Remark. *Let $E \to F$ be a nuclear map of Fréchet spaces. Then $A\hat{\otimes}_{\mathbb{C}}E \to A\hat{\otimes}_{\mathbb{C}}F$ is A-nuclear.* Refn

This follows from the Definition. $\square$

3.4 Proposition. *Let S be a Stein space and let be $V \subset\subset U$ two Stein subspaces. Then the natural restriction* Puvr

$$\mathcal{O}(S \times U) \longrightarrow \mathcal{O}(S \times V)$$

is an $\mathcal{O}(S)$-linear nuclear map.

Let $\mathcal{M}$ be a coherent sheaf on $S \times U$ and let $U' \subset\subset U$, $V' \subset\subset U'$ be Stein subspaces, then

$$\mathcal{M}(S \times U') \longrightarrow \mathcal{M}(S \times V')$$

is $\mathcal{O}(S)$-subnuclear.

4. The lemma of Schwartz

The classical lemma of Schwartz can be formulated as follows.

4.1 Lemma of Schwartz. *Let E, F be two Fréchet spaces and let $f : E \to F$ be surjective and $u : E \to F$ be a compact map. Then $E/(f+u)F$ is finite dimensional.* LS

Proof in the nuclear case. We will not need this lemma in the following. Nevertheless we want to prove it in the special case that E, F are nuclear spaces and that u is a nuclear map. (This special case can be generalized to nuclear modules.) In a first step we prove that there exists a nuclear map $v : E \to E$ such that the following diagram is commutative.

$$\begin{tikzcd} E \arrow[r, "f"] & F \\ E \arrow[u, "v"] \arrow[ur, "u"'] & \end{tikzcd}$$

To prove this, we use Definition 3.1 of a nuclear map

$$u(x) = \sum_{n=0}^{\infty} \lambda_n L_n(x) y_n.$$

Recall that the series $\sum \lambda_n$ converges absolutely. It is possible to decompose $\lambda_n = \lambda_n' \lambda_n''$ such that $\sum_n \lambda_n'$ still converges absolutely and $\lambda_n'' \to 0$. We can replace λ_n by λ_n' and y_n by $\lambda_n'' y_n$. This consideration shows that we can assume without loss of generality that $y_n \to 0$. From the surjectivity of f in combination with the open mapping theorem we deduce the existence of a sequence $x_n \in E$, $x_n \to 0$ and $f(x_n) \to y_n$. Now we can define v.

$$v(x) = \sum_n \lambda_n L_n(x) x_n.$$

This makes the diagram commutative.

In a second step we reduce the lemma to the case $E = F$, $f = \mathrm{id}$. As in the first step we decompose $u = f \circ v$. The natural map

$$\mathrm{Coker}(\mathrm{id} - v) \longrightarrow \mathrm{Coker}(f - u)$$

is surjective which gives the desired result.

In the third and last step it remains to treat the case $E = F$ and $f = \mathrm{id}$. Notice that an analogous result has been proved for Banach spaces E instead of nuclear spaces (Proposition III.6.9). The following argument shows how one can reduce to this case. For this we consider again a presentation as in Definition 3.1.

$$f(x) = \sum_{n=0}^{\infty} \lambda_n L_n(x) y_n.$$

For each $x \in E$ the sequence

$$\alpha(x) := (\lambda_n L_n(x))$$

is absolutely convergent. So we get a map $\alpha : E \to \ell^1$. There is another map

$$\beta : \ell^1 \longrightarrow E, \quad (x_n) \longmapsto \sum x_n y_n.$$

The diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\alpha"'] & E \\ \ell^1 \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes, $f = \beta \circ \alpha$. We also can consider $\gamma := \alpha \circ \beta : \ell^1 \to \ell^1$. Obviously α is nuclear. Hence γ is nuclear and we can apply Proposition III.6.9 to show that the cokernel of $\mathrm{id}_{\ell^1} + \gamma$ is finite dimensional. It is easy to check that α induces a natural map

$$\mathrm{Coker}(\mathrm{id}_{\ell^1} + \gamma) \longrightarrow \mathrm{Coker}(\mathrm{id}_E + f)$$

which can be checked to be an isomorphism. This finishes the proof of the Lemma of Schwartz in the nuclear case. □

There is another way to express the Lemma of Schwartz.

4.2 Theorem of Schwartz. *Let $E^\bullet$, $F^\bullet$ be two complexes of Fréchet spaces which are bounded from the right and let $f^\bullet : E^\bullet \to F^\bullet$ be a quasi-isomorphism. Assume that the f^n are compact. Then there exists a complex $L^\bullet$ of finite dimensional vector spaces and a quasi-isomorphism $L^\bullet \to E^\bullet$.* TS

Corollary. *The vector spaces $H^n(E^\bullet) = H^n(F^\bullet)$ are finite dimensional.*

Proof. In a first step we prove the following. Let n be an integer such that $H^k(E^\bullet) = 0$ for $n > k$. Then $H^n(E^\bullet)$ is finite dimensional. We denote the kernel of $E^k \to E^{k+1}$ by $Z(E^\bullet)^k$. In the case $n > k$ the natural sequence

$$0 \longrightarrow Z(E^\bullet)^k \longrightarrow E^k \longrightarrow Z(E^\bullet)^{k+1} \longrightarrow 0.$$

is exact. The image of $Z(E^\bullet)$ with respect to the map $E^n \to F^n$ is contained in $Z(F^\bullet)^n$ and the map

$$Z(E^\bullet)^n \longrightarrow Z(F^\bullet)^n$$

is also nuclear. The map

$$(d, f^n) : F^{n-1} \times Z(E^\bullet)^n \longrightarrow Z(F^\bullet)^n$$

is surjective. It differs from $(d, 0)$ by a nuclear map. Hence the Schwartz lemma applies and shows that the cokernel of $d : F^{n-1} \to Z(F^\bullet)^n$ is finite dimensional.

We have to construct a complex $L^\bullet$. Assume it has been constructed already. Then we can consider the truncated complexes $L^\bullet_{(n)}$

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & L^n & \longrightarrow & L^{n+1} & \longrightarrow & \cdots \\ & & & & \uparrow & & & & \\ & & & & \text{position } n & & & & \end{array}$$

Actually we shall construct L^n and $L^\bullet_{(n)}$ be descending induction. We will construct it together with a complex homomorphism $L^\bullet_n \to E^\bullet$. The following condition has to hold.

> *The maps $L^\bullet_{(n)} \to E^\bullet$ induce isomorphisms for the cohomology groups in degree $\geq n$.*

We denote by $M^\bullet_{(n)}$ the mapping cone of this morphism. Recall

$$M^k_{(n)} = L^{k+1}_{(n)} \oplus E^k.$$

The above condition means that the cohomology groups of $M^\bullet_{(n)}$ vanish in degree $> n$. We have to consider also the composition of $L^\bullet_{(n)} \to E^\bullet \to F^\bullet$ and its mapping cone $N^\bullet_{(n)}$. Its cohomology groups also vanish in degree $> n$. Due to the first step the vector space $H^n(M^\bullet)$ is finite dimensional. Therefore we can find a finite dimensional vector space $L^n \to M^n$ such the image is contained in $Z^n(M^\bullet)$ and such the the composition $L^n \to Z^n(M^\bullet) \to H^n(M^\bullet)$ is surjective. The map $L^n \to M^n = L^{(n+1)} \oplus E^n$ consists of two components. The first one is used to prolongate the complex $L^\bullet_{(n+1)}$ to a complex $L^\bullet_{(n)}$. and is hence denoted by $d_n : L^n \to L^{n+1}$. The second component $L^n \to E^n$ is used to extend the map $L^\bullet_{(n+1)} \to E^\bullet$ to a map $L^\bullet_{(n)} \to E^\bullet$. This solves the problem. Details are left to the reader. □

5. The lemma of Kiehl

Kiehl succeeded to generalize the Schwartz lemma form nuclear spaces to nuclear modules. In a first step one obtains a generalization of Proposition III.6.9 to Fréchet modules over Banach algebras.

5.1 Lemma. *Let E be a Fréchet module over a Banach algebra A and let* LPkc
$u : M \to M$ be an A-nuclear map. Then the cokernel of $\mathrm{id} + u$ *is a finitely generated A-module.*

The proof is the same as that of Proposition III.6.9. □

5.2 Lemma of Kiehl. *Let A be a nuclear algebra and let E, F be two nuclear* LSK
A-modules and let $f : E \to F$ be surjective, continuous and A-linear and let $u : E \to F$ be a A-subnuclear map. Assume that a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_1 \\ B \arrow[ur] & \end{tikzcd}$$

with a Fréchet algebra A_1 and Banach algebra B is given. Then the cokernel of

$$\mathrm{id} \otimes f : A_1 \hat{\otimes}_A E \longrightarrow A_1 \hat{\otimes}_A F$$

is a finitely generated A_1-module.

Proof. As we know there exists a nuclear A-module E_1 and a surjective A-linear and continuous map $E_1 \to E$. The composition with $E_1 \to F$ is nuclear. We also know that there exists a nuclear free A-module E_2 and a surjective and continuous map $E_2 \to E_1$. The composition $E_2 \to E_1 \to E$ is still nuclear. Hence it is sufficient to prove the lemma under the additional assumptions that E is nuclear free and that u is A-nuclear. Hence we can assume that E is nuclear free and that u is nuclear.

Now we can use the same argument as in the first step of the proof of the Lemma of Schwartz 4.1 to reduce to the case $E = F$, nuclear free, and $f = \mathrm{id}$.

Warning

Let A be a Fréchet algebra and let E, F be two A-Fréchet modules, $F_1 \subset F$ a closed submodule. Assume that there is given a commutative diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "f_1"'] & F \\ & F_1 \arrow[u, "\text{(inclusion)}"'] \end{tikzcd}$$

such that f is nuclear. Then f_1 needs not to be nuclear (even if A, E, F and hence F_1 are). Here is a special situation where a deformed f_1 can be shown to be subnuclear.

5.3 Proposition. *Let A, B be nuclear algebras and E, F nuclear A-modules.* Pss
Assume that a continuous homomorphism $\varrho : A \to B$ is given and that this homomorphism is $\mathbb{C}$-nuclear. Let $F_1 \subset F$ be a closed A-submodule and assume that B is A-transversal to E, F and to F/F_1. Let

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "f_1"'] & F \\ & F_1 \arrow[u, "\text{(inclusion)}"'] \end{tikzcd}$$

be a commutative diagram. Assume that f is subnuclear. Then

$$\mathrm{id} \otimes f_1 : B \hat{\otimes}_A E \longrightarrow B \hat{\otimes}_A F_1$$

is subnuclear.

Proof. In a first step we assume that f is nuclear. In this case we can show that $\mathrm{id} \otimes f_1$ is nuclear too. There exists a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varrho"] \arrow[d, "\alpha"'] & B \\ A_1 \arrow[r, "\varrho_1"'] & B_1 \arrow[u, "\beta"'] \end{tikzcd}$$

where $\varrho_1 : A_1 \to B_1$ is a nuclear map of Banach spaces. We have to consider the map

$$\varrho \otimes f : A \otimes_A E \longrightarrow B \otimes_A F.$$

This can be considered as map $E \to B \otimes_A B$. Now we can construct the commutative diagram

$$\begin{tikzcd}
E \arrow[r, "\varrho\otimes f_1"] \arrow[d, "u"'] & B \otimes_A F_1 \\
\ell^\infty(A_1) \arrow[r, "R"] & \ell^1(B_1) \arrow[u, "v"']
\end{tikzcd}$$

where

$$\begin{aligned}
u(x) &= (\alpha(L_n(x))), \\
v((b_n)) &= \sum_n \beta(b_n) \otimes y_n, \\
R((a_n)) &= (\lambda_n a_n).
\end{aligned}$$

The map R is a nuclear map of Banach spaces. Hence $u \otimes f_1$ is nuclear.

Now we come to the general case. We assume that there is a commutative diagram

$$\begin{tikzcd}
M \arrow[d, "h"'] \arrow[dr, "g"] & \\
E \arrow[r, "f"'] & F
\end{tikzcd}$$

where h is surjective and f is nuclear. Then we can consider the diagram

$$\begin{tikzcd}
B\hat{\otimes}_{\mathbb{C}} M \arrow[r] \arrow[d] & B\hat{\otimes}_{\mathbb{C}}(B\hat{\otimes}_A F_1) \arrow[d] \\
B\hat{\otimes}_A E \arrow[r] & B\hat{\otimes}_A F_1
\end{tikzcd}$$

The first row is nuclear and the first column is surjective. Hence the second row is subnuclear. $\square$

We need the notion of a *nuclear chain.*

5.4 Definition. *A nuclear chain is a family* (A_t), $0 \le t \le 1$ *of nuclear* DnK
algebras together with a family

$$\varrho_{t_1}^{t_2} : A_{t_1} \longrightarrow A_{t_2}, \quad 0 \le t_1 \le t_2 \le 1,$$

of algebra homomorphisms with the properties

$$\varrho_t^t = \mathrm{id}, \quad \varrho_{t_2}^{t_3} \circ \varrho_{t_1}^{t_2} = \varrho_{t_1}^{t_3}.$$

We assume that each $\varrho_{t_1}^{t_2}$ *is nuclear and that it factorizes through a Banach algebra.*

5.5 Theorem of Kiehl. *Let $(A_t)_{t\in[0,1]}$ be a nuclear chain and let $E^\bullet$, $F^\bullet$ be two complexes of nuclear A_0-modules which are bounded from the right and let $f^\bullet : E^\bullet \to F^\bullet$ be a quasi-isomorphism and such that all f^n are nuclear. Let A_t be transversal to all E^n, F^n for all t . Then there exists a complex $L^\bullet$ of finitely generated free A_1-modules and a quasi-isomorphism of complexes of A_1-modules $L^\bullet \to A_1\hat{\otimes}_{\mathbb{C}} E^\bullet$.* TSK

6. The proof of the finiteness theorem

There is another tool which we need for the proof.

6.1 Lemma. *Let $E^\bullet$ be a complex, bounded from the right, of nuclear A-modules and let F be a nuclear A-module, transverse to all E^n. Assume that $E^\bullet$ is exact. Then $F\hat{\otimes}_A E^\bullet$ is exact too.* LAtF

Proof. Let $Z^n := \mathrm{Kernel}(E^n \to E^{n+1})$. Since $E^\bullet$ is exact, the short exact sequences $0 \to Z^n \to E^n \to Z^{n+1}$. Descending induction shows that all Z^n are transverse to F. The sequence remains exact if one tensors it with F. □

6.2 Corollary. *Let $E^\bullet, F^\bullet$ be two complexes, bounded from the right, of nuclear A-modules and let $f : E^\bullet \to F^\bullet$ be a quasi-isomorphism. Furthermore let M be a nuclear A-module that is transverse to all E^n, F^n. Then* Cefm

$$\mathrm{id}\otimes f : M\hat{\otimes}_A E^\bullet \longrightarrow M\hat{\otimes}_A F^\bullet$$

is a quasi-isomorphism too.

Proof. Apply the previous lemma to the mapping cone of f. □

6.3 Definition. *Let $\pi : X \to P$ a holomorphic map of a complex space into a polydisk. A* ***relative chart*** *is a triple (U, Q, φ) where U is an open subset of X, Q is a polydisk and $\varphi : U \to Q \times P$ is a closed embedding such the diagram commutes.* Dech

We show that for each point $a \in X$ there exists a relative chart (U, Q, φ) with $x \in U$. To prove this, we take an open neighborhood U von x that admits a closed embedding into a polydisk, $\psi : U \to Q$. Then we consider

$$\varphi : U \to Q \times P, \quad \varphi(x) = (\pi(x), \psi(x)).$$

This is the desired relative chart.

Let $X \to P$ be a holomorphic map of a complex space into a polydisk. Sometimes we have to replace P by smaller polydisk $P_0 \subset\subset P$ and X by the inverse image of P_0 in X. We express this by saying that we allow shrinking of P.

6.4 Lemma. *Let $\pi : X \to P$ be a proper holomorphic map of a complex space into a polydisk. After possible shrinking of P there exist* Lcsh

1) *finitely many relative charts (X_i, Q_i, φ_i),*
2) *shrinks $Q_i'' \subset\subset Q_i' \subset\subset Q_i$,*
3) *a shrink $P(r) \subset\subset P$,*

such that the followings holds. Let $\mathcal{X}''$ be the family of relative charts

$$(X_i'', Q_i'', \varphi_i); \quad \textit{where } X_i'' = \varphi^{-1}(Q_i'' \times P(r))$$

(similarly $\mathcal{X}'$), then $\mathcal{X}''$ is an open covering of $\pi^{-1}(P(r))$.

Proof of Grauert's finiteness theorem. Let $\pi : X \to S$ a proper holomorphic map between complex spaces and let $\mathcal{M}$ be a coherent sheaf on X. We have to prove that the higher direct images of $\mathcal{M}$ are coherent. This is a local result in S. That means that we can replace a given point $s \in S$ by an open neighbourhood U and X be the inverse image. Hence we can assume that S admits a closed embedding into a polydisk P. The higher direct images of $\mathcal{M}$ in P are the direkt images of the direct images in S. Hence it is sufficient to treat the case $S = P$ and, even more, we are allowed to shrink P (and to replace X by the inverse image.

So we can apply Lemma 6.4. We use the notations of this lemma. We use the notation

$$\begin{aligned} E^\bullet &= \text{Čech complex of the covering } \mathcal{X}'', \\ F^\bullet &= \text{Čech complex of the covering } \mathcal{X}', \\ A_t &= \mathcal{O}(P(r(t-1/2)), \quad 0 \le t \le 1. \end{aligned}$$

So A_t is a nuclear chain starting from $A_0 = \mathcal{O}(P(r))$ and ending with $A_1 = \mathcal{O}(P(r/2))$. There is a natural restriction map $f : E^\bullet \to F^\bullet$. It is a quasi-isomorphism.

All conditions of the Schwartz-like theorem are fulfilled. Hence there exists a complex $L^\bullet$ of A_1-modules and a quasi-isomorphism

$$L^\bullet \longrightarrow A_1 \otimes_{A_0} E^\bullet,$$

such that each L^n is a finitely generated free A_1-module.

Now we consider the free sheaves $\mathcal{L}^n$ over $P(r/2)$ such that

$$\mathcal{L}^n(P(r/2) = L^n.$$

We can extend $L^\bullet$ to a complex $\mathcal{L}^\bullet$ of free $\mathcal{O}_{P(r/2)}$-modules. We can also sheafify the Čech complex $E^\bullet$ as follows. For an open $U \subset P(r/2)$ one can define in the obvious way the covering

$$\mathcal{X}'' \cap \varphi^{-1}(U)$$

We denote its Čech complex by $\mathcal{E}^{\bullet}(U)$. Then

$$U \longmapsto \mathcal{E}^n(U)$$

is a sheaf and $\mathcal{E}^{\bullet}$ is a complex of sheaves. The cohomology sheaves of this complex are $R^n\pi_*\mathcal{M}$. Now we assume that U is Stein. Then we have a natural isomorphism

$$\mathcal{L}^n(U) \stackrel{\sim}{\longrightarrow} \mathcal{E}^n(U).$$

The open Stein subsets give a basis of the topology. Due to the transversality results this map extends to a quasi-isomorphism of complexes of sheaves

$$\mathcal{L}^n \stackrel{\sim}{\longrightarrow} \mathcal{E}^n.$$

Hence the higher direct images of $\mathcal{M}$ are isomorphic to the cohomology sheaves of the complex $\mathcal{L}^n$. But these are clearly coherent. $\square$

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