Complex Spaces

Lecture 2017

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Chapter I. Local complex analysis

1. The ring of power series

All rings are assumed to be commutative and with unit element. Homomorphisms of rings are assumed to map the unit element into the unit element. Modules M over a ring R are always assumed to be unitary, $1_R m = m$.

Recall that an algebra over a ring A by definition is a ring B together with a distinguished ring homomorphism $\varphi : A \to B$. This ring homomorphism can be used to define on B a structure as A-module, namely

$$ab := \varphi(a)b \quad (a \in A,\ b \in B).$$

Let B, B' be two algebras. A ring homomorphism $B \to B'$ is called an algebra homomorphism if it is A-linear. This is equivalent to the fact that

$$\begin{array}{ccc} B & \longrightarrow & B' \\ & \nwarrow \quad \nearrow & \\ & A & \end{array}$$

commutes.

The notion of a formal power series can be defined over an arbitrary ring R. A formal power series just is an expression of the type

$$P = \sum_\nu a_\nu z^\nu, \quad a_\nu \in R,$$

where ν runs through all multi-indices (tuples of nonnegative integers). Here $z = (z_1, \dots, z_n)$ and $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ just have a symbolic meaning. Strictly logically, power series are just maps $\mathbb{N}_0^n \to R$. Power series can be added and multiplied formally, i.e.

$$\sum_\nu a_\nu z^\nu + \sum_\nu b_\nu z^\nu = \sum_\nu (a_\nu + b_\nu) z^\nu,$$
$$\sum_\nu a_\nu z^\nu \cdot \sum_\nu b_\nu z^\nu = \sum_\nu \Big(\sum_{\nu_1 + \nu_2 = \nu} a_{\nu_1} b_{\nu_2} \Big) z^\nu.$$

The inner sum is finite. In this way we get a ring $R[[z_1,\dots,z_n]]$. Polynomials are just power series such that all but finitely many coefficients are zero. In this way, we can consider the polynomial ring $R[z_1,\dots,z_n]$ as subring of the ring of formal power series. The elements of R can be identified with polynomials such that all coefficients a_ν with $\nu \neq 0$ vanish. We recall that for a non-zero polynomial $P \in R[z]$ in one variable the degree $\deg P$ is well-defined. It is the greatest n such the the nth coefficient is different from 0. Sometimes it is useful to define the degree of the zero polynomial to be $-\infty$. If R is an integral domain, the rule $\deg(PQ) = \deg P + \deg Q$ is valid.

There is a natural isomorphism

$$R[[z_1,\dots,z_{n-1}]][[z_n]] \xrightarrow{\sim} R[[z_1,\dots,z_n]]$$

whose precise definition is left to the reader. In particular, $R[[z_1,\dots,z_{n-1}]]$ can be considered as a subring of $R[[z_1,\dots,z_n]]$. One can use this to show that $R[[z_1,\dots,z_n]]$ is an integral domain if R is so.

Let now R be the field of complex numbers $\mathbb{C}$. A formal power series is called convergent if there exists a small neighborhood (one can take a polydisk) of the origin where it is absolutely convergent. It is easy to show that this means just that there exists a constant C such that $|a_\nu| \le C^{\nu_1+\cdots+\nu_n}$. The set

$$\mathcal{O}_n = \mathbb{C}\{z_1,\dots,z_n\}$$

of all convergent power series is a subring of the ring of formal power series. There is a natural homomorphism

$$\mathcal{O}_n \longrightarrow \mathbb{C}, \quad P \longmapsto P(0),$$

that sends a power series to its constant coefficient. Its kernel $\mathfrak{m}_n$ is the set of all power series whose constant coefficient vanishes. The power $\mathfrak{m}_n^k$ is the ideal generated by $P_1 \cdots P_k$ where $P_i \in \mathfrak{m}_n$. It is easy to see that a power series P belongs to $\mathfrak{m}_n^k$ if and only if

$$a_\nu \neq 0 \Longrightarrow \nu_1 + \cdots + \nu_n \ge k.$$

As a consequence we have

$$\bigcap \mathfrak{m}_n^k = 0.$$

2. Holomorphic Functions and Power Series

Complex analysis deals with holomorphic functions. They can be introduced as complex differential functions. These can be considered as special real differentiable functions if one identifies $\mathbb{C}^n$ and $\mathbb{R}^{2n}$.

2.1 Definition. *A map $f : U \to V$ between open subsets $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ is called complex differentiable if it is (totally) differentiable in the sense of real analysis and if the Jacobi map $J(f,a) : \mathbb{C}^n \to \mathbb{C}^m$ is $\mathbb{C}$-linear for all $a \in U$.*

Instead of "complex differentiable", one uses the notation "analytic" or "holomorphic". From definition 2.1 some basic facts can be deduced from real analysis, for example that the composition of complex differentiable maps is complex differentiable and that the chain rule holds. The same is true for the theorem of invertible functions.

2.2 Proposition. *Let $f : U \subset \mathbb{C}^n$, $U \subset \mathbb{C}^n$ open, be a holomorphic map, and let $a \in U$ be a point such that the Jacobi matrix at a is invertible. Then there exists an open neighborhood $a \in V \subset U$ such that $f(V)$ is open and such that f induces a biholomorphic map $V \to f(V)$.*

The theorem of inverse functions is a special case of the theorem of implicit functions (but the latter can be reduced to the first one). We formulate a geometric version of it:

2.3 Proposition. *Let*

$$f : U \longrightarrow V, \quad U \subset \mathbb{C}^n,\ V \subset \mathbb{C}^m \ \textit{open},$$

be a holomorphic map and let $a \in U$ be a point. We assume that the Jacobi matrix $J(f,a)$ has rank n (in particular $n \le m$). After replacing U, V by smaller open neighborhoods of $a, f(a)$ if necessary the following holds: There exists a biholomorphic map $\varphi : V \to W$ onto some open subset $W \subset \mathbb{C}^m$ such that

$$\varphi(f(U)) = \big\{z \in W;\ z_{n+1} = \cdots = z_m = 0\big\}.$$

Moreover, the following is true: If one sets $W' = \{z \in \mathbb{C}^n, \quad (z,0) \in W\}$ then $\varphi \circ f$ induces a biholomorphic map $V \to W'$.

We also mention that the map $f : U \to \mathbb{C}^m$ is complex differentiable if and only if its components f_ν $(f(z) = (f_1(z), \dots, f_m(z)))$ are so.

It is a basic fact that these are functions that locally can be expanded into power series. More precisely this is true on any polydisk. By definition, a polydisk is a cartesian product of discs:

$$U_r(a) = U_{r_1}(a) \times \ldots \times U_{r_n}(a) = \big\{z \in \mathbb{C}^n; \quad |z_\nu - a_\nu| < r_\nu \quad (1 \le \nu \le n)\big\}.$$

Here $r = (r_1, \dots, r_n)$ is a tuple of positive numbers. It is called the multi-radius of the polydisk. The basic fact is:

2.4 Proposition. *Every complex differentiable function f on a polydisk $U_r(a)$ can be expanded in the whole polydisk as an absolutely convergent power series*

$$f(z) = \sum_{\nu} a_\nu (z-a)^\nu.$$

Conversely any power series that converges absolutely on the polydisk is an analytic function there. The coefficients a_ν are uniquely determined by f.

Here $\nu = (\nu_1, \dots, \nu_n)$ runs through all multi-indices. This means that ν_i are nonnegative integers. We use the usual multi-index notation

$$(z-a)^\nu := (z_1 - a_1)^{\nu_1} \cdots (z_n - a_n)^{\nu_n}.$$

The proof of 2.4 is the same as in the case $n = 1$. We just sketch it: One can assume $a = 0$. We choose positive numbers. $\varrho_\nu < r_\nu$. It is sufficient to prove the expansion in the smaller polydisk $U_\varrho(0)$. First we apply the usual Cauchy integral formula to the analytic function in the single variable z_1,

$$z_1 \longmapsto f(z_1, \dots, z_n),$$

keeping $z_2, \dots, z_n$ fixed. We obtain

$$f(z_1, \dots, z_n) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta_1| = \varrho_1} \frac{f(\zeta_1, z_2, \dots, z_n)}{\zeta_1 - z_1} d\zeta_1.$$

Now we apply the Cauchy integral formula step by step for the variables $(z_1, \dots, z_n)$. We obtain the

Cauchy integral formula in several variables

$$f(z_1, \dots, z_n) = \frac{1}{(2\pi \mathrm{i})^n} \oint_{|\zeta_1| = \varrho_1} \cdots \oint_{|\zeta_n| = \varrho_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \dots d\zeta_n.$$

Now the power series expansion of f can be obtained as in the case $n = 1$. One expands the integrand into a *geometric series* and interchanges integration and summation.

We give two simple applications.

2.5 Lemma. *An element $P \in \mathcal{O}_n$ is a unit (i.e. an invertible element) if an only if $P(0) \neq 0$.*

Proof. The power series defines a holomorphic function f in a small neighbourhood U of the origin without zeros. The function $1/f$ is holomorphic as well and can be expanded into a power series. □

As a consequence of Lemma 2.5, the ring $\mathcal{O}_n$ is a local ring. Recall that a ring R is called local if the sum of two non-units is a non-unit. Then the set of all non-units is an ideal, obviously the only maximal ideal.

2.6 Remark. *The ring $\mathcal{O}_n$ is a local ring. The maximal ideal $\mathfrak{m}_n$ consists of all P with $P(0) = 0$.*

The ring $\mathcal{O}_n$ contains $\mathbb{C}$ as a subring (constant power series). Hence it is a $\mathbb{C}$-algebra.

Our next task is to describe the algebra homomorphisms $f : \mathcal{O}_m \to \mathcal{O}_n$. First we claim that non-units are mapped to non-units. This means that f is a local homomorphism. Otherwise there would be non-unit $P \in \mathcal{O}_m$ such that $Q = f(P)$ is a unit. Then we would have $f(P - Q(0)) = Q - Q(0)$. The element $P - Q(0)$ is a unit but its image $Q - Q(0)$ is not. This is not possible.

There is a special kind of such a homomorphism which we call a "substitution homomorphism". It is defined by means of elements $P_1, \dots, P_n \in \mathcal{O}_m$ that are contained in the maximal ideal. If $P(z_1, \dots, z_n)$ is an element of $\mathcal{O}_n$, one can substitute the variables z_i by the power series P_i. For a precise definition one can interprets P, P_i as holomorphic maps, and use then that the composition of holomorphic maps is holomorphic In this way, we obtain a power series $P(P_1, \dots, P_n)$. Another way to see this is to apply standard rearrangement theorems. This substitution gives a homomorphism

$$\mathcal{O}_n \longrightarrow \mathcal{O}_m, \quad P \longmapsto P(P_1, \dots, P_n).$$

We call it a *substitution homomorphism.*

2.7 Lemma. *Each algebra homomorphism $\mathcal{O}_n \to \mathcal{O}_m$ is a substitution homomorphism.*

Proof. Let $\varphi : \mathcal{O}_n \to \mathcal{O}_m$ an algebra homomorphism. Since it is local, the elements $P_i := \varphi(z_i)$ are contained in the maximal ideal. Hence one can consider the substitution homomorphism ψ defined by them. We claim $\varphi = \psi$. At the moment we only know that φ and ψ agree on $\mathbb{C}[z_1, \dots, z_n]$. Let $P = \sum_\nu a_\nu z^\nu \in \mathcal{O}_n$. We claim $\varphi(P) = \psi(Q)$. For this we decompose for a natural number k

$$P = P_k + Q_k, \quad P_k = \sum_{\nu_1 + \dots + \nu_n \leq k} a_\nu z^\nu.$$

Then Q_k is contained in the k-the power $\mathfrak{m}^k$ of the maximal ideal. (Obviously $\mathfrak{m}^k$ is generated by all z^ν where $\nu_1 + \dots + \nu_n \geq k$.) We get

$$\varphi(P) - \psi(P) = \varphi(Q_k) - \psi(P_k) \in \mathfrak{m}^k.$$

This is true for all k. But the intersection of all This argument shows more. There is a big difference between analysis of one and many complex variables.

2.8 Proposition. *Assume $n > 1$. Let ϱ and r be two multi-indices such that $0 < \varrho_i < r_i$. Every holomorphic function f on $U_r(0) - \overline{U_\varrho(0)}$ extends to a holomorphic function on $U_r(0)$.*

As a consequence any holomorphic function on $U_r(0) - \{0\}$ extends to $U_r(0)$. This means: In more than one complex variables there are no isolated singularities.

Proof of 2.8. Since we could make r a little smaller, we can assume that f is holomorphic on some $U_R(0) - \overline{U_\varrho(0)}$ with $R_i > r_i$. To simplify notation we assume $n = 2$. Let z_1 be a fixed number with $\varrho_1 < |z_1| < r_1$. Then all (z_1, z_2) with $|z_2| < r_2$ are contained in $U_r(0) - \overline{U_\varrho(0)}$. Applying Cauchy's integral formula to this z_2-disk we get

$$f(z_1, z_2) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=r_2} \frac{f(z_1,\zeta)}{\zeta - z_2}\, d\zeta \qquad (\varrho_1 < |z_1| < r_1,\ |z_2| < r_2).$$

The point now is that this formula defines an analytic function in the bigger domain $U_r(0)$. □

3. The Preparation and the Division Theorem

We mentioned already that polynomials are special power series and constants can be considered as special polynomials. Hence we have natural inclusions of $\mathbb{C}$-algebras

$$\mathbb{C} \subset \mathbb{C}[z_1, \dots, z_n] \subset \mathbb{C}\{z_1, \dots, z_n\} \subset \mathbb{C}[[z_1, \dots, z_n]].$$

The ring $\mathcal{O}_{n-1}$ can be identified with the intersection $\mathcal{O}_n \cap \mathbb{C}[[z_1, \dots, z_{n-1}]]$ and there is a natural inclusion

$$\mathcal{O}_{n-1}[z_n] \longrightarrow \mathcal{O}_n.$$

We have to introduce the fundamental notion of a *Weierstrass polynomial.* This notion can be defined for every local ring R. Let P be a normalized polynomial of degree d over R. This polynomial is called a Weierstrass polynomial if its image in $R/\mathfrak{m}[X]$ is X^d. For the ring of our interest $R = \mathcal{O}_n$ this means:

3.1 Definition. *A polynomial*

$$P \in \mathcal{O}_{n-1}[z] = \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$$

is called a Weierstrass polynomial of degree d, if it is of the form

$$P = z_n^d + P_{d-1} z_n^{d-1} + \ldots + P_0$$

where all the coefficients besides the highest one (which is 1) vanish at the origin,

$$P_0(0) = \ldots = P_{d-1}(0) = 0.$$

To formulate the Weierstrass preparation theorem we need the notion of a z_n*-general* power series.

3.2 Definition. *A power series*

$$P = \sum a_\nu z^\nu \in \mathbb{C}\{z_1, \dots, z_n\}$$

is called $\boldsymbol{z_n}$***-general****, if the power series $P(0, \dots, 0, z_n)$ does not vanish. It is called z_n-general of order d if*

$$P(0, \dots, 0, z_n) = b_d z_n^d + b_{d+1} z_n^{d+1} + \dots \quad \textit{where } b_d \neq 0.$$

A power series is z_n-general, if it contains a monomial which is independent of $z_1, \dots z_{n-1}$. For example $z_1 + z_2$ is z_2-general, but $z_1 z_2$ not. A Weierstrass polynomial is of course z_n-general and its degree and order agree. If P is a z_n-general power series and U is a unit in $\mathcal{O}_n$ then PU is also z_n-general of the same order.

3.3 Weierstrass preparation theorem. *Let $P \in \mathcal{O}_n$ be a z_n-general power series. There exists a unique decomposition*

$$P = UQ,$$

where U is a unit in $\mathcal{O}_n$ and Q a Weierstrass polynomial.

There is an division algorithm in the ring of power series analogous to the Euclidean algorithm in a polynomial ring. We recall this Euclidean algorithm.

The Euclidean Algorithm for Polynomials

let R be an integral domain and let

a) *$P \in R[X]$ be an arbitrary polynomial,*
b) *$Q \in R[X]$ be a normalized polynomial, i.e. the highest coefficient is 1.*

Then there exists a unique decomposition

$$P = AQ + B.$$

where $A, B \in R[X]$ are polynomials and

$$\deg(B) < d.$$

This includes the case $B = 0$ if one defines $\deg(0) = -\infty$. The proof of this result is trivial (induction on the degree of P). □

3.4 Division theorem. *Let $Q \in \mathcal{O}_{n-1}[z]$ be a Weierstrass polynomial of degree d. Every power series $P \in \mathcal{O}_n$ admits a unique decomposition*

$$P = AQ + B \quad \text{where} \quad A \in \mathcal{O}_n, \quad B \in \mathcal{O}_{n-1}[z_n], \quad \deg_{z_n}(B) < d.$$

Here $\deg_{z_n}(B)$ means the degree of the polynomial B over the ring $\mathcal{O}_{n-1}$ (again taking $-\infty$ if $B = 0$).

The preparation theorem is related to the division theorem which – not correctly – sometimes is also called Weierstrass preparation theorem. Its first prove is due to Stickelberger.

The division theorem resembles the Euclidean algorithm. But there is a difference. In the Euclidean algorithm we divide through arbitrary normalized polynomials. In the division theorem we are restricted to divide through Weierstrass polynomials. But due to the preparation theorem, Weierstrass polynomials are something very general. This also shows the following simple consideration.

Let $A = (a_{\mu\nu})_{1\le\mu,\nu\le n}$ be an invertible complex $n \times n$-matrix. We consider A as linear map

$$A : \mathbb{C}^n \longrightarrow \mathbb{C}^n \quad z \longmapsto w, \qquad w_\mu = \sum_{\nu=1}^{n} a_{\mu\nu} z_\nu.$$

For a power series $P \in \mathcal{O}_n$, we obtain by substitution and reordering the power series

$$P^A(z) := P(A^{-1}z).$$

Obviously the map

$$\mathcal{O}_n \xrightarrow{\sim} \mathcal{O}_n, \qquad P \longmapsto P^A,$$

is an ring automorphism, i.e.

$$(PQ)^A = P^A Q^A.$$

The inverse map is given by A^{-1}.

3.5 Remark. *For every finite set of convergent power series $P \in \mathcal{O}_n$, $P \neq 0$, there exists an invertible $n \times n$-matrix A, such that all P^A are z_n-general.*

Proof. There exists a point $a \neq 0$ in a joint convergence poly-disk, such that $P(a) \neq 0$ for all P. After the choice of suitable coordinate transformation (choice of A), one can assume $A(0, \dots, 0, 1) = a$. Then all P^A are z_n-general. □

Proof of the Preparation and the Division Theorem

The prove that we will give here, depends on the Cauchy integral. The preparation theorem is related to the shape of the zero set of a power series. For Weierstrass polynomials, the following Lemma is obvious. Our proof of the division- and preparation theorem makes use of the fact that it can be proven for z_n-general series directly.

3.6 Lemma. *Let P be a z_n-general power series of order $d > 0$. The number $r > 0$ can be chosen, such that P converges absolutely for $|z_\nu| \le r$ and such that $P(0, \dots, 0, z_n)$ has in the disk $|z_n| \le r$ besides 0 no zero. Then there exists a number ε, $0 < \varepsilon < r$ with the following properties:*

1. *One has $P(z_1, \dots, z_{n-1}, z_n) \ne 0$ for $|z_n| = r$ and $|z_\nu| < \varepsilon$ $(1 \le \nu \le n-1)$.*
2. *For fixed $(z_1, \dots, z_{n-1})$ with $|z_\nu| < \varepsilon$ the function $z_n \mapsto P(z_1, \dots, z_n)$ has precisely d zeros for $|z_n| < r$ (counted with multiplicity).*

Proof. The first statement is clear for each fixed chosen z_n by a continuity argument. For the general case one has to use a simple compactness argument.

The second statement follows by means of the zero counting integral of usual complex analysis ([FB], III.5.7). This integral shows that the number of zeros depends continuously on $z_1, \dots, z_{n-1}$. Since it is integral, it must be constant, and hence be equal to the value for $z_1 = \dots = z_{n-1} = 0$. □

Lemma 3.6 states that the set of zeros of an analytic function is "something $(n-1)$-dimensional". Later we will give a precise formulation for this (Proposition II.5.3).

3.7 Lemma. *Let $Q \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial and let $A \in \mathcal{O}_n$ be a power series with the property*

$$P = AQ \in \mathcal{O}_{n-1}[z_n].$$

Then $A \in \mathcal{O}_{n-1}[z_n]$ too.

For arbitrary polynomials $Q \in \mathcal{O}_{n-1}[z_n]$ instead of Weierstrass polynomials this statement is false as the example

$$(1 - z_n)(1 + z_n + z_n^2 + \cdots) = 1$$

shows.

Proof of 3.7. By means of polynomial division, we can assume that the degree of P (as polynomial over $\mathcal{O}_{n-1}$) is smaller than the degree of Q. In this case we show $A = 0$. We choose $r > 0$ small enough, such that all occurring power series converge for $|z_\nu| < r$. Then we choose $\varepsilon = \varepsilon(r) > 0$ small enough, such that $\varepsilon \le r$ and such that each zero

$$Q(z_1, \dots, z_n) = 0, \quad |z_\nu| < \varepsilon \text{ for } \nu = 1, \dots, n-1,$$

automatically has the property $|z_n| < r$. Then the polynomial

$$z_n \longmapsto P(z_1, \dots, z_n)$$

has for each $(n-1)$-tuple $(z_1, \dots, z_{n-1})$, $|z_\nu| < \varepsilon$, as Q, at least $d = \deg Q$ zeros, counted with multiplicity. Because of $\deg P < \deg Q$ we get

$$P(z_1, \dots, z_n) \equiv 0 \text{ for } |z_\nu| < \varepsilon,\ \nu = 1, \dots, n-1.$$

We obtain $P = 0$ and $A = 0$. □

Proof of the division theorem.

For an arbitrary power series $P \in \mathcal{O}_n$ we have to construct a decomposition

$$P = AQ + B, \quad B \in \mathcal{O}_{n-1}[z_n], \quad \deg B < \deg Q,$$

and to show that it is unique.

Uniqueness. From $AQ + B = 0$, we get that $A \in \mathcal{O}_{n-1}[z_n]$ because of 3.7. Comparing degrees we get $A = B = 0$.

Existence. We want to define

$$A(z_1, \dots, z_n) := \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=r} \frac{P(z_1, \dots, z_{n-1}, \zeta)}{Q(z_1, \dots, z_{n-1}, \zeta)} \frac{d\zeta}{\zeta - z_n}.$$

For this we have to explain how $r > 0$ has to be chosen. It has to be so small that the power series P and Q converge in

$$U = \{\, z; \quad \|z\| < 2r \,\}, \quad \|z\| := \max\{|z_\nu|,\ \nu = 1, \dots, n\,\}.$$

Then their exists a number $\varepsilon,\ 0 < \varepsilon < r$, such that

$$Q(z_1, \dots, z_n) \neq 0 \text{ for } \ |z_n| \geq r, \quad |z_\nu| < \varepsilon \text{ for } 1 \leq \nu \leq n-1.$$

The function A is analytic in $\|z\| < \varepsilon$ and can be expanded into a power series there. We denote this power series by A again. What we have to show now is that

$$B := P - AQ$$

is a polynomial in z_n, and that its degree is smaller than that of Q. By means of the Cauchy integral formula for P one obtains (with $z := (z_1, \dots, z_{n-1})$)

$$\begin{aligned} B(z, z_n) =& \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=r} \frac{P(z,\zeta)}{\zeta - z_n} d\zeta - \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=r} Q(z, z_n) \frac{P(z,\zeta)}{Q(z,\zeta)} \frac{d\zeta}{\zeta - z_n} \\ =& \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=r} \frac{P(z,\zeta)}{Q(z,\zeta)} \left[\frac{Q(z,\zeta) - Q(z, z_n)}{\zeta - z_n} \right] d\zeta. \end{aligned}$$

The variable z_n only occurs inside the big brackets. For fixed $z_1, \dots, z_{n-1}, \zeta$ we know that $Q(z_1, \dots, z_{n-1}, \zeta) - Q(z_1, \dots, z_n)$ is a polynomial of degree $d = \deg Q$ in z_n. This has a zero at $z_n = \zeta$ and hence is divisible by $z_n - \zeta$, such that the quotient is a polynomial of degree $d - 1$. Hence B is a polynomial of degree $< d$ in z_n. □

Now we prove a special case of the preparation theorem. Let $P \in \mathcal{O}_n$ an arbitrary z_n-general power series and Q a Weierstrass polynomial. Both are assumed to converge in $\|z\| \le r$. We also assume that there exist a number ε, $0 < \varepsilon < r$, such that for each fixed $(z_1, \dots, z_{n-1})$ with $|z_\nu| < \varepsilon$ for $(1 \le \nu \le n-1)$ the functions

$$z_n \longmapsto Q(z_1, \dots, z_n), \quad z_n \longmapsto P(z_1, \dots, z_n)$$

have the same zeros —counted with multiplicity— in the disk $|z_n| < r$. We claim that

$$P = UQ$$

with a unit U.

Proof. We can choose ε so small that all d zeros of Q are contained in $|z_n| < r$. By the division theorem we have

$$P = AQ + B, \quad B \in \mathcal{O}_{n-1}[z_n], \quad \deg B < \deg Q.$$

We can assume that A and B both converge in $|z_n| < r$. The polynomial

$$z_n \longmapsto B(z_1, \dots, z_n)$$

has for each $(z_1, \dots, z_{n-1})$, $|z_\nu| < \varepsilon$, $1 \le \nu \le n-1$, more zeros than its degree predicts. Hence it is identically zero. The same consideration shows that A is a unit. □

Now we are able to prove the preparation theorem in full generality. Let P be a z_n-general power series, and d, $0 < d < \infty$ the zero order of $P(0, \dots, 0, z_n)$ at $z_n = 0$. The numbers $0 < \varepsilon < r$ are chosen as in 3.6. We consider the functions

$$\sigma_k(z_1, \dots, z_{n-1}) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta| = r} \zeta^k \frac{\partial P(z, \zeta)}{\partial \zeta} \frac{d\zeta}{P(z, \zeta)}, \quad k = 0, 1, 2, \dots.$$

They are analytic in the domain

$$z \in \mathbb{C}^{n-1}, \quad \|z\| < \varepsilon.$$

By the residue theorem from complex analysis (zero-counting integral) in one variable, we know that $\sigma_0(z_1, \dots, z_{n-1})$ is the number of zeros

$$z_n \longmapsto P(z_1, \dots, z_n)$$

in $|z_n| < r$ (counted with multiplicity). As a consequence σ_0 is integral, hence constant. We order the $d = \sigma_0(z_1, \dots, z_{n-1})$ zeros arbitrarily,

$$t_1(z), \dots, t_d(z).$$

Of course we can not expect that $t_\nu(z)$ are analytic functions in z. But a simple generalization of the zero-counting integral gives

$$\sigma_k(z) = t_1(z)^k + \dots + t_d(z)^k.$$

Therefore the symmetric expressions $t_1(z)^k + \dots + t_d(z)^k$ are analytic functions in z. By a result of elementary algebra, which we want to use without proof, we have:

The νth elementary symmetric polynomial $(1 \le \nu \le d)$

$$E_\nu(X_1, \dots, X_d) = (-1)^\nu \sum_{1 \le j_1 < \dots < j_\nu \le d} X_{j_1} \dots X_{j_\nu}$$

can be written as polynomial (with rational coefficients) in the

$$\sigma_k(X_1, \dots, X_d) = \sum_{j=1}^{d} X_j^k$$

($1 \le k \le d$ is enough).

Example.

$$E_2(X_1, X_2) = X_1 X_2 = \frac{1}{2}\left[(X_1 + X_2)^2 - (X_1^2 + X_2^2)\right] = \frac{1}{2}[\sigma_1^2 - \sigma_2].$$

Especially the elementary symmetric functions $t_1(z), \dots, t_d(z)$ are analytic. We use them to define the Weierstrass polynomial

$$Q(z_1, \dots, z_{n-1}, z_n) = z_n^d + E_1\left(t_1(z), \dots, t_d(z)\right) z_n^{d-1} + \dots + E_d\left(t_1(z), \dots, t_d(z)\right).$$

For fixed $\|z\| < \varepsilon$ the zeros of these polynomials are $t_1(z), \dots, t_d(z)$ by the (trivial) "Vieta theorem". By the second step P and Q differ only by a unit. This proves the preparation theorem. □

4. Algebraic properties of the ring of power series

The ring $\mathcal{O}_0$ just coincides with the field of complex numbers. The ring $\mathcal{O}_1$ is also very simple. Every element can be written in the form $z^n P$ where P is a unit and $n \geq 0$ an integer. It follows that each ideal of $\mathcal{O}_1$ is of the form $\mathcal{O}_1 z^n$. The rings $\mathcal{O}_n$, $n > 1$, are much more complicated.

Let $Q \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial. We can consider the natural homomorphism

$$\mathcal{O}_{n-1}[z_n]/Q\mathcal{O}_{n-1}[z_n] \longrightarrow \mathcal{O}_n/Q\mathcal{O}_n.$$

The division theorem implies that this is an isomorphism.

4.1 Theorem. *For a Weierstrass polynomial $Q \in \mathcal{O}_{n-1}[z_n]$ the natural homomorphism*

$$\mathcal{O}_{n-1}[z_n]/Q\mathcal{O}_{n-1}[z_n] \longrightarrow \mathcal{O}_n/Q\mathcal{O}_n$$

is an isomorphism.

Proof. The surjectivity is an immediate consequence of the existence statement in the division theorem. The injectivity follows from the uniqueness statement in this theorem as follows. Let $P \in \mathcal{O}_{n-1}[z_n]$ a polynomial that goes to 0, i.e. $P = SA$, $Q \in \mathcal{O}_n$. We have to show that S is a polynomial in z_n. We compare with the elementary polynomial division $P = AQ+B$. The uniqueness statement in the division theorem shows $A = S$ and $B = 0$. □

Recall that an element $a \in R$ is a prime element if and only if Ra is a non-zero prime ideal. (A prime ideal $\mathfrak{p} \subset R$ is an ideal such that $R/\mathfrak{p}$ is an integral domain.) By our convention the zero ring is no integral domain. Hence prime ideals are proper ideals and prime elements are non-units. From Theorem 4.1 we obtain the following result.

4.2 Lemma. *A Weierstrass polynomial $P \in \mathcal{O}_{n-1}[z_n]$ is a prime element in $\mathcal{O}_n$, if and only if it is a prime element in $\mathcal{O}_{n-1}[z_n]$.*

We recall that an integral domain R is called a UFD-domain if every nonzero and non-unit element of R can be written as a finite product of prime elements. This product then is unique in an obvious sense. Every principal ideal domain is UFD. As a consequence every field is UFD. But also $\mathbb{Z}$ and $\mathcal{O}_1$ are principal ideal rings and hence UFD. A famous result of Gauss states that the polynomial ring over a UFD domain is UFD. A non-unit and non-zero element a of an integral domain is called *indecomposable* if it cannot be written as product of two non-units. Primes are indecomposable. The converse is true in UFD-domains. It is often easy to show that any element of an integral domain is the product of finitely many indecomposable elements. For example this is case in $\mathcal{O}_n$. On can prove this by induction on

$$o(P) := \sup\{k; \quad P \in \mathfrak{m}_n^k\}.$$

An integral domain is UFD if and only of every element is the product of finitely many indcomposable elements and if each indecomposable element is a prime.

4.3 Theorem. *The ring $\mathcal{O}_n$ is a UFD-domain.*

Proof. We have to show that every indecomposable element $P \in \mathcal{O}_n$ is a prime. The proof is given by induction on n. By the preparation theorem one can assume that $P \in \mathcal{O}_{n-1}[z_m]$ is a Weierstrass plynomial. It can be checked that P is also indecomposable in $\mathcal{O}_{n-1}[z_n]$. By induction assumption $\mathcal{O}_{n-1}$ is UFD. The theorem of Gauss implies that $\mathcal{O}_{n-1}[z_n]$ is UFD. Hence P is a prime element in $\mathcal{O}_{n-1}[z_n]$. By theorem 4.1 then P is prime in $\mathcal{O}_n$. □

Recall that a ring R is called noetherian if each ideal $\mathfrak{a}$ is finitely generated, $\mathfrak{a} = Ra_1 + \cdots + Ra_n$. Then any sub-module of a finitely generated module is finitely generated.

4.4 Theorem. *The ring $\mathcal{O}_n$ is noetherian.*

Proof. Again we argue by induction on n. Let $\mathfrak{a} \subset \mathcal{O}_n$ be an ideal. We want to show that it is finitely generated. We can assume that $\mathfrak{a}$ is non-zero. Take any non-zero element $P \in \mathfrak{a}$. By the preparation theorem we can assume that P is a Weierstrass polynomial. It is sufficient to show that the image of $\mathfrak{a}$ in $\mathcal{O}_n/(P)$ is finitely generated. This is the case, since $\mathcal{O}_{n-1}$ is noetherian by induction hypothesis and then $\mathcal{O}_{n-1}[z_n]$ is noetherian by Hilbert's basis theorem. □

5. Hypersurfaces

Under a hypersurface we understand here the set of zeros of a non-zero analytic function on a domain $D \subset \mathbb{C}^n$. For their study we will make use the theory of the discriminant. It can be used to characterize square free elements of a polynomial ring over factorial rings.

An element a of an integral domain is called square free if $a = bc^2$ implies that c is a unit. Primes are square free. Notice our convention: units are square free but they are no primes.

There is a close relation between the question of divisibility of power series and their zeros.

5.1 Proposition. *Let $P, Q \in \mathcal{O}_n$, $Q \neq 0$, be two power series. We assume that there exists a neighborhood of the origin in which both series converge and such that every zero of Q in this neighborhood is also a zero of P. Then there exist a natural number n such that P^n is divisible by Q,*

$$P^n = AQ, \quad A \in \mathcal{O}_n.$$

If Q is square free, one can take $n = 1$, i.e. then P is divisible by Q.

Proof. Because of the existence of the prime decomposition, we can assume that Q is square free. By our standard procedure, we can assume that Q is a Weierstrass polynomial. From the division theorem we obtain

$$P = AQ + B, \quad B \in \mathcal{O}_{n-1}[z_n], \quad \deg_{z_n} B < d.$$

By assumption we know in a small neigborhood of the origin

$$Q(z) = 0 \Longrightarrow B(z) = 0.$$

Now we make use of the fact that Q is a square free element of $\mathcal{O}_n$. We know then that Q is square free in $\mathcal{O}_{n-1}[z_n]$. Hence the discriminant of Q is different from 0. Now we consider the polynomial

$$Q_a(z) = Q(a_1, \dots, a_{n-1})(z) \in \mathbb{C}[z]$$

for fixed sufficiently small $a = (a_1, \dots, a_{n-1})$. The discriminant d_{Q_a} can be obtained from d_Q by specializing $z_1 = a_1, \dots, z_{n-1} = a_{n-1}$. This follows for example from the existence of the universal polynomial Δ_n. Therefore there exists a dense subset M of a small neighborhood of 0 such that d_{Q_a} is different from 0 for $a \in M$. This means that Q_a is a square free element from $\mathbb{C}[z]$. Since $\mathbb{C}$ is algebraically closed, this means nothing else that Q_a has no multiple zeros. Hence Q_a has d pairwise distinct zeros (for $a \in M$). As we pointed out several times the d zeros are arbitrarily small if a is sufficiently small. We obtain that $z \mapsto B(a, z)$ has d pairwise distinct zeros if a lies in a dense subset of a sufficiently small neighborhood of the origin. It follows that B_a vanishes for these a. By a continuity argument we obtain $B = 0$. □

5.2 Definition. *A holomorphic function*

$$f : D \longrightarrow \mathbb{C} \quad (D \subset \mathbb{C}^n \text{ open})$$

is called ***reduced*** *at a point $a \in D$ if the power series of f at a is a square free element of $\mathbb{C}\{z_1 - a_1, \dots z_n - a_n\}$.*

(The notation $\mathbb{C}\{z_1 - a_1, \dots z_n - a_n\}$ has been introduced for the same time. This ring is just the usual ring of power series. The notation just indicates that the elements now are consider as functions around a. It is the same to consider $f(z - a)$ and then to take the power series expansion around 0.) If a is a non-zero element of an UFC-domain one can define its "square free part" b. This is a square free element which divides a and such that a divides a suitable power of a. The square free part is determined up to a unit of R. The definition of b is obvious fron the prime decomposition of a. For example the square free part of $z_1^2 z_2^3$ is $z_1 z_2$. If we want to investigate local properties of a hypersurface A around a given point $a \in A$ we can assume that the defining equation $f(z) = 0$ in a small neighborhood of a is given by a function f which is reduced at a.

5.3 Proposition. *Let f be a holomorphic function on an open set $U \subset \mathbb{C}^n$. The set of all points $a \in U$ in which f is reduced is an open set.*

For the prove of 5.3 we need the following two remarks:

5.4 Remark. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be a normalized polynomial, which is square free in the ring $\mathcal{O}_{n-1}[z_n]$. Then P is square free in the bigger ring $\mathcal{O}_n$.*

We already used this result for Weierstrass polynomials where it is a consequence of 4.2. For the general case, we use the preparation theorem

$$P = UQ, \quad U \text{ unit in } \mathcal{O}_n, \quad Q \text{ Weierstrass polynomial.}$$

We know that U is a polynomial in z_n (3.7). This implies that Q is square free in the ring $\mathcal{O}_{n-1}[z_n]$ and therefore in $\mathcal{O}_n$. But U is a unit in $\mathcal{O}_n$ ist.Therefore P is square free in $\mathcal{O}_n$. □

The same argument shows:

5.5 Remark. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be a normalized polynomial which is prime in the ring $\mathcal{O}_{n-1}[z_n]$. Then P either is a unit in $\mathcal{O}_n$ or it is a prime in $\mathcal{O}_n$*

Proof of 5.3. Let $a \in D$ be a point in which f is reduced. We can assume $a = 0$ and that the power series $P = f_0$ is a Weierstrass polynomial. We consider the power series of f in all points b in a small polydisk around 0.

$$f_b \in \mathbb{C}\{z_1 - b_1, \dots, z_n - b_n\}$$

This power series is still a normalized polynomial in $\mathbb{C}\{z_1 - b_1, \dots, z_{n-1} - b_{n-1}\}[z_n - b_n]$ but usually not a Weierstrass polynomial. By assumption P is square free (in $\mathcal{O}_n$ but then also in $\mathcal{O}_{n-1}[z_n]$ since it is a Weierstrass polynomial). Therefore the discriminant does not vanish. This (and the universal formula for the discriminant) shows that the discriminant of P_b does dot vanish if b is close to 0. This means that P_b is square free in the polynomial ring and square free in $\mathcal{O}_a$ by 5.4. □

6. Analytic Algebras

All rings are assumed to be commutative and with unit element. Homomorphisms of rings are assumed to map the unit element into the unit element.

Recall that an algebra over a ring A by definition is a ring B together with a distinguished ring homomorphism $\varphi : A \to B$. This ring homomorphism can be used to define on B a structure as A-module, namely

$$ab := \varphi(a)b \quad (a \in A,\ b \in B).$$

Let B, B' be two algebras. A ring homomorphism $B \to B'$ is called an algebra homomorphism if it is A-linear. We will consider $\mathbb{C}$-algebras A. If A is different form zero then the structure homomorphism $\mathbb{C} \to A$ is injective. Usually identify complex numbers with their image in A. So each non-zero $\mathbb{C}$-algebra contains the field of complex numbers as sub-field.

6.1 Definition. *An analytic algebra A is a $\mathbb{C}$-algebra which is different from the zero algebra and such there exist an n and a surjective algebra homomorphism $\mathcal{O}_n \to A$.*

A ring R is called a *local ring* if it is not the zero ring and if the set of non-units is an ideal. This ideal is then a maximal ideal and moreover, it is the only maximal ideal. We denote this ideal by $\mathfrak{m}(R)$. Hence $R - \mathfrak{m}(R)$ is the set of units of R. The algebra $\mathcal{O}_n$ is a local ring. The maximal ideal $\mathfrak{m}_n$ consists of all P with $P(0) = 0$.

Let A be a local ring and $\mathcal{A} \subset \mathfrak{m}$ be a proper ideal. Then $A/\mathfrak{a}$ is a local ring too and the maximal ideal of $A/\mathfrak{a}$ is the image of $\mathfrak{a}$. The shows the following. If A is a local ring and $A \to B$ is a surjective homomorphism onto a non-zero ring, then B is also a local ring and the maximal ideal of A is mapped onto the maximal ideal of B. In general a homomorphism $A \to B$ between local rings is called local if it maps the maximal ideal of A into the maximal ideal of B. The natural map $A/\mathfrak{m}(A) \to B/\mathfrak{m}(B)$ is an isomorphism.

In particular, analytic algebras are local rings and the homomorphism $\mathcal{O}_n \to A$ in Definition 6.1 is a local homomorphism. The natural maps

$$\mathbb{C} \longrightarrow \mathcal{O}_n/\mathfrak{m}_n \longrightarrow A/\mathfrak{m}(A)$$

are isomorphisms. For $a \in A$ we denote by $a(0)$ the its image in $A/\mathfrak{m}(A)$ by $a(0)$. Recall that we identify this with a complex number. The maximal ideal of A consists of all $a \in A$ such that $a(0) = 0$.

We notice that an arbitrary algebra homomorphism $f : A \to B$ between analytic analytic algebras is local. Otherwise there would be non-unit $a \in A$ such that $b = f(a)$ is a unit. Then we would have $f(a - b(0)) = b - b(0)$. The element $a - b(0)$ is a unit but its image $b - b(0)$ is not. This is not possible.

Our next task is to describe the homomorphisms $\mathcal{O}_m \to \mathcal{O}_n$. In Lemma 1.6.4 we already introduced substitution homomorphisms They are defined by means of elements $P_1, \ldots, P_n \in \mathcal{O}_m$ that are contained in the maximal ideal. If $P(z_1, \ldots, z_n)$ is an element of $\mathcal{O}_n$, one can substitute the variables z_i by the power series P_i. This substitution gives a homomorphism

$$\mathcal{O}_n \longrightarrow \mathcal{O}_m, \quad P \longmapsto P(P_1, \ldots, P_n).$$

6.2 Lemma. *Each algebra homomorphism $\mathcal{O}_n \to \mathcal{O}_m$ is a substitution homomorphism.*

Proof. Let $\varphi : \mathcal{O}_n \to \mathcal{O}_m$ an algebra homomorphism. Since it is local, the elements $P_i := \varphi(z_i)$ are contained in the maximal ideal. Hence one can consider the substitution homomorphism ψ defined by them. We claim $\varphi = \psi$. At the moment we only know that φ and ψ agree on $\mathbb{C}[z_1, \dots, z_n]$. Let $P = \sum_\nu a_\nu z^\nu \in \mathcal{O}_n$. We claim $\varphi(P) = \psi(Q)$. For this we decompose for a natural number k

$$P = P_k + Q_k, \quad P_k = \sum_{\nu_1+\cdots+\nu_n \le k} a_\nu z^\nu.$$

Then Q_k is contained in the k-the power $\mathfrak{m}^k$ of the maximal ideal. (Obviously $\mathfrak{m}^k$ is generated by all z^ν where $\nu_1 + \cdots + \nu_n \ge k$.) We get

$$\varphi(P) - \psi(P) = \varphi(Q_k) - \psi(P_k) \in \mathfrak{m}^k.$$

This is true for all k. But the intersection of all $\mathfrak{m}^k$ is zero. This proves 6.4. □

We have to generalize 6.4 to homomorphisms $\varphi : A \to B$ of arbitrary analytic algebras A, B. There is one problem. Let $\mathfrak{m}(B)$ be the maximal ideal of B. It is not obvious that the intersection of all powers of $\mathfrak{m}(B)$ is zero. But it is true by general commutative algebra (Krull's intersection theorem, Theorem VII.5.3).

6.3 Lemma. *Let $A \to B$ a homomorphism of analytic algebras. Assume that surjective algebra homomorphism $\mathcal{O}_n \to A$ and $\mathcal{O}_m \to B$ are given. There exists a (substitution) homomorphism $\mathcal{O}_n \to \mathcal{O}_m$ such the the diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathcal{O}_n & \longrightarrow & \mathcal{O}_m \end{array}$$

commutes.

The proof should be clear. The variable $z_i \in \mathcal{O}_n$ is mapped to an element of A then of B. Consider in $\mathcal{O}_m$ an inverse image P_m. These elements define a substitution homomorphism $\mathcal{O}_n \to \mathcal{O}_m$. From Krull's intersection theorem follows that the diagram commutes. □

From 6.3 follows:

6.4 Lemma. *Let $f_1, \dots, f_m$ be elements of the maximal ideal of an analytic algebra A. There is a unique homomorphism $\mathbb{C}\{z_1, \dots, z_n\} \to A$ such that $z_i \mapsto f_i$.*

We denote the image by $\mathbb{C}\{f_1, \dots, f_n\}$ and call it the analytic algebra generated by $f_1, \dots, f_n$. We want to derive a criterion that $\mathbb{C}\{f_1, \dots, f_n\} = A$. A necessary condition is that $f_1, \dots, f_n$ generate the maximal ideal. Actually it is also sufficient:

6.5 Lemma. *Let $f_1, \dots, f_n$ be elements of the maximal ideal of an analytic algebra A. Then the following conditions are equivalent:*

a) *They generate the maximal ideal.*
b) $A = \mathbb{C}\{f_1, \dots, f_n\}$.

It is easy to reduce this to the ring $A = \mathbb{C}\{z_1, \dots, z_n\}$. Let $P_1, \dots, P_m$ be generators of the maximal ideal. We can write

$$z_i = \sum_{ij} A_{ij} P_j.$$

Taking derivatives and evaluating at 0 we get: The rank of the Jacobian matrix of $P = (P_1, \dots, P_m)$ is n. We can find an system consisting of n elements, say $P_1, \dots, P_n$, such that the Jacobian is invertible. Now one can apply the theorem of invertible functions. □

7. Noether Normalization

We consider ideals $\mathfrak{a} \in \mathcal{O}_n$ and their intersection $\mathfrak{b} := \mathfrak{a} \cap \mathcal{O}_{n-1}$ with $\mathcal{O}_{n-1}$.

7.1 Lemma. *Let $\mathfrak{a} \subset \mathcal{O}_n$ ba a z_n-general ideal. Then $\mathcal{O}_n/\mathfrak{a}$ is a $\mathcal{O}_{n-1}/\mathfrak{b}$-module of finite type with respect to the natural inclusion*

$$\mathcal{O}_{n-1}/\mathfrak{b} \hookrightarrow \mathcal{O}_n/\mathfrak{a} \quad (\mathfrak{b} = \mathcal{O}_{n-1} \cap \mathfrak{a}).$$

Additional remark. *If $\mathfrak{a}$ contains a Weierstrass polynomial of degree d, then $\mathcal{O}_n/\mathfrak{a}$ is generated as $\mathcal{O}_{n-1}/\mathfrak{b}$-module by the images of the powers*

$$1, z_n, \dots, z_n^{d-1}.$$

The proof is an immediate consequence of the division theorem. □

7.2 Noether normalization theorem. *Let A be an analytic algebra. There exists an injective homomorphism of analytic algebras*

$$\mathbb{C}\{z_1, \dots, z_d\} \hookrightarrow A \quad (d \text{ suitable})$$

such that A is a module of finite type over $\mathbb{C}\{z_1, \dots, z_d\}$. The number d is unique (it is the Krull dimension).

Proof. The existence of such an embedding follows from 7.1 by repeated application. One makes use of the following simple fact. If $A \subset B$ and $B \subset C$ are finite then $A \subset C$ is finite too. The essential point is the uniqueness of d. It follows from the characterization as Krull dimension. □

The Noether normalization admits a refinement if the starting ideal $\mathfrak{a}$ is a prime ideal. Recall that an ideal $\mathfrak{p} \subset R$ in a ring R is called a prime ideal if the factor ring is an integral domain.

So let $\mathfrak{P} \subset \mathcal{O}_n$ be a prime ideal and $\mathfrak{p} = \mathcal{O}_{n-1} \cap \mathfrak{P}$. We have an injective homomorphism

$$\mathcal{O}_{n-1}/\mathfrak{p} \hookrightarrow \mathcal{O}_n/\mathfrak{P}$$

which shows that $\mathfrak{p}$ is also a prime ideal. Let K resp. L be the field of quotients of $\mathcal{O}_{n-1}/\mathfrak{p}$ resp. $\mathcal{O}_n/\mathfrak{p}$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{n-1}/\mathfrak{p} & \hookrightarrow & \mathcal{O}_n/\mathfrak{P} \\ \cap & & \cap \\ K & \hookrightarrow & L\,. \end{array}$$

We distinguish two cases which behave completely different:

7.3 Theorem, the first alternative. *Let $\mathfrak{P} \subset \mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$ be a z_n-general prime ideal. Assume*

$$\mathfrak{P} \cap \mathcal{O}_{n-1} = \{0\}.$$

Then $\mathfrak{P}$ is a principal ideal (i.e. generated by one element).

Proof. Let $Q \in \mathfrak{P}$ be a z_n-general element. One of the prime divisors of Q must be contained in $\mathfrak{P}$. It is z_n-general too. Hence we can assume that Q is prime. We will show that Q generates $\mathfrak{P}$. By the preparation theorem we can assume that Q is a Weierstrass polynomial. Let $P \in \mathfrak{P}$ be an arbitrary element. From 7.1 applied to the ideal $\mathfrak{a} = (Q)$ we get an equation

$$P^k + A_{k-1}P^{k-1} + \ldots + A_0 \equiv \operatorname{mod}(Q), \quad A_i \in \mathcal{O}_{n-1}\ (0 \le i < k).$$

The equation shows that A_0 is contained in $\mathfrak{P}$, hence in $\mathfrak{P} \cap \mathcal{O}_{n-1}$. By assumption this ideal is 0 and we obtain $A_0 = 0$. We see

$$P \cdot (P^{k-1} + \ldots + A_1) \equiv 0 \operatorname{mod} Q.$$

But (Q) is a prime ideal and we get

$$\text{either} \qquad P \in (Q) \qquad \text{or} \qquad P^{k-1} + \ldots + A_1 \equiv \operatorname{mod}(Q).$$

Repeated application of this argument shows $P \in (Q)$ in any case. □

7.4 Theorem, the second alternative. *Let*

$$\mathfrak{P} \subset \mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$$

be a prime ideal which is not a principal ideal. After a suitable linear transformation of the coordiantes we can obtain:

a) $\mathfrak{P}$ *is* z_n*-general.*
b) *The rings*

$$\mathcal{O}_{n-1}/\mathfrak{p} \hookrightarrow \mathcal{O}_n/\mathfrak{P} \quad (\mathfrak{p} = \mathcal{O}_{n-1} \cap \mathfrak{P})$$

have the same field of quotients $K = L$.

"After a suitable linear transformation of the coordinates" means that we allow to replace $\mathfrak{P}$ by its image under the automorphism

$$\mathcal{O}_n \to \mathcal{O}_n, \quad P(z) \mapsto P(Az),$$

for suitable $A \in \mathrm{GL}(n, \mathbb{C})$.

Proof of 7.4. We may assume that $\mathfrak{P}$ is already z_n-general. From 7.3 we know that

$$\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_{n-1}$$

is different from 0. After a linear transformation of the variables $(z_1, \ldots, z_{n-1})$ we can assume that $\mathfrak{p}$ is z_{n-1}-general. The ideal $\mathfrak{P}$ remains z_n-general. Now we consider

$$\mathfrak{q} = \mathfrak{p} \cap \mathcal{O}_{n-2} = \mathfrak{p} \cap \mathcal{O}_{n-2}.$$

The extension

$$\mathcal{O}_{n-2}/\mathfrak{q} \subset \mathcal{O}_n/\mathfrak{P}$$

is of finite type. We denote the fields of fractions by $K \subset L$. This is a finite algebraic extension and we have $L = K[\bar{z}_{n-1}, \bar{z}_n]$. The bar indicates that we have to take cosets mod P. From elementary algebra we will use

Theorem of primitive element. *Let* $K \subset L$ *be a finite algebraic extension of fields of characteristic zero, which is generated by two elements,* $L = K[a, b]$. *Then for all* $x \in K$ *but a finite number of exceptions one has*

$$L = K[a + xb].$$

As a consequence every finite algebraic extension of fields of characteristic zero is generated by one element. This is the usual formulation of this theorem. The above variant is contained in the standard proofs.

We obtain that

$$L = K[\bar{z}_{n-1} + a\bar{z}_n].$$

for almost all $a \in \mathbb{C}$. We consider now the following (invertible) linear transformation of variables,

$$w_{n-1} = z_{n-1} + az_n, \qquad w_j = z_j \text{ for } j \neq n-1.$$

We have to take care that $\mathfrak{P}$ remains general in the new coordinates, which now means w_n-general. This possible because we have infinitely many possibilities for a.

Thus we have proved that we can assume without loss of generality $L = K[\bar{z}_{n-1}]$. But then the quotient fields of $\mathcal{O}_{n-1}/\mathfrak{p}$ and $\mathcal{O}_n/\mathfrak{p}$ agree. □

8. Geometric Realization of Analytic Ideals

Let $\mathfrak{a} \subset \mathcal{O}_n$ be a proper ideal of power series. The ring $\mathcal{O}_n$ being noetherian we can choose a finite system of generators $\mathfrak{a} = (P_1, \dots, P_m)$. The generators converge in a common polydisk U around 0 and in this polydisk the set

$$X := \{z \in U; \quad P_1(z) = \cdots = P_m(z) = 0\}$$

is well defined. We call X a *geometric realization* of $\mathfrak{a}$. This realization depends on the choice of the generators and of U. But it is clear that two geometric realizations X, Y agree in a small neighborhood of the origin. This means that for all local questions around the origin the geometric realization behaves as if it were unique.

The technique of the last section was to consider the intersection $\mathfrak{b} = \mathfrak{a} \cap \mathcal{O}_{n-1}$. Let X resp. Y be geometric realizations of $\mathfrak{a}$ resp. $\mathfrak{b}$. We consider the projection (cancelation of the last variable)

$$\mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}, \quad (z_1, \dots, z_n) \longmapsto (z_1, \dots, z_{n-1}).$$

The generators of $\mathfrak{b}$ can be expressed by means of the generators of $\mathfrak{a}$. Therefore a point $a \in X$ which is sufficiently close to the origin will be mapped to a point of Y. If we replace X by its intersection with a small polydisk around Y, we obtain a map

$$X \longrightarrow Y$$

induced by the projection. We call this map the geometric realization of the pair $(\mathfrak{a}, \mathfrak{b} = \mathfrak{a} \cap \mathcal{O}_{n-1})$. Again this realization is uniquely determined in an obvious local sense around 0.

An ideal $\mathfrak{a} \subset \mathcal{O}_n$ is called z_n-general if it contains a z_n-general element. For the theory of ideals in $\mathcal{O}_n$ it is sufficient to restrict to z_n-general ideals, since every non-zero ideal can be transformed into a z_n-general one by means of linear change of coordinates.

8.1 Remark. *Let $\mathfrak{a}$ be a z_n-general ideal in $\mathcal{O}_n$ and $\mathfrak{b} = \mathfrak{a} \cap \mathcal{O}_{n-1}$. There exists a geometric realization $f : X \to Y$ of $(\mathfrak{a}, \mathfrak{b})$ such that the inverse image of $0 \in Y$ consists of only one point, namely $0 \in X$. Furthermore for every neighborhood $0 \in U \subset X$ there exists a neighborhood $0 \in V \subset Y$ such that*

$$f^{-1}(V) \subset U.$$

Proof. There exists a Weierstrass polynomial $P \in \mathfrak{a}$. Close to the origin the inverse image is contained in the set of zeros of $P(0, \dots, 0, z_n) = 0$. But $P(0, \dots, 0, z_n) = z_n^d$ implies that 0 is the only solution. The rest comes from the frequently used argument of "continuity of zeros" of a Weierstrass polynomial. □

We want to mention here an important result, which we cannot prove at the moment but which is always behind the scenes and motivates our constructions:

The geometric realization $X \to Y$ (under the assumption that $\mathfrak{a}$ is z_n-general) can be chosen such that it is surjective and proper and such that the fibres are finite.

We consider now the case that $\mathfrak{P} \subset \mathcal{O}_n$ is a prime ideal of the second alternative, i.e. it is z_n-general and $\mathcal{O}_n/\mathfrak{P}$ and $\mathcal{O}_{n-1}/\mathfrak{p}$ ($\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_{n-1}$) have the same field of fractions. We consider a geometric realization $f : X \to Y$ of the pair $(\mathfrak{P}, \mathfrak{p})$. We may assume that X is closed in the polydisk $U_{(\varrho_1,\dots,\varrho_n)}(0)$ and Y is closed in $U_{(\varrho_1,\dots,\varrho_{n-1})}(0)$.

8.2 Proposition. *Let $\mathfrak{P} \subset \mathcal{O}_n$ be a z_n-general prime ideal and $\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_{n-1}$. We assume that the fieds of fractions of $\mathcal{O}_n/\mathfrak{P}$ and $\mathcal{O}_{n-1}/\mathfrak{p}$ agree (second alternative). There exists a geometric realization $f : X \to Y$ of the pair $(\mathfrak{P}, \mathfrak{p})$ such the following holds:*

There exists a power series $A \in \mathcal{O}_{n-1}$ which is not contained in $\mathfrak{p}$ and which converges in a polydisk containing Y. Let be

$$S := \{\, z \in Y; \quad A(z) = 0 \,\} \quad \text{and} \quad T := f^{-1}(S).$$

The restriction

$$f_0 : X - T \longrightarrow Y - S$$

of f is topological.

Proof. We make use of the fact that the two fields of fractions agree. Expressing the coset of z_n as a fraction we obtain:

There exist power series $A, B \in \mathcal{O}_{n-1}$ with the properties

$$A \notin \mathfrak{p} \qquad Az_n - B \in \mathfrak{P}.$$

We can choose our realization $X \to Y$ such that A and B both converge in a polydisk containing Y. Especially the sets S and T are defined now. All points $z \in X$ satisfy

$$z_n A(z_1, \dots, z_{n-1}) = B(z_1, \dots, z_{n-1}).$$

This means

$$z_n = \frac{B(z_1, \dots, z_{n-1})}{A(z_1, \dots, z_{n-1})}$$

if z is not contained in T. So we have proved the injectivitiy of the map $f_0 : X - T \to Y - S$.

It remains to show that f_0 is surjective for properly chosen X and Y. To do this we choose X and Y as closed subsets of polydisks. This is of course possible,

$$Y \subset U_{(\varrho_1, \dots, \varrho_{n-1})}(0), \quad X \subset U_{(\varrho_1, \dots, \varrho_n)}(0) \qquad \text{(both closed)}.$$

We assume furthermore $\varrho_1 = \dots = \varrho_{n-1}$ and write

$$r := \varrho_1 = \dots = \varrho_{n-1}, \quad \varepsilon := \varrho_n.$$

We define

$$g(z_1, \dots, z_{n-1}) := (z_1, \dots, z_n), \quad z_n := \frac{B(z_1, \dots, z_{n-1})}{A(z_1, \dots, z_{n-1})}.$$

What we need is $g(z) \in X$ for $z \in Y - S$. In a first step we show:

8.3 Lemma. *Let $P \in \mathfrak{P} \cap \mathcal{O}_{n-1}[z_n]$. There exists a r', $0 < r' \le r$, such that*

$$P(g(z)) = 0 \quad \text{for all} \quad z \in Y - S, \ ||z|| < r'.$$

($||\cdot||$ denotes the maximum norm.)
Proof. We choose r' small enough such that the coefficients of P converge in the polydisk with multiradius $(r', \dots, r')$. Let d be the degree of P. Then $A^d P$ can be written as polynomial in Az_n with coefficients from $\mathcal{O}_{n-1}$. By means of $Az_n = (Az_n - B) + B$ we can rearrange P as polynomial in $Az_n - B$,

$$P = \sum_{j=0}^{d} (Az_n - B)^d P_j \quad (P_j \in \mathcal{O}_{n-1}).$$

We want to show $P(g(z)) = 0$ which is equivalent to $P_0(z) = 0$. But this is clear because $P_0 \in \mathfrak{P} \cap \mathcal{O}_{n-1} = \mathfrak{p}$. This completes the proof of the Lemma. □

We continue the proof of 8.2 and claim:

There exists r', $0 < r' \le r$, *such that*

$$|z_n| < \varepsilon \quad \text{for} \quad ||(z_1, \dots, z_{n-1})|| < r'.$$

One applies the Lemma 8.3 to a Weierstrass polynomial Q contained in $\mathfrak{P}$ and uses the standard argument of "continuity of roots".

The set X can be defined by a finite number of equations $P_1(z) = \cdots = P_m(z)$, $P_j \in \mathfrak{P}$, which converge in the polydisk of multiradius $(r, \dots, r, \varepsilon)$. By means of the division theorem ($P_j = A_jQ + B_j$) and the above lemma 8.3 we obtain $P_j(g(z)) = 0$ and hence $g(z) \in X$ for $||z|| < r'$ and suitable $r' \le r$. If we replace Y resp. X by their intersections with the polydisks of multiradius $(r', \dots, r')$ resp. $(r', \dots, r', \varepsilon)$ we obtain that f_0 is surjective and then that f_0 is bijective. The above formula for z_n shows that the inverse of f_0^{-1} is continuous. □

Lemma 8.2 should be interpreted as a result which states that the realization $X \to Y$ in case of the second alternative is close to a biholomorphic map. One could say that f is *bimeromorphic*. But there is a big problem up to now. In principle it could be that S equals the whole Y. The *Rückert Nullstellensatz* will show that this is not the case. This nullstellensatz will be the goal of the next section.

9. The Nullstellensatz

A subset $X \subset \mathbb{C}^n$ is called an analytic subset, if for every point $a \in X$ there exists an open neighborhood $a \in U \subset \mathbb{C}^n$ and finitely many holomorphic functions $f_1, \dots, f_m : U \to \mathbb{C}$ such that

$$X \cap U = \{ z \in U; \quad f_1(z) = \cdots = f_m(z) = 0. \}.$$

A pointed analytic set (X, a) is an analytic set with a distinguished point a. We are interested in local properties of X at a and can assume for this purpose that $a = 0$ is the origin. For an ideal $\mathfrak{a} \subset \mathcal{O}_n$ that is contained in the maximal ideal we considered the notion of a geometric realization X. This is an analytic set with distinguished point 0.

We associate to $\mathfrak{a}$ an ideal $\mathfrak{A} \subset \mathcal{O}_n$. A power series $P \in \mathcal{O}_n$ belongs to $\mathfrak{A}$ if there exists a small polydisk U around 0 such that P converges in U and such that p vanishes on $X \cap U$. The ideal $\mathfrak{A}$ is called the *vanishing ideal* of $(X, 0)$. It is a proper ideal, i.e. contained in the maximal ideal $\mathfrak{m}_n$. It is clear that the vanishing ideal $\mathfrak{A}$ of the realization only depends on $\mathfrak{a}$ and that $\mathfrak{a} \subset \mathfrak{A}$. We call $\mathfrak{A}$ the *saturation* of $\mathfrak{a}$.

The Radical of an Ideal

Let R be a ring. The radical $\operatorname{rad}\mathfrak{a}$ of an ideal $\mathfrak{a}$ is the set of all elements $a \in R$ such that a suitable power a^n, $n \geq 1$ is contained in $\mathfrak{a}$. It is easy to prove that $\operatorname{rad}\mathfrak{a}$ is an ideal which contains $\mathfrak{a}$. Furthermore $\operatorname{rad}\operatorname{rad}\mathfrak{a}=\operatorname{rad}\mathfrak{a}$. An ideal is called *radical ideal* is it coincides wit its radical. This equivalent with the property that $R/\mathfrak{a}$ is a reduced ring, i.e. a ring which contains no nilpotent elements differen form 0. Let R be a UFD-domain. A principal ideal Ra, $a \neq 0$ is a radical ideal if and only if a is square free. We are able to state and prove a fundamental result of local complex analysis:

9.1 The Rückert nullstellensatz. *The saturation* $\mathfrak{A}$ *of a proper ideal* $\mathfrak{a} \subset \mathcal{O}_n$ *is the radical of* $\mathfrak{a}$*,*

$$\mathfrak{A} = \operatorname{rad}\mathfrak{a}.$$

Proof. We want to reduce the nullstellensatz to prime ideals $\mathfrak{a}$. Prime ideals are of course radical ideals. The easiest way to do this reduction is to use a little commutative algebra, namely:

Every proper radical ideal in a noetherian ring is the intersection of finitely many prime ideals.

We use this and write the radical of our given ideal as intersection of prime ideals:

$$\operatorname{rad}\mathfrak{a} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m.$$

The saturation $\mathfrak{A}$ of $\mathfrak{a}$ is contained in the intersection of the of the saturations of the prime ideals. If we assume the nullstellensatz for prime ideals we obtain

$$\mathfrak{A} \subset \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m = \operatorname{rad}\mathfrak{a}$$

This implies $\mathfrak{A} = \operatorname{rad}\mathfrak{a}$ because the converse inclusion is trivial.

Now we can assume that $\mathfrak{P} := \mathfrak{a}$ is a prime ideal. We have to distinguish the two alternatives:

First alternative. The ideal $\mathfrak{P}$ is principal, $\mathfrak{P} = (P)$. The element P is a prime element in $\mathcal{O}_n$. In this case the nullstellensatz is a consequence of the theory of hypersurfaces (5.1).

Second alternative. $\mathfrak{P}$ is not a principal ideal. Then we can assume that $\mathfrak{P}$ is z_n-general, that the extension

$$\mathcal{O}_{n-1}/\mathfrak{p} \hookrightarrow \mathcal{O}_n/\mathfrak{P} \qquad (\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_{n-1}).$$

is module-finite and that the two rings have the same field of fractions. We make use of the geometric realization 8.2.

$$\begin{array}{rccc} f: & X & \longrightarrow & Y \\ & \cup & & \cup \\ f_0: & X - T & \xrightarrow{\sim} & Y - S\ . \end{array}$$

We indicated already in the last section that in principle S could be the whole Y before the nullstellensatz is known. But now we are in better situation. We can prove the nullstellensatz by induction on n and therefore assume:

The nullstellensatz is true for $\mathfrak{p}$.

From this we derive:

Let $P_0 \in \mathcal{O}_{n-1}$ *be a power series which converges in a small polydisk* V *around* 0 *and vanishes on* $(Y - S) \cap V$. *Then* P_0 *is contained in* $\mathfrak{p}$.

This is quite clear, because AP_0 (A as in 8.2) vanishes on $Y \cap V$. The nullstellensatz for $\mathfrak{p}$ gives $AP_0 \in \mathfrak{p}$ and get $P_0 \in \mathfrak{p}$ because $\mathfrak{p}$ is a prime ideal and A is not contained in $\mathfrak{p}$.

So in some sense the set S is negligible. The proof of the nullstellensatz now runs as follows. We take an element P from the saturation of $\mathfrak{P}$. The claim is $P \in \mathfrak{P}$. The idea is to use an integral equation

$$P^m + P_{m-1}P^{m-1} + \ldots + P_0 \in \mathfrak{P}, \quad P_i \in \mathcal{O}_{n-1}\ (0 \le i < m).$$

We take a minimal degree m. We distinguish two cases:

First case. P_0 is contained in $\mathfrak{p}$: Then

$$P \cdot (P^{m-1} + P_{m-1}P^{m-2} + \ldots + P_1) \in \mathrm{P}.$$

Because of the minimality of M the expression in the bracket is not contained in $\mathfrak{P}$. But $\mathfrak{P}$ is a prime ideal and we obtain $P \in \mathfrak{P}$ what we wanted to show.

Second case. P_0 is not contained in $\mathfrak{p}$: We know that P vanishes on X in a neighborhood of 0. We can assume that P vanishes on the whole X (use 8.1). Using the bijection $X - T \to Y - S$ we obtain that P_0 vanishes on $Y - S$. But as we have seen this implies $P_0 \in \mathfrak{p}$ which is a contradiction. This completes the proof of the nullstellensatz. □

We want to introduce the notion "thin at" which reflects that the set S is negligible in Y in a certain sense.

9.2 Definition. *Let* $Y \subset X \subset \mathbb{C}^n$ *be analytic sets and* $a \in Y$ *a distinguished point. We call* Y ***thin at*** a *if the following is true:*

> *If* f *is an analytic function on a neighborhood* $a \in U \subset \mathbb{C}^n$ *which vanishes on* $(X - Y) \cap U$ *then* f *vanishes on* X *in a (possibly smaller) neighborhood of* a.

So the essential part of the proof of the nullstellensatz was to show:

9.3 Remark. *Let* $\mathfrak{P} \subset \mathcal{O}_n$ *be a prime ideal with geometric realization* X. *Let* $P \in \mathcal{O}_n$ *be a power series which is not contained in* $\mathfrak{P}$. *Assume that* P *converges in a polydisk around* 0 *which contains* X. *Then* $Y := \{z \in X;\ P(z) = 0\}$ *is thin at* 0.

Again we get an obvious problem. On should expect that the property "thin at a" extends to a full neighborhood of a and that Y is thin in the usual topological sense in X (in this neighborhood). At the moment we are not able to prove this. This needs the principle of *coherence* which will be our next goal. Before we have developed this basic tool we must (and can) be content with the notion "thin at". But the reader should have in mind that "thin at" is in reality the same as thin in a neighborhood.

10. Oka's Coherence Theorem

We introduced already the ring

$$\mathbb{C}\{z_1 - a_1, \dots, z_n - a_n\}$$

of power series. Every holomorphic function f on an open neighborhood of a has a power series expansion in this ring. Instead of this one could consider the function $f(z-a)$ and take its power series expansion around 0. We have a natural injection

$$\mathbb{C}\{z_1 - a_1, \dots, z_{n-1} - a_{n-1}\} \longrightarrow \mathbb{C}\{z_1 - a_1, \dots, z_n - a_n\}$$

and can define the ring

$$\mathbb{C}\{z_1 - a_1, \dots, z_{n-1} - a_{n-1}\}[z_n - a_n] \subset \mathbb{C}\{z_1 - a_1, \dots, z_n - a_n\}$$

in an obvious way. An element P of this ring is called a Weierstrass polynomial, if it is normalized as polynomial in $z_n - a_n$ and if it has the property $P(a_1, \dots, a_{n-1}, z_n - a_n) = (z_n - a_n)^d$, where d is the degree of P in the variable $z_n - a_n$.

Let m be a natural number. We are interested in $\mathcal{O}_{U,a}$-submodules of the free module $\mathcal{O}_{U,a}^m$. In the case $m = 1$. Such a submodule is nothing else but an ideal and ideals are the modules in which we are interested. For technical reasons it is important to allow arbitrary m. Every submodule of $\mathcal{O}_{U,a}^m$ is finitely generated because the ring of power series is noetherian.

We are not only interested in individual modules but in *systems of modules*. This means that we assume that for every $a \in U$ a submodule

$$\mathcal{M}_a \subset \mathcal{O}_{U,a}^m$$

is given. We denote this system usually by a single letter,

$$\mathcal{M} = (\mathcal{M}_a)_{a \in U}.$$

If V is an open subset of U one defines in an obvious way the restricted system $\mathcal{M}|V := (\mathcal{M}_a)_{a \in V}$.

10.1 Definition. *A system*

$$\mathcal{M} = (\mathcal{M}_a)_{a\in U}, \quad \mathcal{M}_a \subset \mathcal{O}^m_{U,a},$$

is called ***finitely generated****, if there exist finitely many vectors of holomorphic functions*

$$f^{(j)} \in \mathcal{O}(U)^m \quad \textit{for} \quad 1 \le j \le k,$$

such that the $\mathcal{O}_a$-module $\mathcal{M}_a$ is generated by the germs

$$(f^{(1)})_a, \dots, (f^{(k)})_a.$$

The germs are taken of course componentwise.

10.2 Definition. *The system $\mathcal{M} = (\mathcal{M}_a)_{a\in U}$ is called* ***coherent****, if it is locally finitely generated, which means that every point $a \in U$ admits an open neighborhood $a \in V \subset U$ such that $\mathcal{M}|V$ is finitely generated.*

Let p, q be natural numbers and let

$$F = \begin{pmatrix} F_{11} & \dots & F_{1p} \\ \vdots & & \vdots \\ F_{q1} & \dots & F_{qp} \end{pmatrix}$$

by a matrix of holomorphic functions on U. We can consider the $\mathcal{O}(U)$-linear map

$$F : \mathcal{O}(U)^p \longrightarrow \mathcal{O}(U)^q$$

which is defined by

$$Ff := g; \quad g_i := \sum_{j=1}^{p} F_{ij} f_j \quad (1 \le i \le q).$$

As the notation indicates we identify the matrix and the linear map. For every point $a \in U$ we can consider the germ $F_a = ((F_{ik})_a)$ and the corresponding map

$$F_a : \mathcal{O}^p_{U,a} \to \mathcal{O}^q_{U,a}.$$

10.3 Oka's coherence theorem. *Let*

$$F : \mathcal{O}(U)^p \to \mathcal{O}(U)^q \quad (U \subset \mathbb{C}^n \text{ open})$$

be an $\mathcal{O}(U)$–linear map. The system

$$\mathcal{M} = (\mathcal{M}_a)_{a\in U} \quad \mathcal{M}_a := \operatorname{kernel}(F_a)$$

is coherent.

The proof will be given in three steps:
First step, reduction to the case $q = 1$. This will be done by induction on q. So let's assume $q > 1$ and that the theorem is proved for $q-1$ instead of q. Let $a_0 \in U$ be a distinguished point. We want to prove that $\mathcal{M}$ is finitely generated in a neighborhood of a. For this purpose we can replace U by a smaller neighborhood of a_0. We consider the two projections

$$\mathcal{O}(U)^q = \mathcal{O}(U)^{q-1} \times \mathcal{O}(U) \quad \begin{array}{l} \xrightarrow{\alpha} \mathcal{O}(U) \\ \xrightarrow{\beta} \mathcal{O}(U)^{q-1}. \end{array}$$

By the induction hypothesis, applied to

$$\alpha \circ F : \mathcal{O}(U)^p \to \mathcal{O}(U)^{q-1}$$

we can assume that there exist a finite system

$$A^{(1)}, \dots, A^{(m)} \in \mathcal{O}(U)^p,$$

such that the germs $A_a^{(1)}, \dots, A_a^{(m)}$ generated the kernel of $(\alpha \circ F)_a$ for each point $a \in U$. Now we consider the linear map

$$G : \mathcal{O}(U)^m \longrightarrow \mathcal{O}(U)^p, \quad (f_1, \dots, f_m) \longmapsto f_1 A^{(1)} + \dots + f_m A^{(m)},$$

and compose it with the projection β,

$$\beta \circ A : \mathcal{O}(U)^m \to \mathcal{O}(U).$$

We assumed that the case $q = 1$ is proved and can therefore assume that there exists a finite system

$$B^{(1)}, \dots, B^{(l)} \in \mathcal{O}(U)^m,$$

whose germs in an arbitrary point $a \in U$ generate $(\beta \circ A)_a$. It is easy to see that the germs of

$$C^{(i)} = G(B^{(i)}) \in \mathcal{O}(U)^p \quad (1 \le i \le m).$$

generate the kernel of our original F_a. Thus we have show:
If Oka's theorem is true for $q = 1$ in a given dimension n then it is true for all q in this dimension.

Second step. The proof of Oka's theorem rests on *Oka's Lemma*, which is a lemma for an individual ring of power series (not a system). Before we cam formulate it, we need a notation:

$$\mathcal{O}_{n-1}[z_n : m] = \left\{ P \in \mathcal{O}_{n-1}[z_n]; \quad \deg_{z_n} P < m \right\}.$$

This is a free module over $\mathcal{O}_{n-1}$ with basis $1, z_n, \dots, z_n^{m-1}$,

$$\mathcal{O}_{n-1}[z_n : m] \cong \mathcal{O}_{n-1}^m.$$

10.4 Oka's Lemma. *Let*

$$F : \mathcal{O}_n^p \to \mathcal{O}_n$$

be a $\mathcal{O}_n$-linear map and let K be its kernel.

Assumption. *The components of the matrix F are normalized polynomials in $\mathcal{O}_{n-1}[z_n]$ of degree $< d$ (in the variable z_n).*

We consider the restriction of F

$$\mathcal{O}_{n-1}[z_n : m]^p \to \mathcal{O}_{n-1}[z_n : m + d]$$

and denote by K_m its kernel.

Claim. *The $\mathcal{O}_n$-module K is generated by K_m for $m \geq 3d$.*

Proof. In a first step we assume that the first component of the map $F = (F_1, \dots, F_p)$ is aWeierstrass polynomial (and not only a normalized polynomial). We will prove Oka's Lemma in this case with the better bound $2d$ instead of $3d$. Let $G = (G_1, \dots, G_p) \in K$ be an element of the kernel. The division theorem gives

$$G = F_1 A + B, \quad A \in \mathcal{O}_n^p, \quad B \in \mathcal{O}_{n-1}[z_n : d]^p.$$

We notice that the elements

$$H^{(j)} = (-F_j, 0, \dots, 0, F_1, 0, \dots, 0) \quad (1 < j \leq p)$$

are contained in the kernel. The trivial formula

$$F_1 A = \sum_{j=2}^{p} A_j H^{(j)} + (A_1 F_1 + \dots + A_p F_p, 0, \dots, 0)$$

shows that besides G also the element $H := B + (A_1 F_1 + \dots + A_p F_p, 0, \dots, 0)$ is contained in the kernel, i.e.

$$F_1(B_1 + A_1 F_1 + \dots + A_p F_p) + F_2 B_2 + \dots + F_p B_p = 0.$$

This equation shows

$$F_1(A_1 F_1 + \dots + A_p F_p) \in \mathcal{O}_{n-1}[z_n : 2d].$$

Using again that F_1 is a Weierstrass polynomial we obtain

$$A_1 F_1 + \dots + A_p F_p \in \mathcal{O}_{n-1}[z_n : 2d].$$

Now we see that the components of H are contained in K_{2d}. The trivial formula

$$G = \sum_{j=2}^{p} A_j H^{(j)} + H$$

finally shows that G is contained in the module which generated by the $H^{(j)}$ and H, which are elements of K_{2d}.

Now we treat the general case where F_1 is not necessarily a Weierstrass polynomial. We apply the preparation theorem

$$F_1 = Q \cdot U, \quad Q \text{ Weierstrass polynomial}, \quad U \text{ unit in } \mathcal{O}_n.$$

We are interested in the solutions of the equation $F_1P_1 + F_2P_2 + \ldots + F_pP_p = 0$ or equivalently

$$Q\tilde{P}_1 + F_2P_2 + \ldots + F_pP_p = 0 \quad (\tilde{P}_1 = UP_1).$$

Since Q is a Weierstrass polynomial, this system is generated by solutions of z_n-degree $< 2d$. But UP_1 is of degree $< 3d$ if P_1 is of degree $< 2d$. This completes the proof of Oka's lemma.

Third step, the proof of Oka's theorem in the case $q = 1$.
The proof now is given by induction on n. As beginning of the induction can be taken the trivial case $n = 0$. We have to consider a $\mathcal{O}(U)$-linear map

$$F : \mathcal{O}(U)^p \longrightarrow \mathcal{O}(U),$$

which is given by a vector $(F_1, \ldots, F_P)$. We want to show that the kernel system is finitely generated in a neighborhood of a given point and can assume that this point is the origin 0 and that U is a polydisk with center 0. After a suitable linear coordinate transformation we can assume that the power series expansions of $F_1, \ldots, F_p$ in the origin are z_n-general. By the preparation theorem we can assume that the all are Weierstrass polynomials. If we consider the power series expansions in other points $a \in U$ we still have normalized polynomials

$$(F_i)_a \in \mathbb{C}\{z_1 - a_1, \ldots, z_{n-1} - a_{n-1}\}[z_n - a_n].$$

(but usually not Weierstrass polynomials). The degree of all those polynomials is bounded by a suitable number d. We write U in the form

$$U = V \times (-r, r) \quad (V \subset \mathbb{C}^{n-1})$$

and denote by $\mathcal{O}(V)[z_n : m]$ the set of all holomorphic functions on U which are polynomials in z_n of degree $< m$ with coefficients independent of z_n. This is a free $\mathcal{O}(V)$ module,

$$\mathcal{O}(V)[z_n : m] \cong \mathcal{O}(V)^m.$$

Our given map F induces an $\mathcal{O}(V)$-linear map

$$\begin{array}{ccc} \mathcal{O}(V)[z_n : m]^p & \longrightarrow & \mathcal{O}(V)[z_n : m + d] \\ \| & & \| \\ \mathcal{O}(V)^{mp} & \longrightarrow & \mathcal{O}(V)^{m+d}. \end{array}$$

From the induction hypothesis we can assume that the kernel of this map is finitely generated. From Oka's lemma we obtain that the kernel system of F is finitely generated. Oka's theorem is proved. □

Some Important Properties of Coherent Systems

The following trivial property of coherent systems will be used frequently:

10.5 Remark. *Let $\mathcal{M}, \mathcal{N}$ be two coherent systems on an open set $U \subset \mathbb{C}^n$. Assume $\mathcal{M}_{a_0} \subset \mathcal{N}_{a_0}$ for a distinguished point a_0. Then $\mathcal{M}_a \subset \mathcal{N}_a$ in a complete neighborhood of a_0 holds.*

Corollary. *$\mathcal{M}_{a_0} = \mathcal{N}_{a_0}$ implies $\mathcal{M}|V = \mathcal{N}|V$ for an open neighborhood V of a_0.*

Another trivial observation is

10.6 Remark. *Let*

$$F : \mathcal{O}(U)^m \to \mathcal{O}(U)^l \quad (U \subset \mathbb{C}^n \text{ open})$$

be an $\mathcal{O}(U)$ linear map and let

$$\mathcal{M} = (\mathcal{M}_a)_{a \in U}, \quad \mathcal{M}_a \subset \mathcal{O}^m_{U,a},$$

be a coherent system. The the image system

$$\mathcal{N} = (\mathcal{N}_a)_{a \in U}, \quad \mathcal{N}_a := F_a(\mathcal{M}_a) \subset \mathcal{O}^l_{U,a}.$$

is coherent. (The same is true already for "finitely generated" instead for "coherent".)

The next result is not trivial, it uses Oka's theorem:

10.7 Proposition. *Let $\mathcal{M}, \mathcal{N}$ be two coherent systems on the open set $U \in \mathbb{C}^n$,*

$$\mathcal{M}_a, \mathcal{N}_a \subset \mathcal{O}^m_{U,a} \quad (a \in U).$$

The the intersection system $\mathcal{M} \cap \mathcal{N}$ which is defined by

$$(\mathcal{M} \cap \mathcal{N})_a := \mathcal{M}_a \cap \mathcal{N}_a \quad (a \in U)$$

is coherent too.

Proof. The idea is to write the intersection as a kernel. We explain the principle for individual modules $M, N \subset R^n$ of finite type over a ring R instead of a system: We can write M resp. N as image of a linear map $F : R^p \to R^m$ resp. $G : R^q \to R^m$. We denote by K the kernel of the linear map

$$R^{p+q} \longrightarrow R^m, \quad (m, n) \longmapsto F(m) - G(n).$$

The image of K under the map

$$R^{p+q} \longrightarrow R^m, \quad (m, n) \longmapsto F(m).$$

is precisely the intersection $M \cap N$. The proof of 10.7 is clear now. On "reads" M, N as coherent systems. By Oka's theorem K now stands for a coherent system and the image $M \cap N$ is is coherent by 10.6. □

10.8 Proposition. *Let*

$$F : \mathcal{O}(U)^m \to \mathcal{O}(U)^l \quad (U \subset \mathbb{C}^n \text{ open})$$

be an $\mathcal{O}(U)$*-linear map and let*

$$\mathcal{N} = (\mathcal{N}_a)_{a\in U}, \quad \mathcal{N}_a \subset \mathcal{O}^l_{U,a},$$

be a coherent system. The inverse image system

$$\mathcal{M} = (\mathcal{M}_a)_{a\in U}, \quad \mathcal{M}_a := F_a^{-1}(\mathcal{N}_a) \subset \mathcal{O}^m_{U,a},$$

is coherent.

In the special case $\mathcal{N} = 0$ this is Oka's theorem.

Proof. We explain again the algebra behind this result. Let $F : R^m \to R^l$ be a R-linear map and $N \subset R^l$ be a R-module of finite type. We assume that $F(R^m) \cap N$ is finitely generated. Then there exists a finitely generated submodule $P \subset R^m$ such that $F(P) = F(R^m) \cap N$. We also assume that the kernel K of F is finitely generated. It is easily proved that $F^{-1}(N) = P + K$ and we obtain that the inverse image is finitely generated. Thees argument works in an obvious way for coherent systems and gives a proof of 10.8.

□

11. Rings of Power Series are Henselian

The fact that power series are henselian rings can be considered as an abstract formulation of the Weierstrass theorems. We don't need the notion of a henselian ring to formulate this result, but for sake of completeness we give the definition of this property.

A local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$ is called a ***henselian ring*** *if the following is true:*

Let $P \in R[X]$ be a normalized polynomial. We denote by p its image in $k[X]$. Assume that $a, b \in k[X]$ are two coprime normalized polynomials with the property $p = ab$. Then there exist normalized polynomials $A, B \in R[X]$ with cosets a, b such that $P = AB$.

We recall that the polynomial ring in one variable over a field is a principal ideal ring. Therefore two polynomials a, b are coprime if and only if they generate the unit ideal $k[X]$.

We consider the special case where k is algebraically closed. Then every normalized polynomial $p \in k[X]$ is a product of linear factors, if $b_1, \dots b_m$ are the pairwise distinct zeros and $d_1, \dots, d_m$ their multiplicities then

$$p(X) = \prod_{j=1}^{m} (X - b_j)^{d_j}.$$

This is a decomposition of p into m pairwise coprime factors. So the henselian property means in this case:

There exists a decomposition $P = P_1 \cdots P_m$ of P as product of normalized polynomials such that $p_j(X) = (X - b_j)^{d_j}$ where p_j denotes the image of P_j in $k[X]$.

We want to show that the ring of power series $\mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$ is henselian. The residue field $\mathcal{O}_n/\mathfrak{m}_n$ can be identified with $\mathbb{C}$ and the projection $\mathcal{O}_n \to \mathcal{O}_n/\mathfrak{m}_n$ corresponds to the map

$$\mathbb{C}\{z_1, \dots, z_n\} \longrightarrow \mathbb{C}, \quad P \longmapsto P(0).$$

We have to consider the polynomial ring over $\mathbb{C}\{z_1, \dots, z_n\}$. Therefore we need a letter for the variable. To stay close to previous notations we consider $\mathcal{O}_{n-1}$ instead of $\mathcal{O}_n$ and formulate the Hensel property for this ring. Then we have the letter z_n free for the variable of the polynomial ring. After this preparation we see that the following theorem expresses precisely that the rings of power series are henselian.

11.1 Theorem. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be a normalized polynomial of degree $d > 0$ and let β be a zero with multiplicity d_β of the polynomial $z \longmapsto P(0, \dots, 0, z)$. Then there exists a unique normalized polynomial $P^{(\beta)} \in \mathcal{O}_{n-1}[z_n]$ which divides P and such that*

$$P^{(\beta)}(0, \dots, 0, z) = (z - \beta)^{d_P}.$$

Moreover

$$P = \prod_{P(0,\dots,0,\beta)=0} P^{\beta}.$$

(Here β runs through the zeros of $z \mapsto P(0, \dots, 0, z)$.)

For the proof of this theorem we need three lemmas:

11.2 Lemma. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be an **irreducible** normalized polynomial with the property $P(0) = 0$. Then P is a Weierstrass polynomial.*

Proof. By the preparation theorem we have $P = UQ$ with a Weierstrass polynomial and a unit U. We know that U is a polynomial. But P is irreducible. We obtain $U = 1$ and $P = Q$. □

11.3 Lemma. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be an irreducible normalized polynomial of degree $d > 0$. Then*

$$P(0, \dots, 0, z) = (z - \beta)^d$$

with a suitable complex number β.

Proof. Let β be a zero of the polynomial $z \mapsto P(0, \dots, 0, z)$. We rearrange P as polynomial in $z_n - \beta$ and obtain by 11.2 a Weierstrass polynomial in $\mathcal{O}_{n-1}[z_n - \beta]$. □

11.4 Lemma. *Let P, Q be two normalized polynomials in $\mathcal{O}_{n-1}[z]$. The polynomials $p(z) = P(0, \dots, 0, z)$, $q(z) = P(0, \dots, 0, z)$ are assumed to be coprime. (This means that have no common zero.) Then P and Q generate the unit ideal ,*

$$(P, Q) = \mathcal{O}_{n-1}[z_n].$$

Proof. The proof will use the theorem of Cohen Seidenberg: The ring polynomial in one variable over a field is a principal ideal ring. Therefore

$$(p, q) = \mathbb{C}[z].$$

We obtain that P and Q together with the maximal ideal $\mathfrak{m}_{n-1} \subset \mathcal{O}_{n-1}$ generate the unit ideal,

$$(P, Q, \mathfrak{m}_{n-1}) = \mathcal{O}_{n-1}[z_n].$$

Now we consider the natural homomorphism

$$\mathcal{O}_{n-1} \longrightarrow \mathcal{O}_{n-1}[z_n]/(P, Q).$$

This ring extension is module-finite. This follows immediately if one applies the Euclidean algorithm to one of the polynomials P, Q. The theorem of Cohen Seidenberg deals with module finite ring extensions. We give here a formulation which is not the standard one but usually a lemma during the proof:

Let A be a noetherian local ring and $A \to B$ a ring homomorphism such that B is an A-module of finite type. We assume that B is different from the zero ring ($1_B \neq 0_B$). Then there exists a proper ideal in B which contains the image of the maximal ideal of A.

(One can take the ideal which is generated by the image of the maximal ideal of A. The problem is to show that this is different form B.)

We continue the proof of 11.4. We want to show that P and Q generate the unit ideal. We give an indirect argument and assume that this is not the case. Then by Cohen Seidenberg we obtain that the image of $\mathfrak{m}_{n-1}$ in $\mathcal{O}_{n-1}[z_n]/(P, Q)$ does not generate the unit ideal. This means the same that $(P, Q, \mathfrak{m}_{n-1})$ is not the unit ideal, which gives a contradiction. □

Proof of theorem 11.1. Let P be a normalized polynomial of degree $d > 0$ in $\mathcal{O}_{n-1}[z_n]$. We decompose P into a product of irreducible normalized polynomials

$$P = P_1 \cdot \cdot P_m.$$

From 11.3 we obtain

$$P_i(0, \dots, 0, z) = (z - \beta_i)^{d_i} \quad (1 \leq i \leq m).$$

The numbers β_i are the zeros of the polynomial $P(0, \dots, 0, z)$. There is no need that the β_i are pairwise distinct. But we can collect the P_i for a fixed zero and multiply them together. □

We need a further little lemma from algebra:

Let R be a UFD-domain and a, b two coprime elements. The natural homomorphism

$$R/(ab) \longrightarrow R/(a) \times R/(b)$$

is injective. It is an isomorphism if a and b generate the unit ideal.

We apply this to thorem 11.1 and obtain:

11.5 Proposition. *(We use the notations of 11.1.) The natural homomorphism*

$$\mathcal{O}_{n-1}[z_n]/(P) \xrightarrow{\sim} \prod_{\beta} \mathcal{O}_{n-1}[z_n]/(P^{(\beta)})$$

is an isomorphism.

We recall the the P^{β} are Weierstrass polynomials in the ring $\mathcal{O}_{n-1}[z_n - \beta]$. From the division theorem we obtain

$$\mathcal{O}_{n-1}[z_n]/(P^{(\beta)}) = \mathbb{C}\{z_1, \dots, z_{n-1}, z_n - \beta\}/(P^{(\beta)}).$$

Now we can conclude from the Hensel property the following generalization of the division theorem for normalized polynomials instead of Weierstrass polynomials:

11.6 Proposition. *(We use the notations of 11.1.) The natural homomorphism*

$$\mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]/(P) \xrightarrow{\sim} \prod_{\beta} \mathbb{C}\{z_1, \dots, z_{n-1}, z_n - \beta\}/(P^{(\beta)})$$

is an isomorphism. This remains true if one replaces $P^{(\beta)}$ by the power series expansion of P in $(0, \dots, 0, \beta)$.

The last statement uses the decomposition $P = \prod_{\gamma} P^{(\gamma)}$ and the fact that all factors besides the considered $P^{(\beta)}$ do not vanish at $(0, \dots, 0, \beta)$ and hence define units in $\mathbb{C}\{z_1, \dots, z_{n-1}, z_n - \beta\}$.

12. A Special Case of Grauert's Projection Theorem

Let $U \subset \mathbb{C}^n$ be an open domain. We consider a coherent system of ideals

$$\mathfrak{a} = (\mathfrak{a}_a)_{a\in U}, \quad \mathfrak{a}_a \in \mathcal{O}_{U,a}.$$

12.1 Definition. *The **support** of a coherent system $\mathfrak{a}$ of ideals is the of all $a \in U$ such that $\mathfrak{a}_a$ is different from the unit ideal.*

If the system is finitely generated, let's say by $f_1, \dots, f_m$ then the support is nothing else but the set of common zeros as follows from the nullstellensatz. So we see:

12.2 Remark. *The support of a coherent system of ideals is a closed analytic subset of U.*

Conversely analytic sets can be obtained at least locally as the support of coherent systems.

Now we assume that $U = V \times \mathbb{C}$ with a polydisk $V \subset \mathbb{C}^{n-1}$. We consider the *projection*

$$\pi : U \longrightarrow V, \quad (z, z_n) \longmapsto Z.$$

It may happen that the image of an closed analytic set $X \subset U$ in V is a closed analytic set $Y \subset V$ but this must be not the case. We want to give a sufficient condition where it is the case. The idea is to consider rather coherent systems than analytic sets. So let's assume that X is the support of the coherent system $\mathfrak{a}$. We expect that in good situations Y is the support of certain coherent system on V. It's not difficult to guess what this system should be.

12.3 Defintion. *Let $V \subset \mathbb{C}^{n-1}$ be a polydisk and $\mathfrak{a}$ a coherent system of ideals on $U \times V$. We define for a point $b \in V$ the ideal*

$$\mathfrak{b}_b := \mathcal{O}_{V,b} \cap \bigcap_{a\in U,\ \pi(a)=b} \mathfrak{a}_a.$$

and call $\mathfrak{b} := (\mathfrak{b}_b)_{b\in V}$ the projected system.

We recall that the projection π defines a natural inclusion $\mathcal{O}_{V,b} \hookrightarrow \mathcal{U}_{U,a}$ for all a, b with $\pi(a) = b$.

Projections of analytic sets of the above kind can be very bad and similarly the projected systems can be bad and need not to be coherent. But there exist "good" projections:

12.4 Theorem. *Assume that $V \subset \mathbb{C}^{n-1}$ is a polydisk and that $\mathfrak{a}$ is a coherent system on $U = V \times \mathbb{C}$, which can be generated by finitely many functions $f_1, \dots, f_m \in \mathcal{O}(V)[z_n]$. We assume that $P := f_1$ is a normalized polynomial. Then the projected system $\mathfrak{b}$ is coherent.*

Additional remark. *If X is the support of $\mathfrak{a}$, then $Y = \pi(X)$ is the support of $\mathfrak{b}$. Especially $\pi(X)$ is a closed analytic subset of V. The map $\pi : X \to Y$ has finite fibres.*

(The truth is that the projection $X \to V$ is a *proper analytic map with finite fibres*. The above theorem can be considered as a special case of the deep projection theorem of Grauert.)

Proof of 12.4. The proof will use Oka's coherence theorem and the Hensel property of rings of power series. The ideal $\mathfrak{a}_a$ is the unit ideal if $P(a) \neq 0$. For every $b \in V$ the number of $a \in U$ with $\pi(a) = b$ and $P(a) = b$ is finite. Therefore $\mathfrak{b}_b$ is the intersection of *finitely many* ideals:

$$\mathfrak{b}_b := \mathcal{O}_{V,b} \cap \bigcap_{\pi(a)=b,\ P(a)=0} \mathfrak{a}_a.$$

The ideal $\mathfrak{b}_b$ contains 1 if and only this is the case for all $\mathfrak{a}_a$, $\pi(a) = b$. We see that the additional remark will follow automatically from the coherence of $\mathfrak{b}$.

We want to consider the ideal

$$\mathcal{I}_b \subset \mathcal{O}_{V,b}[z_n]/(P_b),$$

which is generated by the $f_1, \dots, f_n$ (more precisely by their images). We have to consider this ideal also as $\mathcal{O}_{V,b}$-module. It is of finite type over this ring, more precisely it is generated as module over this ring by the elements

$$f_i z_n^j \qquad (1 \le i \le m, \quad 0 \le j < d).$$

This uses the Euclidean algorithm, which gives an isomorphism

$$\mathcal{O}_{V,b}^d \xrightarrow{\sim} \mathcal{O}_{V,b}[z_n]/(P_b).$$

A vector $(H_0, \dots, H_d)$ is mapped to $\sum H_j z_n^j$. We take the inverse image of $\mathcal{I}_b$ and get a submodule

$$\mathcal{M}_b \subset \mathcal{O}_{V,b}^d.$$

From the given generators we see that the system $\mathcal{M} = (\mathcal{M}_b)_{b \in V}$ is finitely generated hence coherent on V. This system is closely related to our projected ideals $\mathfrak{b}_b$:

Claim. *The projected ideal $\mathfrak{b}_b$ is precisely the inverse image of $\mathcal{I}_b$ with respect to the natural map*

$$\mathcal{O}_{V,b} \longrightarrow \mathcal{O}_{V,b}[z_n]/(P_b).$$

We assume for a moment that the claim is proved. Then $\mathfrak{b}$ can be considered as inverse image of the coherent system $\mathcal{M}$. But Oka's coherence theorem (10.8) then implies that $\mathfrak{b}$ is coherent. So it remains to prove the claim:

Proof of the claim. In this proof the Hensel property of rings of power series will enter. We have to make further use of our normalized polynomial $P \in \mathcal{O}(V)[z_n]$,

$$P = z_n^d + P_{d-1} z_n^{d-1} + \ldots + P_0.$$

Its coefficients P_j are holomorphic functions on V. We will use the power series expansion $(P_j)_b \in \mathcal{O}_{V,b}$ for varying points $b \in V$. We have to consider the image of P in $\mathcal{O}_{V,b}[z_n]$,

$$P_b = z_n^d + (P_{d-1})_b z_n^{d-1} + \ldots + (P_0)_b \in \mathcal{O}_{V,b}[z_n].$$

We also have to use the ring $\mathcal{O}_{V,b}[z_n]/(P_b)$. The Hensel property of rings of power series gave us important information for this ring. Applying 11.6 we obtain a natural isomorphism*)

$$\mathcal{O}_{V,b}[z_n]/(P_b) \xrightarrow{\sim} \prod_\beta \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)}).$$

Here β runs over the zeros $P(b,\beta) = 0$. The elements $P_b^{(\beta)} \in \mathcal{O}_{V,b}$ come from the "Hensel decomposition"

$$P_b = \prod_\beta P_b^{(\beta)}, \qquad P_b^{(\beta)}(b, z_n) = (z_n - \beta)^{d_\beta}.$$

We determine the image of $\mathcal{I}_b$ under this isomorphism. For this we use the simple fact that an ideal $\mathfrak{c} \subset A \times B$ in the cartesian product of two rings always is the direct product of two ideals, $\mathfrak{c} = \mathfrak{a} \times \mathfrak{b}$, where $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ are the projections of $\mathfrak{c}$. Using this and the definition (12.4) of $\mathfrak{a}$ we see:

The image of the ideal $\mathcal{C}_b$ in $\prod_\beta \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)})$ is the direct product of the ideals $\bar{\mathfrak{a}}_{(b,\beta)}$, which mean the images of $\mathfrak{a}_{(b,\beta)}$ in $\mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)})$.

We have to determine the inverse image of this ideal under the natural map

$$\mathcal{O}_{V,b} \longrightarrow \prod_\beta \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)}).$$

The claim states that this inverse image is the projection ideal $\mathfrak{b}_b$. But this inverse image is the intersection of the inverse images of $\bar{\mathfrak{a}}_{(b,\beta)}$ under

$$\mathcal{O}_{V,b} \longrightarrow \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)}).$$

But $P_b^{(\beta)}$ is contained in $\mathfrak{a}_{(b,\beta)}$ (s. 11.6). Therefore it is the same to take the inverse image of $\mathfrak{a}_{(b,\beta)}$ under

$$\mathcal{O}_{V,b} \longrightarrow \mathcal{O}_{U,(b,\beta)}.$$

This is $\mathcal{O}_{V,b} \cap \mathfrak{a}_{(b,\beta)}$ and the intersection of all of then is $\mathfrak{b}_b$. □

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*) In 11.6 the result has been formulated only for $b = 0$ which is no loss of generality.

13. Cartan's Coherence Theorem

There is a second basic coherence theorem. Oka contributes this theorem to Cartan, but as Grauert and Remmert pointed out in there book „Coherent analytic sheaves", the essential parts of the proof are already in Oka's papers. We give three different formulations for Cartan's theorem:

13.1 Cartan's coherence theorem. *Let $\mathfrak{a} = (\mathfrak{a}_a)_{a\in U}$ be a coherent system of ideals on an open domain $U \subset \mathbb{C}^n$. Then its radical*

$$\operatorname{rad}\mathfrak{a} := (\operatorname{rad}\mathfrak{a}_a)_{a\in U}$$

is coherent too.

let $X \subset U$ be a closed analytic subset. The vanishing ideal system $\mathfrak{A}_X$ is the system of ideals $\mathfrak{A}_a$, $a \in U$ which consists of all elements from $\mathcal{O}_{U,a}$, which vanish in a small neighborhood of a on X. If a is not in X then $\mathfrak{A}_a = \mathcal{O}_{U,a}$. For this one has to use that X is closed in U. A second form of Cartan's theorem is:

13.2 Cartan's coherence theorem. *Let $X \subset U$ be a closed analytic subset of an open set $U \subset \mathbb{C}^n$. The vanishing ideal system $\mathfrak{A}$ is coherent.*

To see the equivalence one has to have in mind that the support of a coherent ideal system $\mathfrak{a}$ is a closed analytic set and that by the nullstellensatz the radical of $\mathfrak{a}$ is the complete vanishing ideal system $\mathfrak{A}$. One also has to use the trivial fact the every analytic set locally is the support of a coherent system. Another formulation is

13.3 Cartan's coherence theorem. *Let $\mathfrak{a}$ be coherent system of ideals. The set of all points a such that $\mathfrak{a}_a = \operatorname{rad}\mathfrak{a}_a$ is open.*

We show that 13.3 implies 13.2. Let $a \in U$ a point. The ideal $\operatorname{rad}\mathfrak{a}_a$ is finitely generated. Therefore there exists a coherent system $\mathfrak{b}$ on an open neighborhood $a \subset V \subset U$ auch that $\mathfrak{b}_a = \operatorname{rad}\mathfrak{a}_a$ and $\mathfrak{a}_b \subset \mathfrak{b}_b \subset \operatorname{rad}\mathfrak{a}_b$. Now 13.3 implies that in a full neighborhood $\mathfrak{b}_b = \operatorname{rad}\mathfrak{a}_b$. The conclusion 13.2 $\Rightarrow$ 13.3is also clear. One uses the fact that two coherent systems which agree in a point agree in a full neighborhood.

The rest of this section is dedicated the proof of Cartan's theorem. We need some preparations:

In a first step we give a reduction. We can assume that the origin is contained U and that $\mathfrak{a}_0 = \operatorname{rad}\mathfrak{a}_0$. We have to prove that $\mathfrak{a}_a = \operatorname{rad}\mathfrak{a}_a$ in a full neighborhood of 0. We want to show that it is enough to treat the case of a prime ideal $\mathfrak{a}_0$. For this we use again the fact that any reduced ideal is the intersection of finitely many prime ideals. Because any ideal is finitely generated we can find (in a small neighborhood of 0) coherent systems $\mathfrak{a}^{(1)}, \ldots, \mathfrak{a}^{(m)}$ such that

$$\mathfrak{a}_0 = \mathfrak{a}_0^{(1)} \cap \ldots \cap \mathfrak{a}_0^{(m)}.$$

From our assumption we know that the ${}^{(j)}_{a}$ are reduced (in a small neighborhood). We also know from Oka's coherence theorem that the intersection system $\mathfrak{a}^{(1)} \cap \ldots \cap \mathfrak{a}^{(m)}$ is coherent. Tis intersection system and $\mathfrak{a}$ agree in the origin and hence in a full neighborhood,

$$\mathfrak{a}_a = \mathfrak{a}_a^{(1)} \cap \ldots \cap \mathfrak{a}_a^{(m)}.$$

Using the trivial fact that the intersection of reduced ideals is reduced we obtain that the $\mathfrak{a}_a$ are reduced.

From now on we assume that $0 \in U$ and that

$$\mathfrak{P} := \mathfrak{a}_0$$

is a prime ideal. We will show that $\mathfrak{a}_a$ is reduced in a neighborhood of 0. We need some preparations for the proof:

An element a of a ring R is called non-zero-divisor if multiplication with a

$$R \longrightarrow R, \quad x \longmapsto ax,$$

is injective.

13.4 Lemma. *Let $\mathfrak{a}$ be a coherent system on an open set $U \subset \mathbb{C}^n$ and let $f \in \mathcal{O}(U)$ be an analytic function on U. The set of all points $a \in U$ such that the germ f_a is a non-zero-divisor in $\mathcal{O}_{U,a}$ is open*

Proof. We denote the map "multiplication by a" by

$$m_f : \mathcal{O}_{U,a} \longrightarrow \mathcal{O}_{U,a}.$$

The element f_a is non-zero-divisor if and only if

$$m_f^{-1}(\mathfrak{a}_a) = \mathfrak{a}_a.$$

From Oka's coherence theorem we know that the system $\big(m_f^{-1}(\mathfrak{a}_a)\big)_{a\in U}$ is coherent. The coincidence set of two coherent systems is open. □

After this preparations the proof of Cartan's theorem runs as follows. Recall that $0 \in U$ and that $\mathfrak{P} = \mathfrak{a}_0$ is a prime ideal. We have to show that $\mathfrak{a}_a$ is reduced in a full neighborhood of 0. We distinguish the two "alternatives".

1. *Alternative.* $\mathfrak{P} = (P)$ is a principal ideal. The element P is a prime element, especially square free. The theory of the *discrimant* gave us that there exists a small polydisk around 0 in which P converges and such that

P_a is square free in this polydisk. Coherence gives us that $\mathfrak{a}_a = (P_a)$ in a full neighborhood. But a principal ideal generated ba a square free element is reduced. What we see that in the case of hypersurfaces the properties of the discriminant imply Cartan's theorem. □

2. *Alternative*. This case is more involved. We will have to use the special case od Grauert's projection theorem. As usual we can assume that $\mathfrak{P} = gota_0$ is z_n-general and that

$$\mathcal{O}_{n-1}/\mathfrak{p} \longrightarrow \mathcal{O}_n/\mathfrak{P} \qquad (gotp := \mathcal{O}_{n-1} \cap \mathfrak{P})$$

have the same field of fractions. The ideal $\mathfrak{P}$ is finitely generated,

$$\mathrm{P} = (Q_1, Q_2, \dots, Q_m).$$

We can assume that $Q := Q_1$ is a Weierstrass polynomial and then by the division theorem that all Q_i are polynomials over $\mathcal{O}_{n-1}$. We can take U in the form $U = V \times (-r, r)$, where $V \subset \mathbb{C}^n$ is a polydisk around 0. We can assume that the coefficients of the Q_j converge in V and that the zeros of the polynomial $z \mapsto Q(b, z)$ for all $b \in V$ have absolute value $< r$. From the special case of Grauert's projection theorem we obtain that the system

$$\mathfrak{b}_b = \mathcal{O}_{V,b} \cap \bigcap_{a=(b,\beta),\ Q(a)=0} \mathfrak{a}_a$$

is coherent on V. Because Q is a Weierstrass polynomial we have

$$\mathrm{b}_0 = \mathfrak{p}.$$

We want to prove Cartan's theorem by induction on n. Therefore we can assume that alle the projected ideals $\mathfrak{b}_b$ are reduced. We will make use of the natural homomorphism

$$\mathcal{O}_{V,b}/\mathfrak{b}_b \longrightarrow \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

It is quite clear that that this homomorphism is an injection. In the case $a = 0$ this is the homomorphism

$$\mathcal{O}_{n-1}/\mathfrak{p} \longrightarrow \mathcal{O}_n/\mathfrak{P}.$$

Now we make use of the basic fact that the fields of fractions of both rings agree. We find elements

$$A, B \in \mathcal{O}_{n-1}, \quad A \notin \mathfrak{p}, \quad Az_n - B \in \mathfrak{P}.$$

We can assume that A and B converge in V and furthermore because of coherence

$$A_b(z_n - a) - B_b \in \mathfrak{a}_a \qquad (a = (b, \beta) \in U).$$

We have to combine this fact that $\mathcal{O}_{U,a}/\mathfrak{a}_a$ is a module of finite type*) over $\mathcal{O}_{V,b}/\mathfrak{b}_b$. More precisely it is generated by the powers

$$(z_n - a_n)^\nu, \quad 0 \le \nu < d,$$

where d is the z_n-degree of Q. Now we consider the analytic function $f := A^d$ on U. The germ f_0 defines a non-zero element of $\mathcal{O}_{U,0}/gotP$ and hence non-zero-divisor, because this ring is an integral domain. Because of the coherence result 13.4 we can assume that the multiplication map $m_f : \mathcal{O}_{U,a}/\mathfrak{a}_a \to \mathcal{O}_{U,a}/\mathfrak{a}_a$ is injective of all a. This map is no ring homomorphism but it is good enough to test nilpotency: First we collect all points $a = (b, \beta)$ over a given b and consider

$$m_f : \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a \longrightarrow \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

The construction of A shows that the image of m_f is already contained in the subring

$$\mathcal{O}_{V,b}/\mathfrak{b}_b \hookrightarrow \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

The proof of Cartan's theorem now can be completed as follows: Let $C \in \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a$ be a nilpotent element, $C^k = 0$. Then $m_f(C^k) = f_a C^k = 0$. But this implies $(f_a C)^k = 0$. We recall that $m_f(C) = f_a C$ is contained in the subring $\mathcal{O}_{V,b}/\mathfrak{b}_b$. But this ring is reduced (by our induction hypothesis). Hence $m_f(C) = 0$. But m_f is injective (!) and we obtain $C = 0$. Hence the ring $\prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a$ is free of nilpotents and the same is true for each of its factors. This completes the proof of Cartan's coherence theorem. □

Because of the importance of this theorem we formulate again the decisive consequence:

13.5 Theorem. *Every analytic set can be written locally as the set of common zeros of a finite system of analytic functions*

$$f_1, \dots, f_m : U \longrightarrow \mathbb{C} \qquad (U \subset \mathbb{C}^n \text{ open}),$$

such that the germs in any point $a \in U$ generate the ***full vanishing ideal*** *in $\mathcal{O}_{U,a}$.*

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*) This true because $Q_a \in \mathfrak{a}_a$ is a normalized polynomial, hence z_n-general, hence the product of a unit and a Weierstrass polynomial of degree $\le d$.

Chapter II. Local theory of complex spaces

1. The notion of a complex space in the sense of Serre

We introduce the notion of a concrete ringed space.

1.1 Definition. *A concrete ringed space $(X, \mathcal{O}_X)$ is a topological space together with a subsheaf of $\mathbb{C}$-algebras $\mathcal{O}_X \subset \mathcal{C}_X$.*

Here $\mathcal{C}_X$ denotes the sheaf of complex valued continuous functions.

1.2 Definition. *A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of concrete ringed spaces is a continuous map with the property. For all open subsets $V \subset Y$ and all $h \in \mathcal{O}_Y(V)$ one has $h \circ f \in \mathcal{O}_X(f^{-1}(U))$.*

It is clear that the identity map defines a morphism $\mathrm{id} : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$. The composition of two morphisms is a morphism. Hence the notion of isomorphism of concrete ringed spaces is explained. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an isomorphism if and only if it is topological and if it induces for any open subset $U \subset X$ a bijection between $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(V)$ where $V = f(V)$.

Let Y be a subset of a concrete ringed space $(X, \mathcal{O}_X)$. We equip Y with a structure $\mathcal{O}_Y$ of a concrete ringed space. The topology of Y is the induced topology. Let $V \subset Y$ an open subset. We define:

A function $f : V \to \mathbb{C}$ on some open subset of X belongs to $\mathcal{O}_Y(V)$ if for every $a \in W$ there exists an open neighborhood $a \in \tilde{W} \subset X$ and a function $h \in \mathcal{O}_X(W)$ such that $f(x) = h(x)$ for all $x \in U \cap W$.

It is clear that that $(Y, \mathcal{O}_Y)$ is a concrete ringed space. Such a space is called a *subspace* of $(X, \mathcal{O}_X)$. In the case that Y is an open subset of X the definition can be made easier. In this case on has $\mathcal{O}_Y(V) = \mathcal{O}_X(V)$.

The canonical injection $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism and moreover the following is true:

1.3 Remark. *Let $(X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z)$ be concrete ringed spaces and $(Y, \mathcal{O}_Y)$ a geometric subspace of $(X, \mathcal{O}_X)$. Let $f : Z \to Y$ be a continuous map. Then $f : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ is a morphism if and only if the composition with the canonical injection is a morphism $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$.*

We mention without proof that the notion of subspace is transitive: If $(Z, \mathcal{O}_Z)$ is a geometric subspace of $(Y, \mathcal{O}_Y)$ and $(Y, \mathcal{O}_Y)$ is a geometric subspace of $(X, \mathcal{O}_X)$ then $(Z, \mathcal{O}_Z)$ is a geometric subspace of $(X, \mathcal{O}_X)$.

The following simple remark indicates that the notion of morphism of a concrete ringed space is useful:

1.4 Remark. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open subset. We equip them with the sheaves $\mathcal{O}_U, \mathcal{O}_V$ of holomorphic functions in the usual sense. A map $f : U \to V$ is holomorphic in the usual sense if and only if it as morphism $(U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$.*

In the following we understand by $\mathcal{O}_{\mathbb{C}^n}$ always the sheaf of holomorphic function in the usual sense. Moreover for an analytic subset $A \subset \mathbb{C}^n$ we denote denote by $\mathcal{O}_A$ always the geometric structure that defines $(A, \mathcal{O}_A)$ as geometric subspace of $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$.

1.5 Definition. *A complex space $(X, \mathcal{O}_X)$ in the sense of Serre is a concrete ringed space such that for each point the exists an open neighborhood U and an analytic subset $A \subset \mathbb{C}^n$ for suitable n such that the geometric subspace $(U, \mathcal{O}_U)$ and $(A, \mathcal{O}_A)$ are isomorphic concrete ringed spaces.*

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be complex spaces. A map $f : X \to Y$ is called a holomorphic map if it is a morphism of concrete ringed spaces. Clearly $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}$ is a complex space. For a function $f : X \to \mathbb{C}$ the following two conditions are equivalent:

1) $f \in \mathcal{O}_X(X)$,
2) $f : (X, \mathcal{O}_X) \to (\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ is a morphism.

The simple proof is left to the reader. Since we use the notation $\mathcal{O}_X$ for the structure sheaf of a complex space, there is no need to mention it always. Hence we frequently write X instead of $(X, \mathcal{O}_X)$.

Another simple property of complex spaces is the following. Let $(X, \mathcal{O}_X)$ be complex space, let $U \subset X$ be an open subset and $f \in \mathcal{O}_X(U)$ a holomorphic function without zeros. Then $1/f \in \mathcal{O}_X(U)$.

The notation of analytic subsets generalizes to complex spaces: A subset $Y \subset X$ of a complex space is called analytic if the following condition is satisfied: For each $a \in Y$ there exists an open neighborhood $a \in W \subset X$ and finitely many $f_1, \dots, f_n \in \mathcal{O}_X(W)$ such that

$$Y \cap W = \{x \in W; \quad f_1(x) = \cdots = f_n(x) = 0.\}.$$

Open subsets are very special cases of analytic subsets. (Take $f_i = 0$.)

1.6 Lemma. *If Y is an analytic subset of an complex space. Then one easily can show that the subspace $(Y, \mathcal{O}_Y)$ is a complex space as well.*

Such a space is called a complex subspace of X.

Usually one only considers only complex subspaces that either are open or closed. This is sufficient because of the following simple

1.7 Remark. *Let $Y \subset X$ be an analytic subset of the complex space X. There exists an open subset $U \subset X$ that contains Y and such that Y is closed in U.*

1.8 Lemma. *Let A, B be two closed analytic subsets of a complex space. Then $A \cap B$ and $A \cup B$ are analytic too.*

1.9 Lemma. *Let $f : X \to Y$ be a holomorphic map of complex spaces and $B \subset Y$ a closed analytic subset. Then $f^{-1}(B)$ is analytic too.*

2. The general notion of a complex space

We introduce the general notion of a complex space in the sense of Grothendieck. This is not really necessary for what follows, so the reader can skip this section.

2.1 Definition. *A ringed space $(X, \mathcal{O}_X)$ is a topological space together with a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_X$.*

2.2 Definition. *A morphism*

$$(f, \varphi) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

between ringed spaces is a pair, consisting of a continuous map $f : X \to Y$ and a homomorphism $\varphi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of $\mathbb{C}$-algebras.

It is clear that the identity map is a morphism and how one composes two morphisms $(f, \varphi) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$, $(g, \psi) : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$. Let $U \subset \mathbb{C}^n$ be an open domain and $f_1, \dots, f_m$ analytic functions on U. We consider the ideal sheaf $\mathcal{J}$ generated by $f_1, \dots, f_m$ in $\mathcal{O}_U$. The support of the sheaf $\mathcal{O}_U/\mathcal{J}$ is the set X of joint zeros of the f_i. We restrict the sheaf to X and define

$$\mathcal{O}_X = (\mathcal{O}_U/\mathcal{J})|X.$$

This is a sheaf of $\mathbb{C}$-algebras on X.

2.3 Definition. *A complex space $(X, \mathcal{O}_X)$ is a ringed space which is locally isomorphic to a model space. A morphism between complex spaces is simply called a holomorphic map.*

Id $U \subset X$ is an open subspace then $(U, \mathcal{O}_X|U)$ is a complex space too. We call it a open analytic subspace.

2.4 Remark. *Let $(X, \mathcal{O}_X)$ be a complex space and $\mathcal{J} \subset \mathcal{O}_X)$ a coherent ideal sheaf. The support Y of $\mathcal{O}_X/\mathcal{J}$ is a closed subset and $(Y, \mathcal{O}_Y)$ where $\mathcal{O}_Y = (\mathcal{O}_X/\mathcal{J})|Y$ is complex space too.*

We call such a space that is defined through a coherent ideal sheaf a closed analytic subspace.

3. Complex spaces and holomorphic functions

Let X be a topological space and let $\mathcal{O}_X$ be a sheaf of $\mathbb{C}$-algebras. We assume that $\mathcal{O}_{X,a}$ are local rings with maximal ideal $\mathfrak{m}_a$ and that

$$\mathbb{C} \longrightarrow \mathcal{O}_{X,a} \longrightarrow \mathcal{O}_{X,a}/\mathfrak{m}_a$$

is an isomorphism. Then we can associate to any section $f \in \mathcal{O}_X(U)$, $U \subset X$ open, a function $f' : U \to \mathbb{C}$ which assigns to each point $a \in U$ the image $f(a)$ of f in $\mathcal{O}_{X,a}/\mathfrak{m}_a$. These functions are continuous. So we obtain a morphism

$$\mathcal{O}_X \longrightarrow \mathcal{C}_X.$$

We denote by $\mathcal{O}'_X$ the image (in the sense of sheaves) of this morphism. If we denote the kernel of this morphism by $\mathcal{J}_X$ we get a canonical isomorphism $\mathcal{O}'_X = \mathcal{O}_X/\mathcal{J}_X$. There are two basic results about this morphism.

Let R be a ring. The nilradical $\mathfrak{n}$ is the set of all nilpotent elements a ($a^n = 0$ for som natural number). It is easy to see that $\mathfrak{n}$ is an ideal. Let $\mathcal{O}$ be a sheaf of rings on a topological space X. The nilradical of $\mathcal{O}$ is the sheaf $\mathcal{J} \subset \mathcal{O}$ generated by $\mathfrak{n}(\mathcal{O}(U)$. Concretely this is

$$\mathcal{J}(U) = \{f \in \mathcal{O}(U); \quad f_a \text{ nilpotent in } \mathcal{O}_a \text{ for all } a \in U\}.$$

3.1 Theorem (Rückert). *The ideal sheaf $\mathcal{J}$ is the nilradical of $\mathcal{O}_X$.*

3.2 Theorem (Cartan). *Let $(X, \mathcal{O}_X)$ be a complex space. The nilradical is coherent.*

A complex space $(X, \mathcal{O}_X)$ is called *reduced* if the natural map $\mathcal{O}_X \to \mathcal{C}_X$ is injective. By Rückert's theorem this means that the nilradical is zero. For a reduced complex space we can consider $\mathcal{O}_X$ as a subsheaf of $\mathcal{C}_X$, i.e. the sections are functions. Reduced complex spaces are also called complex spaces in the sense of Serre.

Let $X = (X, \mathcal{O}_X)$ be a complex space. Then $X^{\text{red}} = (X, \mathcal{O}'_X)$ is a complex space in the sense of Serre. This defines a functor from the category of complex spaces into the category of complex spaces in the sense of Serre.

Why nilpotents?

Consider the topological space p_n consisting of one point and equip it with the sheaf that is associated to the $\mathbb{C}$-algebra $\mathbb{C}^n$ (pointwise multiplication, $\mathbb{C} \to \mathbb{C}^n$ the diagonal embedding. We claim that this is a complex space. To see this we consider the complex plane $(\mathbb{C}, \mathcal{O}_\mathbb{C})$ and the ideal sheaf $\mathcal{J}$ generated by z^{n+1}. The associated complex space is isomorphic to p_n. The associated reduced complex space is the p_1.

3.3 Lemma. *Let $(X, \mathcal{O}_X)$ be a complex space. The morphisms $p_2 \to X$ are in one-to one correspondence with the pairs (a, T) where a is a point in X and $T \in \big(\mathfrak{m}(\mathcal{O}_{X,a})/\mathfrak{m}(\mathcal{O}_{X,a})^2\big)^*$.*

4. Germs of complex spaces

We consider the category of complex spaces. A pointed complex space (X, a) is a complex space with a distinguished point $a \in X$. We can consider also the category of pointed complex spaces. Morphisms are morphisms of complex spaces that map the distinguished point to the distinguished point.

4.1 Theorem. *The category of germs of complex spaces is dual to the category of analytic algebras.*

5. The singular locus

Complex manifolds

A point $a \in X$ of a complex space is called smooth or regular if there exists an open neighborhood $a \subset U \subset X$ such that U is bihomorphically equivalent to a an open subset of some $\mathbb{C}^n$. This is equivalent to the fact that $\mathcal{O}_{X,a}$ is isomorphic to the ring of power series. A point is called singular if it is not regular. The set S of singular points is called the singular locus of X. This is a closed subset. A complex manifold is a complex space such that the singular locus is empty. Hence complex manifolds are locally biholomorphic to open subsets of suitable $\mathbb{C}^n$.

There is another possible introduction to smoothness. It rests on the following:

5.1 Proposition. *Let X be a complex manifold and Y a complex subspace that is smooth at some point $a \in X$. Then there exists a biholomorphic map $f : U \to V$ of an open neighborhood of a onto some open subset $V \subset \mathbb{C}^n$ such that*

$$f(Y \cap V) = \{\, z \in V;\ z_1 = \cdots = z_m = 0 \,\} \qquad (m \text{ suitable}).$$

Proof. The proof is a consequence of the implicit function theorem I.2.3. On can assume that $X = \mathbb{C}^n$ and $a = 0$. We also can assume that there exist an open subset $0 \in W \subset \mathbb{C}^d$ and a biholomorphic map $\varphi : W \to Y$, $\varphi(0) = 0$. The components of φ^{-1} can be assumed to be restrictions of holomorphic functions f_i on some neighborhood of $0 \in \mathbb{C}^n$. We get

$$f(\varphi(z)) = z \qquad (f = (f_1, \dots, f_n)).$$

Now the chain rule shows that the rank of the Jacobi matrix $J(f, 0)$ is d. The claim now follows from the implicit function theorem. □

It is basic to have a dimension theory for complex spaces. Historically one used quite involved topological concepts (Hausdorff dimension). We use here an algebraic approach. It has the advantage to be very clear, but it needs some commutative algebra.

5.2 Definition. *Let X be a complex space. The dimension X at some point $a \in X$ is the Krull dimension of $\mathcal{O}_{X,a}$. The space X is called pure dimensional if this dimension is independent of a. In this case we call this number the dimension of X.*

It is clear that for complex manifolds we get the usual notion of dimension. Connected complex manifolds are pure dimensional. But this is not true for arbitrary complex spaces. Take for example the analytic set $z_1 z_2 = z_1 z_3 = 0$ in $\mathbb{C}^3$. This is the union of a line and a plane that intersect in the origin. In this example the local ring at the origin is not an integral domain.

One hint to the correctness of our definition is:

5.3 Proposition. *Let $f : U \to \mathbb{C}$ be a holomorphic function on some open connected subset $U \subset \mathbb{C}^n$. We assume that f doesn't vanish identically. Then the analytic set $\{z \in U;\ f(z) = 0\}$ is of pure dimension $n - 1$.*

This is an application of the theorem of Cohen Seidenberg. □

We want to study the local behavior of the dimension $\dim_a X$ for varying a.

5.4 Lemma. *Let (X, a) be a pointed complex space. There exists a neighborhood $a \in U \subset X$ such that*

$$\dim_b X \le \dim_a X \quad \text{for all} \quad b \in U.$$

Proof. The proof uses Noether normalization: We can assume $X \subset \mathbb{C}^n$ and $a = 0$. The vanishing ideal $\mathfrak{a}$ of X in a can be assumed z_n-general, We denote by $\mathfrak{b} = \mathfrak{a} \cap \mathcal{O}_{n-1}$ the projected ideal. let Y be a geometric realization of Y. We can assume that the projection (cancellation of the last variable) defines a mapping $f : X \to Y$. We can assume that $\mathfrak{a}$ contains a Weierstrass polynomial $Q \in \mathcal{O}_{n-1}[z_n]$, whose coefficients converge on a polydisc which contains Y. Furthermore we can assume that for all $a \in X$ the polynomal $Q(a_1, \dots, a_{n-1}, z_n - a_n)$ is not identical 0, hence general in $\mathbb{C}\{z_1 - a_1, \dots, z_{n-1} - a_{n-1}\}[z_n - a_n]$. This implies that the ring homomorphism

$$f_a^* : \mathcal{O}_{Y,f(a)} \longrightarrow \mathcal{O}_{X,a}$$

is module-finite for all $a \in X$. This homomorphism is not surjective but from Cohen Seidenberg we obtain still

$$\dim \mathcal{O}_{Y,f(a)} \geq f_a^*(\dim \mathcal{O}_{Y,f(a)}) = \mathcal{O}_{X,a}.$$

For $a = 0$ the homomorphism is injective, i.e.

$$\dim \mathcal{O}_{Y,0} = f_a^*(\dim \mathcal{O}_{Y,0}) = \mathcal{O}_{X,0}.$$

This comes from the fact that $\mathfrak{b}$ is a radical ideal and hence the full vanishing ideal. We will proof 5.4 by induction on n and can therefore assume

$$\dim_0 Y \geq \dim_b Y \qquad (b \in Y).$$

We obtain

$$\dim_0 X = \dim_0 Y \geq \dim_{f(a)} Y \geq \dim_a X,$$

which completes the proof of lemma 5.4. □

□

Assume that $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of concrete ringed spaces and that $\mathcal{M}$ is an $\mathcal{O}_X$-module. Then $f_*\mathcal{M}$ carries an obvious structure as $\mathcal{O}_Y$-module.

We can reformulate Cartan's coherence theorem as follows.

5.5 Cartan's coherence theorem. *Let $\mathcal{J} \subset \mathcal{O}_U$ be a coherent sheaf of ideals in the structure sheaf of an open subset $U \subset \mathbb{C}^n$. Then the radical of $\mathcal{J}$ is coherent too.*

The Hilbert Rückert vanishing theorem implies now that the full vanishing ideal sheaf of a closed analytic subset $X \subset$ of an open subset $U \subset \mathbb{C}^n$ is coherent. From this we derive the following generalization of Oka's theorem.

5.6 Theorem. *The structure sheaf of a complex space is coherent.*

Proof. We can assume that the complex space X is a closed analytic subset of an open subset $U \subset \mathbb{C}^n$. We have to show that the kernel of an $\mathcal{O}_X$-linear map $\mathcal{O}_X^p \to \mathcal{O}_X^q$ is locally finitely generated. We consider the natural embedding $i : X \to U$. We have an obvious exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_U \longrightarrow \rangle_* \mathcal{O_X}.$$

This shows that $i_*\mathcal{O}_X$ is coherent. As a consequence the kernel of the induced map $i_*\mathcal{O}_X^p \to i_*\mathcal{O}_X^q$ is coherent. We can assume that it is finitely generated. We choose a finite system of generators in $i_*\mathcal{O}_X^p(X) = \mathcal{O}_X^p(X)$. Obviously they generate the kernel of $\mathcal{O}_X^p \to \mathcal{O}_X^q$. □

In Lemma 5.7.9 we have seen that the support of a coherent sheaf is closed. For complex spaces a better result holds.

5.7 Remark. *Let $\mathcal{M}$ be a coherent sheaf on a complex space. Then the support of $\mathcal{M}$ is a closed analytic subset.*

Proof. We can assume that there exists a presentation

$$\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \mathcal{M} \longrightarrow 0.$$

Recall that there is an underlying matrix $F \in \mathcal{O}_X(X)^{q \times p}$. The stalk $\mathcal{M}_a$ is not zero of and only if

$$[F]_a : \mathcal{O}_{X,a}^p \longrightarrow \mathcal{O}_{X,a}^q$$

is not surjective.

Claim. The map $[F]_a$ is surjective if and only if

$$F(a) : \mathbb{C}^p \longrightarrow \mathbb{C}^q.$$

is surjective.

If $[F]_a$ is surjective then $F(a)$ clearly is surjective. Hence it is sufficient to proof the converse. Assume that $F(a)$ is surjective. This means that the matrix $F(a)$ has rank q and that $p \geq q$. We can select an invertible $q \times q$-sub-matrix of $F(a)$. This shows that F has a $q \times q$-sub-matrix whose determinant at a is not zero. By continuity the determinant is different from zero for all points in a full neighbourhood U of a. By Cramer's rule this sub-matrix of F is invertible in $\mathcal{O}_X(U)^{q \times q}$. But then it follows that $[F]_b$ is surjective for all $b \in U$. This proves the claim.

The rest of the proof is clear now. A matrix $A \in \mathbb{C}^{p,q}$ has rank $< q$ if and only of all determinants of $q \times q$-matrices vanish. Hence the locus where $\mathcal{M}_a$ is different from zero equals the locus of all a where all $q \times q$-sub-determinants of F vanish. This an analytic locus. □

We give a typical application of coherence:

5.8 Lemma. *Let (X, a) be a pointed complex space and $f : X \to \mathbb{C}$ an analytic function on X. We assume that the germ f_a is a non-zero divisor in $\mathcal{O}_{X,a}$. Then there exists an open neighborhood $a \in U \subset X$ such f_b is not a zero divisor in $\mathcal{O}_{X,b}$.*

Corollary. *The zero locus*

$$Y := \{ x \in U; \quad f(x) = 0 \}$$

is thin in U.

Proof. We consider the map that is induced by multiplication with f. It can be considered as a map $\mathcal{O}_X$-linear map of sheaves $\mathcal{O}_X \to \mathcal{O}_X$. By assumption the stalk of the kernel at a is zero. By coherence this remains true in a full neighborhood. Especially the germ f_b is non-zero for $b \in U$. Hence in any neighborhood of of b there exist points which belong to Y but not to X.

□

5.9 Lemma. *Let Y be a complex subspace of the complex space Y and $a \in Y$ a distinguished point. We assume*

a) *$\mathcal{O}_{X,a}$ is an integral domain.*
b) *The vanishing ideals of Y and X in a are different.*

Then there exists an open neighborhood $a \in U \subset \mathbb{C}_n$, such that $Y \cap U$ is thin in $X \cap U$.

One can assume that there exists a holomorphic function f on X whose germ in a is not contained in the vanishing ideal of (Y, a). Since $\mathcal{O}_{X,a}$ is an integral domain, f_a can not be a zero divisor. Now we can apply 5.8. □

5.10 Proposition. *Let a be a point in a complex space X such that $\mathcal{O}_{X,a}$ is an integral domain. Then there exists a pure dimensional open neighborhood U of a.*

Proof. We can assume that $0 \in X \subset \mathbb{C}^n$ is defined by a prim ideal $\mathfrak{P}$. We use induction by n. we can assume $a = 0$. We distinguish the “two alternatives”.

1. *Alternative.* $\mathfrak{P}$ is a principal ideal. Then we can use the theory of hypersurfaces.

2. *Alternative.* $\mathfrak{P}$ is not a principal ideal. We can assume (5.4) $\dim_a X \leq \dim_0 X$ for all $a \in X$ and by induction $\dim_b Y = \dim_0 Y$ for all $b \in Y$. Let now $a \in X$ be an arbitrary point. Because T is thin, we find in any neighborhood of a a point $x \in X - T$. Because of 5.4 we can assume $\dim_x X \leq \dim_a X$. We obtain

$$\dim_0 X \geq \dim_a X \geq \dim_x X = \dim_{f(x)} Y = \dim_0 Y = \dim_0 X. \qquad \square$$

An important result of Krull dimension theory is:

5.11 Proposition. *Let $Y \subset X$ be analytic sets and $a \in Y$ a point such that $\mathcal{O}_{X,a}$ is an integral domain. Assume*

$$\dim_a Y \geq \dim_a X.$$

Then X and Y agree in a full neighborhood of a.

We have seen that it is often useful to reduce statements about radical ideals to prime ideals. This is possible because every radical ideal is the intersection of finitely many prime ideals. We describe the geometric counterpart of this algebraic fact in more detail:

Local irreducible components

Let R be a noetherian ring. A prime ideal $\mathfrak{p}$ which contains a given ideal $\mathfrak{a}$ is called *minimal* with this property, if any prime ideal $\mathfrak{q}$, $\mathfrak{a} \subset \mathfrak{q} \subset \mathfrak{p}$, agrees with $\mathfrak{p}$. A refinement of the already used statement about radical ideals is:

Let $\mathfrak{a}$ be an ideal in a noetherian ring R. There exist only finitely many minimal prime ideals containing $\mathfrak{a}$. Their intersection is $\operatorname{rad}\mathfrak{a}$. Every prime ideal that contains $\mathfrak{a}$ contains one of the minimals.

Now we consider the geometric counter part of this decomposition: Let X be a complex space. We want to study local properties of X at a given point $a \in X$ (and allow therefore to replace U by a smaller neighborhood if necessary). Since $\mathcal{O}_{X,a}$ is reduced, the zero ideal is a radical ideal. We can write it as the intersection of pairwise distinct minimal prime ideals

$$(0) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m.$$

We allow that X is replaced by a small neighborhood of a. Therefore we can assume that there are closed analytic sets $X_j \subset X$ whose vanishing ideals at a are $\mathfrak{p}_j \subset \mathcal{O}_{X,a}$. Again replacing X by a smaller neighborhood if necessary we can assume

$$X = X_1 \cup \ldots \cup X_m.$$

We call the X_j the local irreducible components of X at a. They are unique up to ordering and in an obvious local sense.

5.12 Lemma. *Let (X, a) be a pointed analytic set and*

$$X = X_1 \cup \ldots \cup X_m$$

be a decomposition into the local irreducible components of X at a. Then

$$\dim_a X = \max_{1 \leq j \leq m} \dim_a X_j.$$

if $Y \subset X$ be an analytic subset which contains a and such that $\mathcal{O}_{Y,a}$ is integral. After replacing X by a small neighborhood of a if necessary, the set Y is contained in one of the components X_j.

Proof. The dimension of X at a is defined by means of sequences of prime ideals in $\mathcal{O}_{X,a}$. Let $\mathfrak{a} \subset \mathcal{O}_n$ be the vanishing ideal of X at a. The chains of prime ideals in $\mathcal{O}_{X,a}$ correspond to chains

$$\mathfrak{a} \subset \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_m \subset \mathcal{O}_n.$$

The ideal $\mathfrak{p}_0$ must contain one of the minimal prime ideals containing $\mathfrak{a}$. This proofs the statement about the dimension. The last statement is also clear because the vanishing ideal of Y in a must contain one of the minimal prime ideals containing $\mathfrak{a}$. □

Now we are in the state to prove a main result of local complex analysis:

5.13 Theorem. *The singular locus S of a complex space is a thin closed analytic subset of X.*

Proof. Since the statement is of local nature we can replace X be a small open neighborhood of a given point. Therefore we can assume that $X = X_1 \cup \ldots \cup X_m$ is a decomposition into local irreducible components at a. We can assume that the X_i are pure dimensional. The points the intersections of two different X_i are singular points since the local rings there are not integral domains. Hence the singular locus of X is the union of the pairwise intersections and the singular loci of the X_i. Since the finite union of closed analytic subsets is analytic we reduced 5.13 to the pure dimensional case.

In the pure dimensional case we will make use of a differential criterion of smoothness: This rests on the implicit function theorem. One version of it states:

Let X be the zero set of m holomorphic functions $f_1, \ldots, f_m$ on some open subset $U \subset \mathbb{C}^n$. Assume that the (complex) Jacobian matrix $J(f,a)$ has rank r at some point $a \in X$. Then a is a smooth point of X and $\dim_a X = n - r$.

There is an immediate consequence:

5.14 Lemma. *Let $X \subset \mathbb{C}^n$ be an analytic set that is defined by analytic equations*

$$f_1(z) = \cdots = f_m(z) = 0$$

in some open neighborhood $0 \in U \subset \mathbb{C}^n$. Let $a \in X$ be a point. The rank r of the Jacobian of $f = (f_1, \ldots, f_m)$ at a is $r \le n - d$, where $d = \dim X$. In the case $r = n - d$ the point a is smooth.

Proof of the lemma. We can choose r of the functions f_i whose Jacobi matrix has rank r at a. We can assume that $f_1, \ldots, f_r$ is this system. The set of zeros of this system is a analytic set $\tilde{X}$ that is smooth and of dimension $n - r$ at a. Since $X \subset \tilde{X}$ we have $d \le n - r$ or equivalently $r \le n - d$. When equality holds X and $\tilde{X}$ agree close to a. Hence X is smooth in a like $\tilde{X}$. □

The converse of 5.14 is not true in general. Consider for example the equation $z^2 = 0$ in $\mathbb{C}$. The dimension d is zero but the rank r of the Jacobi matrix at $a = 0$ is 0. Hence the equation $d + r = n$ is false. The reason is that $z^2 = 0$ is the false description. One should better use the equation $z = 0$. The correct converse of 5.14 is:

5.15 Lemma. *Let $X \subset \mathbb{C}^n$ be an analytic set that is defined by analytic equations*

$$f_1(z) = \cdots = f_m(z) = 0$$

in some open neighborhood $0 \in U \subset \mathbb{C}^n$. Let $a \in X$ be a point. Assume that the germs of the f_i generate the full vanishing ideal of X in $\mathcal{O}_{\mathbb{C}^n,a}$. Then a is a smooth point of X if and only if the Jacobi matrix $J(f, a)$ has the correct rank $n - \dim_a X$.

Proof. It remains to proof that the condition is necessary. So let's assume that a is smooth. Due to the implicit function theorem 5.1 we can assume that X is given by equations $z_{d+1}, \dots, z_n$=0. For these equations the rank condition is trivial. But we may have different equations. From the assumption about the vanishing ideal we know that both generate the same ideal. Hence the statement follows from

5.16 Lemma. *Let $P = (P_1, \dots, P_m)$ and $Q = (Q_1, \dots, Q_l)$ be two systems of power series which generate the same ideal in $\mathcal{O}_n$. Then the Jacobians of P and Q at the origin have the same rank.*

The easy proof is left to the reader. □

Now we are able to prove the main result 5.13. We reduced already to the case of a pure dimension case $d = \dim X$. We can assume that X is defined inside some open subset $U \subset \mathbb{C}^n$ as zero set of a finite number of holomorphic functions $f_1, \dots, f_n$. We choose some point $a \in X$. We can replace U be a smaller neighborhood since the question is od local nature. Since $\mathcal{O}_a$ is noetherian we can assume that $(f_1, \dots, f_n)_a$ is a radical ideal in the point a. By Cartan's coherence theorem this then is true in a full neighborhood. We can assume that this is true in U. Now the singular locus is described as set of all $z \in U$ such $f_i(z) = 0$ and such that the rank of $J(f, z)$ is smaller than $r = n - d$. This means that all determinants of $r \times r$-matrices vanish. Hence the singular locus can be defined by a finite set of analytic equations. □

6. Finite maps

A holomorphic map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_X)$ is called finite, if the underlying map between topological spaces is finite. This means that it is proper and the fibres are finite sets. A holomorphic map $f : X \to Y$ is locally finite at a point $a \in X$ if there exist open sets $a \in U \subset X$ and $f(a) \in V \subset Y$ such that $f(U) \subset V$ and that $f : U \to V$ is finite.

6.1 Theorem. *A holomorphic map $f : X \to Y$ is locally finite at a if an only if the corresponding map of analytic algebras $\mathcal{O}_{Y,f(a)} \to \mathcal{O}_{X,a}$ is finite.*

An important result of Grauert states.

6.2 Theorem. *Let $X \to Y$ be a finite holomorphic map between complex spaces. Let $\mathcal{M}$ be a coherent sheaf on X. Then the direct image $f_*\mathcal{M}$ is coherent too. The functor $\mathcal{M} \mapsto f_*\mathcal{M}$, starting from the category of coherent sheaves on X is exact.*

Chapter III. Stein spaces

1. The notion of a Stein space

Probably the reader knows that on a connected compact complex manifold any holomorphic function is constant. Assume that the dimension is > 1. If one removes from this manifold a single point the situation does not remedy, since in more than one variable there do not exist isolated singularities. Hence there exist also non-compact manifolds that admit no non-constant analytic function. Stein spaces are opposite to this situation. They are spaces that admit many holomorphic functions. We are going to explain in which sense this has to be understood.

Let K be a non-empty compact subset of a topological space X. We use the notation

$$||f||_K := \max\{|f(x)|;\ x \in K\}$$

for a continuous function f on X.

1.1 Definition. *Let K be a non-empty compact subset of a complex space. The* ***holomorphic convex hull*** *$\hat{K}$ of K is the set of all $x \in X$ such that $|f(x)| \leq ||f||_K$ for all $f \in \mathcal{O}_X(X)$.*

1.2 Definition. *A complex space is called* ***holomorphically convex*** *if the holomorphic convex hull of any compact subset is compact.*

Assume that X is a complex space with the following property: For every infinity closed discrete subset $S \subset X$ there exists a holomorphic function $f : X \to \mathbb{C}$ that is unbounded on S. Then X is holomorphically convex. This can be seen by an indirect argument. Let K be a compact subset such that $\hat{K}$ is not compact. Then their exists a sequence in $\hat{K}$ with no convergent subsequence. This gives an infinite subset $S \subset \hat{K}$ that is closed in X and discrete. Then there exists a global holomorphic function which is unbounded on $\hat{K}$. This is not possible.

From this observation we can deduce that open subsets U of the plane $\mathbb{C}$ are holomorphically convex. To show this we consider an infinity closed discrete subset S. If S is unbounded then we take $f(z) = z$. In the case that S is bounded their must be an accumulation point a of S which lies on the boundary of U. Then take $f(z) = 1/(z - a)$.

In more then two variables the situation is completely different. Let $U = U_r(0)$ be a polydisk around zero. We claim that $U - \{0\}$ is not holomorphically convex. For this we consider the subset K consisting of all z with $|z_i| = r_i/2$. We know that every holomorphic function f on $U - \{0\}$ extends holomorphically to U. From the maximum principle one deduces $\hat{K} = \{z \in U;\ |z_i| \leq r_i/2\}$. This set is not compact.

1.3 Definition. *A complex space X is called a Stein space if the following conditions are satisfied:*

1) *It is holomorphically convex.*
2) *(Point separation) For two different points $x, y \in X$ there exists a global $f \in \mathcal{O}_X(X)$ with $f(x) = 0$, $f(y) = 1$.*
3) *(Infinitesimal point separation) For any point $a \in X$ there exist global $f_1, \dots, f_m \in \mathcal{O}_X(X)$ whose germs generate the maximal ideal of $\mathcal{O}_{X,x}$.*

It is clear that open subsets of the complex plane are Stein spaces. More generally it is clear that a cartesian product $D = D_1 \times \cdots \times D_n$ of open subsets $D_i \subset \mathbb{C}$ is Stein. It is already a deep result that all non-compact connected Riemann surfaces are Stein spaces. We will not proof this result here completely. A proof can be found in [Fo]. As we have seen it is false that open subsets of $\mathbb{C}^n$ are always Stein in the case $n > 1$.

1.4 Remark. *Let X be a Stein space. Then every closed analytic subset is a Stein space too.*

1.5 Definition. *An Oka domain in a complex space X is an open subset $U \subset X$ and such that there exists a closed analytic subset A of a polydisk in some $\mathbb{C}^n$ such there exist holomorphic functions $f_1, \dots, f_n$ on the whole X whose restriction define a biholomorphic map $U \to A$.*

The basic exhaustion theorem states:

1.6 Theorem. *Let X by a Stein space. Any compact subset K is contained in an Oka domain U.*

Additional remark. *In the case that $K = \hat{K}$ and W is some open subset containing K one can get $U \subset W$.*

Before we start with the proof we formulate a technical lemma:

1.7 Lemma. *Let $f : X \to Y$ be a holomorphic map of complex spaces. We make two assumptions:*

a) *The induced map $X \to f(X)$ is topological.*
b) *For each point $a \in X$ there exists an open neighborhood U such that $f(U)$ is an analytic subset of Y and such that $U \to f(U)$ is biholomorphic.*

Then $f(X)$ is an analytic set and $X \to f(X)$ is biholomorphic.

Proof of the lemma. Let $a \in X$ be a point and U a neighborhood with property b). Then we know from a) that $f(U)$ is an open subset from $f(X)$. By assumption b) $f(U)$ is analytic. This means that every point of $f(X)$ admits an open neighborhood that is analytic. But then $f(X)$ is analytic. The inverse map $f(X) \to X$ is analytic since this is locally the case. □

Property a) has been used essentially in the proof. So one should have in mind that bijective continuous maps between topological spaces need not to be topological. There is an exceptional case where the situation is better.

Recall that a continuous map $f : X \to Y$ between locally compact Hausdorff spaces is called *proper*, if the inverse image of any compact set $K \subset Y$ is compact. Proper maps have the basic property that they are closed. This means that the images of closed subsets of X are closed in Y. This immediately gives:

Let $f : X \to Y$ be a bijective continuous and proper map between complex Hausdorff spaces. Then f is topological.

This is clear: The inverses under f^{-1} are the images under f. Hence the assumption says that the inverse images of closed sets under f^{-1} are closed. This means that f^{-1} is open.

Proof of 1.6 continued. We can assume that $K = \hat{K}$. We will prove the sharpened form where we have to consider an open neighborhood W of K. For each $a \in K$ we can choose finitely many global functions that map an open neighborhood $U(a)$ of a biholomorphically onto an analytic subset of some $\mathbb{C}^n$. The compact subset K can be covered by finitely many of these neighborhoods, $K \subset U(a_1) \cup \cdots \cup U(a_m)$. We collect the functions for each a and obtain a holomorphic map such is locally biholomorphic on $U(a_1) \cup \cdots \cup U(a_m)$. We choose an open neighborhood U of W whose closure is compact and contained in $U(a_1) \cup \cdots \cup U(a_m)$. We would like to manage that f is injective on U. For this we consider the set A of all $(a, b) \in \bar{U} \times \bar{U}$ such that $f(a) = f(b)$. The diagonal Δ of $\bar{U} \times \bar{U}$ is contained in A. Actually Δ is an open subset of A. To show this we consider some diagonal point (a, a). Then $a \in U(a_i)$ for some i. Then all points. Then $U(i) \times U(i)$ is an open neighborhood of (a, a) in X. Its intersection with A is contained in Δ since f is injective on $U(a_i)$. Since Δ is open in A we get that the complement $A - \Delta$ is compact. For each pair $(a, b) \in A - \Delta$ we can choose a global holomorphic function h with $h(a) \neq h(b)$. Then $h(x) \neq h(y)$ for all (x, y) in a full open neighborhood of (a, b) in $A - \Delta$. We can cover $A - \Delta$ by finitely many such open sets. We add the finitely man functions h as new components to the map f. In this way we produce a globally defined map that is injective. Without loss of generality we can assume that f is injective on $\bar{U}$ (and locally biholomorphic on U).

It remains to manage that f defines a proper map of U onto an analytic set of some polydisk. The polydisk we want to take is just the product of unit discs $(|z_j| < 1)$. For this we can assume without loss of generality $|f_i(z)| \leq 1$ for z in K. One just has to multiply f with a suitable constant. Now we will make use of the holomorphic convexity: For each boundary point $a \in \partial U$ we can choose a global holomorphic function g such that $||g||_K < g(a)$. Multiplying with a suitable constant we can get $||g||_K < 1 < |g(a)|$. This inequality remains true in a full open neighborhood of a. We can cover ∂U with finitely many of these neighborhoods. We add the corresponding functions g as additional components to f. Now we modify U. We replace U by the set of all $x \in U$ such that $|f_i(x)| < 1$. We still have that f is injective and locally biholomorphic on this new $U \supset K$. But now we have the advantage that f defines a proper map of U into the polydisk. For this one has just to show that the inverse image of the compact set $|z_i| \le \varrho < 1$ is compact in U. This is clear since this set is away from the boundary of U. □

2. Approximation theorems for cuboids

In the theory of Stein spaces it turned out to be of some advantage to work with rectangles of the form

$$Q = \{z \in \mathbb{C}; \quad a_1 < x_1 < a_2;\ b_1 < y_1 < b_2\}.$$

Here $a_1 < a_2$ and $b_1 < b_2$ real numbers. In the following we understand by an open *cuboid* a set $Q = Q_1 \times \cdots \times Q_n$, where the Q_i are rectangles in the above sense. Cuboids are Stein spaces and every closed analytic subset of a cuboid is Stein.

A very special case of the so-called Runge approximation theorem states:

2.1 Runge's approximation theorem (special case). *Every holomorphic function on a cuboid is the locally uniform limit of a sequence of polynomials.*

We just give a hint to the proof in the one-dimensional case. Let $f : Q \to \mathbb{C}$ be a holomorphic map on a rectangle Q. We have to show that for each shrunken rectangle $Q_0 \subset Q$ and for each $\varepsilon > 0$ there exists a polynomial P with $|P(z) - f(z)| < \varepsilon$ for all $z \in Q_0$. Cauchy's integral formula gives for $z \in Q_0$

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial Q_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Here we have chosen some cuboid Q_1 between Q_0 and Q. Using an approximation by step functions we can approximate f by functions of the type $C/(z-a)$ where a is on the boundary of Q_1. Hence it is sufficient to assume $f(z) = C/(z-a)$. Since a is outside of the closure of Q_0, we find a disk that contains the closure of Q_0 but not a. In this disc we can expand $C/(z-a)$ into a power series and then approximate it by its Taylor polynomials.

We need a certain matrix valued version of this approximation theorem. For this it is convenient to use the matrix norm for a square matrix A.

$$|A| = \max\{|Az|; \quad |z| = 1\}.$$

Here $|z|$ denotes the Euclidean norm of a vector z. This matrix norm has the properties:

a) $|a_{ik}| \le |A|$.
b) $|AB| \le |A||B|$.
c) $|A+B| \le |A| + |B|$.

From these inequalities immediately follows that the series

$$e^A := \sum_{\nu=0}^{\infty} \frac{A^\nu}{\nu!},$$
$$\log(E-A) := -\sum_{\nu=1}^{\infty} \frac{A^\nu}{\nu} \quad \text{for} \quad |A| < 1$$

converge. The rule

$$A^{\log(E-A)} = E - A$$

holds. It follows from the known case $n = 1$ since it can be expressed as a formal relation in factorials. We have to give some warning. The rule $e^{A+B} = e^A e^B$ is usually false. It holds if the matrices A, B commute.

We will have to consider matrix valued function $F : D \to \mathbb{C}^{(m,m)}$ on open subsets $D \subset \mathbb{C}^n$. Of course holomorphy means that each component of is holomorphic. An immediate application of the above consideration is:

2.2 Lemma. *Let $F : D \to \mathbb{C}^{(m,m)}$ be some matrix valued holomorphic function on an open subset $D \subset \mathbb{C}^n$. Assume that $|F(z) - E| < 1$ for all $z \in D$, Then there exists a holomorphic function $G : D \subset \mathbb{C}^{(m,m)}$ with the property $F = e^G$.*

In contrast to the case $m = 1$ it is very difficult to get holomorphic logarithms without an estimate as in 2.2. This will cause some difficulties. To come around them we prove:

2.3 Lemma. *Let $F : D \to \mathrm{GL}(m, \mathbb{C})$ be an invertible holomorphic matrix valued function on an open convex subset $D \subset \mathbb{C}^n$. Let $K \subset D$ be a compact subset and $\varepsilon > 0$. Then $F = F_1 \cdots F_k$ can be written as finite product of holomorphic functions*

$$F_i : D \to \mathrm{GL}(m, \mathbb{C}), \quad |F_i(z) - E| \le \varepsilon \quad \textit{for} \quad z \in K.$$

Proof. We will use a simple fact about togological groups. Let G be the set of all holomorphic maps $F : D \to \mathrm{GL}(m, \mathbb{C})$. This is a group under multiplication. For any holomorphic $F : D \to \mathbb{C}^{(m,m)}$ and a compact subset $K \subset D$ we define

$$||F||_K = \max\{|F(z)|,\ z \in K\}.$$

Eventually replacing K by a bigger compact set (with non-empty interior) we can assume that $||\cdot||$ is definite. Then $||F - G||_K$ defines a metric on G and G gets a topological space. It is clear that multiplication $G \times G \to G$ and inversion $G \to G$ are continuous. This means that G is a topological group. We claim that this topological space is arcwise connected. To show this we can assume that $0 \in D$. For any $F \in G$ we can consider $F_t(z) = F(tz)$, $0 \le t \le 1$. Notice that $F_t \in G$ and that $t \to F_t$ is continuous. Hence it defines a curve in G that combines F with the constant function F_0. Now the connectedness of G follows from the known fact that $\mathrm{GL}(m, \mathbb{C})$ is connected. For sake of completeness we recall the argument. Any invertible matrix can be written as finite product of diagonal matrices and strict triangular matrices. Each of them, hence also an finite product of them can be combined with the unit matrix. This follows just from the connectedness of $\mathbb{C}^{\bullet}$ and $\mathbb{C}$.

Proof of 2.3 continued. We denote by $U \subset G$ the set of all $F \in G$ with $||F||_K < \varepsilon$ and $||F^{-1}||_K < \varepsilon$. This is an open subset. Then we denote by G_0 the subgroup of G generated by U. It consists of all finite products of elements of U. Since G_0 is the union of translates of G it is an open subgroup of G. But an open subgroup is automatically closed. This follows from the decomposition of G into (say right-) cosets $G_0 g$. The complement of G_0 is the union of all cosets different from G_0 and hence open. From the fact that G is arcwise connected we get $G = G_0$. This finishes the proof of 2.3. □

Now we are able to prove a multiplicative analogue of Runge's approximation theorem.

2.4 Multiplicative version of Runge's approximation theorem. *Let $F : Q \to \mathrm{GL}(m, \mathbb{Z})$ be an invertible holomorphic matrix valued function on a cuboid $Q \subset \mathbb{C}^n$. There exists a sequence $F_\nu : \mathbb{C}^n \to \mathrm{GL}(n, \mathbb{C})$ of invertible holomorphic matrix valued functions on the whole $\mathbb{C}^n$ that converges on Q locally uniformly to F.*

Proof. Let $K \subset Q$ be a compact subset and $\varepsilon > 0$. We have to construct a holomorphic $G : \mathbb{C}^n \to \mathrm{GL}(m, \mathbb{C})$ such that $||F - G||_K < \varepsilon$. We choose a cuboid $K \subset Q_0$ whose compact closure is contained in Q. Because of 2.3 we can restrict to the case $|F(z) - E| < 1$ for $z \in Q_0$. Then there exists a holomorphic logarithm $e^H = F$ on Q_0. By Runge's approximation theorem we can approximate H by a polynomial function P. Hence we can manage $||F - e^P||_K < \varepsilon$. □

The usual theory of infinite products can be generalized to matrix valued functions. Recall that an infinite product $(1+a_1)(1+a_2)\cdots$ is called absolutely convergent if the series $|a_1| + |a_2| + \cdots$ converges. It is known that then the limit

$$\lim_{\nu\to\infty}(1 + a_1)\cdots(1 + a_\nu)$$

exists and that it is zero if and only of one of the factors $1 + a_i$ is zero. Here is a matrix valued variant.

2.5 Lemma. *Let G_ν be a sequence of holomorphic matrix valued functions on some open domain in $\mathbb{C}^n$ such that there exists a convergent series $a_1 + a_2 + \cdots$ of numbers with the property $|G_\nu(z)| \le a_\nu$ for all z. Then the limit*

$$F(z) = \lim_{m\to\infty} F_1 \cdots F_m, \qquad F_\nu := E + G_\nu,$$

exists an is a holomorphic function. It is invertible if all F_ν are.

Proof. The usual theory of infinite products shows that $(1 + a_1)\cdots(1 + a_\nu)$ converges, say to a. $P_\nu = F_1 \cdots F_\nu$ are bounded ba a in the sense $|P_\nu(z)| \le a$ for all z. This follows from $|E + G_i(z)| \le 1 + a_i$. Now we get

$$|P_{\nu+1}(z) - P_\nu(z)| = |P_\nu(z)G_{\nu+1}(z)| \le a \cdot a_\nu$$

From this follows that P_ν is a uniform Cauchy sequence. Hence its limit F exists and is a holomorphic function. We have still to show that it is invertible if all F_ν are. For this it is sufficient to show that the product of the $\det F_\mu$ converges absolutely in the sense of infinite products. This means the the series $\sum(1 - \det F_\nu)$ converges absolutely. Since $1 - \det F_\nu$ is polynomial without constant coefficient in the entries of F_ν it can be bounded for all ν with $a_\nu < 1$ by a bound $C|a_\nu|$. This shows the convergence. □

3. Cartan's gluing lemma

We consider two rectangles $R', R'' \subset \mathbb{C}$ in a very special position. We identify $\mathbb{C}$ with $\mathbb{R}^2$. In fact we assume that there are real numbers $a < b < c < d$ such that the rectangles are of the form $R' = (a, c) \times I$ and $R'' = (b, d) \times I$, where $I \subset \mathbb{R}$ is a bounded open interval.

For a cuboid $D \subset \mathbb{C}^{n-1}$ we can consider $Q' = R' \times D$ and $Q'' = R'' \times D$.

3.1 Cousin's additive gluing lemma. *Let Q', Q'' be two cuboids in $\mathbb{C}^n$ in the special position $Q' := R' \times D$, $Q'' := R'' \times D$, where $R', R'' \subset \mathbb{C}$ are rectangles of the form*

$$R' = (a, c) \times I,\ R'' = (b, d) \times I \quad (a < b < c < d).$$

Furthermore let f be an analytic function on $Q' \cap Q''$ Then one has:
There exist analytic functions

$$f' : Q' \longrightarrow \mathbb{C}, \quad f'' : Q'' \longrightarrow \mathbb{C}$$

with the property

$$f(z) = f'(z) + f''(z) \text{ for } z \in Q' \cap Q''.$$

Proof. We know that the cohomology of $\mathcal{O}$ on a cuboid vanishes. By Leray's Lemma the cohomology $H^1(Q, \mathcal{O})$ can be computed by means of the Čech cohomology with respect to the covering $Q = Q' \cup Q''$. Its vanishing is just the statement of Lemma 3.1. □

We give a second proof of Lemma 3.1 under a slightly stronger assumption. We assume that f can be extended to a an analytic function on an open set U which contains $\overline{Q' \cap Q''}$.

This proof uses the Cauchy integral formula applied to f as function of z_1. During the proof, $z_2, \dots, z_n$ will kept fixed. The integrals in consideration will depend analytically on $z_2, \dots, z_n$ by Leibniz's criterion. Hence it is sufficient to restrict to the case $n = 1$. The Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial(R' \cap R'')} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for } z \in Q' \cap Q''.$$

It is clear that the boundary $\partial(R' \cap R'')$ is the composition of two paths W' and W'', where W' is contained in the boundary of R' and W'' in the boundary of R''.

Then one has

$$f(z) = f'(z) + f''(z) \text{ for } z \in Q' \cap Q''$$

with

$$f'(z) := \oint_{W'} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

and similarly f''. The functions f', f'' are analytic in the complements of W', W'', hence in the whole Q', Q'' (actually in a much bigger domain!)

This second proof of the gluing lemma has the advantage to admit estimates for the functions f', f''. For this improvement we assume that the set U is bounded and also that the function f is bounded on U. Recall that the construction of the gluing functions is given by a Cauchy integral along $f(z)/(\zeta_1 - z_1)$. This integral can be estimated by the standard estimate of line integrals. This estimate involves the length if the curve. This is bounded by the bounds of the domain U. We obtain.

3.2 Lemma. *Assume that a bounded open set $U \subset \mathbb{C}^n$ which contains the closure of $Q' \cap Q''$ is given. There exists a constant M depending only on U such that for each bounded holomorphic function f on U the solution $f' : Q' \to \mathbb{C}$, $f'' : Q'' \to \mathbb{C}$ of the additive gluing lemma can be obtained with the estimate*

$$|f'(z)| \le \frac{M||f||}{\delta'(z)} \qquad (z \in Q').$$

Here $||f||$ denotes the supremum of $|f(z)|$ on U and $\delta'(z)$ denotes the minimal distance of z to a boundary point of Q' (similarly for f'').

Supplement. *For M one can take 3 times the diameter of U. (The diameter is the supremum of the Euclidean lengths of line segments contained in U.)*

There is a multiplicative version of the gluing lemma that produces a decomposition of the typ $f(z) = f'(z)f''(z)$. The proof is easy for scalar valued functions. One takes a holomorphic logarithm of f and applies the additive lemma to the logarithm and exponentiates then. The result follows then from the rule $e^{a+b} = e^a e^b$. Due to Cartan the multiplicative lemma is also valid for matrix valued f. But the proof is more involved. One reason is that the rule $e^{a+b} = e^a e^b$ is false for matrices a, b.

3.3 Lemma (Cartan's multiplicative gluing lemma). *We take the same assumptions as in 3.1. Furthermore let $F : U \to \mathrm{GL}(m, \mathbb{C})$ be a holomorphic function on an open set U which contains the closure of $Q' \cap Q''$. Then there exist holomorphic functions $F' : Q' \to \mathrm{GL}(m, \mathbb{C})$, $F'' : Q'' \to \mathrm{GL}(m, \mathbb{C})$ such that*

$$F(z) = F'(z) \cdot F''(z) \text{ for } z \in Q' \cap Q''.$$

Proof. In a first step we mention that for the proof of the gluing lemma we can assume that $F(z)$ is close to the identity matrix (in the sense $|F(z)| < \varepsilon$ for a given $\varepsilon > 0$). The reason is that be the multiplicative Runge approximation we can choose for an arbitrary F a $G : \mathbb{C}^n \to \mathrm{GL}(m, \mathbb{C})$ such that FG^{-1} is small in the sense we need. So we get a decomposition $FG^{-1} = F'F''$ and then a decomposition $F = F' \cdots (F''G)$. □

In the next step we will explain the strategy of the proof (which only will work if F is close enough to the unit matrix). We write $F(Z) = E + G(Z)$ where E is the unit matrix. Then we apply the additive lemma to the components of G to produce a decomposition $G(Z) = G'(Z) + G''(Z)$, where G', G'' are holomorphic on Q', Q''. Then as a first trial we set $F' = E + G'$, $F'' = E + G''$. Then

$$F'F'' = (E + G')(E + G'') = E + G'G'' + G'G'' = F + G'G''.$$

The term $G'G''$ is a failure term. We want to get rid of it through an approximation method. What we described is only the first step of an approximation. Hence we set

$$G_0 = G, \quad G'_0 = G', \quad G''_0 = G''.$$

By induction we will define a sequence G_ν, G'_ν, G''_ν. Here G_ν should be an invertible matrix valued function on some open neighborhood of $\overline{Q' \cap Q''}$ and $G_\nu = G'_\nu + G''_\nu$ a decomposition in sense of the additive lemma. The basic formula for the procedure is

$$(E + G'_\nu)(E + G_{\nu+1})(E + G''_\nu) = (E + G_\nu).$$

Assume that we have constructed this sequence. Then we can define

$$F'_\nu = E + G'_\nu, \quad F''_\nu = E + G''_\nu.$$

Then we have

$$F = [F'_1 F'_2 \cdots F'_\nu] F_{\nu+1} [F''_1 F''_2 \cdots F''_\nu] \quad \text{on} \quad Q' \cap Q''$$

and the solution of the multiplicative decomposition should be obtained by

$$F' := \lim_{\nu \to \infty} [F'_1 \cdots F'_\nu]$$

and similarly F''. Of course the hope is that G_ν tends to zero for $\nu \to \infty$ and that the infinity products converge.

Before we start with the proof of the convergence, we have to overcome a small technical difficulty. Of course we can define $G_{\nu+1}$ through the equation $(E + G'_\nu)(E + G_{\nu+1})(E + G''_\nu) = E + G_\nu$ if $E + G_\nu$ is invertible and we get a function that is holomorphic on $Q' \cap Q''$. But to apply the additive gluing lemma we should have a holomorphic function on some open neighborhood of $\overline{Q' \times Q''}$. We will overcome this difficult through a small modification. We enlarge the cuboid a little bit: We write Q as the intersection of a decreasing sequence of cuboids (all contained in U) $Q_1 \supset Q_2 \supset \cdots$ such that each Q_ν contains the (compact) closure of $Q_{\nu+1}$. We define the decomposition $Q_\nu = Q'_\nu \cup Q''_\nu$ into two sub-cuboids in the obvious way such that $Q'_\nu \cap Q = Q'$ and $Q''_\nu = Q''$.

Now we can define the functions G_ν inductively as holomorphic functions on $Q'_\nu \cap Q''_\nu$ and then apply the additive gluing lemma to define G'_ν, G''_ν on $Q'_{\nu+1}$, $Q''_{\nu+1}$. So lets recall:

The functions G_ν are holomorphic on $Q'_\nu \cap Q''_\nu$. One has the decomposition $G_\nu = G'_\nu + G''_\nu$ on $Q'_{\nu+1} \cap Q''_{\nu+1}$. Moreover one has (by definition of $G_{\nu+1}$)

$$(E + G'_\nu)(E + G_{\nu+1})(E + G''_\nu) = (E + G_\nu) \quad \text{on} \quad Q'_{\nu+1} \cap Q''_{\nu+1}.$$

Of course the start is $G_0 = E - F$.

Now we come to the problem of convergence of $F'_1 \cdots F'_\nu$ (where $F'_\nu = E + G'_\nu$). We want to use a standard criterion for convergence of infinite products.

Proof of 3.3 continued. The strategy to enforce convergence is to construct the G'_ν with an estimate. What we finally want to have is an estimate of the forms

$$\begin{aligned} &|G_\nu(z)| \le \varrho \cdot 4^{-\nu} \text{ for } z \in Q'_\nu \cap Q''_\nu, \\ &|G'_\nu(z)| \le C \cdot 2^{-\nu} \text{ for } z \in Q'_{\nu+1} \end{aligned}$$

with certain constants $C < 1/2$, ϱ. The condition on C will ensure that $E + G'_\nu + G''_\nu$ is invertible. If we succeed to get such an estimate we are obviously through.

Estimates for the gluing functions

We will obtain the estimates for $G_{\nu+1}$ from estimates of the G'_ν, G''_ν inductively. But this demands also an estimate for the G_ν. Recall that $G_{\nu+1}$ is defined by

$$(E + G'_\nu)(E + G_{\nu+1})(E + G''_\nu) = (E + G'_\nu + G''_\nu) \quad \text{on} \quad Q'_{\nu+1} \cap Q''_{\nu+1}.$$

3.4 Lemma. *Let A, B be $m \times m$-matrices such that $|A| \le 1/2$ and $|B| \le 1/2$ and let be C a matric such that*

$$(E + A)(E + C)(E + B) = E + A + B.$$

There exists a constant P depending only on m such that

$$|C| \le P|A||B|.$$

Proof. The set of all A with $|A| \le 1/2$ is compact. The matrix $E + A$ is invertible for these A. This can be shown by means of the geometric series. The function $|(E+A)^{-1}|$ takes a maximum on $|A| \le 1/2$. Let P be the square of this maximum. An easy computation gives

$$C = (E + A)^{-1}(-AB)(E + B)^{-1}.$$

This shows $|C| \le P|A||B|$. □

Proof of 3.3 continued. It is our goal to get an estimate for G'_ν, G''_ν. To apply this lemma to our situation we make an assumption about our system of enlarged cuboids. We assume that the minimal distance of any point of $Q_{\nu+1}$ to a boundary point of Q_ν is $\ge \delta 2^{-\nu}$ with some positive constant δ. It is clear such a constant δ exists (depending on the shape of $\overline{Q \cap Q'} \subset U$).

We will proceed by induction to produce

$$\begin{aligned} &|G_\nu(z)| \le \varrho \cdot 4^{-\nu} \text{ for } z \in Q'_\nu \cap Q''_\nu, \\ &|G'_\nu(z)| \le C2^{-\nu} \text{ for } z \in Q'_\nu \\ &|G''_\nu(z)| \le C2^{-\nu} \text{ for } z \in Q''_\nu \end{aligned}$$

The constants C, ϱ will be determined during the proof. Whatever the constants will be, we can get the beginning of the induction G_0, G'_0, G''_0 since, as we mentioned at the beginning of the proof, G can be assumed as small as we want. Assume that G_ν and G'_ν, G''_ν have been constructed. Then we construct $G_{\nu+1}$ and then the decomposition $G_{\nu+1} = G''_{\nu+1} + G''_{\nu+1}$. For $G_{\nu+1}$ we get the estimate (Lemma 3.4)

$$|G_{\nu+1}(z)| \le PC^2 4^{-\nu}.$$

So, if we make the choice

$$\varrho := 4PC^2,$$

we get the desired inequality $|G_{\nu+1}(z)| \le \varrho \cdot 4^{-(\nu+1)}$. For $G'_{\nu+1}$ (similarly $G''_{\nu+1}$) we get from Lemma 3.2 the estimate

$$|G'_{\nu+1}(z)| \le \frac{2^\nu M}{\delta} \cdot \varrho \cdot 4^{-(\nu+1)} = \frac{2MPC^2}{\delta} 2^{-(\nu+1)}.$$

So alle we need is the estimate

$$2MPC^2 \le \delta C.$$

This is true if C is small enough.

4. The syzygy theorem

We need a sheaf theoretic version of a famous result, namely Hilbert's syzygy theorem. Hilbert expressed this theorem for the polynomial ring but the proof works literally also for the ring of power series. It states:

4.1 Hilbert's syzygy theorem. *Let M be a finitely generated module over the ring $R = \mathbb{C}\{z_1, \dots, z_n\}$ of convergent power series in n variables. Let*

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

be a an exact sequence where the modules F_i are finitely generated free modules. Then the kernel of $F_n \to F_{n-1}$ is free.

Corollary. *For any finitely generated module M there exists an exact sequence*

$$0 \longrightarrow F_{n+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

with free modules F_i.

There is an immediate sheaf theoretic consequence.

4.2 Remark. *Let $\mathcal{M}$ be a coherent sheaf on some open subset $U \subset \mathbb{C}^n$ and $a \in U$ a point. There exists an open neighborhood $a \in V \subset U$ and an exact sequence*

$$0 \longrightarrow \mathcal{F}_{n+1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{M}|V \longrightarrow 0$$

where $\mathcal{F}_i \cong \mathcal{O}_V^{n_i}$ are free sheaves on V.

Proof. We choose a resolution of the module $\mathcal{M}_a$

$$0 \longrightarrow F_{n+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow M_a \longrightarrow 0$$

by free $\mathcal{O}_{U,a}$-modules. We can extend this sequence using V.7.10 and V.7.11. □

There is a much better result:

4.3 Proposition. *Let $\mathcal{M}$ be a coherent sheaf on a cuboid Q and $Q_0 \subset Q$ a shrunken cuboid. Then there exists an exact sequence*

$$0 \longrightarrow \mathcal{F}_{n+1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{M}|Q_0 \longrightarrow 0$$

where $\mathcal{F}_i \cong \mathcal{O}_{Q_0}^{n_i}$ are free sheaves on Q_0.

The proof of this proposition rests on the *Cartan gluing lemma* 3.3. During the proof we use the following short notation. Let $\mathcal{M}$ be a coherent sheaf on some open subset $U \subset \mathbb{C}^n$. The sheaf $\mathcal{M}$ admits a free resolution over a compact subset $K \subset U$ if there exists an open set $K \subset V \subset U$ and an exact sequence

$$0 \longrightarrow \mathcal{F}_{n+1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{M}|V \longrightarrow 0$$

with free $\mathcal{O}_V$-modules $\mathcal{F}_i$.

4.4 Lemma. *As in 3.1 we consider two rectangles in the position*

$$R' = (a,c) \times I, \ R'' = (b,d) \times I \quad (a < b < c < d)$$

and then the cuboids $Q' = R' \times D$, $Q'' = R'' \times D$ *with a cuboid* $D \subset \mathbb{C}^n$. *Let* $\mathcal{M}$ *be a coherent sheaf over some open neighborhood of* $\overline{Q' \cup Q''}$.

Assume that $\mathcal{M}$ *admits free resolutions over* $\bar{Q}'$ *and* $\bar{Q}''$. *Then* $\mathcal{M}$ *admits a free resolution over* $\overline{Q' \cup Q''}$.

Before we prove this lemma we show that 4.3 follows from it.

We decompose the cuboid Q_0 into N^{2n} closed small sub-cubes, by dividing each edge into N equidistant sub-cuboids as indicated in the figure.

By means of 4.2 and a simple compactness argument this can be done in such a way that $\mathcal{M}$ admits a free resolution over the closure of each small sub-cuboid. Application of the gluing lemma 4.4 several times leads to a free resolution over $\bar{Q}_0$. We describe this in more detail in the case $n = 1$: In the first step one produces a resolution over the first row of squares in the above figure

Then we do the same with the second row and then glue the first with the second row. This gives a free resolution over

It should be clear that this argument works in arbitrary dimension. So we are reduced to the

Proof of 4.4. The resolutions over $\bar{Q}'$ and $\bar{Q}''$ give two different resolutions over the intersection. So we need a method to compare two different resolutions. The principle can be understood already in the local case. So let us assume that we have a finitely generated module M over a ring R and that we have two different free resolutions

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F_{n+1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ 0 & \longrightarrow & G_{n+1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Two such resolutions are called isomorphic if there is a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F_{n+1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & G_{n+1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the vertical arrows are isomorphisms. It is not true that two resolutions are isomorphic. The reason that there exist trivial resolutions of 0. By a trivial resolution of 0 we understand a resolution of the form

$$0 \cdots \longrightarrow F \xrightarrow{\text{id}} F \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

with a free module F. One can define the direct sum of a resolution with such a trivial resolutions. (The direct sum two resolutions

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F_{n+1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ 0 & \longrightarrow & G_{n+1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

is

$$0 \longrightarrow F_{n+1} \oplus G_{n+1} \longrightarrow \cdots \longrightarrow F_1 \oplus G_1 \longrightarrow M \oplus N \longrightarrow 0$$

with obvious arrows. In the case $N = 0$ we can identify $M \oplus 0$ and M.)

By an *elementary modification* of a free resolution we understand a new free resolution which one obtains if one takes the direct sum with a trivial resolution of 0 as described above.

4.5 Lemma. *Two free resolutions of an R-module M get isomorphic after performing a finitely many elementary modifications (to both of them).*

Proof. The proof is given by some induction. The first step is to modify F_1, G_1 if necessary. We take free generators of F_1 and consider their images in M. Taking inverse images of them in G_1 we construct an R-linear map $\sigma : F_1 \to G_1$ and similarly $\tau : G_1 \to F_1$ such that the diagrams

$$\begin{array}{ccc} F_1 & \longrightarrow & M \\ \sigma \downarrow & & \downarrow \| \\ G_1 & \longrightarrow & M \end{array} \quad , \qquad \begin{array}{ccc} F_1 & \longrightarrow & M \\ \tau \uparrow & & \uparrow \| \\ G_1 & \longrightarrow & M \end{array}$$

commute. It may be that σ and τ^{-1} are isomorphisms. Then we do nothing. Otherwise we add to the F-resolution the trivial resolution $0 \to G_1 \to G_1 \to 0$ and to the G-resolution the trivial resolution $0 \to F_1 \to F_1 \to 0$. We get new resolutions

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & F_3 & \longrightarrow & F_2 \oplus G_1 & \longrightarrow & F_1 \oplus G_1 & \longrightarrow & M & \longrightarrow 0 \\ & & & & & & \downarrow & & \| & \\ \cdots & \longrightarrow & G_3 & \longrightarrow & G_2 \oplus F_1 & \longrightarrow & G_1 \oplus F_1 & \longrightarrow & M & \longrightarrow 0 \end{array}$$

where the vertical arrows have to be explained. The map $F_1 \oplus G_1 \to G_1 \oplus F_1$ is defined by means of the matrix

$$\begin{pmatrix} \sigma & 1 - \sigma\tau \\ 1 & -\tau \end{pmatrix}.$$

This has to be understood as follows. The action on a pair (f, g) is given by

$$\begin{pmatrix} \sigma & 1-\sigma\tau \\ 1 & -\tau \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \sigma(f)+g-\sigma\tau(g) \\ f-\tau(g) \end{pmatrix}.$$

The essential point is that this map is an isomorphism. The inverse map is given though the matrix

$$\begin{pmatrix} -\tau & -1+\sigma\tau \\ -1 & \sigma \end{pmatrix}.$$

One checks that the above diagram is commutative. This shows that we can reduce to the situation

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F_{n+1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & G_{n+1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the vertical arrow is an isomorphism. This was the first step of the induction. We explain, how to continue. It might happen that $F_\nu = G_\nu = 0$ for $\nu \geq 0$. Then F_2, G_2 can be considered as submodules of F_1, G_1. The map $F_1 \to G_1$ maps F_2 into G_2 and conversely. Hence we have isomorphic resolutions $0 \to F_2 \to F_1 \to M \to 0$ and $0 \to G_2 \to G_1 \to M$ and we are done. Otherwise we construct now a linear map $\sigma : F_2 \to G_2$ such the diagram

$$\begin{array}{ccccccc} F_2 & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ \downarrow & & \downarrow & & \| & & \\ G_2 & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

commutes. This can easily done by means of the free generators. Similarly we construct $\tau : G_2 \to F_2$. We modify now with the complexes $\cdots 0 \to G_2 \to G_2 \to 0 \to 0$ and $\cdots 0 \to F_2 \to F_2 \to 0 \to 0$ and reduce to a situation

$$\begin{array}{cccccccc} \cdots & F_2 & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0, \\ & \downarrow & & \downarrow & & \| & & \\ \cdots & G_2 & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where both vertical arrows are isomorphism. I should be clear now how the induction runs and terminates.

Proof of 4.4 continued. We come back to the resolutions of $\mathcal{M}$ over $\bar{Q}'$ and $\bar{Q}''$. This means that there are two cuboids $\bar{Q}' \subset \tilde{Q}'$ and similarly $\tilde{Q}''$ that are located similarly as described in 4.4 and such that the resolutions of $\mathcal{M}$ are defined over $\tilde{Q}', \tilde{Q}''$. After finitely many modifications they are isomorphic over the intersection. This means that the resolutions are of the form

$$0 \longrightarrow \mathcal{O}_{\tilde{Q}'}^{m_{n+1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\tilde{Q}'}^{m_1} \longrightarrow \mathcal{M}|\tilde{Q}' \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{\tilde{Q}''}^{m_{n+1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\tilde{Q}''}^{m_1} \longrightarrow \mathcal{M}|\tilde{Q}'' \longrightarrow 0$$

and over $\tilde{Q}' \cap \tilde{Q}''$ there are isomorphisms σ such that the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{O}^{m_{n+1}}_{\tilde{Q}'\cap\tilde{Q}''} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}^{m_1}_{\tilde{Q}'\cap\tilde{Q}''} & \longrightarrow & \mathcal{M}|(\tilde{Q}'\cap\tilde{Q}'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\sigma} & & & & \downarrow{\scriptstyle\sigma} & & & & \\
0 & \longrightarrow & \mathcal{O}^{m_{n+1}}_{'\cap\tilde{Q}''} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}^{m_1}_{\tilde{Q}'\cap\tilde{Q}''} & \longrightarrow & \mathcal{M}|(\tilde{Q}'\cap\tilde{Q}'') & \longrightarrow & 0
\end{array}$$

gets commutative.

The isomorphisms σ are given by invertible holomorphic functions $\tilde{Q}'\cap\tilde{Q}'' \to \mathrm{GL}(m_i, \mathcal{O}(\tilde{Q}' \cap \tilde{Q}''))$. Now can Cartan's gluing lemma to write σ as product $\sigma = \sigma'\sigma''$, where σ' is a holomorphic map from Q' to $\mathrm{GL}(m_i, \mathcal{O}(Q'))$ and similarly σ''. To be precise we first have to shrink $\tilde{Q}'$ and $\tilde{Q}''$ a little. We use the isomorphisms σ', σ'' to modify the resolution of $\mathcal{M}|\tilde{Q}', \mathcal{M}|\tilde{Q}''$ in such a way that now the two resolutions over $\tilde{Q}' \cap \tilde{Q}''$ are identical. If this is the case they glue to single resolution of $\mathcal{M}$ over $\tilde{Q}' \cup \tilde{Q}''$. This finishes the proof of 4.4 and then of 4.3. □

5. Theorem B for cuboids

We know form the lemma of Dolbeault VI.6.9 that the cohomology groups $H^q(Q, \mathcal{O}_Q)$, $q > 0$, vanish for a poly disk Q. Since every rectangle is is biholomorphic equivalent to the unit disk this is also true for cuboids. This section is devoted to the proof of

5.1 Theorem B for cuboids. *Let $\mathcal{M}$ be a coherent sheaf on a cuboid Q. Then*

$$H^q(Q, \mathcal{M}) = 0 \quad \textit{for} \quad q > 0.$$

Corollary. *Theorem B is true for polydisks.*

The corollary follows since each rectangle in the complex plane is biholomorphic equivalent to a disk. Technically it has advantages to work with cuboids instead of polydisks.

It will be necessary to shrink Q a little. This means that we have to consider a cube Q_0 whose (compact) closure is contained in Q. We write $Q_0 \subset\subset Q$ to indicated this. There are two different steps. In the first basic step we will prove

5.2 Theorem. *let $\mathcal{M}$ be a coherent sheaf on a cuboid Q and Q_0 a shrunken cuboid. Then*

$$H^q(Q_0, \mathcal{M}|Q_0) = 0 \quad \textit{for} \quad q > 0.$$

Proof. We just have to show: Let

$$0 \to F_{n+1} \longrightarrow \cdots F_1 \longrightarrow F \longrightarrow 0$$

be an exact sequence of sheaves such that all F_i are acyclic. (This means that the higher cohomology groups vanish). Then F acyclic. For the proof one considers the sequence

$$0 \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow\longrightarrow 0.$$

(So K is a co-kernel). From the long exact cohomology seqence follows that K is acyclic. There is an obvious exact sequence

$$0 \longrightarrow K \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow K.$$

Now we can argue by induction on n. □

Proof of 5.1. The proof for arbitrary cubes uses an exhaustion argument. This argument also will work in the general case of arbitrary Steil spaces. But in the case of a cuboid is it is technically easier. Hence we give the details already in the case of the cuboid.

If X is an complex manifold we know that $\mathcal{O}_X(X)$ gets the structure as a Frèchet space if one equips it with the topology of uniform convergence on compact subsets. Slightly more generally $\mathcal{O}_X(X)^n$ gets a Frèchet space if we equip it with the product topology. Our starting point for constructing topologies is:

5.3 Lemma. *Let X be a complex manifold and $\mathcal{M} \subset \mathcal{O}_U^m$ be a coherent subsheaf of a free sheaf. Then $\mathcal{M}(X)$ is a closed subspace of $\mathcal{O}_X^m(X)$.*

Proof. Let s_k be sequence in $\mathcal{M}(X)$ that converges (uniformly on compact subsets) to $s \in \mathcal{O}_X(X)^n$. We have to show that $s \in \mathcal{M}(X)$. This means that for any point $a \in X$ we have $s_a \in \mathcal{M}_a$. We use the notation $F = \mathcal{O}_{X,a}^n$ and $M = \mathcal{M}_a$. We consider the maximal ideal $\mathfrak{m}$ in the local ring $\mathcal{O}_{X,a}$. The vector space $F/\mathfrak{m}^m F$ is finite dimensional for any m. Hence it carries a natural topology. Now we consider the images $\bar{s}_n, \bar{s}$ of s_n, s in $F/\mathfrak{m}^m F$. The essential point is that $\bar{s}_n$ converges to $\bar{s}$ in this finite dimensional vector space. We have to explain the reason for this. Taking coordinates we can identify $\mathcal{O}_{X,a}$ with the ring of power series. Then $\mathcal{O}_{X,a}/\mathfrak{m}^m$ can be identified with a $\mathbb{C}^N$ where the map $\mathcal{O}_{X,a}/\mathfrak{m}^m \to \mathbb{C}^N$ associates to a power series the vector of coefficients a_ν of degree $\leq m$. Now we have to use from complex analysis that the locally uniform convergence $s_k \to s$ implies the locally uniform convergence of all partial derivatives and hence also of the Taylor coefficients. This proves that $\bar{s}_n \to \bar{s}$ in $F/\mathfrak{m}^m F$. Now we use that every sub-vector space of a finite dimensional vector space is closed. This gives us that $\bar{s}$ is in the image of M since this is a sub-vector space. This can be expressed as

$$s \in M + \mathfrak{m}^m F.$$

From this follows $s \in M$ by a pure algebraic argument: One has to use Krull's intersection theorem. Application of the Krull intersection theorem completes the proof of 5.3. □

Let $\mathcal{M}$ be a coherent sheaf on a cuboid Q. We shrink Q to a cuboid Q_0. We want to construct a topology on $\mathcal{M}(Q_0)$ For this purpose we slightly enlarge the shrink. That is we choose a cuboid Q_1 such that Q_0 is a shrink of Q_1 and Q_1 is a shrink of Q. We know that $\mathcal{M}|Q_1$ is finitely generated. This means that there exists a surjective map $\mathcal{O}_{Q_1}^n \to \mathcal{M}|Q_1$. From the weak form of Theorem B (5.2) we we get the surjectivity

$$\mathcal{O}_Q(Q_0)^n \longrightarrow \mathcal{M}(Q_0).$$

From 5.3 follows that the kernel is closed. In this way we get a structure as Frèchet space on $\mathcal{M}(Q_0)$. It is rather clear that this structure is independent of the presentation $\mathcal{O}_{Q_1}^n \to \mathcal{M}|Q_1$. Hence we obtain:

5.4 Lemma. *Let Q be a cuboid and $\mathcal{M}$ a finitely generated coherent sheaf on Q. Let Q_0 be a shrunken cuboid. Then $\mathcal{M}(Q_0)$ carries a unique structure as Frèchet space with the following property. For each cuboid $Q_0 \subset\subset Q_1 \subset\subset Q$ and all surjective maps $\mathcal{O}_{Q_1}^n \to \mathcal{M}|Q_1$ the induced map $\mathcal{O}_Q^n(Q_0) \to \mathcal{M}(Q_0)$ is continuous.*

A direct consequence of Runge's approximation theorem 2.1 is:

5.5 Runge approximation theorem for coherent sheaves (weak form). *Let $Q_0 \subset\subset Q_1 \subset\subset Q$ be cuboids and let $\mathcal{M}$ be a coherent sheaf on Q. The image of the restriction map $\mathcal{M}(Q_1) \to \mathcal{M}(Q_0)$ is dense.*

Now we collected all tools for:

Proof of Theorem B for cubes 5.1. We choose a sequence of cuboids

$$Q_1 \subset\subset Q_2 \subset\subset Q_3 \subset\subset \cdots \subset\subset Q$$

whose union is Q. This is an open covering $\mathfrak{U}$ of Q. We know $H^q(Q_\nu, \mathcal{M}|Q_\nu) = 0$ for $q > 0$. We want to show that $H^q(Q, \mathcal{M}) = 0$ for $q > 0$. From Leray's theorem follows that this cohomology group can be computed by means of Čech cohomology

$$H^q(\mathfrak{U}, \mathcal{M}) = H^q(Q, \mathcal{M}).$$

Similarly we get

$$H^q(\mathfrak{U}_m, \mathcal{M}|Q_m) = H^q(Q_m, \mathcal{M}|Q_m) \qquad (= 0),$$

where $\mathfrak{U}_m$ denotes the (finite) covering of U_m by $U_1, \dots, U_m$. We recall that the Čech complex has been denoted by $C^q(\mathfrak{U}, \mathcal{M})$. For sake of simplicity we use the notation $C^q(\mathfrak{U}_m,\mathcal{M}) := C^q(\mathfrak{U}_m,\mathcal{M}|Q_m)$. There are natural restriction maps

$$C^q(\mathfrak{U},\mathcal{M}) \longrightarrow C^q(\mathfrak{U}_m,\mathcal{M}) \longrightarrow C^q(\mathfrak{U}_k,\mathcal{M}) \quad \text{for} \quad m > k$$

and there is a natural (injective) extension map

$$C^q(\mathfrak{U}_k,\mathcal{M}) \longrightarrow C^q(\mathfrak{U}_m,\mathcal{M}) \quad \text{for} \quad m > k,$$

a cochain s is extended by the definition $s(i_0,\dots,i_q) = 0$ if one of the indices is out of the range (greater than k).

We consider now a cochain $s \in C^q(\mathfrak{U},\mathcal{M})$, $q > 0$, with the property $ds = 0$ We have to show that there is cochain $t \in C^{q-1}(\mathfrak{U},\mathcal{M})$ with $dt = s$. We denote by $s^{(m)} \in C^q(\mathfrak{U}_m,\mathcal{M})$ for some $m > 0$ the restriction of s. Since $ds = 0$ implies $ds^{(m)} = 0$ we get $s^{(m)} = dt^{(m)}$ with $t^{(m)} \in C^{q-1}(\mathfrak{U}_m,\mathcal{M})$. We can restrict $t^{(m)}$ to $C^{q-1}(\mathfrak{U}_{m-1},\mathcal{M})$. From the restriction we can subtract $t^{(m-1)}$. We denote the result simply by $t^{(m)} - t^{(m-1)}$. We know $d(t^{(m)} - t^{(m-1)}) = 0$. There are two different cases. The case $q > 1$ is very easy, the difficult part will be the case $q = 1$.

First case, $q > 1$. In this case we have still $H^{q-1}(\mathfrak{U}_{m-1},\mathcal{M}) = 0$. Hence there exists

$$\alpha_{m-1} \in C^{q-2}(\mathfrak{U}_{m-1},\mathcal{M}) \quad \text{such that} \quad t^{(m)} - t^{(m-1)} = d\alpha_{m-1}.$$

We denote the natural extension of α_{m-1} to $C^{q-2}(\mathfrak{U}_k,\mathcal{M})$, $k > m-1$, by the same letter. Then we can define

$$T^{(m)} := t^{(m)} - d\Big(\sum_{k=1}^{m-1} \alpha^{(k)}\Big) \qquad \big(\in C^{q-1}(\mathfrak{U}_m,\mathcal{M})\big).$$

The $T^{(m)}$ are modifications of the $t^{(m)}$ in the sense that the satisfy $s^{(m)} = dT^{(m)}$. The advantage of the modification is that we now have that the system $T^{(m)}$ is compatible. We omit the simple calculation for it. This means that the restriction of $T^{(m)}$ to $\mathfrak{U}_{m-1}$ is $T^{(m-1)}$. This implies that they glue to a cochain $t \in C^{q-1}(\mathfrak{U},\mathcal{M})$. But with this cochain we clearly have $dt = s$. This is want we wanted to prove.

Second case, $q = 1$. We consider the sequence of Frèchet spaces

$$\mathcal{M}(Q_1) \longleftarrow \mathcal{M}(Q_2) \longleftarrow \mathcal{M}(Q_2) \longleftarrow \cdots$$

The image of each arrow is dense. Recall that we have chosen $t^{(m)}$ with $dt^{(m)} = s^{(m)}$. Now $q = 1$. The elements of $C^0(\mathcal{U}_m,\mathcal{M})$ attach to each index $k \le m$ a section from $\mathcal{M}(Q_k)$. If the element is closed, then these sections glue to a section from $\mathcal{M}(Q_m)$. Hence $\mathcal{M}(Q_m)$ can be identified with the closed elements from $C^0(\mathcal{U}_m,\mathcal{M})$. In this way $t^{(m)} - t^{(m-1)}$ can be considered as element of

$\mathcal{M}(Q_{m-1})$. As in the first case we will have to replace $t^{(m)}$ by some other $T^{(m)} = t^{(m)} + \alpha^{(m)}$. Here $\alpha^{(m)}$ should be a zero cochain with the property $d\alpha^{(m)} = 0$. As we explained this can be considered as element of $\mathcal{M}(Q_m)$. The construction of $\alpha^{(m)}$ will use Runge approximation. The aim of the construction is that the sequence $T^{(m)}$ converges. Since the entries of this sequence are in different spaces, we have to explain what convergence means: It means that there exist an $T \in \mathcal{M}(Q)$ such that for each k the sequence $(t^{(m)})_{\geq k}$, more precisely its image in $\mathcal{M}(Q_k)$ converges to $T|\mathcal{M}(Q_k)$. To prove the convergence, we will use the Cauchy criterion: For each k we will have to show:

For each neighborhood $0 \in U \subset \mathcal{M}(Q_k)$ there exists an N such that the image of $T^{(\mu)} - T^{(\nu)}$ in $\mathcal{M}(Q_k)$ is contained in U for $\mu > \nu \geq 0$.

We will use also that each space $\mathcal{M}(Q_m)$ has a countable fundamental system of neighborhoods of the origin (Frèchet spaces are metrizable).

For each m we choose a fundamental system of neighborhoods of the origin as indicated in the figure

$$\begin{array}{ccccccc}
\mathcal{M}(Q_1) & \longleftarrow & \mathcal{M}(Q_2) & \longleftarrow & \mathcal{M}(Q_3) & \longleftarrow & \cdots \\
\cup & & \cup & & \cup & & \\
U_{11} & \longleftarrow & U_{21} & \longleftarrow & U_{31} & & \cdots \\
\cup & & \cup & & \cup & & \\
U_{12} & \longleftarrow & U_{22} & \longleftarrow & U_{32} & & \cdots \\
\cup & & \cup & & \cup & & \\
U_{13} & \longleftarrow & U_{23} & \longleftarrow & U_{33} & \longleftarrow & \cdots \\
\cup & & \cup & & \cup & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

The horizontal arrows indicate that U_{km} is mapped to $U_{k,m-1}$ under the restriction map $\mathcal{M}(Q_m) \to \mathcal{M}(Q_{m-1})$. We also want to have that the neighborhoods shrink rapidly in the sense $U_{m,k+1} + U_{m,k+1} \subset U_{m,k}$. It clear that such a system of neighborhoods can be constructed. Then induction shows

$$\overbrace{U_{m,k+\nu} + \cdots + U_{m,k+\nu}}^{\nu+1 \text{ copies}} \subset U_{m,k}.$$

After this preparation we come the construction of $T^{(m)} = t^{(m)} + \alpha^{(m)}$. What we want to have is $T^{(m+1)} - T^{(m)} \in U_{m,m}$. It is now problem to construct this by induction. One starts with $T^{(1)} = t^{(1)}$. Assume that $T^{(1)}, \dots T^{(m)}$ have been constructed. We construct $T^{(m+1)}$. For this we consider $T^{(m+1)} - t^{(n)} \in \mathcal{M}(Q_m)$. By the approximation theorem there exists an element $\alpha^{(m+1)} \in \mathcal{M}(Q_{m+1})$ such that $T^{(m)} - t^{(m+1)} - \alpha^{(m+1)} \in U_{m,m}$. Now $T^{(m+1} = t^{(m+1)} + \alpha^{(m+1)}$ has the desired property.

We have to check that $T^{(m)}$ is a Cauchy sequence in the described sense. For this wa have to fix an k and to consider a neighborhood of the origin in

$\mathcal{M}(Q_k)$. We can take this neighborhood in the form $U = U_{k,l}$ with some l. We have to construct an $N \geq k$ such that

$$T^{(\mu)} - T^{(\nu)} \longrightarrow U \quad \text{for} \quad \mu \geq \nu \geq N.$$

(The arrow just indicates that —after restriction to $\mathcal{M}(Q_k)$— the element should be contained in U.) We claim that a possible choice is $N = \max(k, l+2)$. For this we decompose

$$T^{(\mu)} - T^{(\nu)} = (T^{(\mu)} - T^{(\mu-1)}) + \cdots + (T^{(\nu+1)} - T^{(\nu)}).$$

We can consider this element in $\mathcal{M}(Q_\nu)$. There it lies in

$$U_{\nu,\mu-1} + U_{\nu,\mu-2} + \cdots + U_{\nu,\nu} \subset \overbrace{U_{\nu,\nu} + \cdots + U_{\nu,\nu}}^{\mu-\nu-2 \text{ copies}} \subset U_{\nu,\nu-(\mu-\nu-2)} = U_{\nu,2\nu-\mu-2}.$$

Hence the image of $T^{(\mu)} - T^{(\nu)}$ in $\mathcal{M}(Q_k)$ is in $U_{k,2\nu-\mu-2}$. Since $2\nu - \mu - 2 \geq \nu - 2 \geq N - 2 \geq l$ we obtain $T^{(\mu)} - T^{(\nu)} \in U_{k,l}$ as desired. So the global section T "$= \lim T^{(m)}$" has been constructed.

Finally we claim $dT = s$ (globally). Since $dT^{(k)} = s^{(k)}$ we only have to show that $T|U_k - T^{(k)}$ is closed. From construction is a limit of the close elements. Now d is clearly a continuous operator. This finishes the proof of 5.1. □

6. Theorem A and B for Stein spaces

The basic theorems about Stein spaces are

6.1 Theorem A for Stein spaces. *Let X be a Stein space and $\mathcal{M}$ a coherent sheaf. For each $a \in X$ the stalk $\mathcal{M}_a$ can be generated by (the germs of) finitely many global sections.*

6.2 Theorem B for Stein spaces. *Let X be a Stein space and $\mathcal{M}$ a coherent sheaf. Then*

$$H^q(X, \mathcal{M}) = 0 \quad \textit{for} \quad q > 0.$$

The formulation seems to indicate that we have two independent theorems. Actually theorem A is an easy consequence of theorem B. To prove this we consider the vanishing ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$ of the point a and then for an arbitrary natural number Then we use the exact sequence

$$0 \longrightarrow \mathcal{J}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{J} \longrightarrow 0.$$

From theorem B we get that $\mathcal{M}(X) \to (\mathcal{M}/\mathcal{J})(X)$ is surjective. Notice that $(\mathcal{M}/\mathcal{J})(X) = \mathcal{M}_a/\mathfrak{m}\mathcal{M}_a$. Here $\mathfrak{m}$ means the maximal ideal of $\mathcal{O}_{X,a}$. We denote by M the submodule of $\mathcal{M}_a$ that is generated by the image of $\mathcal{M}(X)$ and by $N = \mathcal{M}_a/M$ the factor module. The above argument shows $\mathcal{M}_a = M + \mathfrak{m}\mathcal{M}_a$ or $\mathfrak{m}N = N$. The proof now follows from the lemma of Nakayama.

For the proof of theorem B we will use an exhaustion by Oka domains. So the procedure is similar as in the proof of Theorem B for cuboids. But there are some technical difficulties that arise. One of them is to define a structure as Frèchet space on $\mathcal{M}(X)$ for singular X. Actually this is possible for each coherent sheaf on an arbitrary complex space in a natural way. But the construction is difficult. This is already visible for the structure sheaf. Actually on can try to equip $\mathcal{O}_X(X)$ with the topology of uniform convergence on compact sets. To make this work correctly one needs that the limit of a sequence of analytic functions that converges uniformly on each compact subset is analytic too. Actually this is true but unfortunately rather deep and not at reach at the moment. Hence we restrict to topologize $\mathcal{M}(X)$ only in special cases.

We will use 5.3 to construct a Frèchet topology on $\mathcal{O}_X(X)$ for special non-smooth complex spaces. Let $P \subset \mathbb{C}^n$ be a polydisk and $X \subset P$ a closed analytic subset. We have a natural map $\mathcal{O}_P(P) \to \mathcal{O}_X(X)$. This map is surjective by Theorem B for polydisks. To see this just consider the ideal sheaf $\mathcal{J} \subset \mathcal{O}_P$ corresponding to X. Then we have a short exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_P/\mathcal{J} \longrightarrow 0.$$

From theorem B we get $H^1(P, \mathcal{J}) = 0$ and form this the surjectivity of $\mathcal{O}_P(P) \to \mathcal{O}_P(P)/\mathcal{J}(P)$. There is a natural isomorphism $\mathcal{O}_X(X) \cong \mathcal{O}_P/\mathcal{J}(P)$. This gives the claimed surjectivity $\mathcal{O}_P(P) \to \mathcal{O}_X(X)$. We know from 5.3 that the kernel is closed. Hence the factor space of $\mathcal{X}_P(P)$ by this kernel carries a natural structure as Frèchet space. We transport this structure to $\mathcal{O}_X(X)$ to get a structure as Frèchet space there.

6.3 Proposition. *Let X be a complex space such there exists polydisk P and a closed holomorphic embedding $\alpha : X \to \mathcal{P}$. There exists a unique structure as Frèchet space on $\mathcal{O}_X(X)$ such that the induced map $\mathcal{O}_P(P) \to \mathcal{O}_X(X)$ is continuous. This structure is independent of the choice of the embedding α.*

The open mapping theorem for Frèchet spaces shows that $\mathcal{O}_P(P)$ must carry the quotient topology of $\mathcal{O}_P(P)$. Hence we only have to show the independence of the choice of the embedding α. Let $\beta : X \to P'$ be another closed embedding. We connect both embeddings to an embedding

$$(\alpha, \beta) : X \to P \times P'.$$

We consider the natural maps

$$\mathcal{O}_P(P) \longrightarrow \mathcal{O}_{P\times P'}(P \times P') \longrightarrow \mathcal{O}_X(X).$$

the first one is associated to the projection $P \times P' \to P$. Now we first equip $\mathcal{O}_X(X)$ with the quotient topology of $\mathcal{O}_{P\times P'}(P \times P')$. Since $\mathcal{O}_P(P) \to \mathcal{O}_{P\times P'}(P \times P')$ is continuous by trivial reasons we get with this topology that $\mathcal{O}_P(P) \to \mathcal{O}_X(X)$ is continuous. By the open mapping theorem $\mathcal{O}_X(X)$ must carry the quotient topology of $\mathcal{O}_P(P)$. So we see that $\mathcal{O}_P(P)$ and $\mathcal{O}_{P\times P'}(P \times P')$ induce the same topology. Since the roles of P and P' can be interchanged, we see that $\mathcal{O}_P(P)$ and $\mathcal{O}_{P'}(P')$ induce the same topology.

□

We get a first version of a variant of Runge's approximation theorem.

6.4 Approximation theorem, first version. *Let X' be a complex space that can be embedded into some polydisk $\beta : X' \hookrightarrow Q' \subset \mathbb{C}^n$ as closed complex subspace. Let X be an open subset of X that also can be embedded into some polydisk $\alpha : X \hookrightarrow Q \subset \mathbb{C}^m$ as closed complex subspace. We assume that the function α extends to a holomorphic map $X' \to \mathbb{C}^m$. Then the following holds:*

1) *The natural map $\mathcal{O}_{X'}(X') \to \mathcal{O}_X(X)$ is continuous.*
2) *The image of this map is dense.*

Proof. We denote the extension of α also by $\alpha : X' \to \mathbb{C}^m$. The two polydisks can be very different and not be compared directly. We improve this by modifying them. Instead of $\alpha : X \to Q$ we consider

$$X \longrightarrow Q \times Q', \quad x \longmapsto (\alpha(x), \beta(x)).$$

This is also an closed embedding. Similarly we consider

$$X' \longrightarrow \mathbb{C}^m \times Q', \quad x \longmapsto (\alpha(x), \beta(x)).$$

which is also a closed embedding. Since the topologies don't depend on the choice of the embeddings, we can assume from advance.

The polydisks Q, Q' are in the same $\mathbb{C}^n$ and we have $Q \subset Q'$. The diagram

$$\begin{array}{ccc} X & \hookrightarrow & Q \\ \cap & & \cap \\ X' & \hookrightarrow & Q' \end{array}$$

commutes.

From this diagram we get a map $\mathcal{O}_{Q'}(Q') \to \mathcal{O}_Q(Q) \to \mathcal{O}_X(X)$ that clearly is continuous. From the universal property of the quotient topology we get that $\mathcal{O}_{X'}(X') \to \mathcal{O}_X(X)$ is continuous. The claimed density now follows from the density of the image of $\mathcal{O}_{Q'}(Q') \to \mathcal{O}_Q(Q)$. This is a consequence of the possibility power series expansions in polydisk.

We need an extension of 6.4 to finitely generated coherent sheaves. For this we need a generalization of 5.3.

6.5 Lemma. *Let X be a complex space that can be embedded as closed analytic subset into a polydisk. Let $\mathcal{M} \subset \mathcal{O}_X^n$ be a coherent subsheaf of a free sheaf. Then $\mathcal{M}(X)$ is closed in $\mathcal{O}_X(X)^n$.*

Proof. Let $X \to P$ be the closed embedding into a polydisk. It is sufficient to show that inverse image of $\mathcal{M}(X)$ in $\mathcal{O}_P(P)^m$ is closed. But $\mathcal{M}$ is the module of global sections of a coherent sub-sheaf of $\mathcal{O}_P^n$. Hence we can apply 5.3.

□

6.6 Lemma. *Let X be a complex space that is embeddable as as closed analytic subset into a polydisk. Let $\mathcal{M}$ be a finitely generated coherent sheaf on X. Then there exists a unique structure as Frèchet space on $\mathcal{M}(X)$ such that for each presentation $\mathcal{O}_X^n \to \mathcal{M}$ the map $\mathcal{O}_X(X)^n \to \mathcal{M}(X)$ is continuous.*

The approximation theorem 6.4 now has an obvious generalization.

6.7 Runge's approximation theorem, second version. *Let X' be a complex space that can be embedded into some polydisk $\beta : X' \hookrightarrow Q' \subset \mathbb{C}^n$ as closed complex subspace. Let X be an open subset of X that also can be embedded into some polydisk $\alpha : X \hookrightarrow Q \subset \mathbb{C}^m$ as closed complex subspace. We assume that the function α extends to a holomorphic map $X' \to \mathbb{C}^m$. Assume that $\mathcal{M}$ is a finitely generated coherent sheaf on X'. Then the following holds:*

1) *The natural map $\mathcal{M}(X') \to \mathcal{M}(X)$ is continuous.*
2) *The image of this map is dense.*

With the so far developed tools the proof of theorem B is literally the same as for a cube 5.1. So can keep short. Using 1.6 we can construct an exhaustion

$$U_1 \subset\subset U_2 \subset\subset U_3 \subset\subset \cdots \quad \subset\subset X$$

by Oka domains. We know that the cohomology of $\mathcal{M}$ vanishes on each U_m. This follows from Theorem B for polydisks. We also have a Frèchet space structure on $\mathcal{M}(U_m)$ such the image in $\mathcal{M}(U_m)$ is dense. (Notice that 5.5 can be applied since Oka domains are embedded into polydisks by global functions.) So we have produced the analogue situation as we had in the case of a cuboid. The proof that we started behind 5.5 now works literally. □

7. Meromorphic functions

An element a of a ring R is called a non-zero divisor if $ax = 0 \Rightarrow x = 0$. Let S be a set of all non-zero divisors. Assume that $1 \in S$ and that $s, t \in S$ implies $st \in S$. Then we call S a multiplicative subset. There exists a ring R_S that contains R as subring such that the elements of S are invertible in R_S and such that each element of R_S can be written in the form a/s, $a \in R$, $s \in S$.

Such a ring is uniquely determined up to canonical isomorphism. It is called the total quotient ring of R. In the case that R is an integral domain, one can take for S the set of all non-zero elements and R_S then is the quotient field of R. Let $f : R_1 \to R_2$ be a ring homomorphism and let $S_1 \subset R_1$, $S_2 \subset R_2$ be multiplicative subsets such that $f(S_1) \subset S_2$ then the homomorphism f extends in a natural way to a homomorphisme $R_{S_1} \to R_{S_2}$.

Let $\mathcal{O}$ be sheaf of rings. For an open subset U we consider the set $S(U)$ of all $f \in \mathcal{O}(U)$ such that $f|V$ is a non-zero divisor in $\mathcal{O}(V)$ for each open $V \subset U$. In particular, the elements of $S(U)$ are non-zero divisors in $\mathcal{O}(U)$. Hence one can consider

$$\mathcal{O}(U)_{S(U)} = \{\ f/g; \quad f \in \mathcal{O}(U),\ g \in S(U)\ \}.$$

There are obvious restriction maps, such that this assignment gives a pre-sheaf. We denote the generated sheaf of rings by $\mathcal{M}$. The natural map $\mathcal{O} \to \mathcal{M}$ is injective since the functor „generated sheaf“ is exact. Hence we can consider $\mathcal{O}$ as a subsheaf of $\mathcal{M}$.

There is a natural map $\mathcal{O} \to \mathcal{M}$ of sheaves of rings and this map is injective, since the functor “generated sheaf” is exact.

We call $\mathcal{M}$ the sheaf of meromorphic sections of $\mathcal{O}$. The construction of $\mathcal{M}$ is compatible with restriction to open subsets U. This means that $\mathcal{M}|U$ can be identified with the sheaf of meromorphic sections of $\mathcal{O}|U$. Let $f \in \mathcal{M}(X)$ be a section of $\mathcal{M}$. Consider the set of all open subsets $U \subset X$ such that $f|U \in \mathcal{O}(U)$. The union of all these U is an open subset U_f of X. Clearly $f|U_f \in \mathcal{O}(U_f)$. We call U_f the domain of holomorphy of f.

Let a be a point. There is a natural map from $\mathcal{M}_a$ into the total quotient ring of $\mathcal{O}_a$. Clearly this is injective.

7.1 Lemma. *Let X be a topological space and let $\mathcal{O}$ be a coherent sheaf of rings. The natural homomorphism of $\mathcal{M}_a$ into the total quotient ring of $\mathcal{O}_{X,a}$ is an isomorphism.*

Proof. Let $f \in \mathcal{O}_X(U)$ be an element such that f_a is a non-zero divisor in $\mathcal{O}_{X,b}$. We know from the coherence theorems that then f_b is a non-zero divisor in a full neighborhood. This implies Lemma 7.1. □

Let $f \in \mathcal{M}(X)$ be a section of $\mathcal{M}$. Consider the set of all open subsets $U \subset X$ such that $f|U \in \mathcal{O}(U)$. The union of all these U is an open subset U_f of X. Clearly $f|U_f \in \mathcal{O}(U_f)$. We call U_f the domain of holomorphy of f.

So far $\mathcal{M}$ is a rather abstract object, even if $\mathcal{O} \subset \mathcal{C}_X$ is a sheaf of continuous functions, for example if $(X, \mathcal{O}_X)$ is a complex space in the sense of Serre. To remedy this situation, we make the following assumption.

7.2 Assumption. *Assume that $\mathcal{O}$ is a subsheaf of rings of $\mathcal{C}_X$. Assume that for each open subset $U \in \mathcal{O}$ and that each $f \in \mathcal{O}(U)$ with the property $f(x) \neq 0$ is invertible in $\mathcal{O}(U)$.*

This property is fulfilled of course for complex spaces in the sense of Serre.

7.3 Proposition. *Assume that the assumption above is fulfilled. Let f a global section of $\mathcal{M}_X$. The domain of holomorphy U_f is open and dense in X. Assume that there exists an open and dense subset U that is contained in U_f and such that $f(x) = 0$ for all $x \in U$. Then $f = 0$.*

Proof. Let $a \in X$ be a point in the complement of U. There exists a non-zero divisor $h_a \in \mathcal{O}_{X,a}$ such that $h_a f_a \in \mathcal{O}_{X,a}$. This implies $hf =$ in a full neighborhood W of a where $h \in \mathcal{O}_X(W)$ is a representative of h_a. The function fh is zero in $U \cap W$. By continuity it is zero in W. We can assume that h_b is a non-zero divisor of all $b \in W$. We obtain $f|W = 0$. This shows $f = 0$. □

So we see that global sections of $\mathcal{M}_X$ can be considered as holomorphic functions on dense open subsets with additional properties. For this reason we call sections of $\mathcal{M}_X$ simply "meromorphic functions".

8. Cousin problems

An additive Cousin datum on a complex X space is an open covering $\mathfrak{U} = (U_i)_{i\in I}$ on X and a collection of meromorphic functions $f_i \in \mathcal{M}_X(U_i)$ for all indices i such that for each two indices the difference $f_i - f_j$ is holomorphic on $U_i \cap U_j$. One then can ask whether there exists a global meromorphic function $f \in \mathcal{M}_X(X)$ such that $f - f_i$ is holomorphic on U_i for all i.

8.1 The first Cousin problem. *Let $X = (X, \mathcal{O}_X)$ be a complex space. Does any additive Cousin datum admit a solution?*

In standard courses about complex analyis one proves the Mittag-Leffler theorem which in a constructive way gives an positive answer in the case $X = \mathbb{C}$. Again we consider an open covering $\mathfrak{U} = (U_i)_{i\in I}$ of a complex space.

A multiplicative Cousin datum is a collection of holomorphic functions f_i with the following property:

a) The set of zeros of f_i is thin in U_i.
b) There exists a holomorphic function f_{ij} on $U_i \cap U_j$ without zeros such that $f_i = f_{ij} f_j$ on $U_i \cap U_j$.

This means that the zeros of f_i and f_j in $U_i \cap U_j$ are the same. Hence a multiplicative Cousin datum should be considered as prescription of zeros. On can ask whether there exists a global holomorphic function $f : X \to \mathbb{C}$ such that $f = \varphi_i f_i$ on U_i with a holomorphic function $\varphi : U_i \to \mathbb{C}$ without zeros.

8.2 Second Cousin problem. *Let X be a complex space. Does every multiplicative Cousin datum admit a solution?*

We are able to prove now:

8.3 Theorem. *Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open covering of a Stein space X $f_i \in \mathcal{M}_X(U_i)$ collection of meromorphic functions for all indices i such that for each two indices the difference $f_i - f_j$ is holomorphic on $U_i \cap U_j$. Then there exists a global meromorphic function $f \in \mathcal{M}_X(X)$ such that $f - f_i$ is holomorphic on U_i for all i.*

Proof. We consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{M}_X \longrightarrow \mathcal{M}_X/\mathcal{O}_X \longrightarrow 0.$$

We consider the images s_i of f_i in $(\mathcal{M}_X/\mathcal{O}_X)(U_i)$. By assumption they agree in the intersections and hence define a global section $(\mathcal{M}_X/\mathcal{O}_X)(X)$. Now from the long exact cohomology sequence follows that $\mathcal{M}(X) \longrightarrow (\mathcal{M}_X/\mathcal{O}_X)(X)$ is surjective. Choose $f \in \mathcal{M}(X)$ with image s. Then clearly $f_a - (f_i)_a$ is contained in $\mathcal{O}_{X,a}$ for all $a \in U_i$. This shows that $f - f_i$ is holomorphic on U_i. □

The second Cousin problem

Again we consider an open covering $\mathfrak{U} = (U_i)_{i\in I}$ of a complex space. A multiplicative Cousin datum is a collection of holomorphic functions f_i with the following property:

a) The set of zeros of f_i is thin in U_i.
b) There exists a holomorphic function f_{ij} on $U_i \cap U_j$ without zeros such that $f_i = f_{ij} f_j$ on $U_i \cap U_j$.

This means that the zeros of f_i and f_j in $U_i \cap U_j$ are the same. Hence a multiplicative Cousin datum should be considered as prescription of zeros. On can ask whether there exists a global holomorphic function $f : X \to \mathbb{C}$ such that $f = \varphi_i f_i$ on U_i with a holomorphic function $\varphi : U_i \to \mathbb{C}$ without zeros.

For a solution of this Cousin problem we need the sheaf $\mathbb{Z}_X$ of locally constant functions with values in $\mathbb{Z}$.

8.4 Theorem. *Let X be a Stein space with the property $H^2(X, \mathbb{Z}_X) = 0$. Then any multiplicative Cousin problem has a solution.*

Proof.. For any open subset $U \subset X$ we consider the set $\mathcal{O}_X^*(U)$ of holomorphic functions without zeros on U. This is a group under multiplication. (This statement easily can be reduced to the case $\mathbb{C}^n$ where it is known.) With usual restriction maps we get a sheaf $\mathcal{O}_X^*$ of abelian groups. Let $f \in \mathcal{O}_X(U)$. Then $e^{2\pi \mathrm{i} f}$ is holomorphic too. (Again this follows from the case $X = \mathbb{C}^n$.) We claim that the sequence

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

is exact. The only problem is the surjectivity. For this one has to show: Let $a \in X$ be a point and f a holomorphic function without zeros on some open neighborhood of a. Then there is a holomorphic function g in a maybe smaller open neighborhood of a with the property $e^g = f$. This also can be reduced easily to the case $X = \mathbb{C}^n$. For the construction of g one may assume that $|f(a) - 1| < 1$. Then one can make use of the logarithm series.

Now the proof of 8.4 is easy. From the assumptions and the long exact cohomology sequence we get $H^1(X, \mathcal{O}_X^*) = 0$. A Cousin distribution is nothing but a Čech cocycle. The solution of the Cousin problem means that this cocycle is trivial. Hence we have to show $\check{H}^1((\mathfrak{U}, \mathcal{O}_X^*) = 0$. But we know that the first Čech cohomology groups are embedded into the true cohomology. □

One should investigate now what it means that $H^2(X, \mathbb{Z}_X)$ is zero. Clearly this depends only on the topological nature of X. Hence it is more a problem of topology than of complex analysis. Hence we only mention

1) $H^2(\mathbb{C}^n, \mathbb{Z}_{\mathbb{C}^n}) = 0$. (This has been proved in VI.6.8.)
2) $H^2(U, \mathbb{Z}_U) = 0$ for an open subset $U \subset \mathbb{C}$. (We will not prove it here.)

We recall that in standard courses on complex functions the solution of the multiplicative Cousin problem for $X = \mathbb{C}$ is given in a constructive way by means of Weierstrass products. So we obtained a very remarkable generalization using cohomological methods.

Chapter IV. Finiteness Theorems

1. Compact Complex Spaces

The aim of this section is:

1.1 Theorem. *Let X be a compact complex space and $\mathcal{M}$ a coherent sheaf. The cohomology group $H^p(X, \mathcal{M})$ is a finite dimensional complex vector space for each p.*

Since the sheaf of holomorphic differential forms on a complex manifold is locally free and hence coherent we get

1.2 Corollary. *Let X be a compact complex manifold of (pure) dimension n. Then the cohomology groups*

$$H^p(X, \Omega_X^q)$$

are finite dimensional. Moreover the vanish for $p + q > n$.

Another approach uses Hodge theory.

Before we start with the proof we formulate to lemmas:

1.3 Lemma. *Let U_1, U_2 be two open Stein subspaces of complex space X. Then $U_1 \cap U_2$ is Stein.*

Proof. Point separation and infinitesimal point separation are clear. Hence let $K \subset U_1 \cap U_2$ be a compact subset. We have to show that $\hat{K}$ (the holomorphic hull taken in $U_1 \cap U_2$) is compact. We denote the holomorphic hulls taken in U_1 resp U_2 by $\hat{K}_1$ resp. $\hat{K}_2$. Then $\hat{K}$ is a closed subset of $\hat{K}_1 \cap \hat{K}_2$. This shows that K is compact. □

Chapter V. Sheaves

1. Presheaves

1.1 Definition. *A presheaf F (of abelian groups) on a topological space X is a map which assigns to every open subset $U \subset X$ an abelian group $F(U)$ and to every pair U, V of open subsets with the property $V \subset U$ a homomorphism*

$$r_V^U : F(U) \longrightarrow F(V)$$

such that for three open subsets U, V, W with the property $W \subset V \subset U$

$$r_W^U = r_W^V \circ r_V^U$$

holds:

Example: $F(U)$ is the set al continuous functions $f : U \to \mathbb{C}$ and $r_V^U(f) := f|V$ (restriction).

Many presheaves generalize this example. Hence the maps r_V^U are called "restrictions" in general and one uses the notation

$$s|V := r_V^U(s) \qquad \text{for} \quad s \in F(U).$$

The elements of $F(U)$ sometimes are called "sections" of F over U. In the special case $U = X$ they are called "global" sections.

1.2 Definition. *Let X be a topological space. A homomorphism of presheaves*

$$f : F \longrightarrow G$$

is a family of group homomorphisms

$$f_U : F(U) \longrightarrow G(U),$$

such that the diagram

$$\begin{array}{ccc} F(U) & \longrightarrow & G(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & G(V) \end{array}$$

commutes for every pair $V \subset U$ of open subsets, i.e. $f_U(s)|_G V = f_V(s|_F V)$.

It is clear how to define the identity map $\mathrm{id}_F : F \to F$ of a presheaf and the composition $g \circ f$ of two homomorphims $f : F \to G$, $g : G \to H$ of presheaves.

There is also a natural notion of a sub-presheaf $F \subset G$. Besides $F(U) \subset G(U)$ for all U one has to demand, that the restrictions are compatible. This means:

The canonical inclusions $i_U : F(U) \to G(U)$ define a homomorphism $i : F \to G$ of presheaves.

When $f : F \to G$ is a homomorphism of presheaves, the images $f_U(F(U))$ define a sub-presheaf of G. We call it the *presheaf-image* and denote it by

$$f_{\mathrm{pre}}(F).$$

It is also clear that the kernels of the maps f_U define a sub-presheaf of F. We denote it by Kernel$(f : F \to G)$. When F is a sub-presheaf of G then one can can consider the factor groups $G(U)/H(U)$. Using VII.1.1 it is clear how to define restriction maps to get a presheaf $G/_{\mathrm{pre}}F$. We call this presheaf the *factor-presheaf.*

Since we have defined Kernel and Image we can also introduce the notion of a *preasheaf-exact sequence.* A sequence $F \to G \to H$ is presheaf-exact if and only if $F(U) \to G(U) \to H(U)$ is exact for all U. What we have said about exact sequences of abelian groups carries literarily over to presheaf-exact sequences of presheaves of abelian groups.

2. Germs and Stalks

let F be a presheaf on a topological space X und let $a \in X$ be a point. We consider pairs (U, s), where U is an open neighpourhood of a and $s \in F(U)$ a section over U. Two pairs (U, s), (V, t) are called equivalent, if there exists an open neighborhood $a \in W \subset U \cap V$, such that $s|W = t|W$. This is an equivalence relation. The equivalence classes

$$[U, s]_a := \{\, (V, t); \quad (V, t) \sim (U, s) \,\}$$

are called *germs* of F in the point a. The set of all germs

$$F_a := \{\, [U, s]_a, \quad a \subset U \subset X, \; s \in F(U) \,\}$$

is the so-called *stalk* of F in a. The stalk carries a natural structure as abelian group. One defines

$$[U, s]_a + [V, t]_a := [U \cap V, s|U \cap V + t|U \cap V]_a.$$

We use frequently the simplified notation

$$s_a = [U, s]_a.$$

For every open neighborhood $a \in U \subset X$ there is an obvious homomorphism

$$F(U) \longrightarrow F_a, \quad s \longmapsto s_a.$$

A homomorphism of presheaves $f : F \to G$ induces natural mappings

$$f_a : F_a \longrightarrow G_a \qquad (a \in X).$$

The image of a germ $[U, s]_a$ is simply $[U, f_U(s)]_a$. It is easy to see that this is well-defined.

2.1 Remark. *Let $F \to G$ and $G \to H$ be homomorphism of presheaves and let $a \in X$ be a point. Assume that every neighborhood of a contains a small open neighborhood U such that $F(U) \to G(U) \to H(U)$ is exact. Then $F_a \to G_a \to H_a$ is exact.*

Corollary. *if $F \to G \to H$ is presheaf-exact then $F_a \to G_a \to H_a$ is exact for all a.*

If F is a preasheaf on X, one can consider for each open subset $U \subset X$

$$F^{(0)}(U) := \prod_{a \in U} F_a.$$

The elements are families $(s_a)_{a \in U}$ with $s_a \in F_a$. There is now coupling between the different s_a. Hence $F^{(0)}(U)$ usually is very giantly.

For open sets $V \subset U$, one has an obvious homomorphism $F^{(0)}(U) \to F^{(0)}(V)$. Hence we obtain a preasheaf $F^{(0)}$ together with a natural homomorphism

$$F \longrightarrow F^{(0)}.$$

3. Sheaves

3.1 Definition. *A presheaf F is called **sheaf**, if the following conditions are satisfied:*

(G1) *When $U = \bigcup U_i$ is an open covering of an open subset $U \subset X$ and if $s, t \in F(U)$ are sections with the property $s|U_i = t|U_i$ fur alle i, then $s = t$.*

(G2) *When $U = \bigcup U_i$ is an open covering of an open subset $U \subset X$ und if $s_i \in F(U_i)$ is a family of sections with the property*

$$s_i|U_i \cap U_j = s_j|U_i \cap U_j \quad \textit{fur alle } i, j,$$

then there exists a section $s \in F(U)$ with the property $s|U_i = s_i$ for all i.

(G3) *$F(\emptyset)$ is the zero group.*

Clearly the presheaf of continuous functions is a sheaf, since continuity is a local property. An example of a presheaf F, which usually is not a sheaf is the presheaf of constant functions with values in $\mathbb{Z}$ ($F(U) = \{f : U \to \mathbb{Z},\ f$ constant$\}$). But the set of *locally constant* functions with values in $\mathbb{Z}$ is a sheaf.

By a subsheaf of a sheaf F we understand a sub-presheaf $G \subset F$ which is already a sheaf. If F, G are presheaves then a homomorphism $f : F \to G$ of presheaves is called also a homomorphism of sheaves.

3.2 Remark. *Let $F \subset G$ be a sub-presheaf. We assume that G (but not necessarily F) is a sheaf. Then there is a smallest subsheaf $\tilde{F} \subset G$ which contains F. For an arbitrary point $a \in X$ the induced map $f_a : F_a \to \tilde{F}_a$ is an isomorphism.*

It is clear, that $\tilde{F}(U)$ has to be defined as set of all $s \in G(U)$, such that:

There exists an open covering $U = \bigcup U_i$, such that $s|U_i$ is in the image of $F(U_i) \to G(U_i)$ for all i.

This is equivalent with:

The germ s_a is in the image of $F_a \to G_a$ for all $a \in U$.

3.3 Definition. *Let $F \to G$ be a homomorphism of sheaves. The sheaf-image $f_{sheaf}(F)$ is the smallest subsheaf of G, which contains the presheaf-image- $f_{pre}(F)$.*

We have to differ between two natural notions of surjectivity.

3.4 Definition.

1) *A homomorphism of preasheaves $f : F \to G$ is called* ***presheaf-surjective*** *if $f_{pre}(F) = G$.*
2) *A homomorphism of sheaves $f : F \to G$ is called* ***sheaf-surjective*** *if $f_{sheaf}(F) = G$.*

Wenn F and G both are sheaves then sheaf-surjectivity and presheaf-surjectivity are different things. We give an example which will be basic:

Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}$, hence $\mathcal{O}(U)$ is the set of all holomorphic functions on an open subset U. This a sheaf of abelian groups (under addition). Similarly we consider the sheaf $\mathcal{O}^*$ of holomorphic functions without zeros. This is also a sheaf of abelain groups (under multiplication). The map $f \to e^f$ defines a sheaf homomorphism

$$\exp : \mathcal{O} \longrightarrow \mathcal{O}^*.$$

The map $\mathcal{O}(U) \to \mathcal{O}^*(U)$ is not always surjective. For example for $U = \mathbb{C}^*$ the function $1/z$ is not in the image. Hence exp is not presheaf-surjective. But it is know from complex calculus that $\exp : \mathcal{O}(U) \to \mathcal{O}^*(U)$ is surjective if U is simply connected, for example for a disc U. Since a point admits arbitrarily small neighborhoods which are discs, it follows that exp is sheaf-surjective.

3.5 Remark. *A homomorphism of sheaves $f : F \to G$ is sheaf-surjective if and only if the maps $f_a : F_a \to G_a$ are surjective for all $a \in X$.*

Fortunately the notion "injective" doesn't contain this difficulty.

3.6 Remark. *Let $f : F \to G$ be a homomorphism of sheaves. The kernel in the sense of presheaves is already a sheaf.*

Hence we don't have to distinguish between presheaf-injective and sheaf-injective and also not between preasheaf-kernel and sheaf-kernel.

3.7 Remark. *A homomorphism of sheaves $f : F \to G$ is injective if and only if the maps $f_a : F_a \to G_a$ are injective for all $a \in X$.*

A homomorphism of presheaves $f : F \to G$ (sheaves) is called an isomorphism if all $F(U) \to G(U)$ are isomorphisms. Their inverses then define a homomorphism $f^{-1} : G \to F$.

3.8 Remark. *A homomorphism of sheafs $F \to G$ is an isomorphism if and only if $F_a \to G_a$ is an isomorphism for all a.*

For presheaves this is false. As counter example on can take for F the presheaf of constant functions and for G the sheaf of locally constant functions.

It is natural to introduce the notion of sheaf-exactness as follows:

3.9 Definition. *A sequence $F \to G \to H$ of sheaf homomorphims is sheaf-exact at G , if the the kernel of $G \to H$ and the sheaf-image of $F \to G$ agree.*

Generalizing 3.5 and 3.7 one can easily show:

3.10 Proposition. *A sequence $F \to G \to H$ is exact if and only if $F_a \to G_a \to H_a$ is exact for all a.*

Our discussion so far has obviously one gap: Let $F \subset G$ be subsheaf of a sheaf G. We would like to have an exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

The sheaf H should be the factor sheaf of G by F. But up to now we only defined the factor-presheaf $G/_{\text{pre}}F$ which usually is no sheaf. In the next section we will give the correct definition for a factor sheaf $G/_{\text{sheaf}}F$.

4. The generated sheaf

For a presheaf F we introduced the monstrous presheaf

$$F^{(0)}(U) = \prod_{a \in U} F_a.$$

Obviously $F^{(0)}$ is a sheaf. Sometimes its is called the "Godement-sheaf" or the "associated flabby sheaf". There is a natural homomorphism

$$F \to F^{(0)}.$$

We can consider its presheaf-image and then the smallest subsheaf which contains it. We denote this sheaf by $\hat{F}$ and call it the "generated sheaf" by F. There is a natural homomorphism

$$F \to \hat{F}.$$

From the construction follows immediately

4.1 Remark. *Let F be a presheaf. The natural maps*

$$F_a \xrightarrow{\sim} \hat{F}_a$$

are isomorphisms.

A homomorphism $F \to G$ of presheaves induces a homomorphism $F^{(0)} \to G^{(0)}$. Clearly $\hat{F}$ is mapped into $\hat{G}$.

4.2 Remark. *Let $f : F \to G$ be a homomorphism of presheaves. There is a natural homomorphism $\hat{F} \to \hat{G}$, such that the diagram*

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ \hat{F} & \longrightarrow & \hat{G} \end{array}$$

commutes.

When F is already a sheaf then $F \to F^{(0)}$ is injective. Then the map of F into the presheaf image is an isomorphism. This implies that the presheaf image is already a sheaf.

4.3 Remark. *Let F be a sheaf. Then $F \to \hat{F}$ is an isomorphism.*

If F is a sub-presheaf of a sheaf G, then the induced map $\hat{F} \to \hat{G} \cong G$ is an isomorphism $\hat{F} \to \tilde{F}$ between $\hat{F}$ and the smallest subsheaf $\tilde{F}$ of G, wich contains F.

We identify $\tilde{F}$ and $\hat{F}$.

Factor sheaves and exact sequences of sheaves

Let $F \to G$ be a homomorphism of presheaves. We introcuced already the factor preshaf $G/_{\rm pre}F$, which asociates to an open U the factor group $G(U)/F(U)$. Even if both F and G are sheaves this will usually not a shesf. Hence we define the factor sheaf as the sheaf generated by the factor-presheaf.

$$G/_{\rm sheaf}F := \widehat{G/_{\rm pre}}F.$$

This called the factor-sheaf. Since we are interested mainly in sheaves, we will write usually for a homomorphism for sheaves $f : F \to G$:

$$\begin{aligned} G/F &:= G/_{\rm sheaf}F \qquad \text{(factor sheaf)} \\ f(F) &:= f_{\rm sheaf}(F) \qquad \text{(sheaf image)} \end{aligned}$$

Notice that there is no need to differ between sheaf- and presheaf-kernel. When we talk about an exact sequence of sheaves

$$F \longrightarrow G \longrightarrow H$$

we usually mean "sheaf exactness". All what we have said about exactness properties of sequences of abelian groups is literally true for sequences of sheaves. For example: A sequence of sheaves $0 \to F \to G$ (0 denotes the zero sheaf) is exact if and only of $F \to G$ is injective. A sequence of sheaves $F \to G \to 0$ is exact if and only if $F \to G$ is surjective (in the sense of sheaves of course). A sequence of sheaves $0 \to F \to G \to H \to 0$ is exact if and only if there is an ismomorphism $H \cong G/F$ which identifies this sequence with

$$0 \longrightarrow F \longrightarrow G \longrightarrow G/F \longrightarrow 0.$$

4.4 Remark. *Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves. Then for open U the sequence*

$$0 \to F(U) \to G(U) \to H(U)$$

is exact.

Corollary. *The sequence*

$$0 \to F(X) \to G(X) \to H(X)$$

is exact.

Usually $G(X) \longrightarrow H(X)$ is not surjective as the example

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O} \overset{f \mapsto e^{2\pi i f}}{\longrightarrow} \mathcal{O}^* \longrightarrow 0$$

shows. Cohomology theory will measure the absence the right exactness. The above sequence will be part of a long exact sequence

$$0 \to F(X) \to G(X) \to H(X) \longrightarrow H^1(X, F) \longrightarrow \cdots$$

5. Direct and inverse image of sheaves

Let $f : X \to Y$ a continuous map of topological spaces and F a pre-sheaf on X. Then $(f_*F)(V) := F(f^{-1}(C))$ with natural restriction maps is a pre-sheaf on Y. It is a sheaf if F is a sheaf. We call it the direct image sheaf. For a point $a \in X$ there is an obvious map $(f_*F)_{f(a)} \to F_a$. Let $X \to Y$ be a closed embedding. This means that the image is closed and that $X \to Y$ is a topological map. In this case the above map induces a bijection $(f_*F)_{f(a)} \cong F_a$.

We use some simple facts about sheaves. Let F be a sheaf on a topological space. We know the trivial procedure of restricting F to an open subset. There is a more general procedure to restrict sheaves to an arbitrary subset $Y \subset X$ (equipped with the induced topology). Even more general, one can define for a continuous map $f : Y \to X$ the inverse image $f^{-1}F$ of a sheaf F on X. First one considers the presheaf

$$G(V) = \varinjlim F(U)$$

where U runs through all open subsets of X that contain $f(V)$. Then one defines $f^{-1}F$ to be its generated sheaf. If $U \subset X$ is open an $\iota : U \to X$ is the canonical injection then $\iota^{-1}F$ can be identified with the restriction. Hence we can use the notation $F|Y = \iota^{-1}F$ for any subset $Y \subset X$, equipped with the induced topology. Again ι denotes the natural injection.

5.1 Lemma. *Let X be a topological space and $Y \subset X$ a closed subspace. Let F be a sheaf on X such that $F|(X - Y)$ is zero and let $\iota : Y \to X$ the natural injection. Then there is a natural isomorphism*

$$\iota_*(F|Y) \xrightarrow{\sim} F.$$

More precisely, the functor $F \mapsto F|Y$ defines an equivalence between the category of sheaves on Y and the category of sheaves on X whose restriction to U vanishes.

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6. Sheaves of rings and modules

A sheaf of A-modules is a sheaf F of abelian groups such that every $F(U)$ carries a structure as A-module and such the the restriction maps $F(U) \to F(V)$ for $V \subset U$ are A-linear. A homomorphism $F \to G$ is called A-linear if all $F(U) \to G(U)$ are so. Then kernel and image carry natural structures of sheafs of A-modules. Also the stalks carry such a structure naturally. Hence the whole canonical flabby resolution is a sequence of sheafs of A-modules.

There is a refinement of this construction: By a sheaf of rings $\mathcal{O}$ we understand a sheaf of abelian groups such that every $\mathcal{O}(U)$ is not only an abelian group but a ring and such that all restriction maps $\mathcal{O}(U) \to \mathcal{O}(V)$ are ring homomorphisms. Then the stalks $\mathcal{O}_a$ carry a natural ring structure such that the homomorphisms $\mathcal{O}(U) \longrightarrow \mathcal{O}_a$ (U is an open neighborhood of a) are ring homomorphisms.

By an $\mathcal{O}$-module we understand a sheaf $\mathcal{M}$ of abelian groups such every $F(U)$ carries a structure as $\mathcal{O}(U)$-module and such that the restriction maps are compatible with the module structure. To make this precise we give a short comment. Let M be an A-module and N be a module over a different ring B. Assume that a homomorphism $r : A \to B$ is given. A homomorphism $f : M \to N$ of abelian groups is called compatible with the module structures if the formula

$$f(am) = r(a)f(m) \qquad (a \in A,\ m \in M)$$

holds. An elegant way to express this is as follows. We can consider N also as an module over A by means of the definition $an := r(a)n$. Sometimes this A-module is written as $N_{[r]}$. Then the compatibility of the map f simply means that it is an A-linear map

$$f : M \longrightarrow N_{[r]}.$$

Usually we will omit the subscript $[r]$ and simply say that $f : M \to N$ is A-linear.

If $\mathcal{M}$ is an $\mathcal{O}$-module then the stalk $\mathcal{M}_a$ is naturally an $\mathcal{O}_a$-module. An $\mathcal{O}$-linear map $f : \mathcal{M} \to \mathcal{N}$ between two $\mathcal{O}$-modules is a homomorphism of sheaves of abelian groups such the maps $\mathcal{M}(U) \to \mathcal{N}(U)$ are $\mathcal{O}(U)$ linear. Then the Kernel and image also carry natural structures of $\mathcal{O}$-modules.

Another standard construction of commutative algebra carries over to the case of modules over sheaves.

An $\mathcal{O}$-submodule $\mathcal{P} \subset \mathcal{M}$ is an sub-sheaf of abelian groups such that $\mathcal{P}(U)$ is an $\mathcal{O}(U)$-submodule of $\mathcal{M}(U)$ for every open U. Then the natural inclusion $\mathcal{P} \hookrightarrow \mathcal{M}$ is $\mathcal{O}$-linear. The factor sheaf $\mathcal{M}/\mathcal{N}$ carries a natural structure as $\mathcal{O}$-module. An ideal sheaf in $\mathcal{O}$ is just an $\mathcal{O}$-submodule of $\mathcal{O}$ (which can be considered as $\mathcal{O}$-module in the obvious way). The factor sheaf of $\mathcal{O}$ by an ideal sheaf carries a natural structure as sheaf of rings.

Let $\varphi : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{O}$-linear map and $\mathcal{P} \subset \mathcal{N}$ an $\mathcal{O}$-submodule. Then $\varphi^{-1}(\mathcal{P})$ is defined in the naiv way: $\varphi^{-1}(\mathcal{P})(U) := \varphi_U^{-1}(\mathcal{P}((U))$. This is already a sheaf, actually an $\mathcal{O}$-submodule of $\mathcal{M}$.

The direct sum $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_m$ of $\mathcal{O}$-modules $\mathcal{M}_i$ can be defined in the naive way.

Clearly the canonical flabby resolution of an $\mathcal{O}$-module is naturally a sequence of $\mathcal{O}$-modules.

Since for every open subset $U \subset X$ we have a ring homomorphism $\mathcal{O}(X) \to \mathcal{O}(U)$ all $\mathcal{M}(U)$ can be considered as $\mathcal{O}(X)$-modules. Hence a $\mathcal{O}$-module can be considered as sheaf of $\mathcal{O}(X)$-modules.

Finitely generated sheaves

Let $\mathcal{M}$ be an $\mathcal{O}$-module and $\mathcal{O}^m \to \mathcal{M}$ an $\mathcal{O}$-linear map. There is an induced map $\mathcal{O}^m(X) \to \mathcal{M}(X)$. Hence there are m distinguished global sections $s_1, \dots, s_m \in \mathcal{M}(X)$ (the images of the elements of the standard basis $e_1, \dots, e_m$ of $\mathcal{O}^m(X)$. These global sections determine the map, since for any open $U \subset X$ an arbitrary section of $\mathcal{O}^m$ can be written in the form $s = f_1 e_1|U + \cdots + f_m e_m|U$. The image of this section is $f_1 s_1|U + \cdots + f_m s_m$. Conversely we obtain an $\mathcal{O}$-linear map through this formula for any choice of global sections $s_1 \dots, s_m$. This shows:

6.1 Lemma. *There is a natural one to one correspondence between $\mathcal{O}$-linear maps $\mathcal{O}^m \to \mathcal{M}$ and m-tuples of global sections of $\mathcal{M}$.*

An $\mathcal{O}$-module is called finitely generated if there is a surjective of $\mathcal{O}$-modules $\mathcal{O}^m \to \mathcal{M}$. Surjectivity of course is understood in the sense of sheaves. So this means that there exist global sections $s_1, \dots, s_m \in \mathcal{M}(X)$ whose germs generate the stalk $\mathcal{M}_a$ for each point $a \in X$.

Let $\mathcal{M}$ be an $\mathcal{O}$-module. Assume that for each point a a submodule $\mathcal{N}_a$ is given. One can ask whether there exists a sub-module $\mathcal{N} \subset \mathcal{M}$ with these stalks. Of course $\mathcal{N}$ is uniquely determined if it exists. For example $\mathcal{N}$ exists if there exist finitely man sections $s_1, \dots, s_m \in \mathcal{M}(X)$ such that $\mathcal{N}_a$ is generated by $s_1, \dots, s_m$. Then $\mathcal{N}$ is generated by these sections. This shows that the notion of "finitely generated system" in connection with Oka's coherence theorem is closely related to the notion of a sheaf.

6.2 Remark. *A finitely generated system* $(\mathcal{M}_a)$ *in the sense of I.10.1 can be considered as a finitely generated submodule of* $\mathcal{O}_U^m$.

The support of a sheaf F is defined as

$$\text{support} F := \{a \in F;\ F_a \neq 0\}.$$

6.3 Lemma. *Let* $\mathcal{M}$ *be a finitely generated* $\mathcal{O}$*-module. The support of* F *is a closed subset.*

Proof. We show that the complement of the support is open. Let a be a point such that $\mathcal{M}_a = 0$. Consider generators $s_1, \dots, s_m$ of $\mathcal{M}$. the germs $(s_i)_a$ are zero. Hence there exists an open neighborhood U such that all $s_i|U = 0$. This shows $\mathcal{M}_b = 0$ for all $b \in U$. □

6.4 Lemma. *Let* $\mathcal{M}, \mathcal{N}$ *be two finitely generated submodules of an* $\mathcal{O}$*-module* $\mathcal{P}$ *Let* a *be a point such that* $\mathcal{M}_a \subset \mathcal{N}_a$ *Then there exists an open neighborhood* $a \in U$ *such that* $\mathcal{M}|U \subset \mathcal{N}|U$.

Proof. Take generators $s_1, \dots, s_m$ of $\mathcal{M}$ and $t_1, \dots, t_n$ of $\mathcal{N}$. Express the germs $(t_i)_a$ by the $(s_j)_a$. Since there are only finitely coefficients involved, these equations extend to a small open neighborhood of a. □

A similar argument gives:

6.5 Lemma. *Let* $\mathcal{M} \to \mathcal{N}$ *be an* $\mathcal{O}$*-linear map of finitely generated* $\mathcal{O}$*-modules. Let* a *be a point such that* $\mathcal{M}_a \to \mathcal{N}_a$ *is surjective. Then there exists an open neighborhood* U *such that* $\mathcal{M}|U \to \mathcal{N}|U$ *is surjective.*

Lifting of maps

There is very simple fact of commutative algebra. Let $M \to N$ be a surjective R-linear map of R-modules and $R^n \to M$ a linear map too. Then there exists a lift. $R^n \to M$. Denote the images of the standard basis $e_1, \dots, e_n$ in N by $b_1, \dots, b_n$ and take pre-images a_i in M. Then map e_i to a_i.

To get an analogue for sheaves we consider a surjective $\mathcal{O}$-linear map $\mathcal{M} \to \mathcal{N}$ of $\mathcal{O}$-modules and an $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{N}$. Now we get a problem since the map $\mathcal{M}(X) \to \mathcal{N}(X)$ needs not to be surjective. So we can not repeat the above argument. We only can say:

6.6 Lemma. *Let $\mathcal{M} \to \mathcal{N}$ be a surjective $\mathcal{O}$-linear map and $\mathcal{O}^n \to \mathcal{N}$ also an $\mathcal{O}$-linear map. For each point a there exists an open neighborhood U and an $\mathcal{O}|U$-linear map such the diagram*

$$\begin{array}{ccc} \mathcal{O}|U^n & & \\ \downarrow & \searrow & \\ \mathcal{M}|U & \longrightarrow & \mathcal{N}|U \end{array}$$

commutes.

7. Coherent sheaves

Let X be a topological space and $\mathcal{O}$ a sheaf of rings. We consider the category of $\mathcal{O}$-modules $\mathcal{M}$. Examples are the free modules $\mathcal{O}^n$. We consider $\mathcal{O}$-linear maps $\mathcal{O}^n \to \mathcal{M}$. It involves a map $\mathcal{O}(X)^n \to \mathcal{M}(X)$. The images of the unit vectors gives n global sections $s_1, \dots, s_n$. These sections determine the whole map of sheaves, since necessarily

$$\mathcal{O}(U)^n \longrightarrow \mathcal{M}(U), \quad (f_1, \dots f_n) \longmapsto f_1 s_1|U + \cdots + f_n s_n|U.$$

Conversely, if n global sections $s_1, \dots, s_n$ are given, then this formula defines an $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{M}$. This means that we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^n, \mathcal{M}) \cong \mathcal{M}(X)^n.$$

An $\mathcal{O}$-module $\mathcal{M}$ is called *finitely generated* if there exists a surjective $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{M}$. "Surjective" is of course understood in the sense of sheaves, i.e. the maps $\mathcal{O}_{X,a}^n \to \mathcal{M}_a$ have to be surjective for all $a \in X$. For the defining sections $s_1, \dots, s_n$ this means that $\mathcal{M}_a$ is generated by $s_{1,a}, \dots, s_{n,a}$. Concretely this means the following:

Let $U \in X$ be open and let $s \in \mathcal{M}(U)$. Then there exists an open covering $U = \bigcup U_i$ such that every $s|U_i$ is a linear combination of the $s_1|U_i, \dots s_n|U_i$ with coefficients in $\mathcal{O}(U_i)$.

If this is the case we also say that $\mathcal{M}$ is generated by the global sections $s_1, \dots, s_n$.

7.1 Lemma. *Let $\mathcal{M} \to \mathcal{N}$ be a surjective $\mathcal{O}$-linear map and $\mathcal{O}^n \to \mathcal{N}$ also an $\mathcal{O}$-linear map. For each point a there exists an open neighborhood U and an $\mathcal{O}|U$-linear map such the diagram*

$$\begin{array}{ccc} \mathcal{O}|U^n & & \\ \downarrow & \searrow & \\ \mathcal{M}|U & \longrightarrow & \mathcal{N}|U \end{array}$$

commutes.

Proof. The map $\mathcal{O}^n \to \mathcal{N}$ corresponds to n global sections $s_1, \dots, s_n$. If we take U small enough there are in the image of $\mathcal{M}(U)$, $t_i \mapsto s_i$. The sections t_i give a map $\mathcal{O}^n \to \mathcal{M}$. □

An $\mathcal{O}$-module $\mathcal{M}$ is called *locally finitely generated* if every point $a \in X$ admits an open neighbourhood such that $\mathcal{M}|U$ is a finitely generated $\mathcal{O}|U$-module.

Let us recall a basic property of noetherian rings R. Let M be a finitely generated module, i.e. there exists a surjective R-linear map $R^n \to M$. Then the kernel K of this map is finitely generated as well. Hence there exists an exact sequence $R^n \xrightarrow{\varphi} R^m \to M$. The map φ determines $M \cong R^n/\mathrm{Im}(\varphi)$. The map φ just given by a matrix with m rows and n columns. This is the way how computer algebra can manage computations for finitely generated modules over neotherian rings as polynomial rings. Serre found a weak substitute for $\mathcal{O}$-modules.

7.2 Definition. *A sheaf of rings $\mathcal{O}$ is called* ***coherent*** *if for any open subset and any surjective $\mathcal{O}^n|U \to \mathcal{O}^m|U$ the kernel is locally finitely generated.*

7.3 Definition. *Let $\mathcal{O}$ be a coherent sheaf of rings. An $\mathcal{O}$-module $\mathcal{M}$ is called coherent if for every point there exists an open neighborhood U and an exact sequence*

$$\mathcal{O}|U^n \longrightarrow \mathcal{O}|U^m \longrightarrow \mathcal{M}|U \longrightarrow 0.$$

Of course $\mathcal{O}$ considered as $\mathcal{O}$-module then is coherent. Just consider $0 \to \mathcal{O} \to \mathcal{O} \to 0$.

An $\mathcal{O}$-module is called a (finitely generated) free sheaf if it is isomorphic to $\mathcal{O}^m$ for suitable m. It is called locally free if every point admits an open neighborhood such that the restriction to it is free. A locally free sheaf is also called a vector bundle. For trivial reasons a free sheaf over a coherent sheaf of rings is coherent. Since coherence is a local property every vector bundle is coherent. The property "coherent" is stable under standard constructions. The proves are not difficult. We will keep them short:

First we treat some special cases for free $\mathcal{O}$-modules. A first trivial observation is that the image of an $\mathcal{O}$-linear map $\mathcal{O}^p \to \mathcal{O}^q$ is coherent. The next observation is that the intersection $\mathcal{M} \cap \mathcal{N}$ of two coherent subsheaves $\mathcal{M}, \mathcal{N}$ of $\mathcal{O}^n$ is coherent. (The intersection $\mathcal{M} \cap \mathcal{N}$ is defined in the naive sense as presheaf and turns to be out a sheaf, more precisely an $\mathcal{O}$-module.) The idea is to write the intersection as a kernel. We explain the principle for individual modules $M, N \subset R^n$ of finite type over a ring R: Let $F : R^p \to R^m$,

$G : R^q \to R^m$ be linear maps and let M, N be their images. We denote by K the kernel of the linear map

$$R^{p+q} \longrightarrow R^m, \quad (m, n) \longmapsto F(m) - G(n).$$

The image of K under the map

$$R^{p+q} \longrightarrow R^m, \quad (m, n) \longmapsto F(m).$$

is precisely the intersection $M \cap N$. The last observation is the following. Let $\mathcal{O}^p \to \mathcal{O}^q$ be $\mathcal{O}$-linear and let $\mathcal{M} \subset \mathcal{O}^q$ be coherent. We claim that its inverse image in $\mathcal{O}^p$ is coherent. We explain again the algebra behind this result. Let $F : R^m \to R^l$ be a R-linear map and $N \subset R^l$ be a R-module of finite type. We assume that $F(R^m) \cap N$ is finitely generated. Then there exists a finitely generated submodule $P \subset R^m$ such that $F(P) = F(R^m) \cap N$. We also assume that the kernel K of F is finitely generated. It is easily proved that $F^{-1}(N) = P + K$ and we obtain that the inverse image is finitely generated.

These observations carry over to arbitrary coherent $\mathcal{O}$-modules.

7.4 Lemma. *Let $\mathcal{M} \to \mathcal{N}$ be an $\mathcal{O}$-linear map of coherent sheaves. The image sheaf is coherent.*

Corollary. *A locally finitely generated sub-sheaf of a coherent sheaf is coherent.*

Proof. It is sufficient to show that the image of a map $\mathcal{O}^m \to \mathcal{M}$ is coherent. By definition of coherence it is sufficient to show that the kernel $\mathcal{K}$ is locally finitely generated. We can assume that there exists an exact sequence

$$\mathcal{O}^p \longrightarrow \mathcal{O}^q \longrightarrow \mathcal{M} \longrightarrow 0.$$

Since $\mathcal{O}^q \to \mathcal{M}$ is surjective we can assume (use Lemma 7.1) that there exists a lift $\mathcal{O}^m \to \mathcal{O}^q$ such that the diagram

$$\begin{array}{ccc} \mathcal{O}^m & \longrightarrow & \mathcal{M} \\ & \searrow & \uparrow \\ & & \mathcal{O}^q \\ & & \uparrow \\ & & \mathcal{O}^p \end{array}$$

commutes. Take the image of $\mathcal{O}^p \to \mathcal{O}^q$ and then its pre-image in $\mathcal{O}^m$ It is easy to check that this is the kernel $\mathcal{K}$. □

7.5 Lemma. *The kernel of a map $\mathcal{M} \to \mathcal{N}$ of coherent sheaves is coherent.*

Proof. Because of Lemma 7.4 we can assume that $\mathcal{M} \to \mathcal{N}$ is surjective. We choose presentations

$$\mathcal{O}^a \longrightarrow \mathcal{O}^b \longrightarrow \mathcal{M}, \quad \mathcal{O}^c \longrightarrow \mathcal{O}^d \longrightarrow \mathcal{N}.$$

We can assume that there is commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}^b & \stackrel{\varphi}{\longrightarrow} & \mathcal{O}^d & & \\
 & & & & \uparrow & & \uparrow \psi & & \\
 & & & & \mathcal{O}^a & \longrightarrow & \mathcal{O}^c & &
\end{array}$$

The existence of φ follows from Lemma 7.1 (after replacing X by a small open neighborhood of a given point). The existence of $\mathcal{O}^a \to \mathcal{O}^c$ is trivial. Then we get a natural surjection $\varphi^{-1}(\psi(\mathcal{O}^c)) \to \mathcal{K}$. □

7.6 Lemma. *The cokernel $\mathcal{N}/\varphi(\mathcal{N})$ of a map $\varphi : \mathcal{M} \to \mathcal{N}$ of coherent sheaves is coherent.*

Proof. We can assume that $\mathcal{N}$ is a sub-sheaf of $\mathcal{M}$ and that φ is the canonical injection. We can assume that a commutative diagram with exact columns exists:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}/\mathcal{N} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}^b & \longrightarrow & \mathcal{O}^d & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}^a & \longrightarrow & \mathcal{O}^c & & & &
\end{array}$$

It is easy to construct from this diagram an exact sequence

$$\mathcal{O}^b \oplus \mathcal{O}^c \longrightarrow \mathcal{O}^d \longrightarrow \mathcal{M}/\mathcal{N} \longrightarrow 0. \qquad \square.$$

7.7 The two of three lemma. *Let $\mathcal{O}$ be a coherent sheaf of rings and*

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

an exact sequence of $\mathcal{O}$-modules. Assume that two of them are coherent than the third is coherent too.

Proof. All what remains to show is that $\mathcal{M}_2$ is coherent if $\mathcal{M}_1, \mathcal{M}_2$ are. We can assume that there is a commutative diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \mathcal{M}_3 & \longrightarrow & 0 \\ & & & & & \nwarrow{\scriptstyle\alpha} & \uparrow & & \\ & & & & & & \mathcal{O}^q & & \\ & & & & & & \uparrow & & \\ & & & & & & \mathcal{O}^p & & \end{array}$$

We use this to produce a map

$$\mathcal{M}_1 \oplus \mathcal{O}^q \longrightarrow \mathcal{M}_2, \quad (x, y) \longmapsto x - \alpha(y).$$

It is easy to check that this map is surjective. The kernel is defined by $x = \alpha(y)$. Hence it can be identified with the part of $\mathcal{O}^q$ that is mapped into $\mathcal{M}_1$ under α. But this precisely the kernel of $\mathcal{O}^q \to \mathcal{M}_3$ hence the image of $\mathcal{O}^p$. We get an exact sequence

$$\mathcal{O}^p \longrightarrow \mathcal{M}_1 \oplus \mathcal{O}^q \longrightarrow \mathcal{M}_2 \longrightarrow 0.$$

This shows that $\mathcal{M}_2$ is coherent (use Lemma 7.6).). □

7.8 Lemma. *The intersection of two coherent subsheaves of a coherent sheaf is coherent.*

Proof. One uses the fact that intersections can be constructed as kernels. Let $\mathcal{M}, \mathcal{N} \subset \mathcal{X}$ be two sub modules of an $\mathcal{O}$-module $\mathcal{X}$. Then $\mathcal{M} \cap \mathcal{N}$ is isomorphic to the kernel of $\mathcal{M} \times \mathcal{N} \to \mathcal{X}$, $(a, b) \mapsto a - b$. □

7.9 Remark. *Let $\mathcal{M}$ be a coherent $\mathcal{O}$-module. Then the support of $\mathcal{M}$ is a closed.*

Proof. We show that the set of all a such that $\mathcal{M}_a = 0$ is open. We can assume that $\mathcal{M}$ is finitely generated by sections $s_1, \dots, s_n$. If there germs at a are zero then $s_1, \dots, s_n$ are zero in a full neighbourhood of a. □

We collect some of the permanence properties of coherent sheaves.

7.10 Proposition.

1) *Let $\mathcal{M}, \mathcal{N}$ be two coherent sub-sheaves of a coherent sheaf. Assume $\mathcal{M}_a \subset \mathcal{N}_a$ for some point a. Then there exists an open neighborhood U such that $\mathcal{M}|U \subset \mathcal{N}|U$.*

2) *Let $\mathcal{M}, \mathcal{N}$ be two coherent sub-sheaves of a coherent sheaf. Assume $\mathcal{M}_a = \mathcal{N}_a$ for some point a. Then there exists an open neighborhood U such that $\mathcal{M}|U = \mathcal{N}|U$.*
3) *Let $f, g : \mathcal{M} \to \mathcal{N}$ be two $\mathcal{O}$-linear maps between coherent sheaves such that $f_a = g_a$ for some point a. Then there exists an open neighborhood U such that $f|U = g|U$.*
4) *Let $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ be $\mathcal{O}$-linear maps of coherent sheaves and a a point. The following two conditions are equivalent:*
   a) *The sequence $\mathcal{M}_a \to \mathcal{N}_a \to \mathcal{P}_a$ is exact.*
   b) *There is an open neighborhood U such that the sequence $\mathcal{M}|U \to \mathcal{N}|U \to \mathcal{P}|U$ is exact.*

Proof.

1) Use that $\mathcal{M}_a \subset \mathcal{N}_a$ is equivalent to $\mathcal{N}_a = \mathcal{M}_a \cap \mathcal{N}_a$ $(= (\mathcal{M} \cap \mathcal{N})_a$.

2) follows from 1).

3) Consider the kernel of $f - g$.

4)Consider the image $\mathcal{A}$ of $\mathcal{M} \to \mathcal{N}$ and the kernel $\mathcal{B}$ of $\mathcal{N} \to \mathcal{P}$. Both are coherent. We can assume that they are finitely generated. From assumption we know $\mathcal{A}_a = \mathcal{B}_a$. □

7.11 Proposition. *Let $\mathcal{M}, \mathcal{N}$ coherent $\mathcal{O}$-modules and $\mathcal{M}_a \to \mathcal{N}_a$ an $\mathcal{O}_a$-linear map. There exists an open neighborhood U and an extension $\mathcal{M}|U \to \mathcal{N}|U$ as $\mathcal{O}|U$-linear map.*

Additional remark. *By Proposition 7.10 this extension is unique in the obvious local sense.*

Proof. We can assume that there is a surjective $\mathcal{O}$-linear map $\mathcal{O}^n \to \mathcal{M}$. We consider the composed map $\mathcal{O}_a^n \to \mathcal{M}_a \to \mathcal{N}_a$. It is no problem to extend to $\mathcal{O}_a^n \to \mathcal{N}_a$ to an open neighborhood $\mathcal{O}|U^n \to \mathcal{N}|U$. We can assume that U is the whole space. The kernel of $\mathcal{O}_a^n \to \mathcal{M}_a$ is contained in the kernel of $\mathcal{O}_a^n \to \mathcal{N}_a$. Since the kernels are coherent this extends to a full open neighborhood U. Hence we get a factorization $\mathcal{M}|U \to \mathcal{N}|U$. The same construction works in the category of $\mathcal{O}_X$-modules.

7.12 Lemma. *Let $\mathcal{O}_X$ be a coherent sheaf of rings on a topological space X. Let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent sheaf of ideals. Let Y be the support of $\mathcal{O}_X/\mathcal{J}$. Then the restriction of $\mathcal{O}_X/\mathcal{J}$ to Y is a coherent sheaf of rings. The category of coherent $\mathcal{Y}$-modules is equivalent to the category of coherent $\mathcal{O}_X$ modules which are annihilated by $\mathcal{J}$.*

□

Chapter VI. Cohomology of sheaves

1. Some homological algebra

A complex $A^\bullet$ is a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \longrightarrow \cdots$$

such that the composition of two consecutive is 0, $d_n \circ d_{n-1} = 0$. Usually one omits indices at the d-s and writes simply $d = d_n$ and hence $d \circ d = 0$, which sometimes is written as $d^2 = 0$. The cohomology groups of $A^\bullet$ are defined as

$$H^n(A^\bullet) := \frac{\text{Kernel}(A^n \to A^{n+1})}{\text{Image}(A^{n-1} \to A^n)} \qquad (n \in \mathbb{Z}).$$

They vanish if and only if the complex is exact. Hence the cohomology groups measure the absence of exactness of a complex.

A homomorphism $f^\bullet : A^\bullet \to B^\bullet$ of complexes is a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & B^{n-1} & \longrightarrow & B^n & \longrightarrow & B^{n+1} & \longrightarrow & \cdots \end{array}$$

It is clear how to compose two complex homomorphisms $f^\bullet; A^\bullet \to B^\bullet$, $g^\bullet; B^\bullet \to C^\bullet$ to a complex homomorphism $g^\bullet \circ f^\bullet : A^\bullet \to C^\bullet$. A sequence of complex homomorphisms

$$\cdots \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow \cdots$$

is called exact, if all the induced sequences

$$\cdots \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow \cdots$$

are exact. There is also the notion of a short exact sequence of complexes

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

Here 0 stands for the zero-complex ($0^n = 0$, $d^n = 0$ for all n).

A homomorphism of complexes $A^\bullet \to B^\bullet$ induces naturally homomorphisms

$$H^n(A^\bullet) \longrightarrow H^n(B^\bullet)$$

of the cohomology groups (use VII.1.1). These homomorphisms are compatible with the composition of complex-homomorphisms. A less obvious construction is as follows: Let

$$0 \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow 0$$

be a short exact sequence of complexes. We construct a homomorphism

$$\delta : H^n(C^\bullet) \longrightarrow H^{n+1}(A^\bullet).$$

Let $[c] \in H^n(C^\bullet)$ be represented by an element $c \in C^n$. Take a pre-image $b \in B^n$ and consider $\beta = db \in B^{n+1}$. Since β goes to $d(c) = 0$ in C^{n+1} there exists a pre-image $a \in A^{n+1}$. This goes to 0 in A^{n+2} (because A^{n+2} is imbedded in B^{n+2} and b goes to $d^2(b) = 0$ there). Hence a defines a cohomology class $[a] \in H^{n+1}(A^\bullet)$. It is easy to check that this class doesn't depend on the above choices.

1.1 Fundamental lemma of homological algebra. *Let*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

be a short exact sequence of complexes. Then the long sequence

$$\cdots \to H^{n-1}(C^\bullet) \xrightarrow{\delta} H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(C^\bullet) \to \cdots$$

is exact.

We leave the details to the reader. □

There is a second lemma of homological algebra which we will need.

1.2 Lemma. *Let*

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{00} & \longrightarrow & A^{01} & \longrightarrow & A^{02} & \longrightarrow & A^{03} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{10} & \longrightarrow & A^{11} & \longrightarrow & A^{12} & \longrightarrow & A^{13} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{20} & \longrightarrow & A^{21} & \longrightarrow & A^{22} & \longrightarrow & A^{23} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{30} & \longrightarrow & A^{31} & \longrightarrow & A^{32} & \longrightarrow & A^{33} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

be a commutative diagram where all lines and columns are exact besides the first column and the first row (those containing A^{00}). Then there is a natural isomorphism between the cohomology groups of the first row and the first column,

$$H^n(A^{\bullet,0}) \cong H^n(A^{0,\bullet})$$

For $n = 0$ this is understood as

$$\text{Kernel}(A^{00} \longrightarrow A^{01}) = \text{Kernel}(A^{00} \longrightarrow A^{10}).$$

The proof is given by "diagram chasing". We only give a hint how it works. Assume $n = 1$. Let $[a] \in H^1(A^{0,\bullet})$ be a cohomology class represented by an element $a \in A^{0,1}$. This element goes to 0 in $A^{0,2}$. As a consequence the image of a in $A^{1,1}$ goes to 0 in $A^{1,2}$. Hence this image comes from an element $\alpha \in A^{1,0}$. Clearly this element goes to zero in $A^{2,0}$ (since it goes to 0 in $A^{2,1}$.) Now α defines a cohomology class $[\alpha] \in H^1(A^{\bullet,0})$. There is some extra work to show that this map is well-defined. □

2. The canonical flabby resolution

A sheaf F is called *flabby,* if $F(X) \to F(U)$ is surjective of all U. Then $F(U) \to F(V)$ is surjective for all $V \subset U$. An example for a flabby sheaf is the Godement sheaf $F^{(0)}$. Recall that we have the exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)}.$$

We want to extend this sequence. For this we consider the sheaf $F^{(0)}/F$ and embed it into its Godement sheaf,

$$F^{(1)} := (F^{(0)}/F)^{(0)}.$$

In this way we get a long exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)} \longrightarrow F^{(1)} \longrightarrow F^{(2)} \longrightarrow \cdots$$

If $F^{(n)}$ has been already constructed then we define

$$F^{(n+1)} := \big(F^{(n)}/F^{(n-1)}\big)^{(0)}.$$

The sheaves $F^{(n)}$ are all flabby. We call this sequence the *canonical flabby resolution* or the *Godement resolution.* Sometimes it is useful to write the resolution in the form

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & F & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & F^{(0)} & \longrightarrow & F^{(1)} & \longrightarrow & F^{(2)} & \longrightarrow & F^{(3)} & \longrightarrow & \cdots
\end{array}$$

Both lines are complexes. The vertical arrows can be considered as a complex homomorphism. The induced homomorphism of the cohomology groups are isomorphisms. Notice that only the 0-cohomology group of both complexes is different from 0. This zero cohomology group is naturally isomorphic F.

Now we apply the global section functor Γ to the resolution. This is

$$\Gamma F := F(X).$$

We obtain a long sequence

$$0 \longrightarrow \Gamma F \longrightarrow \Gamma F^{(0)} \longrightarrow \Gamma F^{(1)} \longrightarrow \Gamma F^{(2)} \longrightarrow \cdots$$

The essential point is that this sequence is no longer exact. we only can say that it is a complex. We prefer to write in the form

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \Gamma F & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \Gamma F^{(0)} & \longrightarrow & \Gamma F^{(1)} & \longrightarrow & \Gamma F^{(2)} & \longrightarrow & \Gamma F^{(3)} & \longrightarrow & \cdots \end{array}$$

The second line is

$$\begin{array}{l} \cdots \longrightarrow 0 \longrightarrow \Gamma F^{(0)} \longrightarrow \Gamma F^{(1)} \longrightarrow \Gamma F^{(2)} \longrightarrow \cdots \\ \qquad\qquad\quad \uparrow \\ \qquad\quad \text{zero position} \end{array}$$

Now we define the cohomology groups $H^{\bullet}(X, F)$ to be the cohomology groups of this complex:

$$H^n(X, F) := \frac{\text{Kernel}(\Gamma F^{(n)} \longrightarrow \Gamma F^{(n+1)})}{\text{Kernel}(\Gamma F^{(n-1)} \longrightarrow \Gamma F^{(n)})}$$

(We define $\Gamma F^{(n)} = 0$ for $n < 0$.) Clearly

$$H^n(X, F) = 0 \qquad \text{for} \quad n < 0.$$

Next we treat the special case $n = 0$,

$$H^0(X, F) = \text{Kernel}(\Gamma F^{(0)} \longrightarrow \Gamma F^{(1)}).$$

Since the kernel can be taken in the presheaf sense, we can write

$$H^0(X, F) = \Gamma\text{Kernel}(F^{(0)} \longrightarrow F^{(1)}).$$

Recall that $F^{(1)}$ is a sheaf, which contains $F^{(0)}/F$ as subsheaf. We obtain

$$H^0(X, F) = \Gamma\text{Kernel}(F^{(0)} \longrightarrow F(0)/F)$$

This is the image of F in $F^{(0)}$ an hence a sheaf which is canonically isomorphic to F.

2.1 Remark. *There is a natural isomorphism*

$$H^0(X, F) \cong \Gamma F = F(X).$$

If $F \to G$ is a homomorphism of sheaves, then the homomorphism $F_a \to G_a$ induce a homomorphism $F^{(0)} \to G^{(0)}$. If $F \to G \to H$ is an exact sequence. Then $F^{(0)} \to G^{(0)} \to H^{(0)}$ is also exact (already as sequence of presheaves). More generally

2.2 Lemma. *Let $0 \to F \to G \to H \longrightarrow 0$ be an exact sequence of sheaves. Then the induced sequence $0 \to F^{(n)} \to G^{(n)} \to H^{(n)} \to 0$ is exact for every n.*

The proof is by induction. One needs the following lemma about abelian groups:

Let

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{00} & \longrightarrow & A^{01} & \longrightarrow & A^{02} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{10} & \longrightarrow & A^{11} & \longrightarrow & A^{12} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{20} & \longrightarrow & A^{21} & \longrightarrow & A^{22} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

be a commutative diagram such that the three columns and the first to lines are exact. Then the third line is also exact.

This follows from 1.2. □

Before we continue we need a basic lemma:

2.3 Lemma. *Let $0 \to F \to G \to H \to 0$ be a short exact sequence of sheaves. Assume that F is flabby. Then*

$$0 \to \Gamma F \to \Gamma G \to \Gamma H \to 0$$

is exact.

Proof. Let $h \in H(X)$. We have to show that h is the image of an $g \in G(X)$. For the proof one considers the set of all pairs (U, g), where U is an open subset and $g \in G(U)$ and such that g maps to $h|U$. This set is ordered by

$$(U, g) \geq (U', g') \iff U' \subset U \text{ and } g|U' = g'.$$

From the sheaf axioms follows that every inductive subset has an upper bound. By Zorns's lemma there exists a maximal (U, g). We have to show $U = X$. If this is not the case, we can find a pair (U', g') in the above set such that U' is not contained in U. The difference $g - g'$ defines a section in $F(U \cap U')$. Since F is flabby, this extends to a global section. This allows us to modify g' such that it glues with g to a section on $U \cup U'$. □

An immediate corollary of 2.3 states:

2.4 Lemma. *Let $0 \to F \to G \to H \to 0$ an exact sequence of sheaves. If F and G are flabby then H is flabby too.*

Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheafs. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n-1)} & \longrightarrow & G^{(n-1)} & \longrightarrow & H^{(n-1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n)} & \longrightarrow & G^{(n)} & \longrightarrow & H^{(n)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(n+1)} & \longrightarrow & G^{(n+1)} & \longrightarrow & H^{(n+1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

From 2.2 we know that all lines of this diagram are exact From 2.3 follows that they remain exact after applying Γ. Hence the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n-1)} & \longrightarrow & \Gamma G^{(n-1)} & \longrightarrow & \Gamma H^{(n-1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n)} & \longrightarrow & \Gamma G^{(n)} & \longrightarrow & \Gamma H^{(n)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma F^{(n+1)} & \longrightarrow & \Gamma G^{(n+1)} & \longrightarrow & \Gamma H^{(n+1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

can be considered as a short exact sequence of complexes. We can apply 1.1 to obtain the long exact cohomology sequence:

2.5 Theorem. *Every short exact sequence* $0 \to F \to G \to H \to 0$ *induces a natural long exact cohomology sequence*

$$0 \to \Gamma F \longrightarrow \Gamma G \longrightarrow \Gamma H \xrightarrow{\delta} H^1(X,F) \longrightarrow H^1(X,G) \longrightarrow H^1(X,H)$$
$$\xrightarrow{\delta} H^2(X,F) \longrightarrow \cdots$$

The next Lemma shows that the cohomology of flabby sheaves is trivial.

2.6 Lemma. *Let*

$$0 \to F \longrightarrow F_0 \to F_1 \longrightarrow \cdots$$

be an exact sequence of flabby sheaves (finite or infinite). Then

$$0 \to \Gamma F \longrightarrow \Gamma F_0 \to \Gamma F_1 \longrightarrow \cdots$$

is exact.

Corollary. *For flabby* F *one has:*

$$H^i(X,F) = 0 \quad \textit{for} \quad i > 0.$$

Proof. We use the so-called splitting principle. The long exact sequence can be splitted into short exact sequences

$$0 \longrightarrow F \longrightarrow F_0 \longrightarrow F_0/F \longrightarrow 0, \quad 0 \longrightarrow F_0/F \longrightarrow F_1 \longrightarrow F_1/F_0 \longrightarrow 0, \ldots .$$

From 2.4 we get that the $F_0/F, F_1/F_0, \ldots$ are flabby. The claim now follows from 2.3. □

A sheaf F is called *acyclic* if $H^n(X,F) =$ for $n > 0$. Hence flabby sheaves are acyclic. By an *acyclic* resolution of a sheaf we understand an exact sequence

$$0 \longrightarrow F \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

with acyclic F_i.

2.7 Proposition. *Let* $0 \to F \to F_0 \to F_1 \to \cdots$ *be an acyclic resolution of* F. *Then there is a natural isomorphism between the* n*-the cohomology group* $H^n(X,F)$ *and the* n*-th cohomology group of the complex*

$$\cdots \longrightarrow 0 \longrightarrow \underset{\substack{\uparrow \\ \textit{zero position}}}{\Gamma F_0} \longrightarrow \Gamma F_1 \longrightarrow \Gamma F_2 \longrightarrow \cdots$$

Proof. Taking the canonical flabby resolutions of F and of all F_n on gets a diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & F_0 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(0)} & \longrightarrow & F_0^{(0)} & \longrightarrow & F_1^{(0)} & \longrightarrow & F_2^{(0)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(1)} & \longrightarrow & F_0^{(1)} & \longrightarrow & F_1^{(1)} & \longrightarrow & F_2^{(1)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^{(2)} & \longrightarrow & F_0^{(2)} & \longrightarrow & F_1^{(2)} & \longrightarrow & F_2^{(2)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

All lines and columns are exact. We apply Γ to this complex. Then all lines and columns besides the first ones remain exact. We can apply 1.2. □

One may ask what "natural" means in 2.7 means. It means that certain diagrams in which this ismomorphism appears are commutative. Since it is the best to check this when it is used we give just one example: Consider the above commutative diagram in the following new meaning: All occurring sheaves besides F are acyclic. Then 1.2 gives an isomorphism between the n-th cohomology groups of the complexes $0 \to \Gamma F_0 \to \Gamma F_1 \to \cdots$ and $0 \to \Gamma F^{(0)} \to \Gamma F^{(1)} \to \cdots$. Both are isomorphic to $H^n(X, F)$. This gives a commutative triangle.

3. Paracompactness

We consider a very special case. We take for $\mathcal{O}$ the sheaf $\mathcal{C}$ of continuous functions. There are two possibilities: $\mathcal{C}_{\mathbb{R}}$ is the sheaf of continuous real-valued and $\mathcal{C}_{\mathbb{C}}$ the sheaf of continuous complex-valued functions. If we write $\mathcal{C}$ we mean one of both. The sheaf $\mathcal{C}$ or more generally a module over this sheaf have over paracompact spaces a property which can be considered as a weakened form of flabbyness.

3.1 Remark. *Let X be paracompact space and $\mathcal{M}$ a $\mathcal{C}$-module on X. Assume that U is an open subset and $V \subset\subset U$ an open subset which is relatively compact in U. Assume that $s \in \mathcal{M}(U)$ is a section over U. Then there is a global section $S \in \mathcal{M}(X)$ such that $S|V = s|V$.*

Proof. We choose a continuous real valued function φ on X, which is one on V and whose support is compact and contained in U. Then we consider the open covering $X = U \cup U'$, where U' denotes the complement of the support of φ. On U we consider the section φs and on U' the zero section. Since both are zero on $U \cap U'$ they glue to a section S on X.

3.2 Lemma. *Let X be a paracompact space and $\mathcal{M} \to \mathcal{N}$ a surjective $\mathcal{C}$-linear map of $\mathcal{C}$-modules. Then $\mathcal{M}(X) \to \mathcal{N}(X)$ is surjective.*

Proof. Let $s \in \mathcal{N}(X)$. There exists an open covering $(U_i)_{i\in I}$ of X such that $s|U_i$ is the image of an section $t_i \in \mathcal{M}(U_i)$. We can assume that the covering is locally finite. We take relatively compact open subsets $V_i \subset U_i$ such that (V_i) is still a covering. Then we choose a partition of unity (φ_i) with respect to (V_i). By 3.2 there exists global sections $T_i \in \mathcal{M}(X)$ with $T_i|V_i = t_i|V_i$. We now consider

$$T := \sum_{i\in I} \varphi_i T_i.$$

Since I can be infinite we have to explain what this means. Let $a \in X$ a point. There exists an open neighborhood $U(a)$ such $V_i \cap U(a) \neq \emptyset$ only for a finite subset $J \subset I$. We can define the section

$$T(a) := \sum_{i\in J} \varphi T_i|U(a).$$

The sets $U(a)$ cover X and the sections $T(a)$ glue to a section T. Clearly T maps to s. □

3.3 Lemma. *Let X be a paracompact space and $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ an exact sequence of $\mathcal{C}$-modules. Then $\mathcal{M}(X) \to \mathcal{N}(X) \to \mathcal{P}(X)$ is exact too.*

Proof. The exactness of the sequence implies the exactness of

$$0 \longrightarrow \mathrm{Image}(\mathcal{M} \to \mathcal{N}) \longrightarrow \mathcal{N} \longrightarrow \mathrm{Kernel}(\mathcal{N} \to \mathcal{P}) \longrightarrow 0.$$

From 3.2 we get

$$0 \longrightarrow \mathrm{Image}(\mathcal{M} \to \mathcal{N})(X) \longrightarrow \mathcal{N}(X) \longrightarrow \mathrm{Kernel}(\mathcal{N} \to \mathcal{P})(X) \longrightarrow 0.$$

Applying 3.2 to $\mathcal{M} \to \mathrm{Image}(\mathcal{M} \to \mathcal{N})$ we obtain

$$\mathrm{Image}(\mathcal{M} \to \mathcal{N})(X) = \mathrm{Image}(\mathcal{M}(X) \to \mathcal{N}(X)).$$

Since also

$$\text{Kernel}(\mathcal{N} \to \mathcal{P})(X) = \text{Kernel}(\mathcal{N}(X) \to \mathcal{P}(X))$$

we get the exactness of

$$0 \longrightarrow \text{Image}(\mathcal{M}(X) \to \mathcal{N}(X)) \longrightarrow \mathcal{N}(X) \longrightarrow \text{Kernel}(\mathcal{N}(X) \to \mathcal{P}(X)) \longrightarrow 0.$$

This proves 3.3. □

Let $\mathcal{M}$ b an $\mathcal{C}$-module over a paracompact space. Then the canonical flabby resolution is also a sequence of $\mathcal{C}$-modules. From 3.3 follows that the resolution remains exact after the application of Γ. We obtain.

3.4 Proposition. *Let X be paracompact. Every $\mathcal{C}$-module is acyclic, i.e. $H^n(X, \mathcal{M}) = 0$ for $n > 0$.*

The essential tool of the proofs has been the existence of a partition of unity. Partitions of unity exist also in the differentiable world. Hence there is the following variant of 3.3.

3.5 Proposition. *Let X be a paracompact differentiable manifold, then every $\mathcal{C}^\infty$-modul is acyclic.*

4. Čech Cohomology

We have to work with open coverings $\mathfrak{U} = (U_i)_{i\in I}$ of the given topological space X. For indices $i_0, \dots, i_p$ we use the notation

$$U_{i_0,\dots,i_p} = U_{i_0} \cap \dots \cap U_{i_p}.$$

Let F be sheaf on X. A p-cochain of F with respect to the covering $\mathfrak{U}$ is family of sections is an element of

$$\prod_{(i_0,\dots,i_p)\in I^{p+1}} F(U_{i_0,\dots,i_p}).$$

This means that to any $(p+1)$-tuple of indices $i_0, \dots, i_p$ there is associated a section $s(i_0, \dots, i_p) \in F(U_{i_0,\dots,i_p})$. We denote the group of all cochains by $C^p(\mathfrak{U}, F)$. The derivative ds of a p-cochain the $(p+1)$-cochain defined by

$$ds(s_0, \dots s_{p+1}) = \sum_{j=0}^{p+1} (-1)^j s(i_0, \dots, \hat{i}_j, \dots, i_{p+1})|U_{i_0,\dots,i_{p+1}}.$$

The rule $d^2 = 0$ is obvious, hence we obtain a complex

$$\cdots \longrightarrow C^{p-1}(\mathfrak{U}, F) \longrightarrow C^p(\mathfrak{U}, F) \longrightarrow C^{p+1}(\mathfrak{U}, F) \longrightarrow \cdots$$

Here for negative p we set $C^p(\mathfrak{U}, F) = 0$. The cohomology groups of this complex are the Čech cohomology groups $\check{\mathrm{H}}^p(\mathfrak{U}, F)$.

4.1 Lemma. *There is a natural isomorphism*

$$\check{H}^0(\mathfrak{U}, F) = H^0(X, F) \qquad (= F(X)).$$

Proof. A zero-cochain s is just a family $s_i \in F(U_i)$. The condition $ds = 0$ means $s_i|U_i \cap U_j = s_j|U_i \cap U_j$. By the sheaf axioms they glue to a global section. □

4.2 Remark. *Let F be a flabby sheaf. Then for every open covering*

$$\check{H}^p(\mathfrak{U}, F) = 0 \quad \textit{for} \quad p > 0.$$

Proof. Just to save notation we restrict to the case $p = 1$. The general case works in the same way. We start with a little remark. Assume that the whole space $X = U_{i_0}$ is a member of the covering. Then the Čech cohomology vanishes (for every sheaf): if (s_{ij}) is a cocycle one defines $s_i = s_{i,i_0}$. Then $d((s_i)) = (s_{ij})$.

For the general proof of 4.2 (in the case $p = 1$) we now consider the sequence

$$\begin{array}{ccccccc} 0 \longrightarrow F(X) & \longrightarrow & \prod_i F(U_i) & \longrightarrow & \prod_{ij} F(U_i \cap U_j) & \longrightarrow & \prod_{ijk} F(U_i \cap U_j \cap U_k) \\ s & \longmapsto & (s|U_i) & & & & \\ & & (s_i) & \longmapsto & (s_i - s_j) & & \\ & & & & (s_{ij}) & \longmapsto & (s_{ij} + s_{jk} - s_{ik}) \end{array}$$

We will proof that this sequence is exact. (Then 4.2 follows.) The idea is to sheafify this sequence: For an open subset $U \subset X$ one considers $F|U$ and also the restricted covering $U \cap U_i$. Repeating the above construction for U instead of X on obtains a sequence of sheaves

$$0 \longrightarrow F \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}.$$

Since F is flabby, also $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are flabby. The remark at the beginning of the proof shows that $0 \longrightarrow F(U) \longrightarrow \mathcal{A}(U) \longrightarrow \mathcal{B}(U) \longrightarrow \mathcal{C}(U)$ is exact, when U is contained in some U_i. Hence the sequence of sheaves is exact. From 2.6 follows that the exactness is also true for $U = X$. □

4.3 Theorem of Leray. *Let F be a sheaf on X and $\mathfrak{U} = (U_i)$ an open covering of X. Assume that $H^p(U, F|U) = 0$ for all $p > 0$ and for arbitrary intersection of finitely many U_i. Then there is a natural isomorphism*

$$H^p(X, F) \cong \check{H}^p(\mathfrak{U}, F)$$

for all p.

Proof. We consider a flabby resolution $0 \to F \to F_0 \to F_1 \to \cdots$. There is a natural diagram

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F(X) & \to & F_0(X) & \to & F_1(X) & \to & F_2(X) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C^0(\mathfrak{U},F) & \to & C^0(\mathfrak{U},F_0) & \to & C^0(\mathfrak{U},F_1) & \to & C^0(\mathfrak{U},F_2) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C^1(\mathfrak{U},F) & \to & C^1(\mathfrak{U},F_0) & \to & C^1(\mathfrak{U},F_1) & \to & C^1(\mathfrak{U},F_2) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C^2(\mathfrak{U},F) & \to & C^2(\mathfrak{U},F_0) & \to & C^2(\mathfrak{U},F_1) & \to & C^2(\mathfrak{U},F_2) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

All rows but the first one are exact. Similarly all columns but first one are exact. Now a homological lemma 1.2 gives the desired result. □

5. The first cohomology group

The first Čhech cohomology group has some special properties: We will keep very short, since later we will use it only in applications. Let $f : G \to H$ be a surjective homomorphism of sheaves and $\mathfrak{U} = (U_i)$ an open covering of X. We denote by $H_{\mathfrak{U},f}(X)$ the set of all global sections of H with the following property:

For every index i there is a section $t_i \in G(U_i)$ with $f(t_i) = s|U_i$. By definition of (sheaf-)surjectivity for every global section $s \in H(X)$ there exists an open covering $\mathfrak{U}$ with $s \in H_{\mathfrak{U},f}(X)$. It follows

$$H(X) = \bigcup_{\mathfrak{U}} H_{\mathfrak{U},f}(X).$$

Let $0 \to F \to G \xrightarrow{f} H \to 0$ be an exact sequence and $\mathfrak{U}$ an open covering. There exists a natural homomorphism

$$\delta : H_{\mathfrak{U},f}(X) \longrightarrow \check{H}^1(\mathfrak{U}, F),$$

which is constructed as follows: Let be $s \in H_{\mathfrak{U},f}(X)$. We choose elements $t_i \in G(U_i)$ which are mapped to $s|U_i$. The differences $t_i - t_j$ come from sections $t_{ij} \in F(U_i \cap U_j)$. They define a 1-cocycle $\delta(s)$. It is easy to check that this corresponding element of $\check{H}^1(\mathfrak{U}, F)$ doesn't depend on the choice of the t_i.

5.1 Lemma. *Let* $0 \to F \xrightarrow{f} G \to H \to 0$ *be an exact sequence of sheaves and* $\mathfrak{U}$ *an open covering. The sequence*

$$0 \to F(X) \longrightarrow G(X) \longrightarrow H_{\mathcal{U},f}(X) \xrightarrow{\delta} \check{H}^1(\mathcal{U}, F) \longrightarrow \check{H}^1(\mathcal{U}, G) \longrightarrow \check{H}^1(\mathcal{U}, H)$$

is exact.

The simple proof is left to the reader. □

Let now F be an arbitrary sheaf, $F^{(0)}$ the associated flabby sheaf. We get an exact sequence $0 \to F \to F^{(0)} \to H \to 0$. let $\mathfrak{U}$ be an open covering. We know that $\check{H}^1(\mathfrak{U}, F^{(0)})$ vanishes, 4.2. From 4.2 we obtain an isomorphy

$$\check{H}^1(\mathfrak{U}, F) \cong H_{\mathfrak{U},f}(X)/G(X).$$

From the long exact cohomology sequence we get for the usual cohomology

$$H^1(X, F) \cong H(X)/G(X).$$

This gives an *injective* homomorphism

$$\check{H}^1(\mathfrak{U}, F) \longrightarrow H^1(X, F).$$

In the following we consider $\check{H}^1(\mathfrak{U}, F)$ as subset of $H^1(X, F)$. Now it is easy to check:

5.2 Proposition. *Let* F *be a sheaf. Then*

$$H^1(X, F) = \bigcup_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, F).$$

The following commutative diagram that the Čech combining δ from 5.1 and that of general sheaf theory 2.5 coincide:

5.3 Remark. *For a short exact sequence* $0 \to F \to G \xrightarrow{f} H \to 0$ *the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H_{\mathfrak{U},f}(X) & \xrightarrow{\delta} & \check{H}^1(\mathfrak{U}, F) \\ & & \| & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H(X) & \xrightarrow{\delta} & H^1(X, F) \end{array}$$

is commutative.

The proof is left to the reader. □

Let $\mathfrak{V} = (V_j)_{j \in J}$ be a refinement of $\mathfrak{U} = (U_i)_{i \in I}$ and $\varphi : J \longrightarrow I$ a refinement map $(V_\varphi \subset U_i)$. Using this refinement map one obtains a natural map

$$\check{H}^1(\mathfrak{U}, F) \longrightarrow \check{H}^1(\mathfrak{V}, F).$$

This shows:

5.4 Remark. *Let $\mathfrak{V}$ be an refinement of $\mathfrak{U}$ and $\varphi : J \to I$ a refinement map. The diagram*

$$\begin{array}{ccc} \check{H}^1(\mathfrak{U}, F) & \longrightarrow & \check{H}^1(\mathfrak{V}, F) \\ & \searrow \quad \swarrow & \\ & H^1(X, F) & \end{array}$$

commutes. Especially it doesn't depend on the choice of the refinement map.

We also mention a refinement of Leray's lemma 4.3 in case of the first cohomology group.

5.5 Theorem (refinement of Leray's theorem in case of the first cohomology group). *Let F be a sheaf on X and $\mathfrak{U} = (U_i)$ an open covering of X. Assume that $H^1(U_i, F|U_i) = 0$ for all $\subset\in I$. Then there is a natural isomorphism*

$$H^1(X, F) \cong \check{H}^1(\mathfrak{U}, F).$$

Hint for the proof. One has to show that for any refinement $\mathfrak{V}$ the map $H^1(\mathfrak{U}, F) \to H^1(\mathfrak{V}, F)$ is surjective. The proof is easy and left to the reader. Details can be found in Forster's book "Riemann surfaces", Propostition II.12.8. □

6. Some vanishing results

Let X be a topological space and A an abelian group. We denote by A_X the sheaf of locally constant functions with values in A. This sheaf can be identified with the sheaf which is generated by the presheaf of constant functions. We will write

$$H^n(X, A) := H^n(X, A_X).$$

6.1 Proposition. *Let U be an open and convex subset of $\mathbb{R}^n$. Then for every abelian group A*

$$H^1(U, A) = 0.$$

Actually this is true for all H^n, $n > 0$. The best way to prove this to use the comparison theorem with singular cohomology as defined in algebraic topology. We restrict to H^1.

Proof of 6.1. Every convex open subset of $\mathbb{R}^n$ is topologically equivalent to $\mathbb{R}^n$. Hence it is sufficient to restrict to $U = \mathbb{R}^n$. Just for simplicity we assume $n = 1$. (The general case should then be clear.) We use Čhech cohomology and show that every open covering admits a refinement $\mathfrak{U}$ such that $H^1(\mathfrak{U}, A_X) = 0$. To show this we take a refinement of a very simple nature. It is easy to show that there exists a refinement of the following form. The index set is $\mathbb{Z}$. There exists a sequence of real numbers (a_n) with the following properties:

a) $a_n \leq a_{n+1}$
b) $a_n \to +\infty$ for $n \to \infty$ and $a_n \to -\infty$ for $n \to -\infty$
c) $U_n = (a_n, a_{n+2})$.

Assume that $s_{n,m}$ is a cocycle with respect to this covering. Notice that U_n has non empty intersection only with U_{n-1} and U_{n+1}. Hence only $s_{n-1,n}$ is of relevance. This a locally constant function on $U_{n-1} \cap U_n = (a_n, a_{n+1})$. Since this is connected, the function $s_{n-1,n}$ is constant. We want to show that it is coboundary, i.e. we want to construct constant functions s_n on U_n such that $s_{n-1,n} = s_n - s_{n-1}$ on (a_n, a_{n+1}). This is easy. One starts with $s_0 = 0$ and then constructs inductively $s_1, s_2, \dots$ and in the same way for negative n. □

Consider on the real line $\mathbb{R}$ the sheaf of real valued differentiable functions $\mathcal{C}^\infty$. Taking derivatives one gets a sheaf homomorphism $\mathcal{C}^\infty \to \mathcal{C}^\infty$, $f \mapsto f'$. The kernel is the sheaf of all locally constant functions, which we denote simply by $\mathbb{R}$. Hence we get an sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty \longrightarrow 0.$$

This sequence is exact since every differentiable function has an integral. Hence this sequence can be considered as acyclic resolution of $\mathbb{R}$. We obtain $H^n(\mathbb{R}, \mathbb{R}) = 0$ for all $n > 0$. For $n = 1$ this follows already from 6.1. There is a generalization to higher dimensions. For example a standard result of vector analysis states in the case $n = 2$.

6.2 Lemma. *Let $E \subset \mathbb{R}^n$ be an open and convex subset, $f, g \in \mathcal{C}^\infty$ a pair of differentiable functions with the property*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

Then there is a differentiable function h with the property

$$f = \frac{\partial h}{\partial x}, \quad g = \frac{\partial h}{\partial y}.$$

In the sequence of exact sequences this means:

The sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(E) \times \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(E) \longrightarrow 0$$

$$f \longmapsto \Big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\Big)$$

$$(f,g) \longmapsto \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}$$

is exact. When E is not convex, this sequence needs not to be exact. But since every point in $\mathbb{R}^2$ has an open convex neighborhood, the sequence of sheaves

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \mathcal{C}_X^\infty \longrightarrow \mathcal{C}_X^\infty \times \mathcal{C}_X^\infty \longrightarrow \mathcal{C}_X^\infty \longrightarrow 0$$

is exact. This is an acyclic resolution and we obtain:

6.3 Proposition. *For convex open $E \subset \mathbb{R}^2$ we have*

$$H^i(E, \mathbb{R}) = 0 \quad \textit{for} \quad i > 0.$$

The sequence is a special case of the de-Rham complex

$$0 \longrightarrow \mathbb{R} \longrightarrow A_X^0 \longrightarrow A_X^1 \longrightarrow \cdots \longrightarrow A_X^n \longrightarrow 0$$

Here X is a differentiable manifold of dimension n and A_X^i denotes the sheaf of alternating differential forms of degree i.

6.4 Lemma of Poincaré. *Let $U \subset \mathbb{R}^n$ be an open convex subset. Then*

$$H^p(U, \mathbb{R}) = 0 \quad \textit{for} \quad p > 0.$$

Proof. Let ω be a closed form. We decompose it as

$$\omega = \alpha + \beta \wedge dx_n,$$

where α doesn't contain any term with dx_n. We write

$$\beta = \sum f_a dx_a$$

where a are subsets of $\{1, \dots, n-1\}$ that do nor contain n. (We use the notation $dx_a = dx_{a_1} \wedge \ldots \wedge dx_{a_p}$, where $a_1 < \ldots < a_p$ are the elements of a in their natural order.) Integrating with respect to the last variable we find differentiable functions F_a such that $\partial_n F_a = f_a$. Now the difference $\omega - d\sum_a F_a dx_a$ doesn't contain any term in which dx_n occurs. Hence we can assume that in ω no term with dx_n occurs. We write

$$\omega = \sum_a g_a dx_a,$$

where all a are subsets of $\{1, \dots, n-1\}$. Now we use $d\omega = 0$. We obtain $\partial_n g_a = 0$. Hence g_a do not depend on x_n. But now ω can be considered as differential form in one dimension less (on the image of U with respect to the projection map that cancels the last variable) and an induction argument completes the proof. □

We obtain

6.5 Theorem of de Rham. *For a differentiable manifold X on has*

$$\dim H^i(X,\mathbb{R}) \cong \frac{\mathrm{Kernel}(A^i(X) \longrightarrow A^{i+1}(X))}{\mathrm{Image}((A^{i-1}(X) \longrightarrow A^i(X))}.$$

Applying the Lemma of Poincarè again we obtain:

6.6 Proposition. *For convex open $E \subset \mathbb{R}^n$ on has*

$$H^i(E,\mathbb{R}) = 0 \quad \textit{fur} \quad i > 0.$$

Differential forms can also be considered complex valued. The Lemma of Poincarè remains true by trivial reasons. Hence we see also:

6.7 Proposition. *For convex open $E \subset \mathbb{R}^n$ on has*

$$H^i(E,\mathbb{C}_X) = 0 \quad \textit{fur} \quad i > 0.$$

As an application we prove

6.8 Proposition. *For convex open $E \subset \mathbb{R}^n$ on has*

$$H^2(E,\mathbb{Z}) = 0.$$

Proof. We consider the homomorphism

$$\mathbb{C} \longrightarrow \mathbb{C}^{\bullet}, \qquad z \longmapsto e^{2\pi \mathrm{i} z}.$$

The kernel is $\mathbb{Z}$. This can be considered as a exact sequence of sheaves for example on an open convex $E \subset \mathbb{R}^n$. A small part of the long exact cohomology sequence is

$$H^1(E,\mathbb{C}^*) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow H^2(E,\mathbb{C}).$$

Since the first and the third member of this sequence vanish (6.1 and 6.3) we get the proof of 6.6. □

Next we treat an example of complex analysis. For this wee need the Dolbeault complex

$$0 \longrightarrow \Omega^p(U) \xrightarrow{\bar\partial} A^{p,0}(U) \xrightarrow{\bar\partial} A^{p,1}(U) \xrightarrow{\bar\partial} \cdots$$

for an open subset $U \subset \mathbb{C}^n$.

6.9 Lemma of Dolbeault. *Let $U \subset \mathbb{C}^n$ be a polydisk. The sequence*

$$0 \longrightarrow \Omega^p(U) \xrightarrow{\bar\partial} A^{p,0}(U) \xrightarrow{\bar\partial} A^{p,1}(U) \xrightarrow{\bar\partial} \cdots$$

is exact.

Corollary. *One has*

$$H^q(U, \mathcal{O}_U) = 0 \quad \textit{for} \quad q > 0.$$

6.10 Basic Lemma. *Let $f : E \to \mathbb{C}$ be a $\mathcal{C}^\infty$-function on the unit disk E. Then there exists a $\mathcal{C}^\infty$-function $g : E \to \mathbb{C}$ with the property*

$$\frac{\partial g}{\partial \bar z} = f(z).$$

Additional Remark. *If f depends differentiably on more variables, one can get that the seme is true for g.*

Proof of the basic lemma. In a first step we assume that f is defined on some open neighborhood of $\bar E$. The proof uses Stokes's theorem. The idea is to define g as an surface integral:

$$g(a) = \frac{1}{2\pi \mathrm{i}} \int_E f(z) \frac{dz \wedge d\bar z}{z - a}.$$

Since there is a singular point in the integrand, the integral needs an interpretation. For this we use polar coordinates $z = a + re^{\mathrm{i}\varphi}$ in a small disk around a. We get

$$dz \wedge d\bar z = 2\mathrm{i} dx \wedge dy = 2\mathrm{i} r dr d\varphi.$$

The new integrand is $2\mathrm{i} f(z) e^{-\mathrm{i}\varphi}$. The singularity disappeared!. This considerations shows that as precise definition of the integral one can take

$$g(a) = \frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon \to 0} \int_{E(\varepsilon)} f(z) \frac{dz \wedge d\bar z}{z - a}$$

where $E(\varepsilon)$ denotes the complement of the disk $|z - a| \le \varepsilon$. Here ε should be taken small enough such that this closed disk is contained in E. We will apply the theorem of Stokes to $E(\varepsilon)$ and the differential form

$$\omega := f(z) \log |z - a|^2 d\bar z.$$

Since

$$d\omega = \partial\omega = \frac{\partial f}{z} \log |z - a|^2 + \frac{f(z)}{z - a},$$

we get from Stokes theorem

$$\oint_{|z|=1} f(z)\log|z-a|^2 d\bar z - \oint_{|z-a|=\varepsilon} f(z)\log|z-a|^2 d\bar z$$
$$= \int_{E(\varepsilon)} \frac{\partial f}{\partial z}\log|z-a|^2 dz\wedge d\bar z + \int_{E(\varepsilon)} f(z)\frac{dz\wedge d\bar z}{z-a}.$$

Now we take the limit ε to 0 the integral $\oint_{|z-a|=\varepsilon} f(z)\log|z-a|^2 d\bar z$ tends to 0. This follows from the standard estimate of line curve integrals and the fact $\lim_{\varepsilon\to 0}\varepsilon\log\varepsilon = 0$. Taking the limit now we get

$$2\pi \mathrm{i} g(a) = \oint_{|z|=1} f(z)\log|z-a|^2 d\bar z - \int_E \frac{\partial f}{\partial z}\log|z-a|^2 dz\wedge d\bar z.$$

One should notice that the integrand of the surface integral still has a singularity at a. But this is only a logarithmic singularity and $\log|z-a|$ is Lebesgue integrable over E. It is easy to verify that the Lebesgue limit theorem applies. The same argument applies to show that g is differentiable and that differentiation can be interchanged with integration:

$$2\pi \mathrm{i}\frac{\partial g(a)}{\partial \bar a} = \int_E \frac{\partial f(z)}{\partial z}\frac{dz\wedge d\bar z}{\bar z - a} - \int_{|z|=1} f(z)\frac{d\bar z}{\bar z - a}.$$

Now the proof follows from the *generalized Cauchy integral formula*: Let f be a $\mathcal{C}^\infty$ function on an open neighborhood of $\bar E$. Then

$$2\pi \mathrm{i} f(a) = \int_{|z|=1} \frac{f(z)}{z-a}dz + \int_E \frac{\partial f}{\partial \bar z}\frac{dz\wedge d\bar z}{z-a}.$$

(For holomorphic f this is the usual Cauchy integral formula.

Since this formula may not be standard, we mention that it is also an application of Stokes theorem. One uses the formula

$$d\Big(f(z)\frac{dz}{z-a}\Big) = \frac{\partial f}{\partial \bar z}\frac{dz\wedge d\bar z}{z-a}$$

and again applies Stoke's theorem to the domain $E(\varepsilon)$, introduces polar coordinates and takes the limit $\varepsilon\to 0$.

Now we assume that F is given only on E (and not on a neighborhood of $\bar E$. This needs a new technique. The idea is to use an approximation argument. We choose an exhaustion of E by the sequence of disks $E_n = \{z;\ |z| < 1-1/n\}$. We know already that there exists $g_n \in \mathcal{C}^\infty(E_n)$ such that $\partial g_n/\partial\bar z = f$ on E_n. The functions g_n are not uniquely determined. The idea is to prepare them such that they converge. More precisely we want to have that for each i the sequence $g_n, g_{n+1}, \ldots$ converges on E_n. The limit will be a function on E_n and all these differential forms glue to a function g on the whole E. This will be the solution of our problem. $(\partial g/\partial \bar{z} = f)$.

We have to explain in which sense convergence is understood. For this we use the maximum norm $||h||_{E_n}$ for a function that is continuous on some open neighborhood of E_n. The strategy is to construct the g_n inductively such that

$$||g_{n+1} - g_n||_{E_{n-1}} < 2^{-n}.$$

One starts with arbitrary g_1. The induction step is very easy. Assume that $g_1, \ldots, g_n$ have been constructed. Then choose any h such that $\partial h/\partial \bar{z} = f$ on E_{n+1}. We can modify h by adding function. Hence we try to define $g_{n+1} = h + P$ with an analytic function. Now we use that $h - g_n$ is holomorphic on E_n. We can approximate this function on E_{n-1} by a polynomial P (taking a partial sum of the Taylor expansion). This gives the construction of g_{n+1}.

Now it is easy to show that the limit of the g_n exists. Just write in (on P_n) in the form

$$g = g_n + \sum_{i=n}^{\infty} (f_{i+1} - f_i).$$

The sum is a series of holomorphic functions that converges uniformly on E_n. Hence the limit exists and differentiation can be exchanged with the limit.

This finishes the proof of the basic lemma. □

Proof of 6.9 continued. Now we go to several variables and consider a polydisk P. We assume that ω is a differential form of type (p, q) not only on P but on an open neighborhood of $\bar{P}$. We assume $\partial\omega = 0$ and claim that there exists a $(p, q-1)$-form ω' on P with $\partial\omega' = \omega$. The proof can be given by induction in the same way as in the proof of the lemma of Picareè. The beginning of the induction now is the basic lemma 6.10. We skip details. □

We give a nice application. Let $\bar{\mathbb{C}}$ be the Riemann sphere.

6.11 Theorem. *One has*

$$H^1(\bar{\mathbb{C}}, \mathcal{O}_{\bar{\mathbb{C}}}) = 0.$$

For the proof we use a covering by two disks of the Riemann sphere $U = \{z \in \mathbb{C}; |z| < 2\}$ and $V = \{z \in \bar{\mathbb{C}}; |z| > 1\}$ (including ∞). We can apply the refinement of Leray's theorem 5.5 to obtain $H^1(\mathfrak{U}, \mathcal{O}) = H^1(\bar{\mathbb{C}}, \mathcal{O}_{\bar{\mathbb{C}}})$. A Čech 1-cocycle simply is given by a holomorphic function on the circular ring. We have to show that it can be written as difference $f_1 - f_2$ where f_i is holomorphic on the disc E_i. This is possible by the theory of the *Laurent decomposition*.

□

Chapter VII. Algebraic tools

1. Abelian groups

We assume that the reader is familiar with the notion of an abelian group and homomorphism between abelian groups. If A is a subgroup of an abelian group B, then the factor group B/A is well defined. All what one needs usually is that there is a natural surjective homomorphism $f : B \to B/A$ with kernel A. Let $f : B \to X$ be a homomorphism into some abelian group. Then f factors through a homomorphism $B/A \to X$ if and only if the kernel of f contains A. That f factors means that there is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ A/B & & \end{array}$$

Let $f : A \to B$ be a homomorphism of abelian groups. Then the image $f(A)$ is a subgroup of B. If there is no doubt which homomorphism f is considered, we allow the notation

$$B/A := B/f(A).$$

1.1 Lemma. *A commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & D \end{array}$$

induces homomorphisms

$$B/A \longrightarrow D/C, \quad C/A \longrightarrow D/B.$$

A (finite or infinite) sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \cdots$$

is called exact at B if

$$\mathrm{Kernel}(B \longrightarrow C) = \mathrm{Image}(A \longrightarrow B).$$

It is called exact if it is exact at every place. An exact sequence $A \to B \to C$ induces an injective homomorphism

$$B/A \hookrightarrow C\ .$$

The sequence $0 \to A \to B$ is exact if and only if $A \to B$ is injective. The sequence $A \to B \to 0$ is exact if and only of $A \to B$ is surjective. The sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact if and only if $A \to B$ is injective and if the induced homomorphism $B/A \to C$ is an isomorphism. A sequence of this form is called a *short exact sequence*. Hence the typical short exact sequence is

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0 \qquad (A \subset B).$$

1.2 The five term lemma. *Let*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

be a commutative diagram with exact lines and such that f_1, f_2 and f_4, f_5 are isomorphisms. Then f_3 is an isomorphism too.

The proof is easy and left to the reader. □

2. Modules and ideals

All rings which we consider are assumed to be commutative and with unit elements. Ring homomorphisms are assumed to map the unit element into the unit element. A module M over a ring A is an abelian group together with a map $A \times M \to M$, $(a, m) \longmapsto am$, such that the usual axioms of a vector space are satisfied including $1_A m = m$ for all $m \in M$. The notion of linear maps, kernel, image of a linear map are as in the case of vector spaces. But in contrast to the case of vector spaces, a module has usually no basis. A module which admits a basis is called free. A finitely generated free module is isomorphic to R^n.

If $M \subset N$ is a submodule, then the factor group N/M carries a structure of an A-module.

Recall that an ideal $\mathfrak{a}$ in a Ring R is an abelian subgroup such that $ra \in \mathfrak{a}$ for $r \in R$ and $a \in \mathfrak{a}$. Hence an ideal is nothing but an R-submodule of R. The factor $R/\mathfrak{a}$ is not only a R-module but carries a structure as ring such that $R \to R/\mathfrak{a}$ is a ring homomorphism.

An ideal is called finitely generated if it is finitely generated as module. This means that there are elements $a_1, \dots, a_n$ such that $\mathfrak{a} = Ra_1 + \cdots + Ra_n$. One writes $\mathfrak{a} = (a_1, \dots, a_n)$. The product $\mathfrak{ab}$ of two ideals is the set of all finite sums $\sum_i a_i b_i$ with $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{b}$. Ideal multiplication is associative. Especially powers of ideals are defined.

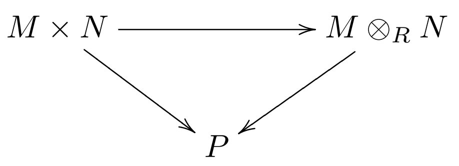
All what we have said about exact sequences of abelian groups is literarily true for A-modules.

Tensor product

Recall that for two modules M, N over a ring R, there exists a module $M \otimes_R N$ together with an R-bilinear map

$$M \times N \longrightarrow M \otimes_R N, \quad (a, b) \longmapsto a \otimes b,$$

such that for each bilinear map $M \times N \to P$ into an arbitrary third module P there exists a unique commutative diagram



with an R-linear map $M \otimes_R N \to P$. The tensor product $M \otimes_R N$ is generated by the special elements $m \otimes n$.

If $f : M \to M'$ and $g : N \to N'$ are R-linear maps, then one gets a natural R-linear map

$$f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N', \quad (a, b) \longmapsto f(a) \otimes g(a).$$

It is clear that this map is uniquely determined by this formula. The existence follows from the universal property applied to the map $(a, b) \mapsto f(a) \otimes f(b)$.

Basic properties of the tensor product

There is a natural isomorphism

$$R \otimes_R M \xrightarrow{\sim} M, \quad (r \otimes m) \longmapsto rm$$

and more generally

$$R^n \otimes M \xrightarrow{\sim} M^n.$$

As a special case we get

$$R^n \otimes_R R^m \cong R^{n \times m}.$$

This is related also to the formula

$$(M \times N) \otimes_R P \cong (M \otimes_R P) \times (N \otimes_R P) \qquad \text{(canonically)}.$$

The tensor product is associative: For usual R-modules M, N, P on has an isomorphism

$$(M \otimes_R N) \otimes_R P \xrightarrow{\sim} M \otimes_R (N \otimes_R P), \quad (m \otimes n) \otimes p \longmapsto m \otimes (n \otimes p).$$

The existence of this map follows from the universal property of the tensor product.

The tensor product is also commutative:

$$M \otimes_R N \xrightarrow{\sim} N \otimes_R M, \quad m \otimes n \longmapsto n \otimes m.$$

Ring extension

Let $A \to B$ be a ring homomorphism and M an A-module. Then $M \otimes_A B$ carries a natural structure as B-module. It is given by $b(m \otimes b') = m \otimes (bb')$. The existence follows from the universal property of the tensor product. A special case is

$$A^n \otimes_A B = B^n.$$

Existence of the tensor product

For an arbitrary set I we define R^I to be the set of all maps $I \to R$, $i \mapsto r_i$ such that r_i is 0 for almost all i. So $R^I = R^n$ for $I = \{1, \dots, n\}$. By definition a module is free if and only if it is isomorphic to an R^I for suitable I. An arbitrary R-module M can be represented by an exact sequence

$$R^J \longrightarrow R^I \longrightarrow M \longrightarrow 0.$$

For another N module we define now the tensor product by the exact sequence

$$N^J \longrightarrow N^I \longrightarrow M \otimes_R N \longrightarrow 0.$$

The bilinear map $M \times N \to M \otimes_R N$ and the universal property are obvious.

Exactness properties

Let $M \to N$ be an injective homomorphism of R-modules. For an R-module P the induced homomorphism $M \otimes_R P \to N \otimes_R P$ needs not to be injective. But when $P \cong R^n$ is free, injectivity is preserved. A slight and trivial extension of this observation is:

2.1 Remark. *Let $M_1 \to M_2 \to M_3$ be an exact exact sequence of R-modules. Then for every **free** module P the sequence $M_1 \otimes_R P \to M_2 \otimes_R P \to M_3 \otimes_R P$ remains exact.*

In this connection we mention some other exactness properties. For two R-modules M, N we denote by $\mathrm{Hom}_R(M, M)$ the set of all R-linear maps $M \to N$. This is an R-module. Let $M \to N$ be an R-linear map. Then for an arbitrary R-module P one has obvious R-linear maps

$$\mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N), \qquad \mathrm{Hom}_R(N, P) \longrightarrow \mathrm{Hom}_R(M, P).$$

Since $\mathrm{Hom}(R^n, M) \cong M^n$, one has:

2.2 Remark. *Let $0 \longrightarrow M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules and P also an R-module, then:*

a) *If P is free then*

$$0 \longrightarrow \mathrm{Hom}_R(P, M_1) \longrightarrow \mathrm{Hom}_R(P, M_2) \longrightarrow \mathrm{Hom}_R(P, M_3) \longrightarrow 0$$

remains exact.

b) *If M_3 is free than*

$$0 \longrightarrow \mathrm{Hom}_R(M_3, P) \longrightarrow \mathrm{Hom}_R(M_2, P) \longrightarrow \mathrm{Hom}_R(M_1, P) \longrightarrow 0$$

remains exact.

We comment shortly b). When M_3 is free one can chose a system of elements in M_2 whose images in M_3 define a basis. This system generates a submodule $M_3' \subset M_2$ which maps isomorphically to M_3. Now it is easy to see that M_2 is isomorphic to $M_1 \times M_3$ and the map $M_1 \to M_2$ corresponds to $m \mapsto (m, 0)$ and the map $M_2 \to M_3$ corresponds to $(m_1, m_3) \longmapsto m_3$. Now the exactness shold be clear.

3. Divisibility

We recall some basic notions of divisibility in rings. Let R be ring (commutative and with unit). An element $a \in R$ of a ring is called a unit if the equation $ax = 1_R$ is solvable in R. Then the solution is unique. The set R^* of units is a group under multiplication. A ring is called an integral domain if $ab = 0 \Rightarrow a = 0$ or $b = 0$.

3.1 Definition. *Let R be an integral domain. An element $a \in R - R^*$ is called*

a) ***indecomposable***, *if one has*

$$a = bc \quad \Longrightarrow \quad b \text{ or } c \text{ is a unit}$$

b) ***prime element,*** *if*

$$a|bc \quad \Longrightarrow \quad a|b \text{ or } a|c$$

($a|b$ means that the equation $b = ax$ is solvable in R). Notice that units are not prime elements.)

Of course prime elements are indecomposable, but usually the converse is false.

Example. Let $R = \mathbb{C}[X]$ be the polynomials ring in one variable over $\mathbb{C}$ and R_0 the sub-ring of all polynomials without linear term. The element X^3 is indecomposable in R_0 but not a prime: $X^3|X^2 \cdot X^4$.

3.2 Definition. *The integral domain R is called **factorial** or **UFD-ring,** if the following two conditions are satisfied:*

1) *Each element $a \in R - R^*$ can be written as product of finitely man indecomposable elements.*
2) *Each indecomposable element is prime.*

In factorial rings the decomposition into primes is unique in the following sense: Let

$$a = u_1 \cdots u_n = v_1 \cdots v_m$$

be two decompositions of $a \in R - R^*$ into primes. Then one has

a) $m = n$.
b) There exists a permutation σ of the digits $1, \dots, n$, such that

$$u_\nu = \varepsilon_\nu v_{\sigma(\nu)}, \quad \varepsilon_\nu \in R^* \quad \text{for } 1 \le \nu \le n.$$

It is easy to prove this by induction.

Examples for factorial rings.

1) Each field is factorial.
2) $\mathbb{Z}$ is factorial
3) By an important *Theorem of Gauss* the polynomial ring $R[z_1, \dots, z_n]$ over a factorial ring is factorial too.

3.3 Theorem of Gauss. *The polynomial ring $R[z_1, \dots, z_n]$ over a factorial ring is factorial too.*

4. The discriminant

The discriminant should be treated in an course of basic algebra: We just recall the basic facts. One constructs for each natural number n a polynomial Δ_n of n variables over the ring $\mathbb{Z}$ of integers. Using this universal polynomial one defines for any normalized polynomial

$$P = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

over a ring R the discriminant

$$d(P) := \Delta_n(a_0, \dots, a_{n-1}) \in R.$$

The basic fact about the discriminant is: Assume that R is factorial. Then P is square free if and only if $d(P) \neq 0$.

We just give a comment. In the case $R = \mathbb{C}$ a polynomial is square free if and only if has no double zero. The discriminant of the quadratic polynomial $X^2 + bX + c$ is $b^2 - 4c$.

5. Noetherian rings

In commutative algebra there is a basic notion of noetherian ring. A ring R (commutative and with unit) is called noetherian, if any ideal $\mathfrak{a} \subset R$ is finitely generated. Noetherian rings have the basic property that a sub-module of a finitely generated module is finitely generated. It is trivial that the factor ring of a noetherian ring is noetherian. The Hilbert basis theorem states the following: The polynomial ring $R[X_1, \dots, X_n]$ over a noetherian ring is noetherian. Hence every finitely generated R-algebra is noetherian.

A ring R is called local if it is not the zero ring and if the set of all non-units $\mathfrak{m}$ is an ideal. Then the factor ring $R/\mathfrak{m}$ is a field. A homomorphism $A \to B$ of local rings is call local, if the maximal ideal of A is mapped into the maximal ideal of B. A field K is a local ring, $\mathfrak{m} = \{0\}$. The ring $\mathbb{Z}$ of integers is not a local ring, since the units are just the elements ± 1. Similarly the ring of polynomials $K[X_1, \dots, X_n]$ $(n \geq 1)$ is not a local ring. The basic example for us the ring $\mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$ of convergent power series. Elements with a non zero constant term are invertible. Elements with zero constant term are not invertible. Obviously they form an ideal $\mathfrak{m}_n$. The residue field $\mathcal{O}_n/\mathfrak{m}_n$ is isomorphic to $\mathbb{C}$. More precisely the composition of the natural homomorphisms

$$\mathbb{C} \longrightarrow \mathcal{O}_n \longrightarrow \mathcal{O}_n/\mathfrak{m}_n$$

is an isomorphism. We can use this isomorphism to identify $\mathbb{C}$ and $\mathcal{O}_n/\mathfrak{m}_n$.

In this connection we want to mention another algebraic result. Let M be a module over a ring R

5.1 Lemma von Nakayama. *Let M be a finitely generated module over a local ring R with maximal ideal $\mathfrak{m}$. Assume $\mathfrak{m}M = M$. Then $M = 0$.*

There is a rather obvious application:

5.2 Lemma. *Let R be a noetherian local ring R and $r_1, \dots, r_n$ elements of the maximal ideal. Assume that their cosets mod $\mathfrak{m}^2$ generated $\mathfrak{m}/\mathfrak{m}^2$ as R-module. Then they generate $\mathfrak{m}$.*

For the prove one applies the lemma of Nakayama to $\mathfrak{m}/(r_1, \dots, r_n)$. We also mention that $\mathfrak{m}/\mathfrak{m}^2$ is not only an R-module but an $R/\mathfrak{m}$ module in a natural way. Hence it is vector space over the field $R/\mathfrak{m}$. A subset of $\mathfrak{m}/\mathfrak{m}^2$ is an R-submodule if and only if it as $R/\mathfrak{m}$-module.

5.3 Krull intersection theorem, first version. *Let R be a local noetherian ring. The intersection of all powers of the maximal ideal is zero.*

The intersection theorem has an important consequence for noetherian local rings:

5.4 Lemma. *Let $f, g : A \to B$ be two local homomorphisms between noetherian local rings. Assume that there exist generators $a_1, \dots, a_n$ of the maximal ideal of A such that $f(a_i) = g(a_i)$. Then $f = g$.*

There is second version of Krull's intersection theorem:

5.5 Krull intersection theorem, second version. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Assume that M is a finitely generated R-module. Then for each submodule $N \subset M$ one has*

$$N = \bigcap_{\nu=1}^{\infty} (N + \mathfrak{m}^{\nu} M).$$

If one applies this version to $M = R$ and $N = 0$, one obtains the first version.

Finiteness properties for algebras

Recall that a an algebra is just a fing homomorphism $\varphi : A \to B$. Then B is called an A-algebra. One can consider B as A-module by $ab =: \varphi(a)b$. An algebra homomorphism $B \to C$ of A-algebras is just a ring homomorphism that is also A-linear.

There are two basic finiteness properties for algebras $A \to B$. The first is: B is finitely generated as A-algebra. This means that there exists a surjective homomorphism of A-algebras of the polynomial ring $A[X_1, \dots, X_n]$ to B. This means that there are finitely many elements $b_1, \dots, b_n$ such that any element of B can be expressed as a polynomial with coefficients in A. There is another much more restrictive finiteness condition: B is finitely generated as A-module. This means that there exist finitely many elements $b_1, \dots, b_n$ such that $B = Ab_1 + \dots + Ab_n$. We call a ring extension $A \to B$ finite, if this second stronger condition is satisfied. A ring extension $A \to B$ is called integral, if any element $b \in B$ satisfies an equation

$$b^n + a_{n-1}b^{n-1} + \dots + a_0 = 0, \qquad a_i \in A.$$

Be aware. This notion of "integral" has nothing to do with "integral domain". Notice that the highest coefficient is one. It is a basic fact that finite extensions are integral. More precisely, a ring extension is finite if and only if it is integral and if it is finitely generated as algebra. The usual noether normalization theorem in commutative algebra states the following:

If $K \to A$ is a finitely algebra over a field K then there exist a subalgebra $A_0 \subset A$ such that A is a finite over A_0 and such that A_0 is isomorphic as K-algebra to a polynomial ring $K[X_1, \dots, X_n]$. The number n is unique. It equals the so-called Krull dimension of A.

An ideal $\mathfrak{p}$ in a ring R is called a prime ideal if $R/\mathfrak{p}$ is an integral domain. Concretely this means

$$ab \in \mathfrak{p} \Longrightarrow a \in \mathfrak{p} \quad \text{or} \quad b \in \mathfrak{p}.$$

The Krull dimension $\dim A$ is a basic notion of commutative algebra. It is defined for any commutative ring with unity and can be an integer ≥ 0 or ∞. By definition is the Supremum of all n such that there exists a chain of prime ideals

$$\mathfrak{p}_0 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} \mathfrak{p}_n.$$

The basic facts about the Krull dimension are:

5.6 Proposition. *Let R be a local noetherian ring such the maximal ideal can be generated by n elements. Then* $\dim R \le n$.

The rings $K[z_1, \dots, z_n]$, $K[[z_1, \dots, z_m]]$ (where K is a field) and the ring $\mathbb{C}\{z_1, \dots, z_n\}$ have Krull dimension n.

A maximal chain of prime ideals in all three cases is

$$0 \subset (z_1) \subset \ldots \subset (z_1, \ldots, z_n).$$

This shows that the dimension is $\geq n$. That the dimension equals n follows in the case $\mathbb{C}\{z_1, \ldots, z_n\}$ from the first part.

5.7 Theorem of Cohen Seidenberg. *If $A \subset B$ is an integral ring extension of noetherian rings then* $\dim A = \dim B$.

An important result of Krull dimension theory is:

5.8 Proposition. *Let R be a noetherian local ring and $a \in R$ a non-zero divisor. Then*

$$\dim R/(a) = \dim R - 1.$$

Corollary. *If $\mathfrak{a}$ is an ideal which contains a non-zero divisor then*

$$\dim R > \dim R/\mathfrak{a}.$$

Recall that a ring R is called an integral domain if $ab = 0 \Rightarrow a = 0$ or $b = 0$. We recall that each integral domain is contained in a field K as subring. One can achieve that K consists of all a/b, $a, b \in R$, $b \neq 0$. Such a field is called a field of fractions. A field of fractions is uniquely determined up to canonical isomorphism in an obvious way. Hence one talks about “the” field of fractions.

A special case of the so-called primary decomposition in noetherian ring states:

5.9 Proposition. *Every proper radical ideal in a noetherian ring is the intersection of finitely many prime ideals.*

Recall that an ideal is called a radical ideal if $a^n = 0$ for some natural number implies $a = 0$. Prime ideals of course are radical ideals. The intersection of radical ideals is a radical ideal.

Chapter VIII. Topological tools

1. Paracompact spaces

A covering $\mathfrak{U} = (U_i)_{i\in I}$ of a topological space is called *locally finite,* if for every point $a \in X$ there exists a neighborhood W, such that the set of indices $i \in I$ with $U_i \cap W \neq \emptyset$ is finite.

A covering $\mathfrak{V} = (V_j)_{j\in J}$ is called a *refinement* of the covering $\mathfrak{U}$ if for every index $j \in J$ there exists an index $i \in I$ with $V_j \subset U_i$. If one chooses for each j such an i one obtains a so-called *refinement map* $J \to I$, which needs not to be unique.

1.1 Definition. *A Hausdorff space is called* ***paracompact*** *if every open covering admits a locally finite (open) refinement.*

We collect some results about paracompact spaces without proofs. Firstly we give examples:

Every metric space is paracompact.

Every locally compact space with countable basis of topology is paracompact.

Next we formulate the basic result about paracompactness: Let $\mathfrak{U} = (U_i)$ be a locally finite covering. A partition of unity with respect to $\mathfrak{U}$ is family φ_i of continuous real valued functions on X with the following property:

a) The support of φ_i is compact and contained in U_i.
b) $0 \le \varphi_i \le 1$,
c) $\sum_{i\in I} \varphi_i(x) = 1$ for all $x \in X$.

(This sum is finite.)

1.2 Proposition. *Let X be a paracompact space. For every locally finite open covering there exists a partition of unity.*

We mention two related results:

1.3 Proposition. *Let X be a paracompact space and $\mathfrak{U} = (U_i)$ a locally finite open covering. There exist open subsets $V_i \subset U_i$ whose closure $\bar{V}_i$ (taken in X) is contained in U_i and such that $\mathfrak{V} = (V_i)$ is still a covering.*

Another related result states:

1.4 Proposition. *Let X be a locally compact paracompact space, U an open subset and $V \subset\subset U$ a relatively compact open subset in U. Then there exists a continuous function on X which is one on V and whose support is compact and contained in U.*

(The symbol $V \subset\subset U$ means that the closure $\bar{V}$, taken in X, is compact and contained in U.)

2. Frèchet spaces

A topological vector space is (complex) vector space E together with a topology such the addition map $E \times E \longrightarrow E$ and the multiplication with scalars $\mathbb{C} \times E \longrightarrow E$ is continuous. It is easy to derive then that or each fixed $a \in E$ the map $E \to E$, $x \mapsto x + a$, is topological. Topological vector spaces very often are constructed by means of semi-norms.

A semi-norm p on a complex vector space E is a map $p : E \to \mathbb{R}$ with the properties

a) $p(a) \geq 0$ for all $a \in E$,
b) $p(ta) = |t|p(a)$ for all $t \in \mathbb{C}$, $a \in E$,
c) $p(a + b) \leq p(a) + p(b)$.

The ball of radius $r > 0$ is defined as

$$U_r(a, p) := \{\, x \in E;\ p(a - x) < r \,\}.$$

Let $\mathcal{M}$ be a set of semi-norms. A subset $B \subset E$ is called a semi-ball around a with respect to $\mathcal{M}$ if there exists a finite subset $\mathcal{N} \subset \mathcal{M}$ and for each $p \in \mathcal{N}$ a number $r_p > 0$ such that

$$B = \bigcap_{p \in \mathcal{N}} U_{r_p}(a, p).$$

A subset U of E is called open (with respect to $\mathcal{M}$) if for every $a \in U$ there exists a semi-ball B around a with $B \subset U$.

It is clear that this defines a topology on E such that all $p : E \to \mathbb{C}$ are continuous. (It is actually the weakest topology with this property.) It is also easy to to see that E is a topological vector space. Moreover a sequence (a_n) in E converges to $a \in E$ if and only if $p(a_n - a) \to 0$ for all $p \in \mathcal{M}$. Obviously the elements $p \in \mathcal{M}$ are continuous. Let $\mathcal{M}_{\max}$ be the set of all continuous semi-norms. Two sets $\mathcal{N}$ and $\mathcal{M}$ define the same topology if and only if $\mathcal{M}_{\max} = \mathcal{N}_{\max}$. Especially $\mathcal{M}_{\max}$ and $\mathcal{M}$ define the same topology.

The set $\mathcal{M}$ is called definit, if

$$p(a) = 0 \quad \text{for all} \quad p \in \mathcal{M} \quad \Longrightarrow \quad a = 0.$$

It is easy to prove that $\mathcal{M}$ is definit if and only if E is a Hausdorff space.

A sequence (a_n) in E is called a *Cauchy sequence* with respect to $\mathcal{M}$, if for every $\varepsilon > 0$ and every $p \in \mathcal{M}$ there exists an $N = N(p, \varepsilon)$ such that

$$p(a_n - a_m) < \varepsilon \quad \text{for} \quad n, m \geq N.$$

Remarkably this notion only depends on the topology. Obviously a sequence is a Cauchy sequence if and only if for every neighborhood U od the origin one has $a_n - a_m \in U$ if noth n, m are sufficiently large.

The set $\mathcal{M}$ is called of countable type, if there exists a countable subset $\mathcal{N} \subset \mathcal{M}$ defining the same topology and the same Cauchy sequences.

2.1 Definition. *A Frèchet space E is a topological vector space whose topology can be defined by a set $\mathcal{M}$ of semi-norms such the following properties are satisfied:*

a) $\mathcal{M}$ is definite.
b) $\mathcal{M}$ is of countable type.
c) Every Cauchy sequence converges.

Notice that a Banach space is a Frèchet space, where $\mathcal{M}$ consists of a single element.

2.2 Lemma. *Frèchet spaces are metrizable.*

Proof. We choose some ordering of $\mathcal{N} = \{p_1, p_2, \dots\}$. Then one defines

$$d(a, b) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(a - b)}{1 + p(a_n) + p(b_n)}.$$

It is easy to show that this is a metric which defines the original topology.

□

An important result about Frèchet spaces is:

2.3 Open mapping theorem. *Any surjective linear continuous map $E \to F$ between Frèchet spaces is open. Especially the topology on F agrees with the quotient topology of E.*

An obvious corollary states that a bijective linear continuous map between Frèchet spaces is topological.

Basic example of Frèchet spaces

Let X be a complex manifold and $\mathcal{O}(X)$ the set of all analytic functions on X. This is a complex vector space. For an arbitrary compact subset $K \subset X$ we define

$$p(f) = p_K(f) := \max_{z \in K} |f(z)|.$$

This is s semi norm. A sequence (f_n) converges with respect to p_K if and only if f_n converges uniformly on K.

2.4 Remark. *Let X be a complex manifold. The vector space $\mathcal{O}(X)$ equipped with the set of all norms of the form p_K, $K \subset X$ compact, is a Frèchet space.*

The set of all p_K is of countable type since X is assumed to have countable basis of topology. This implies that there is a sequence $K_1 \subset K_2 \subset \cdots$ of compact subsets whose union is X and such that K_i is contained in the interior of K_{i+1}. Then every compact subset is contained in one of the K_i. The convergence of Cauchy sequences follows from the theorem of Weierstrass, which states that analyticity is stable under uniform convergence. □

The basic result about this Frèchet space is:

2.5 Theorem of Montel. *Let X be a complex manifild and $C > 0$ a positive constant. The set*

$$\mathcal{O}(X, C) := \left\{ f \in \mathcal{O}(X); \quad |f(z)| \leq C \text{ for } z \in X \right\}$$

is compact in $\mathcal{O}(X)$.

For the proof one has to use the fact that a metric space is compact if every sequence admits a convergent subsequence. Hence the statement follows from the usual theorem of Montel which states that every sequence in $\mathcal{O}(X, C)$ admits a locally convergent sub-sequence. We notice that the analogue for real differentiable functions is false. The proof uses heavily the Cauchy integral.

Compact operators

A well-known fact is that in a Banach space of infinite dimension the closed ball $||a|| \leq 1$ is not compact. This result is also true for Frèchet spaces in the following form:

Assume that the Frèchet space admits a non-empty open subset with compact closure. Then it is of finite dimension.

We need a generalization of this result: A continuous linear map $f : E \to F$ between Frèchet spaces is a *compact operator,* if there exists a non-empty open subset of E such that the closure of its image is compact. It is clear that this is the case if $f(E)$ is of finite dimension.

A linear map $f : E \to F$ is called *nearly surjective* if $F/f(E)$ has finite dimension. This is automatically the case when F is finite dimensional.

2.6 Theorem of Schwartz. *Let $f : E \to F$ be a surjective continuous linear map between Frèchet spaces and let $g : E \to F$ be a compact operator. Then $f + g$ is nearly surjective.*

If one applies Schwartz's theorem in the case $E = F$, $f = -\,\mathrm{id}$ and $g = \mathrm{id}$ on obtains:

2.7 Corollary. *When the identity operator $\mathrm{id} : E \to E$ of a Frèchet space is compact, then E is finite dimensional*

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