Complex Spaces

Lecture 2017

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Chapter I. Complex spaces

1. The ring of power series

All rings are assumed to be commutative and with unit element. Homomorphisms of rings are assumed to map the unit element into the unit element. Modules $M$ over a ring $R$ are always assumed to be unitary, $1_Rm = m$.

Recall that an algebra over a ring $A$ by definition is a ring $B$ together with a distinguished ring homomorphism $\varphi : A \to B$. This ring homomorphism can be used to define on $B$ a structure as $A$-module, namely

$$ab := \varphi(a)b \quad (a \in A, \ b \in B).$$

Let $B, B'$ be two algebras. A ring homomorphism $B \to B'$ is called an algebra homomorphism if it is $A$-linear. This is equivalent to the fact that

$$\begin{array}{ccc}
B & \longrightarrow & B' \\
\downarrow & & \downarrow \\
A & \longrightarrow & B'
\end{array}$$

commutes.

The notion of a formal power series can be defined over an arbitrary ring $R$. A formal power series just is an expression of the type

$$P = \sum_{\nu} a_{\nu} z^{\nu}, \quad a_{\nu} \in R,$$

where $\nu$ runs through all multi-indices (tuples of nonnegative integers). Here $z = (z_1, \ldots, z_n)$ and $z^{\nu} = z_1^{\nu_1} \cdots z_n^{\nu_n}$ just have a symbolic meaning. Strictly logically, power series are just maps $\mathbb{N}_0^n \to R$. Power series can be added and multiplied formally, i.e.

$$\sum_{\nu} a_{\nu}z^{\nu} \cdot \sum_{\nu} b_{\nu}z^{\nu} := \sum_{\nu} \left( \sum_{\nu_1 + \nu_2 = \nu} a_{\nu_1}b_{\nu_2} \right) z^{\nu}. $$
The inner sum is finite. In this way we get a ring $R[[z_1, \ldots, z_n]]$. Polynomials are just power series such that all but finitely many coefficients are zero. In this way, we can consider the polynomial ring $R[z_1, \ldots, z_n]$ as subring of the ring of formal power series. The elements of $R$ can be identified with polynomials such that all coefficients $a_\nu$ with $\nu \neq 0$ vanish. We recall that for a non-zero polynomial $P \in R[z]$ in one variable the degree $\deg P$ is well-defined. It is the greatest $n$ such the the $n$th coefficient is different from 0. Sometimes it is useful to define the degree of the zero polynomial to be $-\infty$. If $R$ is an integral domain, the rule $\deg(PQ) = \deg P + \deg Q$ is valid.

There is a natural isomorphism

$$R[[z_1, \ldots, z_{n-1}]][[z_n]] \xrightarrow{\sim} R[[z_1, \ldots, z_n]]$$

whose precise definition is left to the reader. In particular, $R[[z_1, \ldots, z_{n-1}]]$ can be considered as a subring of $R[[z_1, \ldots, z_n]]$. One can use this to show that $R[[z_1, \ldots, z_n]]$ is an integral domain if $R$ is so.

Let now $R$ be the field of complex numbers $\mathbb{C}$. A formal power series is called convergent if there exists a small neighborhood (one can take a polydisk) of the origin where it is absolutely convergent. It is easy to show that this means just that there exists a constant $C$ such that $|a_\nu| \leq C^{\nu_1 + \cdots + \nu_n}$. The set

$$\mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$$

of all convergent power series is a subring of the ring of formal power series. There is a natural homomorphism

$$\mathcal{O}_n \rightarrow \mathbb{C}, \quad P \mapsto P(0),$$

that sends a power series to its constant coefficient. Its kernel $m_n$ is the set of all power series whose constant coefficient vanishes. The power $m_n^k$ is the ideal generated by $P_1 \cdots P_k$ where $P_i \in m_n$. It is easy to see that a power series $P$ belongs to $m_n^k$ if and only if

$$a_\nu \neq 0 \implies \nu_1 + \cdots + \nu_n \geq k.$$

As a consequence we have

$$\bigcap m_n^k = 0.$$

**Convergent power series and holomorphic functions**

A function $f : D \rightarrow \mathbb{C}$ on some open subset of $\mathbb{C}^n$ is called holomorphic if every point $a \in D$ admits an open neighbourhood $U$ such that $f$ can be expanded in $U$ into an (absolute) convergent series

$$f(z) = \sum_\nu a_\nu (z - a)^\nu.$$

Similar to the case $n = 1$ there is the following result. Assume that $f$ is continuous on the open set $D \subset \mathbb{C}^n$ and that it is holomorphic in each of its variables. Then it is analytic and its power series is given by the Taylor expansion.
1.1 Lemma. An element $P \in \mathcal{O}_n$ is a unit (i.e. an invertible element) if and only if $P(0) \neq 0$.

Proof. The power series defines a holomorphic function $f$ in a small neighbourhood $U$ of the origin without zeros. The function $1/f$ is holomorphic as well and can be expanded into a power series. $\square$

Our next task is to describe the homomorphisms $f : \mathcal{O}_m \rightarrow \mathcal{O}_n$. First we claim that non-units are mapped to non-units. Otherwise there would be non-unit $P \in \mathcal{O}_m$ such that $Q = f(P)$ is a unit. Then we would have $f(P - Q(0)) = Q - Q(0)$. The element $P - Q(0)$ is a unit but its image $Q - Q(0)$ is not. This is not possible. $\square$

There is a special kind of such a homomorphism which we call a “substitution homomorphism”. It is defined by means of elements $P_1, \ldots, P_n \in \mathcal{O}_m$ that are contained in the maximal ideal. If $P(z_1, \ldots, z_n)$ is an element of $\mathcal{O}_n$, one can substitute the variables $z_i$ by the power series $P_i$. For a precise definition one can interprets $P, P_i$ as holomorphic maps. Then one uses that the composition of holomorphic maps are holomorphic. In this way, we obtain a power series $P(P_1, \ldots, P_n)$. Another way to see this is to apply standard rearrangement theorems. This substitution gives a homomorphism

$$\mathcal{O}_n \rightarrow \mathcal{O}_m, \quad P \mapsto P(P_1, \ldots, P_n).$$

1.2 Lemma. Each algebra homomorphism $\mathcal{O}_n \rightarrow \mathcal{O}_m$ is a substitution homomorphism.

Proof. Let $\varphi : \mathcal{O}_n \rightarrow \mathcal{O}_m$ an algebra homomorphism. Since it is local, the elements $P_i := \varphi(z_i)$ are contained in the maximal ideal. Hence one can consider the substitution homomorphism $\psi$ defined by them. We claim $\varphi = \psi$. At the moment we only know that $\varphi$ and $\psi$ agree on $\mathbb{C}[z_1, \ldots, z_n]$. Let $P = \sum_{\nu} a_{\nu} z^{\nu} \in \mathcal{O}_n$. We claim $\varphi(P) = \psi(Q)$. For this we decompose for a natural number $k$

$$P = P_k + Q_k, \quad P_k = \sum_{\nu_1 + \cdots + \nu_n \leq k} a_{\nu} z^{\nu}.$$

Then $Q_k$ is contained in the $k$-the power $m^k$ of the maximal ideal. (Obviously $m^k$ is generated by all $z^{\nu}$ where $\nu_1 + \cdots + \nu_n \geq k$.) We get

$$\varphi(P) - \psi(P) = \varphi(Q_k) - \psi(P_k) \in m^k.$$

This is true for all $k$. But the intersection of all $m^k$ is zero. This proves 1.2. $\square$
2. The Weierstrass Preparation Theorem

We mentioned already that polynomials are special power series and constants can be considered as special polynomials. Hence we have natural inclusions of $\mathbb{C}$-algebras

\[ \mathbb{C} \subset \mathbb{C}[z_1, \ldots, z_n] \subset \mathbb{C}\{z_1, \ldots, z_n\} \subset \mathbb{C}[[z_1, \ldots, z_n]]. \]

The ring $\mathcal{O}_{n-1}$ can be identified with the intersection $\mathcal{O}_n \cap \mathbb{C}[[z_1, \ldots, z_{n-1}]]$ and there is a natural inclusion

\[ \mathcal{O}_{n-1}[z_n] \rightarrow \mathcal{O}_n. \]

We have to introduce the fundamental notion of a Weierstrass polynomial. This notion can be defined for every local ring $R$. Let $P$ be a normalized polynomial of degree $d$ over $R$. This polynomial is called a Weierstrass polynomial if its image in $R/m[X]$ is $X^d$. For the ring of our interest $R = \mathcal{O}_n$ this means:

2.1 Definition. A polynomial

\[ P \in \mathcal{O}_{n-1}[z] = \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n] \]

is called a Weierstrass polynomial of degree $d$, if it is of the form

\[ P = z_n^d + P_{d-1}z_n^{d-1} + \ldots + P_0 \]

where all the coefficients besides the highest one (which is 1) vanish at the origin,

\[ P_0(0) = \ldots = P_{d-1}(0) = 0. \]

To formulate the Weierstrass preparation theorem we need the notion of a $z_n$-general power series.

2.2 Definition. A power series

\[ P = \sum a_\nu z^\nu \in \mathbb{C}\{z_1, \ldots, z_n\} \]

is called $z_n$-general, if the power series $P(0, \ldots, 0, z_n)$ does not vanish. It is called $z_n$-general of order $d$ if

\[ P(0, \ldots, 0, z_n) = b_d z_n^d + b_{d+1}z_n^{d+1} + \ldots \text{ where } b_d \neq 0. \]

A power series is $z_n$-general, if it contains a monomial which is independent of $z_1, \ldots, z_{n-1}$. For example $z_1 + z_2$ is $z_2$-general, but $z_1z_2$ not. A Weierstrass polynomial is of course $z_n$-general and its degree and order agree. Let $U \in \mathcal{O}_n$ be a power series with $U(0) \neq 0$. Then $1/U$ defines a holomorphic function in a small polydisk around 0. This can be expanded into a power series. This shows that the invertible elements in $\mathcal{O}_n$ are precisely those with non-zero constant coefficients. Invertible elements are also called “units”. If $P$ is a $z_n$-general power series and $U$ is a unit in $\mathcal{O}_n$ then $PU$ is also $z_n$-general of the same order.
2.3 Weierstrass preparation theorem. Let $P \in \mathcal{O}_n$ be a $z_n$-general power series. There exists a unique decomposition

$$ P = UQ, $$

where $U$ is a unit in $\mathcal{O}_n$ and $Q$ a Weierstrass polynomial.

There is a division algorithm in the ring of power series analogous to the Euclidean algorithm in a polynomial ring. We recall this Euclidean algorithm.

The Euclidean Algorithm for Polynomials

Let $R$ be an integral domain and let

a) $P \in R[X]$ be an arbitrary polynomial,

b) $Q \in R[X]$ be a normalized polynomial, i.e. the highest coefficient is 1.

Then there exists a unique decomposition

$$ P = AQ + B, $$

where $A, B \in R[X]$ are polynomials and

$$ \deg(B) < d. $$

This includes the case $B = 0$ if one defines $\deg(0) = -\infty$. The proof of this result is trivial (induction on the degree of $P$).

2.4 Division theorem. Let $Q \in \mathcal{O}_{n-1}[z]$ be a Weierstrass polynomial of degree $d$. Every power series $P \in \mathcal{O}_n$ admits a unique decomposition

$$ P = AQ + B \quad \text{where} \quad A \in \mathcal{O}_n, \quad B \in \mathcal{O}_{n-1}[z], \quad \deg_{z_n}(B) < d. $$

Here $\deg_{z_n}(B)$ means the degree of the polynomial $B$ over the ring $\mathcal{O}_{n-1}$ (again taking $-\infty$ if $B = 0$).

The preparation theorem is related to the division theorem which – not correctly – sometimes is also called Weierstrass preparation theorem. Its first prove is due to Stickelberger.

The division theorem resembles the Euclidean algorithm. But there is a difference. In the Euclidean algorithm we divide through arbitrary normalized polynomials. In the division theorem we are restricted to divide through Weierstrass polynomials. But due to the preparation theorem, Weierstrass polynomials are something very general. This also shows the following simple consideration.
Let \( A = (a_{\mu \nu})_{1 \leq \mu, \nu \leq n} \) be an invertible complex \( n \times n \)-matrix. We consider \( A \) as linear map

\[
A : \mathbb{C}^n \to \mathbb{C}^n \quad z \mapsto w, \quad w_\mu = \sum_{\nu=1}^n a_{\mu \nu} z_\nu.
\]

For a power series \( P \in \mathcal{O}_n \), we obtain by substitution and reordering the power series

\[
P^A(z) := P(A^{-1} z).
\]

This is a special case of substitution. Obviously the map

\[
\mathcal{O}_n \xrightarrow{\sim} \mathcal{O}_n, \quad P \mapsto P^A,
\]

is an ring automorphism, i.e.

\[
(PQ)^A = P^AQ^A.
\]

The inverse map is given by \( A^{-1} \).

**2.5 Remark.** For every finite set of convergent power series \( P \in \mathcal{O}_n, \ P \neq 0 \), there exists an invertible \( n \times n \)-matrix \( A \), such that all \( P^A \) are \( z_n \)-general.

**Proof.** There exists a point \( a \neq 0 \) in a joint convergence poly-disk, such that \( P(a) \neq 0 \) for all \( P \). After the choice of suitable coordinate transformation (choice of \( A \)), one can assume \( A(0, \ldots, 0, 1) = a \). Then all \( P^A \) are \( z_n \)-general.

\[\square\]

3. **Algebraic properties of the ring of power series**

Let \( Q \in \mathcal{O}_{n-1}[z_n] \) be a Weierstrass polynomial. We can consider the natural homomorphism

\[
\mathcal{O}_{n-1}[z_n]/Q\mathcal{O}_{n-1}[z_n] \to \mathcal{O}_n/Q\mathcal{O}_n.
\]

The division theorem implies that this is an isomorphism.

**3.1 Theorem.** For a Weierstrass polynomial \( Q \in \mathcal{O}_{n-1}[z_n] \) the natural homomorphism

\[
\mathcal{O}_{n-1}[z_n]/Q\mathcal{O}_{n-1}[z_n] \to \mathcal{O}_n/Q\mathcal{O}_n
\]

is an isomorphism.

**Proof.** The surjectivity is an immediate consequence of the existence statement in the division theorem. The injectivity follows from the uniqueness statement in this theorem.

\[\square\]

Recall that an element \( a \in R \) is a prime element if and only if \( Ra \) is a prime ideal. (A prime ideal \( \mathfrak{p} \subset R \) is an ideal such that \( R/\mathfrak{p} \) is an integral domain.) By our convention the zero ring is no integral domain. Hence prime ideals are proper ideals and prime elements are non-units.
3.2 Theorem. The ring $\mathcal{O}_n$ is a UFD-domain. 

Recall that a ring $R$ is called noetherian if each ideal $a$ is finitely generated, $a = Ra_1 + \cdots + Ra_n$. Then any sub-module of a finitely generated module is finitely generated.

3.3 Theorem. The ring $\mathcal{O}_n$ is noetherian.

4. Analytic Algebras

We will consider $\mathbb{C}$-algebras $A$. If $A$ is different from zero then the structure homomorphism $\mathbb{C} \to A$ is injective. Usually we identify complex numbers with their images in $A$. In this sense each non-zero $\mathbb{C}$-algebra contains the field of complex numbers as sub-field.

4.1 Definition. An analytic algebra $A$ is a $\mathbb{C}$-algebra which is different from the zero algebra and such there exist an $n$ and a surjective algebra homomorphism $\mathcal{O}_n \to A$.

A ring $R$ is called a local ring if it is not the zero ring and if the set of non-units is an ideal. This ideal is then a maximal ideal and moreover, it is the only maximal ideal. We denote this ideal by $m(R)$. Hence $R - m(R)$ is the set of units of $R$. The algebra $\mathcal{O}_n$ is a local ring. The maximal ideal $m_n$ consists of all $P$ with $P(0) = 0$.

Let $A$ be a local ring and $a \subset m$ be an ideal. Then $A/a$ is a local ring too and the maximal ideal of $A/a$ is the image of $a$. The shows the following. If $A$ is a local ring and $A \to B$ is a surjective homomorphism onto a non-zero ring, then $B$ is also a local ring and the maximal ideal of $A$ is mapped onto the maximal ideal of $B$. In general a homomorphism $A \to B$ between local rings is called local if it maps the maximal ideal of $A$ into the maximal ideal of $B$. The natural map $A/m(A) \to B/m(B)$ is an isomorphism.

In particular, analytic algebras are local rings and the homomorphism $\mathcal{O}_n \to A$ in Definition 4.1 is a local homomorphism. The natural maps

$$\mathbb{C} \to \mathcal{O}_n/m_n \to A/m(A)$$

are isomorphisms. For $a \in A$ we denote by $a(0)$ the its image in $A/m(A)$ by $a(0)$. Recall that we identify this with a complex number. The maximal ideal of $A$ consists of all $a \in A$ such that $a(0) = 0$.

We notice that an arbitrary algebra homomorphism $f : A \to B$ between analytic algebra is local.

We have to generalize 1.2 to homomorphisms $\varphi : A \to B$ of arbitrary analytic algebras $A, B$. There is one problem. Let $m(B)$ be the maximal ideal
of $B$. It is not obvious that the intersection of all powers of $\mathfrak{m}(B)$ is zero. But it is true by general commutative algebra (Krull’s intersection theorem, Theorem ??).

4.2 Lemma. Let $A \to B$ a homomorphism of analytic algebras. Assume that surjective algebra homomorphism $\mathcal{O}_n \to A$ and $\mathcal{O}_m \to B$ are given. There exists a (substitution) homomorphism $\mathcal{O}_n \to \mathcal{O}_m$ such the the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\uparrow & & \uparrow \\
\mathcal{O}_n & \longrightarrow & \mathcal{O}_m
\end{array}
$$

commutes.

The proof should be clear. The variable $z_i \in \mathcal{O}_n$ is mapped to an element of $A$ then of $B$. Consider in $\mathcal{O}_m$ an inverse image $P_m$. These elements define a substitution homomorphism $\mathcal{O}_n \to \mathcal{O}_m$. From Krull’s intersection theorem follows that the diagram commutes. $\square$

From ?? follows:

4.3 Lemma. Let $f_1, \ldots, f_m$ be elements of the maximal ideal of an analytic algebra $A$. There is a unique homomorphism $\mathbb{C}\{z_1, \ldots, z_n\} \to A$ such that $z_i \mapsto f_i$.

We denote the image by $\mathbb{C}\{f_1, \ldots, f_n\}$ and call it the analytic algebra generated by $f_1, \ldots, f_n$. We want to derive a criterion that $\mathbb{C}\{f_1, \ldots, f_n\} = A$. A necessary condition is that $f_1, \ldots, f_n$ generate the maximal ideal. Actually it is also sufficient:

4.4 Lemma. Let $f_1, \ldots, f_n$ be elements of the maximal ideal of an analytic algebra $A$. Then the following conditions are equivalent:

a) They generate the maximal ideal.

b) $A = \mathbb{C}\{f_1, \ldots, f_n\}$.

It is easy to reduce this to the ring $A = \mathbb{C}\{z_1, \ldots, z_n\}$. Let $P_1, \ldots, P_m$ be generators of the maximal ideal. We can write

$$z_i = \sum_{ij} A_{ij} P_j.$$

Taking derivatives and evaluating at 0 we get: The rank of the Jacobian matrix of $P = (P_1, \ldots, P_m)$ is $n$. We can find an system consisting of $n$ elements, say $P_1, \ldots, P_n$, such that the Jacobian is invertible. Now one can apply the theorem of invertible functions. $\square$
5. Coherent sheaves

Let $X$ be a topological space and $\mathcal{O}$ a sheaf of rings. We consider the category of $\mathcal{O}$-modules $\mathcal{M}$. Examples are the free modules $\mathcal{O}^n$. We consider $\mathcal{O}$-linear maps $\mathcal{O}^n \to \mathcal{M}$. It involves a map $\mathcal{O}(X)^n \to \mathcal{M}(X)$. The images of the unit vectors gives $n$ global sections $s_1, \ldots, s_n$. These sections determine the whole map of sheaves, since necessarily

$$\mathcal{O}(U)^n \to \mathcal{M}(U), \quad (f_1, \ldots, f_n) \mapsto f_1s_1|U + \cdots + f_ns_n|U.$$ 

Conversely, if $n$ global sections $s_1, \ldots, s_n$ are given, then this formula defines an $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{M}$. This means that we have a canonical isomorphism

$$\text{Hom}_{\mathcal{O}}(\mathcal{O}^n, \mathcal{M}) \cong \mathcal{M}(X)^n.$$ 

An $\mathcal{O}$-module $\mathcal{M}$ is called finitely generated if there exists a surjective $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{M}$. “Surjective” is of course understood in the sense of sheaves, i.e. the maps $\mathcal{O}_{X,a}^n \to \mathcal{M}_a$ have to be surjective for all $a \in X$. For the defining sections $s_1, \ldots, s_n$ this means that $\mathcal{M}_a$ is generated by $s_{1,a}, \ldots, s_{n,a}$. Concretely this means the following:

Let $U \subseteq X$ be open and let $s \in \mathcal{M}(U)$. Then there exists an open covering $U = \bigcup U_i$ such that every $s|U_i$ is a linear combination of the $s_1|U_i, \ldots, s_n|U_i$ with coefficients in $\mathcal{O}(U_i)$. If this is the case we also say that $\mathcal{M}$ is generated by the global sections $s_1, \ldots, s_n$.

5.1 Lemma. Let $\mathcal{M} \to \mathcal{N}$ be a surjective $\mathcal{O}$-linear map and $\mathcal{O}^n \to \mathcal{N}$ also an $\mathcal{O}$-linear map. For each point $a$ there exists an open neighborhood $U$ and an $\mathcal{O}|U$-linear map such the diagram

$$\begin{array}{ccc}
\mathcal{O}|U^n & \to & \mathcal{M}|U \\
\downarrow & & \downarrow \\
\mathcal{M}|U & \to & \mathcal{N}|U
\end{array}$$

commutes.

Proof. The map $\mathcal{O}^n \to \mathcal{N}$ corresponds to $n$ global sections $s_1, \ldots, s_n$. If we take $U$ small enough there are in the image of $\mathcal{M}(U)$, $t_i \mapsto s_i$. The sections $t_i$ give a map $\mathcal{O}^n \to \mathcal{M}$. □

An $\mathcal{O}$-module $\mathcal{M}$ is called locally finitely generated if every point $a \in X$ admits an open neighbourhood such that $\mathcal{M}|U$ is a finitely generated $\mathcal{O}|U$-module.
Let us recall a basic property of noetherian rings $R$. Let $M$ be a finitely generated module, i.e. there exists a surjective $R$-linear map $R^n \to M$. Then the kernel $K$ of this map is finitely generated as well. Hence there exists an exact sequence $R^n \xrightarrow{\varphi} R^m \to M$. The map $\varphi$ determines $M \cong R^n/\text{Im}(\varphi)$. The map $\varphi$ just given by a matrix with $m$ rows and $n$ columns. This is the way how computer algebra can manage computations for finitely generated modules over noetherian rings as polynomial rings. Serre found a weak substitute for $\mathcal{O}$-modules.

5.2 Definition. A sheaf of rings $\mathcal{O}$ is called coherent if for any open subset and any surjective $\mathcal{O}^n|U \to \mathcal{O}^m|U$ the kernel is locally finitely generated.

5.3 Definition. Let $\mathcal{O}$ be a coherent sheaf of rings. An $\mathcal{O}$-module $M$ is called coherent if for every point there exists an open neighborhood $U$ and an exact sequence

$$\mathcal{O}|U^n \to \mathcal{O}|U^m \to M|U \to 0.$$ 

Of course $\mathcal{O}$ considered as $\mathcal{O}$-module then is coherent. Just consider $0 \to \mathcal{O} \to \mathcal{O} \to 0$.

An $\mathcal{O}$-module is called a (finitely generated) free sheaf if it is isomorphic to $\mathcal{O}^m$ for suitable $m$. It is called locally free if every point admits an open neighborhood such that the restriction to it is free. A locally free sheaf is also called a vector bundle. For trivial reasons a free sheaf over a coherent sheaf of rings is coherent. Since coherence is a local property every vector bundle is coherent. The property “coherent” is stable under standard constructions. The proves are not difficult. We will keep them short:

5.4 Lemma. Let $M \to N$ be an $\mathcal{O}$-linear map of coherent sheaves. The image sheaf is coherent. 

Corollary. A locally finitely generated sub-sheaf of a coherent sheaf is coherent.

Proof. It is sufficient to show that the image of a map $\mathcal{O}^m \to M$ is coherent. By definition of coherence it is sufficient to show that the kernel $K$ is locally finitely generated. We can assume that there exists an exact sequence

$$\mathcal{O}^p \to \mathcal{O}^q \to M \to 0.$$ 

Since $\mathcal{O}^q \to M$ is surjective we can assume (use Lemma 5.1) that there exists
a lift $\mathcal{O}^m \to \mathcal{O}^q$ such that the diagram

\[
\begin{array}{c}
\mathcal{O}^m \\
\downarrow \\
\mathcal{M} \\
\downarrow \\
\mathcal{O}^q \\
\downarrow \\
\mathcal{O}^p
\end{array}
\]

commutes. Take the image of $\mathcal{O}^p \to \mathcal{O}^q$ and then its pre-image in $\mathcal{O}^m$. It is easy to check that this is the kernel $\mathcal{K}$.

\begin{flushright}
\boxrule 0.6pt \boxheight 6pt \par
\end{flushright}

\textbf{5.5 Lemma.} The kernel of a map $\mathcal{M} \to \mathcal{N}$ of coherent sheaves is coherent.

\textit{Proof.} Because of Lemma 5.4 we can assume that $\mathcal{M} \to \mathcal{N}$ is surjective. We choose presentations $\mathcal{O}^a \to \mathcal{O}^b \to \mathcal{M}$, $\mathcal{O}^c \to \mathcal{O}^d \to \mathcal{N}$.

We can assume that there is commutative diagram

\[
\begin{array}{c}
0 \\
\mathcal{K} \\
\downarrow \\
\mathcal{M} \\
\downarrow \\
\mathcal{N} \\
\downarrow \\
0
\end{array}
\]

\[
\begin{array}{c}
\mathcal{O}^b \\
\downarrow \varphi \\
\mathcal{O}^d \\
\downarrow \psi \\
\mathcal{O}^a
\end{array}
\]

The existence of $\varphi$ follows from Lemma 5.1 (after replacing $X$ by a small open neighborhood of a given point). The existence of $\mathcal{O}^d \to \mathcal{O}^c$ is trivial. Then we get a natural surjection $\varphi^{-1}(\psi(\mathcal{O}^c)) \to \mathcal{K}$.

\begin{flushright}
\boxrule 0.6pt \boxheight 6pt \par
\end{flushright}

\textbf{5.6 Lemma.} The cokernel $\mathcal{N}/\varphi(\mathcal{N})$ of a map $\varphi : \mathcal{M} \to \mathcal{N}$ of coherent sheaves is coherent.

\textit{Proof.} We can assume that $\mathcal{N}$ is a sub-sheaf of $\mathcal{M}$ and that $\varphi$ is the canonical injection. We can assume that a commutative diagram with exact columns exists:

\[
\begin{array}{c}
0 \\
\mathcal{N} \\
\downarrow \\
\mathcal{M} \\
\downarrow \\
\mathcal{M}/\mathcal{N} \\
\downarrow \\
0
\end{array}
\]

\[
\begin{array}{c}
\mathcal{O}^b \\
\downarrow \\
\mathcal{O}^d \\
\downarrow \\
\mathcal{O}^a
\end{array}
\]
It is easy to construct from this diagram an exact sequence

$$\mathcal{O}^b \oplus \mathcal{O}^c \to \mathcal{O}^d \to \mathcal{M}/\mathcal{N} \to 0.$$  \hfill \Box

5.7 The two of three lemma. Let $\mathcal{O}$ be a coherent sheaf of rings and

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

an exact sequence of $\mathcal{O}$-modules. Assume that two of them are coherent than the third is coherent too.

Proof. All what remains to show is that $\mathcal{M}_2$ is coherent if $\mathcal{M}_1, \mathcal{M}_2$ are. We can assume that there is a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{M}_1 & \to & \mathcal{M}_2 & \to & \mathcal{M}_3 & \to & 0 \\
\downarrow \alpha & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}^q & \to & \mathcal{M}_1 \oplus \mathcal{O}^q & \to & \mathcal{M}_2 & \to & 0 \\
\end{array}
\]

We use this to produce a map

$$\mathcal{M}_1 \oplus \mathcal{O}^q \to \mathcal{M}_2, \quad (x, y) \mapsto x - \alpha(y).$$

It is easy to check that this map is surjective. The kernel is defined by $x = \alpha(y)$. Hence it can be identified with the part of $\mathcal{O}^q$ that is mapped into $\mathcal{M}_1$ under $\alpha$. But this precisely the kernel of $\mathcal{O}^q \to \mathcal{M}_3$ hence the image of $\mathcal{O}^p$. We get an exact sequence

$$\mathcal{O}^p \to \mathcal{M}_1 \oplus \mathcal{O}^q \to \mathcal{M}_2 \to 0.$$  \hfill \Box

This shows that $\mathcal{M}_2$ is coherent (use Lemma 5.6).)

5.8 Lemma. The intersection of two coherent subsheaves of a coherent sheaf is coherent.

Proof. One uses the fact that intersections can be constructed as kernels. Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be two sub modules of an $\mathcal{O}$-module $\mathcal{X}$. Then $\mathcal{M} \cap \mathcal{N}$ is isomorphic to the kernel of $\mathcal{M} \times \mathcal{N} \to \mathcal{X}$, $(a, b) \mapsto a - b$.  \hfill \Box

5.9 Remark. Let $\mathcal{M}$ be a coherent $\mathcal{O}$-module. Then the support of $\mathcal{M}$ is a closed.

Proof. We show that the set of all $a$ such that $\mathcal{M}_a = 0$ is open. We can assume that $\mathcal{M}$ is finitely generated by sections $s_1, \ldots, s_n$. If there germs at $a$ are zero then $s_1, \ldots, s_n$ are zero in a full neighbourhood of $a$.  \hfill \Box

We collect some of the permanence properties of coherent sheaves.
5.10 Proposition.
1) Let $\mathcal{M}, \mathcal{N}$ be two coherent sub-sheaves of a coherent sheaf. Assume $\mathcal{M}_a \subset \mathcal{N}_a$ for some point $a$. Then there exists an open neighborhood $U$ such that $\mathcal{M}|U \subset \mathcal{N}|U$.

2) Let $\mathcal{M}, \mathcal{N}$ be two coherent sub-sheaves of a coherent sheaf. Assume $\mathcal{M}_a = \mathcal{N}_a$ for some point $a$. Then there exists an open neighborhood $U$ such that $\mathcal{M}|U = \mathcal{N}|U$.

3) Let $f, g : \mathcal{M} \to \mathcal{N}$ be two $\mathcal{O}$-linear maps between coherent sheaves such that $f_a = g_a$ for some point $a$. Then there exists an open neighborhood $U$ such that $f|U = g|U$.

4) Let $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ be $\mathcal{O}$-linear maps of coherent sheaves and $a$ a point. The following two conditions are equivalent:
   a) The sequence $\mathcal{M}_a \to \mathcal{N}_a \to \mathcal{P}_a$ is exact.
   b) There is an open neighborhood $U$ such that the sequence $\mathcal{M}|U \to \mathcal{N}|U \to \mathcal{P}|U$ is exact.

Proof. 
1) Use that $\mathcal{M}_a \subset \mathcal{N}_a$ is equivalent to $\mathcal{N}_a = \mathcal{M}_a \cap \mathcal{N}_a (= (\mathcal{M} \cap \mathcal{N})_a)$.
2) follows from 1).
3) Consider the kernel of $f - g$.
4)Consider the image $\mathcal{A}$ of $\mathcal{M} \to \mathcal{N}$ and the kernel $\mathcal{B}$ of $\mathcal{N} \to \mathcal{P}$. Both are coherent. We can assume that they are finitely generated. From assumption we know $\mathcal{A}_a = \mathcal{B}_a$.

5.11 Proposition. Let $\mathcal{M}, \mathcal{N}$ coherent $\mathcal{O}$-modules and $\mathcal{M}_a \to \mathcal{N}_a$ an $\mathcal{O}_a$-linear map. There exists an open neighborhood $U$ and an extension $\mathcal{M}|U \to \mathcal{N}|U$ as $\mathcal{O}|U$-linear map.

Additional remark. By Proposition 5.10 this extension is unique in the obvious local sense.

Proof. We can assume that there is a surjective $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{M}$. We consider the composed map $\mathcal{O}^n_a \to \mathcal{M}_a \to \mathcal{N}_a$. It is no problem to extend to $\mathcal{O}^n_a \to \mathcal{N}_a$ to an open neighborhood $\mathcal{O}|U^n \to \mathcal{N}|U$. We can assume that $U$ is the whole space. The kernel of $\mathcal{O}^n_a \to \mathcal{M}_a$ is contained in the kernel of $\mathcal{O}^n_a \to \mathcal{N}_a$. Since the kernels are coherent this extends to a full open neighborhood $U$. Hence we get a factorization $\mathcal{M}|U \to \mathcal{N}|U$.

6. The general notion of a complex space

We use some simple facts about sheaves. Let $F$ be a sheaf on a topological space. We know the trivial procedure of restricting $F$ to an open subset. There is a more general procedure to
6.1 Lemma. Let $X$ be a topological space and $Y \subset X$ a closed subspace. Let $F$ be a sheaf on $X$ such that $F|(X-Y)$ is zero.

6.2 Lemma. Let $\mathcal{O}_X$ be a coherent sheaf of rings on a topological space $X$. Let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent sheaf of ideals. Let $Y$ be the support of $\mathcal{O}_X/\mathcal{J}$. Then the restriction of $\mathcal{O}_X/\mathcal{J}$ to $Y$ is a coherent sheaf of rings. The category of coherent $\mathcal{Y}$-modules is equivalent to the category of coherent $\mathcal{O}_X$ modules which are annihilated by $\mathcal{J}$.

Let $U \subset \mathbb{C}^n$ be an open domain and $f_1, \ldots, f_m$ analytic functions on $U$. We consider the ideal sheaf $\mathcal{J}$ generated by $f_1, \ldots, f_m$ in $\mathcal{O}_U$. The support of the sheaf $\mathcal{O}_U/\mathcal{J}$ is the set $X$ of joint zeros of the $f_i$. We restrict the sheaf to $X$ and define

$$\mathcal{O}_X = (\mathcal{O}_U/\mathcal{J})|X.$$ 

This is a sheaf of $\mathbb{C}$-algebras on $X$.

6.3 Definition. A complex space $(X, \mathcal{O}_X)$ is a ringed space which is locally isomorphic to a model space.

If $U \subset X$ is an open subspace then $(U, \mathcal{O}_X|U)$ is a complex space too. We call in open analytic subspace.

6.4 Remark. Let $(X, \mathcal{O}_X)$ be a complex space and $\mathcal{J} \subset \mathcal{O}_X$ a coherent ideal sheaf. The support $Y$ of $\mathcal{O}_X/\mathcal{J}$ is a closed subset and $(Y, \mathcal{O}_Y)$ where $\mathcal{O}_Y = (\mathcal{O}_X/\mathcal{J})|Y$ is complex space too.

We call such a space that is defined through a coherent ideal sheaf a closed analytic subspace.

7. Complex spaces and holomorphic functions

Let $X$ be a topological space and let $\mathcal{O}_X$ be a sheaf of $\mathbb{C}$-algebras. We assume that $\mathcal{O}_{X,a}$ are local rings with maximal ideal $m_a$ and that

$$\mathbb{C} \rightarrow \mathcal{O}_{X,a} \rightarrow \mathcal{O}_{X,a}/m_a$$

is an isomorphism. Then we can associated to any section $f \in \mathcal{O}_X(U), U \subset X$ open, a function $f' : U \rightarrow \mathbb{C}$ which assigns to each point $a \in U$ the image $f'(a)$ of $f$ in $\mathcal{O}_{X,a}/m_a$. These functions are continuous. So we obtain a morphism $\mathcal{O}_X \rightarrow \mathcal{C}_X$.

We denote by $\mathcal{O}_X'$ the image (in the sense of sheaves) of this morphism. If we denote the kernel of this morphism by $\mathcal{J}_X$ we get a canonical isomorphism $\mathcal{O}_X' = \mathcal{O}_X/\mathcal{J}$. There are two basic results about this morphism.
7.1 Theorem (Rückert). The ideal sheaf $\mathcal{I}$ is the nilradical of $\mathcal{O}_X$. 

7.2 Theorem (Cartan). Let $(X, \mathcal{O}_X)$ be a complex space. The nilradical is coherent.

8. Germs of complex spaces

We consider the category of complex spaces. A pointed complex space $(X, a)$ is a complex space with a distinguished point $a \in X$. We can consider also the category of pointed complex spaces. Morphisms are morphisms of complex spaces that map the distinguished point to the distinguished point.

8.1 Theorem. The category of germs of complex spaces is dual to the category of analytic algebras.

9. Coherence

A holomorphic map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called finite, if the underlying map between topological spaces is finite. This means that it is proper and the fibres are finite sets. A holomorphic map $f : X \to Y$ is finite at a point $a \in X$ if there exist open sets $a \in U \subset X$ and $f(a) \in V \subset Y$ such that $f(U) \subset V$ and that $f : U \to V$ is finite.

9.1 Theorem. A holomorphic map $f : X \to Y$ is locally finite at $a$ if and only if the corresponding map of analytic algebras $\mathcal{O}_{Y,f(a)} \to \mathcal{O}_{X,a}$ is finite.
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