Algebraic tools

1. Sets and classes

Besides sets their exist classes. Every set is also a class but not conversely. Imagine classes which are not sets as oversized sets which are to big to deserve to be called sets. The main example is the class of all sets. One can do with classes all one does usually with sets besides one exception. Let A be a class and let $E(a)$ be for each $a \in A$ a property that can be true ore false. One would like to define the class of all $a \in A$ for which $E(a)$ is true. But this class usually does not exist. It exists always if A is a set. In this case it is a set too.

2. Categories

A category $\mathcal{A}$ consists of

1) a class whose elements a re called *objects*.

2) For each two objects $A, B \in \mathcal{A}$ there is associated a set $\mathrm{Mor}(A, B)$. Its elements are called morphisms.

3) For each three objects $A, B, C \in \mathcal{A}$ there is associated a map

$$\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \longrightarrow \mathrm{Mor}(A, C), \quad (f, g) \longmapsto g \circ f.$$

It is called the composition of morphisms.

4) For each object A there is a distinguished element $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ called the identity.

There are obvious axioms: the composition is associative in an obvious sense. The identities are neutral in the sense

$$f \circ \mathrm{id}_A = f, \quad \mathrm{id}_A \circ g = g, \qquad (f \in \mathrm{Mor}(A, B),\ g \in \mathrm{Mor}(B, A)).$$

Typical examples of categories are

The category of sets (objects are sets and morphisms are just maps).

The category of groups (objects are groups and morphisms are just homomorphisms).

The category of topological spaces (objects are topological spaces and morphisms are continuous maps).

3. Functors

Let $\mathcal{A}$, $\mathcal{B}$ be two categories. A functor $F : \mathcal{A} \to \mathcal{B}$ consists of

1) a map F that associates to each object $A \in \mathcal{A}$ an object $F(A) \in \mathcal{B}$

2) For each to objects $A, B \in \mathcal{A}$ there is given a distinguished map, also denoted by F

$$F : \mathrm{Mor}(A, B) \longrightarrow \mathrm{Mor}(F(A), F(B)).$$

Again there are obvious assumptions. The identity map goes to the identity map. Let A, B, C be three objects. Then the diagram

$$\begin{array}{ccc} \mathrm{Mor}(A,B) & \longrightarrow & \mathrm{Mor}(B,C) \\ & \searrow \quad \swarrow & \\ & \mathrm{Mor}(A,C) & \end{array}$$

is commutative.

An example: we consider the category $\mathcal{A}$ whose objects are open subsets $U \subset \mathbb{R}^n$ with a distinguished point $a \in U$. Morphisms are (totally) differentiable maps which map the distinguished point to the distinguished point. Let $\mathcal{B}$ be the category whose objects are $\mathbb{R}^n$ for $n \geq 0$ and whose morphisms are linear maps. We define a functor $F : \mathcal{A} \to \mathcal{B}$. For a pointed set (U, a) with open $U \subset \mathbb{R}^n$ we define $F(U, a) = \mathbb{R}^n$. For a morphism $f : (U, a) \to (V, b)$ we define $F(f)$ to be the linear map that is associated to the functional matrix. The chain rule says that this defines a functor.

4. Equivalent categories

Let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be functors. Then the composition $G \circ F$ us defined in an obvious way This composition is associative. There is the identity functor $\mathcal{A} \to \mathcal{A}$.

(One is tempted to talk about the category of categories, but be a aware. The class of all functors $\mathcal{A} \to \mathcal{B}$ needs not to be a set.)

Two functors $F, G : \mathcal{A} \to \mathcal{B}$ are called isomorphic if there can be chosen of each X in $\mathcal{A}$ an isomorphism $F(X) \to G(X)$ such that for all morphisms $X \to Y$ in $\mathcal{A}$ the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & G(X) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & G(Y) \end{array}$$

commutes.

A category $\mathcal{A}$ is called a subcategory of $\mathcal{B}$ if every object of $\mathcal{A}$ is an object of $\mathcal{B}$ and if for all objects $X, Y \in \mathcal{A}$ one has $\mathrm{Hom}_{\mathcal{A}}(X, Y) \subset \mathrm{Hom}_{\mathcal{B}}(X, Y)$. It is called a *full subcategory* if always equality holds.

Definition. *A functor $F : \mathcal{A} \to \mathcal{B}$ is called an equivalence of categories if the following two conditions hold:*

1) *For any two objects $X, Y \in \mathcal{A}$ the natural map*

$$\mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(F(X), F(Y))$$

is bijective.

2) *For each object $Y \in \mathcal{B}$ there exists an object $X \in \mathcal{B}$ such that Y and $F(X)$ are isomorphic.*

One can ask whether there is an inverse functor. Here is a problem with the axiom of choice. In many cases inverse functors exist in the following sense.

Definition. *Let $\mathcal{A}$, $\mathcal{B}$ be categories. Two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are called inverse to each other if the functors $G \circ F$ and $F \circ G$ are isomorphic to the identity functor on $\mathcal{A}$ and $\mathcal{B}$.*

Then F and G are equivalences of categories. One calls G an inverse of F. (Notice that it is not uniquely determined.)