Eberhard Freitag

Analytic number theory

Self-Publishing, 2016

Eberhard Freitag
Universität Heidelberg
Mathematisches Institut
Im Neuenheimer Feld 288
69120 Heidelberg
freitag@mathi.uni-heidelberg.de

Contents

Chapter I. Theta functions

1. Interchanging infinite sums and integration

Since we have to interchange integration and summation several times, we recall without prove some basic results. We need not the most general limit theorems, for example we can restrict completely to continuous functions on some subset $D \subset \mathbb{R}^n$ (this includes $\mathbb{C} = \mathbb{R}^2$) and to sequences of continuous functions (f_n) that converge locally uniform to a (continuous function) f. This means that for every point in the domain of definition there exists a neighbourhood U such that the sequence converges uniformly in $U \cap D$. Assume that the functions f_n are defined on some (finite or infinite) interval $I \subset \mathbb{R}$. First we assume that $I = [a, b]$ is compact. Then integration and limit can be interchanged for trivial reason,

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx.$$

This follows immediately from the standard estimate

$$\Big| \int_a^b f(x)dx \Big| \le (b-a)\|f - f_n\|.$$

For improper integrals, i.e. in the case where I os not compact, this simple argument fails, and, in fact, in first courses of analysis for improper integrals usually there is no limit theorem provided.

We recall that a (continuous) function $f : I \to \mathbb{C}$ on some interval I is called *absolutely integrable* if there exists a bound C such that for all compact intervals $[a, b]$ in the interior of I of has

$$\int_a^b |f(x)|dx \le C.$$

Then it is easy to show that for all sequences of intervals $[a_n, b_n]$ in the interior of I and such that

$$[a_1, b_1] \subset [a_2, b_2] \subset \cdots, \qquad I = \bigcup_{n=1}^{\infty} [a_n, b_n]$$

the limit of the proper integrals $\int_{a_n}^{b_n} f(x)dx$ exists and is independent of the choice of the sequence. The value $\int_I f(x)dx$ then is defined as this limit. If one wants to interchange improper integrals with a locally uniform limit, one has to replace the interval I by a compact interval in its interior, then one can apply the trivial theorem for proper integrals and finally one has to show that the rest can be estimated uniformly in n by some given $\epsilon > 0$. This is possible in all cases which occur in these notes. But this effort is not necessary if one is acquainted with the limit theorems which are provided in the Lebesgue integration theorem. We formulate them, not in the most general form, but in a form which is sufficient for our purpose.

1.1 Theorem of Beppo Levi. *Let $f_n : I \to \mathbb{R}$ be a sequence of absolutely integrable real valued continuous functions on some (not necessarily compact) interval I. We assume that f_n converges pointwise to some continuous function f. Assume furthermore* BepLev

$$f_1(x) \leq f_2(x) \leq \cdots \quad \textit{for all } x \in I$$

and that the sequence of integrals $\int_I f_n(x)dx$ is bounded. Then f is absolutely integrable too and we have

$$\lim_{n\to\infty} \int_I f_n(x)dx = \int_I f(x)dx.$$

1.2 Theorem (Lebesgue limit theorem). *Let $f_n : I \to \mathbb{C}$ be a sequence of absolutely integrable continuous functions on some (not necessarily compact) interval I. We assume that f_n converges pointwise to some continuous function f. Assume furthermore that there exists an absolutely integrable continuous function g with the property* LebLim

$$|f_n(x)| \leq |g(x)| \quad \textit{for all } a \in \mathbb{N},\ x \in I.$$

Then f is absolutely integrable too and we have

$$\lim_{n\to\infty} \int_I f_n(x)dx = \int_I f(x)dx.$$

The limit theorems can be reformulated for infinite series. We formulate a theorem that uses both.

1.3 Theorem. *Let $f_n : I \to \mathbb{C}$ be a sequence continuous and absolutely integrable functions and assume that the series* BepLes

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad h(x) = \sum_{n=1}^{\infty} |f_n(x)|$$

converge to continuous functions f, h. Assume that the partial sums of integrals

$$\sum_{n=1}^{N} \int_I |f_n(x)| dx$$

are bounded. Then $f(x)$ is integrable and summation and integration can be interchanged.

Proof. One applies the Theorem of Beppo Levi to the sequence $\sum_{n=1}^{N} |f_n(x)| dx$ and then the Lebesgue limit theorem to $\sum_{n=1}^{N} f_n(x) dx$. □

2. The simplest theta function

The series

$$\sum_{n=1}^{\infty} q^n = 1 + q + q^2 + \cdots$$

converges for $|q| < 1$. We claim that it has no zero there. This follows from the explicit formula

$$\frac{1}{1-q} = \sum_{n=1}^{\infty} q^n$$

which is known as the *geometric series.* Without knowledge of this formula it my be hard to prove this. Since analytic functions are often defined as infinite series it may be a major problem to determine their zeros.

We give another example. The series

$$\sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

converges also for $|q| < 1$. This function is very interesting for number theory. This can be seen if one takes the k-th power (for some natural number k) and evaluates it by means of the Cauchy multiplication theorem. The result is

$$\Big(\sum_{n=1}^{\infty} q^n\Big)^k = \sum_{(n_1,\dots,n_k) \in \mathbb{Z}^k} q^{n_1^2 + \cdots + n_k^2}.$$

If we collect terms with fixed $n = n_1^2 + \cdots n_k^2$, we get

$$\Big(\sum_{n=-\infty}^{\infty} q^n\Big)^k = \sum A_k(n) q^n,$$
$$A_k(n) := \#\{x \in \mathbb{Z}^k;\quad x_1^2 + \cdots + x_k^2 = n\}.$$

For example $A_3(2) = 12$, since one can place two ± 1 and a 0.

2.1 Theorem. Thenen

$$\sum_{n=-\infty}^{\infty} q^{n^2} \neq 0 \qquad (|q| < 1).$$

We will proof this in Sect. 3.

It turns out to be useful to introduce a new variable z related to q by

$$q = e^{\pi \mathrm{i} z}.$$

Because of

$$|q| = e^{\mathrm{Re}\,\pi \mathrm{i} z} = e^{-\pi y} \qquad (z = x + \mathrm{i}y)$$

we get

$$|q| < 1 \Longleftrightarrow y > 0.$$

Hence we can introduce

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} n^2 z}, \qquad y > 0.$$

This is a special case of the Jacobi theta function which we will study in the next section.

3. The Jacobi theta function

We denote the upper half plane by

$$\mathcal{H} = \{z \in \mathbb{C},\quad y = \mathrm{Im}\, z > 0\}.$$

We introduce a new variable w and define

$$f(w) = \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}(n+w)^2 z}.$$

In fact, this is a function of two variables but for a while we want to keep z fixed and consider it as a function of w. Notice that $f(0) = \vartheta(z)$. We give a short comment on the convergence. For fixed z and w we have

$$\left|e^{\pi(n+w)^2 z}\right| = e^{\alpha n^2+\beta n+\gamma}$$

where $\alpha < 0$. For all but finitely many n we have $\alpha n^2 + \beta n + \gamma \le \alpha n^2/2$ and hence domination by the geometric series. A little more effort shows that this series converges *normal**) on $\mathcal{H} \times \mathbb{C}$. Hence it is continuous and holomorphic in each of the two variables. The basic thing is that f is periodic for trivial reason, $f(w + 1) = f(w)$. Hence we can expand it into a Fourier series [FB], Satz III.5.4.

$$\begin{aligned} f(w) &= \sum_{m=-\infty}^{\infty} a_m e^{2\pi \mathrm{i} m w}, \\ a_m &= \int_0^1 f(w) e^{-2\pi \mathrm{i} m w}\, du \quad (w = u + \mathrm{i} v). \end{aligned}$$

Here v can be taken arbitrarily in $\mathbb{C}$.

During the computation of the coefficients there will occur an improper integral

$$\int_{-\infty}^{\infty} e^{\pi \mathrm{i} z u^2}\, du \qquad (\operatorname{Im} z > 0).$$

We refer to [FB], IV.1 to some comments about such integrals which show as in the case of the Gamma function that this integral converges and depends analytically from z. It is known that the value of the integral is 1 for $z = \mathrm{i}$. The integral transformation $t = u\sqrt{y}$ shows that the formula

$$\int_{-\infty}^{\infty} e^{\pi \mathrm{i} z u^2}\, du = \sqrt{\frac{z}{\mathrm{i}}}^{\,-1}$$

holds for $z = \mathrm{i}y$. By the principle of analytic continuation the formula will hold on all $z \in H$ if can make the choice of the square root such that it is positive for $z = \mathrm{i}y$ and analytic for all z. This is the case if if we use the principal part of the logarithm for the definition of the square root, since z/i is never on the negative of the real axis. Recall that the principal part of the logarithm of a complex number $a \neq 0$ is defined as

$$\operatorname{Log} a = \log|a| + \mathrm{i} \operatorname{Arg} a, \quad -\pi < \operatorname{Arg} a \le \pi.$$

---

*) A series $\sum f_n(z)$ of continuous functions on a subset $D \subset \mathbb{R}^n$ is called normally convergent if every point $a \in D$ admits a neighbourhood $a \in U$ such there exists a convergent series $\sum m_n$ of numbers such that $|f_n(z)| \le m_n$ for all $z \in U \cap D$ and for all n. If D is open, one can demand instead of this that for every compact subset $K \subset D$ there exists a simultaneous majorant for all $a \in K$. Normally convergent series converge locally uniform.

Then the principal value of the square root is defined as

$$\sqrt{a} = e^{(\operatorname{Log} a)/2} \qquad \text{(principal value)}.$$

Its real part is $\operatorname{Re} a = \cos(\operatorname{Arg}(a))$. Hence we see

$$\operatorname{Re} \sqrt{a} > 0 \quad \text{if } a \text{ is not real}.$$

For sake of completeness we mention that the principal value of the square root $\sqrt{a}$ is positive of a is real and positive, and that $\sqrt{a} = \mathrm{i}\sqrt{|a|}$ for negative a.

Now we evaluate the integral

$$a_m = \int_0^1 \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} z(n+w)^2 - 2\pi \mathrm{i} m w} du.$$

We can interchange summation with integration and then shift the integration variable $u \mapsto u - n$. The result is

$$a_m = \int_{-\infty}^{\infty} e^{\pi \mathrm{i}(zw^2 - 2mw)} du = e^{-\pi \mathrm{i} m^2 z^{-1}} \int_{-\infty}^{\infty} e^{\pi \mathrm{i} z(w - m/z)^2} du.$$

Now we make the choice of the imaginary part v of w such that $w - m/z$ is real. After a translation of u we get

$$a_m = e^{\pi \mathrm{i} m^2(-1/z)} \int_{-\infty}^{\infty} e^{\pi \mathrm{i} z u^2} = e^{\pi \mathrm{i} m^2(-1/z)} \sqrt{\frac{z}{\mathrm{i}}}^{-1}.$$

3.1 Theorem. *For* $(z, w) \in \mathcal{H} \times \mathbb{C}$ *the formula* JTF

$$\sqrt{\frac{z}{\mathrm{i}}} \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}(n+w)^2 z} = \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} n^2(-1/z) + 2\pi \mathrm{i} n w}.$$

holds. Both series converge normally on $\mathcal{H} \times \mathbb{C}$*. Here the square root of* z/i *is defined by means of the principal part of the logarithm.*

The function

$$\vartheta(z, w) = \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} n^2 z + 2\pi \mathrm{i} n w}$$

is called the *Jacobi theta function.* Its zero value ($w = 0$) is the function $\vartheta(z)$ of our interest. The theta transformation formula implies

$$\vartheta\Big(-\frac{1}{z}\Big) = \sqrt{\frac{z}{\mathrm{i}}}\,\vartheta(z).$$

Besides $\vartheta(z)$ two similar functions play a role. We set

$$\widetilde{\vartheta}(z)=\sum_{-\infty}^{\infty}(-1)^n e^{\pi\mathrm{i}n^2z},\quad \widetilde{\widetilde{\vartheta}}(z)=\sum_{-\infty}^{\infty}e^{\pi\mathrm{i}(n+1/2)^2z}.$$

As ϑ also $\widetilde{\vartheta}$ can be considered as a special value of the Jacobi theta function

$$\widetilde{\vartheta}(z)=\vartheta(z,1/2),$$

and the theta transformation formula gives

$$\widetilde{\vartheta}(Sz)=\sqrt{\frac{z}{\mathrm{i}}}\,\widetilde{\widetilde{\vartheta}}(z).$$

A trivial formula is

$$\vartheta(Tz)=\widetilde{\vartheta}(z).$$

3.2 Lemma. *We have* LemN

$$\vartheta(Tz)=\widetilde{\vartheta}(z),\quad \vartheta(TSz)=\sqrt{\frac{z}{\mathrm{i}}}\,\widetilde{\widetilde{\vartheta}}(z).$$

4. Some fundamental sets

The group $\mathrm{SL}(2,\mathbb{R})$ consists of all 2×2-matrices with real entries and determinant 1. Each element M of this groups induces a biholomorphic transformation of the upper half plane $\mathcal{H}$ onto itself ([FB], Satz V.7.2)

$$Mz=\frac{az+b}{cz+d},\quad M=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

One has

$$\operatorname{Im}Mz=\frac{y}{|cz+d|^2}$$

and

$$E(z)=z\quad\text{and}\quad (MN)(z)=M(Nz).$$

Let $\Gamma\subset\mathrm{SL}(2,\mathbb{R})$ be a subgroup. A subset $F\subset\mathcal{H}$ is called a fundamental set if for every $z\in\mathcal{H}$ there exists $M\in\Gamma$ such that $Mz\in F$. We are interested in the group Γ that is generated by the two matrices

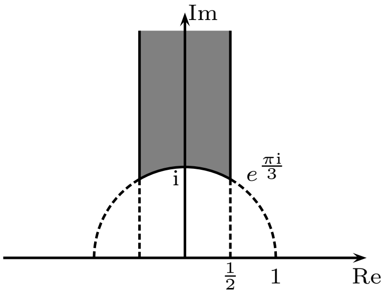
$$S=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad T=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

It consists of all products $M_1\cdots M_n$ where the M_i belong to $\{S,S^{-1},T,T^{-1}\}$.

4.1 Lemma. *The set* fundTh

$$\mathcal{F} = \{z \in \mathcal{H}; \quad |x| \leq 1/2, \ |z| \geq 1\}$$

is a fundamental set of Γ.



Proof. We assume that there is a point $z \in \mathcal{F}$ such that Mz is not in $\mathcal{F}$ for all $M \in \Gamma$. We can assume that $|\operatorname{Re} z| \leq 1/2$. Then necessarily $|z| < 1$. Then $|-z^{-1}| \geq 1$ and, as a consequence, $\operatorname{Im}(-z^{-1}) > \operatorname{Im} z$. This shows that we can find a sequence $M_n \in \Gamma$ such that $\operatorname{Im} M_{n+1}(z) > \operatorname{Im} M_n(z)$, $|\operatorname{Re} M_n(z)| < 1$. Then necessarily $|M_n(z)| < 1$. Hence the sequence $M_n(z)$ converges in the upper half plane. This shows that

$$|c_n z + d_n|, \qquad M_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},$$

converges. Since c_n, d_n are integers, it follows that c_n and d_n are constant for big n. But then the imaginary part of $M_n(z)$ would also be constant for big n. This is a contradiction. □

A similar result can be obtained for the group Γ_ϑ that is generated by

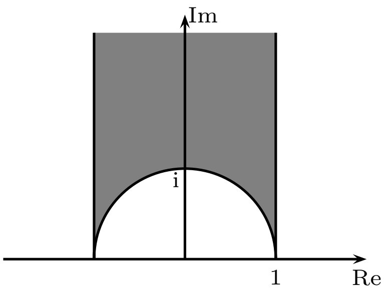
$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The same proof shows the following result.

4.2 Lemma. *The set* fundT

$$\tilde{\mathcal{F}}_\vartheta = \{z \in \mathcal{H}; \quad |x| \leq 1, |z| \geq 1\}$$

is a fundamental set of Γ_ϑ.

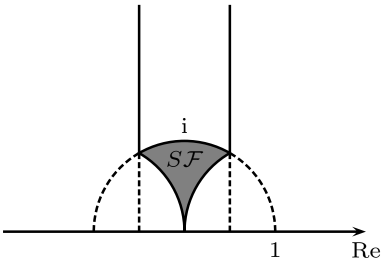


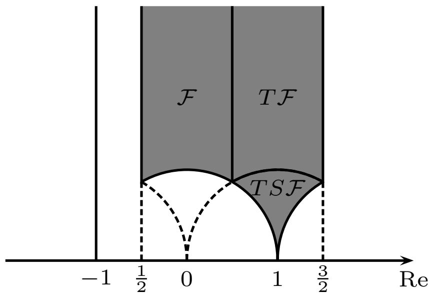
Sometimes it is better to consider a slightly modified fundamental set. If we cut off the part $x \leq -1/2$ of $\tilde{\mathcal{F}}$ and add its translate under $z \mapsto z+2$ we obtain another fundamental set.

4.3 Lemma. *The set* fundTm

$$\mathcal{F}_\vartheta = \mathcal{F} \cup T\mathcal{F} \cup TS\mathcal{F}$$

is a fundamental set of Γ_ϑ*.*





5. Powers of the theta function

We will make use of the transformation formulas of the theta function.

5.1 Lemma. *Assume that* $h(z)$ *is a holomorphic function on the upper half plane with the property* hCon

$$h(z+2) = h(z), \quad h(-1/z) = h(z).$$

Assume that the limits

$$a = \lim_{y\to\infty} h(z), \quad b = \lim_{y\to\infty} h(1-1/z)$$

exist. Then h *is constant.*

Proof. Instead of $h(z)$ we consider $H(z) = (h(z)-a)(h(z)-b)$. We want to a apply the maximum principle and hence have to show that H has a maximum in $\mathcal{H}$. It is sufficient to show that there exists a maximum of $|H(z)|$ in a fundamental set. We take $\mathcal{F}_\vartheta$. The limit behaviour $\lim_{y\to\infty} H(z) = \lim_{y\to\infty} H(1-1/z) = 0$ shows that $|H(z)|$ has a mximimum in $\mathcal{F}_\vartheta$. Since it is invariant under Γ_ϑ, it has a maximum in $\mathcal{H}$. Hence H is constant. The constant must be zero. It follows that $h(z)-a$ or $h(z)-b$ is zero. Hence h is constant. □

5.2 Theorem. *Let* $k \in \mathbb{Z}$ *and let* $f : \mathcal{H} \to \mathbb{C}$ *be a holomorphic function with the properties* ThChar

a) $$f(z+2) = f(z), \quad f\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}^{\,k} f(z),$$

b) $\lim_{y\to\infty} f(z)$ *exists,*

c) $$\lim_{y\to\infty} \sqrt{\frac{z}{\mathrm{i}}}^{\,-k} f\left(1-\frac{1}{z}\right) e^{-\pi \mathrm{i} k z/4} \text{ exists.}$$

Then

$$f(z) = \text{const.}\vartheta(z)^k.$$

$\big($*The constant is* $\lim\limits_{y\to\infty} f(z)\big)$.

Proof. First we notice that the function $\vartheta(z)^r$ has the properties a),b),c). It is enough to verify it in the case $k = 1$. Now we can consider the function $h(z) = f(z)/\vartheta(z)^r$. Since $\vartheta(z)$ has no zeros, this is a holomorphic function on $\mathcal{H}$. Lemma 5.1 shows that it is constant. □

Chapter II. Eisenstein series

1. Convergence of Eisenstein series

It should be known (from Analysis I) that the series

$$\sum_{n=1}^{\infty} n^{-s}$$

converges for real $s > 1$. A possible proof is to compare it with the improper integral

$$\int_1^{\infty} x^{-s}dx.$$

A similar argument shows the following result.

1.1 Lemma. *The series* KonvE

$$\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}-\{(0,0)\}} \frac{1}{(m^2+n^2)^s}$$

converges for real s if and only if $s > 2$.

Proof. Here one has to compare with the integral

$$\int_{x^2+y^2\geq 1} \frac{dxdy}{(x^2+y^2)^s}$$

The integral transformation $x = r\cos\varphi$, $y = r\sin\varphi$ $(dxdy = rdrd\varphi)$ shows that the integral converges if and only if $s > 2$. We leave the details to the reader. □

We mention that we now have to consider series

$$\sum_{s\in S} a_s$$

where S is an arbitrary countable index set, not necessarily $\mathbb{N}$. In this case it only makes sense to talk about *absolute* convergence. It means that the series

$$a_{s_1} + a_{s_2} + \cdots$$

converges absolutely for some ordering of S (which can be considered as a bijective map $\mathbb{N} \to S$). The convergence and the value do not depend on the choice of the ordering (by the rearrangement theorem). The notion of normal convergence carries over to the general case in the obvious way.

Let z be a point in the upper half plane. The series

$$\sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{|cz+d|^{2s}}.$$

is a generalization of the series above (take $z = \mathrm{i}$) To compare the series for arbitrary z with that at i we consider the expression

$$\Big|\frac{cz+d}{c\mathrm{i}+d}\Big|$$

We consider this as a function for *real* (c, d) and fixed z. We can restrict to $c^2 + d^2 = 1$, since this function is invariant under $(x, y) \mapsto t(x, y)$. Since this defines a compact set this function has a maximum and a minimum. Hence we get the convergence of the series for all $z \in \mathcal{H}$ and $s > 2$. The same argument works, if we let vary z in a compact neighbourhood. Hence the argument gives *normal* convergence which shows that we get a holomorphic function on the upper half plane. But there is a better result. Consider a domain $|x| \le C$, $y \ge \delta$ for positive numbers. The estimate

$$|cz+d|^2 = (cx+d)^2 + c^2y^2 \ge |cz_0+d|^2 \quad \text{where } z_0 = x + \mathrm{i}\delta$$

shows that we have uniform converge on these domains.

1.2 Theorem. *The Eisenstein series* KonE

$$G_r(z) = \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} (cz+d)^{-r}, \quad r > 2\ (r \in \mathbb{Z}),$$

converges normal and even more uniformly in the domains $|x| \le C$, $y \ge \delta$.

If (c, d) runs through $\mathbb{Z}^2 - \{(0, 0,)\}$, then also $-(c, d)$ does. This shows that the Eisenstein series is 0 for odd r. Hence we will consider only even $r > 2$. The formula

$$G_r(z+1) = \sum (cz + (c+d))^{-r} = G_r(z)$$

shows the periodicity of G_r and a similar rearrangement shows

$$G_r(-1/z) = z^r \sum (-dz+c)^r = z^r G_r(z).$$

Finally we consider the limit $\lim_{y\to\infty} G_r(z)$. We can take it term by term since the convergence is uniform and obtain $\lim_{y\to\infty} G_r(z) = 2\zeta(r)$.

1.3 Lemma. *The Eisenstein series G_r, r even, $r \geq 4$, has the properties* TraEis

$$G_r(z+1) = G_r(z), \quad G_r\Big(-\frac{1}{z}\Big) = z^r G_r(z)$$

and

$$\lim_{y\to\infty} G_r(z) = 2\zeta(r).$$

2. Fourier expansion of the Eisenstein series

We need a classical identity. We start with the geometric series

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n.$$

Differentiating and multiplication by q gives

$$\frac{q}{(1-q)^2} = \sum_{n=1}^{\infty} nq^n.$$

If we insert

$$q = e^{2\pi \mathrm{i} z}$$

we get

$$\frac{\pi^2}{(\sin \pi z)^2} = (2\pi \mathrm{i})^2 \sum_{n=1}^{\infty} nq^n.$$

A classical identity states

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\Big(\frac{1}{z+n} + \frac{1}{z-n}\Big).$$

Differentiation gives another classical identity.

$$\frac{\pi^2}{(\sin \pi z)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}.$$

Hence we obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2} = (2\pi \mathrm{i})^2 \sum_{n=1}^{\infty} nq^n \quad (y > 0).$$

Repeatedly differentiation gives the following identity.

2.1 Proposition. *For natural $r \geq 2$ we have* Partk

$$(-1)^r \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^r} = \frac{1}{(r-1)!}(2\pi \mathrm{i})^r \sum_{n=1}^{\infty} n^{r-1} q^n \quad (y > 0).$$

We arrange the Eisenstein series in the form

$$G_r(z) = 2\zeta(r) + 2\sum_{c=1}^{\infty}\Big\{\sum_{d=-\infty}^{\infty} \frac{1}{(cz+d)^r}\Big\}.$$

Now we replace in Proposition 2.1 z by cz and insert it.

$$G_r(z) = 2\zeta(r) + \frac{2(2\pi\mathrm{i})^r}{(r-1)!}\sum_{c=1}^{\infty}\sum_{d=1}^{\infty} d^{r-1} q^{cd}.$$

Now we collect all pairs (c, d) for fixed $n = cd$. Introducing the notation

$$\sigma_r(n) = \sum_{0<d|n} d^r$$

we get the Fourier expansion of the Eisenstein series:

$$\boxed{G_r(z) = 2\zeta(r) + \frac{2(2\pi\mathrm{i})^r}{(r-1)!}\sum_{n=1}^{\infty} \sigma_{r-1}(n) q^n}$$

3. Sums of eight squares

We want to relate $\vartheta(z)^k$ to G_r. The transformations under $z \mapsto -1/z$ show that we should take $k = 2r$. We only got Eisenstein series of even weight. Hence we are restricted to the case where k is divisible by 4. (This is not the end of the theory since the Eisenstein series can be generalized to odd and even half integer weight r. We will not treat this here.) One of the characteristic properties is $f(-1/z) = \sqrt{-z/i}^k f(z)$. So we have to divide into two cases.

1) $k \equiv 0 \bmod 8$: In this case $f(-1/z) = z^r f(z)$,

2) $k \equiv 4 \bmod 8$: In this case $f(-1/z) = -z^r f(z)$.

We start with the first case. Here a function with the desired transformation formula is $G_r(z)$. So one might be attempted to suggest that ϑ^k and G_r agree upto a constant factor. But this is not the case, since the Eisenstein series has period 1 but ϑ^k not. Hence we have to modifiy this ansatz.

3.1 Lemma. *The function* ModG

$$f(z) = G_r\Bigl(\frac{z+1}{2}\Bigr), \quad r \geq 4,$$

has the property

$$f(z+2) = f(z), \quad f(-1/z) = z^r f(z).$$

Proof. The following short proof is due to Elstrodt. One uses the formula

$$\frac{-\frac{1}{z}+1}{2} = A\Bigl(\frac{z+1}{2}\Bigr) \quad \text{where} \quad A = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Using $A = T^{-1}ST^2S$ one computes $f(-1/z() = z^r f(z)$. □

Now we found two functions $G_r(z), G_r((z+1)/2)$ which both have the correct transformation formulas a) in Theorem I.5.2. Also b) is satisfied for both,

$$\lim_{y\to\infty} G_r(z) = \lim_{y\to\infty} G_r((z+1)/2) = 2\zeta(r).$$

But the condition c) fails for both. Even the following weaker condition

c′) $$\lim_{y\to\infty} \sqrt{\frac{z}{\mathrm{i}}}^{\,r} f(z) = 0$$

fails for both as one can check easily (using the following calculation). It looks promising to introduce a a linear combination

$$g(z) = aG_r(z) + bG_r\Bigl(\frac{z+1}{2}\Bigr).$$

Then we get

$$z^{-r} g\Bigl(1-\frac{1}{z}\Bigr) = aG_r(z) + 2^r bG_r(2z).$$

The limit for $y \to \infty$ is

$$\zeta(r)(a + 2^r b).$$

This should be 0. We also should have $2\zeta(r)(a+b) = 1$. Hence we get the system

$$\boxed{\begin{aligned} a + 2^r b &= 0 \\ 2\zeta(r)(a+b) &= 1 \end{aligned}}$$

3.2 Theorem. *Assume $r \equiv 0 \bmod 4$, $r > 0$, $k = 2r$. Then* EisNr

$$g(z) = aG_r(z) + bG_r\Big(\frac{z+1}{2}\Big)$$

where

$$a = \frac{2^{r-1}}{\zeta(r)(2^r - 1)}, \quad b = -\frac{1}{2\zeta(r)(2^r - 1)}$$

satisfies the conditions a),b) in Theorem I.5.2 and the weaker condition c'). Even more,

$$\lim_{y \to \infty} g(z) = 1.$$

The identity

$$z^{-r} g\Big(1 - \frac{1}{z}\Big) = aG_r(z) - 2^r bG_r(2z)$$

shows that the left hand side has a Fourier expansion

$$z^{-r} g\Big(1 - \frac{1}{z}\Big) = a_0 + a_1 q + a_2 q^2 + \cdots \qquad (q = e^{2\pi \mathrm{i} z}).$$

The condition c') says that $a_0 = 0$. We then get that the limit

$$z^{-r} g\Big(1 - \frac{1}{z}\Big) e^{-2\pi \mathrm{i} z} = a_1$$

exists. This is exactly condition c) in the special case $r = 4$. With the well-known value $\zeta(4) = \pi^2/90$ we obtain the following beautiful identity.

3.3 Theorem. *We have* AchtQ

$$\vartheta(z)^8 = \frac{3}{\pi^4}\left(16G_4(z) - G_4\Big(\frac{z+1}{2}\Big)\right).$$

A simple calculation now gives the following beautiful formula of Jacobi.

3.4 Theorem. *We have* JacFo

$$A_8(n) = 16 \sum_{d|n} (-1)^{n-d} d^3.$$

Proof. If we write $q = e^{\pi i z}$, the Fourier expansion G_4 reads as

$$G_4(z) = \frac{\pi^4}{45} + \frac{32\pi^4}{6} \sum_{n=1}^{\infty} \sigma_{r-1}(n) q^{2n}.$$

If we define $\sigma_{r-1}(n/2) = 0$ for odd n, we can rewrite this as

$$G_4(z) = \frac{\pi^4}{45} + \frac{32\pi^4}{6} \sum_{n=1}^{\infty} \sigma_{r-1}(n/2) q^{n}.$$

On the other side

$$G_4\Big(\frac{z+1}{2}\Big) == \frac{\pi^4}{45} + \frac{32\pi^4}{6} \sum_{n=1}^{\infty} (-1)^n \sigma_{r-1}(n) q^{n}.$$

So we get

$$A_8(n) = 16(16\sigma_{r-1}(n/2) - (-1)^n \sigma_{r-1}(n)).$$

This equals the expression in the theorem. For example, for odd n, both expressions give $A_8(n) = 16 \sum_{d|n} d^{r-1}$. In the special case of an odd prime p we have

$$A_8(p) = 16(1 + p^3).$$ □

For example, $A_8(5) = 2016$. There are actually solutions with 3 zeros and 5 ones. Their number is 1792, and there are solution with 6 zeros and a one and a four. Their number is 224. The sum is 2016.

4. The Eisenstein series of weight two

In the case $k \equiv 0 \bmod k$ there works a similar ansatz,

$$h(z) = aG_r(2z) + bG_r(z/2).$$

The conditions $\lim h(z) = 1$ and $h(-1/z) = -z^r h(z)$ now lead to

$$\boxed{\begin{aligned} 2\zeta(r)(a+b) &= 1 \\ a2^r + b &= 0 \end{aligned}}$$

Now we obtain in analogy to Theorem 3.2 the following result.

4.1 Theorem. *Assume* $r \equiv 2 \bmod 4$, $r > 2$, $k = 2r$. *Then* EisNrz

$$h(z) = aG_r(z) + bG_r\Big(\frac{z+1}{2}\Big)$$

where

$$b = \frac{2^{r-1}}{\zeta(r)(2^r - 1)}, \quad a = \frac{1}{2\zeta(r)(1 - 2^r)}$$

satisfies the conditions a),b) in Theorem I.5.2 and the condition

c′) $$\lim_{y\to\infty} \sqrt{\frac{z}{\mathrm{i}}}^{\,k} h\Big(1 - \frac{1}{z}\Big) = 0.$$

Even more,

$$\lim_{y\to\infty} h(z) = 1.$$

Proof. It remains to prove c'). We start with a small calculation

$$\begin{aligned} G_r\Big(\frac{z-1}{2z-1}\Big) &= G_r\Big(\frac{z-1}{2z-1} - 1\Big) = G_r\Big(-\frac{z}{2z-1}\Big) \\ &= \Big(\frac{z}{2z-1}\Big)^r G_r\Big(2 - \frac{1}{z}\Big) = (2z-1)^r G_r(z). \end{aligned}$$

We replace z by $z/2 + 1/2$ to obtain

$$G_r\Big(\frac{-1/z + 1}{2}\Big) = z^r G_r\Big(\frac{z}{2} + \frac{1}{2}\Big).$$

This implies

$$z^{-r} h\Big(1 - \frac{1}{z}\Big) = a2^r G_r\Big(\frac{z}{2} + \frac{1}{2}\Big) + bG_r\Big(\frac{z}{2}\Big).$$

Since $a2^r + b = 0$ we obtain c'). Actually we can derive a better result. The function on the right hand side is a Fourier series of period two. Hence we get that

$$\lim_{y\to\infty} z^{-r} h\Big(1 - \frac{1}{z}\Big) e^{-\pi \mathrm{i} z}$$

exists. This is the condition c) in Theorem I.5.2 in the case $k = 4$, $r = 2$. But the Eisenstein series $G_r(z)$ converges absolutely only for $r > 2$.

Ro remedy the situation, we will define the Eisenstein series also in weight $r = 2$. We realize the the Fourier series converges also and makes sense for $r = 2$. Hence we can *define* G_2 through the Fourier series.

4.2 Definition. *We define* DefGz

$$G_2(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

The goal of this section is the proof of the following result.

4.3 Theorem. *We have* TraEz

$$G_2\Big(-\frac{1}{z}\Big) = z^2 G_2(z) - 2\pi\mathrm{i}z.$$

Proof. The formula

$$\sum_{d=-\infty}^{\infty} \frac{1}{(cz+d)^r} = \frac{1}{(r-1)!}(2\pi\mathrm{i})^r \sum_{n=1}^{\infty} n^{r-1} q^n$$

is true for all even $r \geq 2$, including $r = 2$. This implies

$$\sum_{c=1}^{\infty} \Big\{ \sum_{d=-\infty}^{\infty} \frac{1}{(cz+d)^r} \Big\} = \frac{(2\pi\mathrm{i})^r}{(r-1)!} \sum_{c=1}^{\infty} \sum d^{r-1} q^{cd}.$$

The convergence of the left hand side (in this ordering) follows if one looks the right hand-side, since for real q all terms are positive. This gives us

$$G_2(z) = \sum_{c=-\infty}^{\infty} \Big\{ \sum_{\substack{d=-\infty \\ d\neq 0 \text{ if } c=0}}^{\infty} \frac{1}{(cz+d)^2} \Big\}$$

A simple calculation shows

$$G_2\left(-\frac{1}{z}\right) = z^2 G_2^*(z)$$

where

$$G_2^*(z) = \sum_{d=-\infty}^{\infty} \Big\{ \sum_{\substack{c=-\infty \\ c\neq 0 \text{ if } d=0}}^{\infty} \frac{1}{(cz+d)^2} \Big\}.$$

For lack of absolute convergence we cannot conclude that G_2 and G_2^* agree. In fact, this is false, but the following relation is true (and Theorem 4.3 follows from this identity).

$$G_2^*(z) = G_2(z) - \frac{2\pi\mathrm{i}}{z}.$$

For the proof of this identity we introduce two related functions.

$$H(z) = \sum_{c=-\infty}^{\infty} \Big\{ \sum_{\substack{d=-\infty \\ c^2+d(d-1)\neq 0}}^{\infty} \frac{1}{(cz+d)(cz+d-1)} \Big\},$$

$$H^*(z) = \sum_{d=-\infty}^{\infty} \Big\{ \sum_{\substack{c=-\infty \\ c\neq 0, \ \text{ if } \ d\in\{0,1\}}}^{\infty} \frac{1}{(cz+d)(cz+d-1)} \Big\}.$$

We have

$$H(z) - G_2(z) = \sum_{c=-\infty}^{\infty} \Big\{ \sum_{\substack{d=-\infty \\ d\neq 0 \ \text{and} \ d\neq 1, \ \text{if} \ c=0}}^{\infty} \frac{1}{(cz+d)^2(cz+d-1)} \Big\} - 1.$$

Since $\sum_{(c,d)\neq(0,0)} |cz+d|^{-3}$ converges, we get that in $H(z) - G_2(z)$ we have normal convergence without brackets. Hence we can reorder to prove

$$H(z) - G_2(z) = H^*(z) - G_2^*(z).$$

Hence it is sufficient to prove $H(z) - H^*(z) = 2\pi \mathrm{i}/z$. Now the point is that $H(z), H^*(z)$ both can be summed up:

$$H(z) = 2, \quad H^*(z) = 2 - 2\pi \mathrm{i}/z.$$

The advantage of the switch to H, H^* is that they are teloscopic sums, i.e. the formula

$$\sum_{n=1}^{N} (a_n - a_{n+1}) = a_1 - a_N$$

can be applied. It immediately follows

$$\sum_{\substack{d=-\infty \\ c^2+d(d-1)\neq 0}}^{\infty} \left(\frac{1}{cz+d-1} - \frac{1}{cz+d} \right) = \begin{cases} 0, & \text{falls } c \neq 0 \\ 2 & \text{if } c = 0, \end{cases}$$

$$\begin{aligned} H^*(z) &= \sum_{d=-\infty}^{\infty} \Big\{ \sum_{\substack{c=-\infty \\ c\neq 0, \ \text{if} \ d\in[0,1]}}^{\infty} \Big[\frac{1}{cz+d-1} - \frac{1}{cz+d} \Big] \Big\} \\ &= \lim_{N\to\infty} \sum_{d=-N+1}^{N} \Big\{ \sum_{\substack{c=-\infty \\ c\neq 0, \ \text{if} \ d\in[0,1]}}^{\infty} \Big[\frac{1}{cz+d-1} - \frac{1}{cz+d} \Big] \Big\} \end{aligned}$$

$$= \lim_{N\to\infty}\Bigg[\sum_{d=-N+1}^{-1}\sum_{c=-\infty}^{\infty}\left[\frac{1}{cz+d-1}-\frac{1}{cz+d}\right]$$

$$+\sum_{d=2}^{N}\sum_{c=-\infty}^{\infty}\left[\frac{1}{cz+d-1}-\frac{1}{cz+d}\right]$$

$$+\sum_{\substack{c=-\infty\\ c\neq 0}}^{\infty}\left[\frac{1}{cz-1}-\frac{1}{cz}\right]+\sum_{\substack{c=-\infty\\ c\neq 0}}^{\infty}\left[\frac{1}{cz}-\frac{1}{cz+1}\right]\Bigg]$$

$$= \lim_{N\to\infty}\sum_{c=-\infty,\,c\neq 0}^{\infty}\left[\frac{1}{cz-N}-\frac{1}{cz+N}\right]+2.$$

The last sum is related to the cotangens:

$$\sum_{\substack{c=-\infty\\ c\neq 0}}^{\infty}\left[\frac{1}{cz-N}-\frac{1}{cz+N}\right]=\frac{2}{z}\cdot\sum_{c=1}^{\infty}\left[\frac{1}{c-N/z}-\frac{1}{c+N/z}\right]$$

$$=\frac{2}{z}\left[\pi\cot\left(-\pi\,\frac{N}{z}\right)+\frac{z}{N}\right].$$

Wir have to take the limit $N\to\infty$.

$$\frac{2\pi}{z}\lim_{N\to\infty}\cot\left(-\pi\,\frac{N}{z}\right)=\frac{2\pi}{z}\lim_{N\to\infty}\mathrm{i}\,\frac{e^{-2\pi\mathrm{i}N/z}+1}{e^{-2\pi\mathrm{i}N/z}-1}=-\frac{2\pi\mathrm{i}}{z}.$$

The finishes the proof of Theorem 4.3. □

If we consider $f(z)=4G_2(2z)-G_2(z/2)$, in the transformation formula the disruptive term $2\pi\mathrm{i}z$ cancels out and we obtain

$$f(-1/z)=\sqrt{z/i}^{\,2}f(z)$$

just as if in G_2 this term would not occur. The same is true in all other calculations and we obtain the following formula.

4.4 Theorem. *We have* EiszT

$$\vartheta^4=\frac{4G_2(2z)-G_2(z/2)}{\pi^2}.$$

As application on can derive the following formula due to Jacobi (1828).

4.5 Theorem. *For $n\in\mathbb{N}$ we have* SumvQ

$$A_4(n)=8\sum_{4\nmid d\,|\,n}d.$$

In particular, every natural number is a sum of 4 squares (Lagrange, 1770).

5. An asymptotic formula

Assume that $f(z)$ is a holomorphic function in the upper half plane with the properties $f(z+1)=f(z)$, $f(-1/z)=(z/\mathrm{i})^r f(z)$. In addition, assume $\lim_{y\to\infty} f(z)=0$. Then the Fourier series of f must be of the form

$$f(z)=\sum_{n=1}^{\infty} a_n q^n.$$

This shows that f decreases exponentially for $y\to\infty$. In particular,

$$\lim_{y\to\infty} y^{r/2}|f(z)|=0.$$

The essential point now is that the function $g(z)=y^{r/2}|f(z)|$ is invariant, $g(z+1)=g(z)$, $g(-1/z)=g(z)$. Hence g takes all its values already in the fundamental domain $\mathcal{F}$. Since g takes its maximum in $\mathcal{F}$ it has a maximum in $\mathcal{H}$,

$$|f(z)|\le Cy^{-r/2}.$$

A similar result holds if we demand the weaker condition $f(z+2)=f(z)$ and $f(-1/z)=(z/\mathrm{i})^r f(z)$. Then we have to demand

$$\lim_{y\to\infty} f(z)=\lim_{y\to\infty}\Big(\frac{z}{\mathrm{i}}\Big)^r f\Big(1-\frac{1}{z}\Big)=0.$$

The same method will give an estimate $|f(z)|\le Cy^{-k/4}$. We use it to estimate the Fourier coefficients

$$|a_n|\le\int_0^1 |f(z)|e^{2\pi ny}.$$

We specialize this formula to $y=1/n$ to obtain the following result.

5.1 Proposition. *Let $f(z)$ be a holomorphic function on the upper half plane with the properties* HecAb

$$f(z+2)=f(z),\quad f\Big(-\frac{1}{z}\Big)=\Big(\frac{z}{\mathrm{i}}\Big)^r f(z),$$
$$\lim_{y\to\infty} f(z)=\lim_{y\to\infty}\Big(\frac{z}{\mathrm{i}}\Big)^{-r} f\Big(1-\frac{1}{z}\Big)=0.$$

Then the Fourier coefficients satisfy an estimate

$$|a_n|\le Cn^{r/2}.$$

It is of interest to compare this with the Fourier coefficients of the Eisenstein series.

5.2 Lemma. *For each even $r \geq 4$ the estimate* AbTei

$$n^{r-1} \leq \sigma_{r-1}(n) \leq \zeta(r-1)n^{r-1}.$$

holds

Proof. The estimate from below is trivial. The other estimate can be obtained as follows.

$$\frac{\sigma_{r-1}(n)}{n^{r-1}} = \frac{\sum_{d|n}\left(\frac{n}{d}\right)^{r-1}}{n^{r-1}} = \sum_{d|n} d^{1-r} \leq \zeta(r-1). \qquad \square$$

We mention that

$$\lim_{\sigma\to\infty} \zeta(\sigma) = 1.$$

5.3 Theorem. *Let k be natural number that is divisible by 8 and let $r = k/2$. Then the asymptotic formula* AsyDa

$$A_k(n) = \frac{(2\pi)^r}{(r-1)!\zeta(r)(2^r-1)} \sum_{d|n} (-1)^{n-d} d^{r-1} + O(n^{r/2})$$

holds.

We use here the Landau notation. Let $f, g, h : D \to \mathbb{C}$ be functions on some subset $D \subset \mathbb{R}$ which is not bounded above. Then

$$f(x) = h(x) + O(g(x))$$

means that there is a constant C such that

$$|f(x) - h(x)| \leq Cg(x)$$

for big enough x. For example $f(x) = O(1)$ means the f is bounded for big enough x. We leave it as an exercise to show the existence of positive constants A, B such that

$$An^{r-1} \leq \sum_{d|n} (-1)^{n-d} d^{r-1} \leq Bn^{r-1}.$$

Chapter III. Dirichlet series

1. Convergence of Dirichlet series

We consider series of the type

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Such a series is called a Dirichlet series. The most famous Dirichlet series is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

A Dirichlet series is called (absolutley) *convergent* if there exists a point s_0 in the the complex plane such that

$$\sum_{n=1}^{\infty} |a_n n^{-s}|$$

converges.

1.1 Definition. *A sequence (a_n) of complex numbers has at most polynomial growth if there exist constants C, N such that* DefPG

$$|a_n| \leq C n^N$$

for almost all n.

1.2 Lemma. *A Dirichlet series converges if an only if its coefficients have at most polynomial growth.* Dcp

In the following we use the notation

$$s = \sigma + it.$$

Assume that $D(s)$ converges absolutely for s_0. Since

$$|t|^{-s} = t^{-\sigma}$$

it converges for $\sigma > \sigma_0$. For a convergent Dirichlet series we define

$$\sigma_0 = \operatorname{Inf}\{\sigma \in \mathbb{R};\quad \sum |a_n| n^{-\sigma} \text{ converges}\}.$$

Here Inf means the usual infimum inf if the set is bounded from below and $-\infty$ else. We see that the Dirichlet series converges absolutely for $\sigma > \sigma_0$ and not absolutely for any S with $\sigma < \sigma_0$. We can say nothing about the behaviour for $\sigma = \sigma_0$. We call σ_0 the *abscissa of absolute convergence.*

It can happen that a Dirichlet series converges, but converges non absolutely, at some point. For example, the series

$$\sum_{n=1}^{\infty} (-1)^n n^{-s}$$

converges for $s = 1$ but does not converge absolutely there. This can be investigated systematically by means of the rests on the following elementary formula (Abel's partial summation).

1.3 Lemma. *Let $a(n)$ be a sequence of real numbers and let for real x be* AmPS

$$A(x) = \sum_{n \le x} a(n).$$

Then for every continuously differentiable function

$$f : [x, y] \longrightarrow \mathbb{C}, \quad 0 < x < y, \ (x, y \text{ real})$$

the following formula holds.

$$\sum_{x < n \le y} a(n) f(n) = A(y) f(y) - A(x) f(x) - \int_x^y A(t) f'(t) dt.$$

Proof. We give just a sketch. The formula is trivial if the interval $[x, y]$ contains no natural number in its interior. If the formula is true for intervals $[x, y]$, $[y, z]$, then it is true of $[x, z]$. □

1.4 Theorem. *Assume that $D(s) = \sum a_n n^{-s}$ is a Dirichlet series such that* BedK

$$A(x) = \sum_{n \le x} a_n$$

is bounded. Then $D(s)$ converges (non necessarily absolutely) for $\operatorname{Re} s > 0$ and represents an analytic function there.

We apply Abel's partial summation of $f(t) = t^{-s}$ to obtain

$$\sum_{x<n\leq y} a_n n^{-s} = A(y)y^{-s} - A(x)x^{-s} + s\int_x^y A(t)t^{-s-1}dt.$$

Hence we obtain for a given ε and a fixed s with $\operatorname{Re} s > 0$ that

$$\Big|\sum_{x<n\leq y} a_n n^{-s}\Big| \leq \varepsilon \quad \text{for} \quad y > x > N(\varepsilon).$$

By means of the Cauchy convergence criterium we get the convergence. It is clear that this estimate can be given locally uniform. □

As an example this shows that

$$\sum_{n=1}^{\infty}(-1)^n n^{-s}$$

converges and represents an analytic function for $\operatorname{Re} s > 0$. But it converges absolutely only for $\operatorname{Re} s > 1$.

Another remarkable result is the following theorem.

1.5 Theorem. *Let* PosKo

$$\sum_{n=1}^{\infty} a_n n^{-s}, \quad a_n \geq 0,$$

be a Dirichlet series with real, non-negative coefficients and let σ_0 ist abscissa of absolute convergence. Then it is not possible to extend $D(s)$ analytically to any open neighbourhood of σ_0.

Proof. We can assume $\sigma_0 = 0$ (replace s by $s - s_0$). We argue indirectly and assume that $D(s)$ can be extended analytically into a small disk around 0. The there exists a positive number $\varepsilon > 0$ such that $D(s)$ is analytic in the disk $|z-1| < 1 + 2\varepsilon$. In particular, the Taylor sereis at $s = 1$ will converge at $-\varepsilon$. We compute it:

$$D(s) = \sum_{m=0}^{\infty} \frac{1}{m!} D^{(m)}(1)(s-1)^m,$$

$$D^{(m)}(s) = (-1)^m \sum_{n=1}^{\infty} a_n (\log n)^m n^{-s}.$$

This implies

$$D(s) = \sum_{m=0}^{\infty} \left\{ \sum_{n=1}^{\infty} a_n (\log n)^m n^{-1} \right\} \frac{(-1)^m}{m!}(s-1)^m.$$

We can insert $s=-\varepsilon$.

$$D(-\varepsilon)=\sum_{m=0}^{\infty}\left\{\sum_{n=1}^{\infty}a_n(\log n)^m n^{-1}\right\}\frac{(\varepsilon+1)^m}{m!}.$$

All entries in this double series are positive. Hence we can rearrange the sum. We sum first over m and then over n.

$$D(-\varepsilon)=\sum_{n=1}^{\infty}\left\{\sum_{m=0}^{\infty}\frac{(\varepsilon+1)^m}{m!}(\log n)^m\right\}a_n n^{-1}.$$

The inner sum is $e^{(\varepsilon+1)\log n}=n^{\varepsilon+1}$. Hence

$$D(\varepsilon)=\sum_{n=}^{\infty}a_n\frac{1}{n^{-\varepsilon}}.$$

This means that the Dirichlet series converges also absolutely for $s=-\varepsilon$. This is a contradiction to $\sigma_0=0$. □

The essential point now is that all terms in this double series are non-negative real numbers. Hence we can apply the (great) rearrangement theorem on sum first over m and then over n. the result is the series $D(s)$ at $s=-\varepsilon$. Hence this series converges (of course absolutely) ind contrast to the assumption $\sigma_0=1$. □

2. The functional equation for the Riemann zeta function

In this section we will prove the following famous result of Riemann

2.1 Theorem. *The Riemann zeta function* FuRi

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$$

converges absolutely for $\operatorname{Re}s>1$ *and represents there an analytic function. It extends to a meromorphic function in the whole plane which is analytic outside* $s=1$*. Here it has a pole of first order and residue* 1*. The function*

$$\xi(s)=\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)$$

is a meromorphic function that satisfies the functional equation

$$\xi(s)=\xi(1-s).$$

Proof. We will reduce this to the inversion formula for the theta function ϑ. We start with the gamma integral

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t}.$$

We replace the integration variable t by $n^2 t$ for a natural number n. Then we obtain

$$\Gamma(s) n^{-2s} = \int_0^\infty t^s e^{-tn^2} \frac{dt}{t}.$$

Now sum over n on the lest hand side we get $\Gamma(s)\zeta(s)$. One the right hand side one interchanges sum und integration to obtain

$$R(s) := \pi^{-s}\Gamma(s)\zeta(2s) = \int_0^\infty t^s \frac{\vartheta(\mathrm{i}t) - 1}{2} \frac{dt}{t}.$$

So we have $R(s) = \xi(2s)$. We divide this integral into two parts

$$R_1 + R_2 = \int_1^\infty + \int_0^1 .$$

There exists a constant C such that

$$\Big|\frac{\vartheta(it) - 1}{2}\Big| \le Ce^{-t} \quad \text{for } t \ge 1.$$

Hence R_1 extends to an holomorphic function in the whole plane. To treat R_2 we use the transformation $t \to 1/t$ to obtain

$$R_2(s) = \int_0^1 t^{-s} \frac{\vartheta(\mathrm{i}/t) - 1}{2} \frac{dt}{t}.$$

Now we insert the theta inversion formula.

$$\begin{aligned} R_2(s) &= \int_1^\infty t^{-s} \frac{\sqrt{t}\vartheta(\mathrm{i}t) - 1}{2} \frac{dt}{t} = \int_1^\infty t^{-s} \frac{\sqrt{t}(\vartheta(\mathrm{i}t) - 1) + \sqrt{t} - 1}{2} \frac{dt}{t} \\ &= R_1(1/2 - s) + \frac{1}{2} \int_1^\infty t^{1/2 - s} \frac{dt}{t} - \frac{1}{2} \int_1^\infty t^{-s} \frac{dt}{t} \\ &= R_1(1/2 - s) + \frac{1}{2}\Big(\frac{1}{1/2 - s} + \frac{1}{s}\Big). \end{aligned}$$

This gives us

$$R(s) = R_1(s) + R_1(1/2 - s) + \frac{1}{2}\Big(\frac{1}{1/2 - s} + \frac{1}{s}\Big).$$

This formula has been proved under the assumption $\sigma > 1$. But the right hand side is an analytic continuation to the whole plane with two exceptional poles at $s = 0$ and $s = 1/2$. The right hand hand side is invariant under $s \mapsto 1/2 - s$. This means that $\xi(s)$ is invariant under $s \mapsto 1 - s$ which is the functional equation of the zeta function.

The Euler product

We start with the geometric series

$$(1-p^{-s})^{-1}=\sum_{\nu=0}^{\infty}p^{-\nu s}$$

which converges for $\operatorname{Re} s > 1$. We apply this to the first m primes

$$\mathcal{P}_m=\{p_1,\dots,p_m\}$$

and take the product

$$\prod_{k=1}^{m}(1-p_k^{-s})^{-1}=\prod_{k=1}^{m}\sum p_k^{-\nu s}.$$

We apply the Cauchy multiplication formula (term by term) to obtain

$$\prod_{k=1}^{m}(1-p_k^{-s})=\sum_{\nu_1,\dots,\nu_m=0}^{\infty}(p_1^{\nu_1}\cdots p_n^{\nu_m})^{-s}=\sum_{n\in\mathcal{A}(m)}n^{-s}$$

where $\mathcal{A}(m)$ denotes all natural numbers which are not divisible by any prime outside $\mathcal{P}_m$. Here we have to use the result of elementary number theory that every natural number has a unique decomposition into primes. We obtain

$$\lim_{m\to\infty}\prod_{k=1}^{m}(1-p_k^{-s})^{-1}=\sum_{n=1}^{\infty}n^{-s}.$$

2.2 Theorem. *The Riemann zeta function* EulP

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$$

admits for Res > 1 *an expansion into an infinite product*

$$\zeta(s)=\prod_{p}(1-p^{-s})^{-1}\qquad\textit{(extended over all primes).}$$

Here and in the following $\prod_p$ and similarly $\sum_p$ always means that the product or sum has to be taken over all primes. It remains to show that the product converges (normally) in the sense of infinite products. This means that

$$\sum_{p}|1-(1-p^{-s})^{-1}|$$

converges normally. This follows from the estimate

$$\sum_p \sum_m |p^{-ms}| \le \sum_{n=1}^{\infty} |n^{-s}|. \qquad \square$$

Now we restrict to real $s > 1$. Since the real $\log x$ is continuous for $x > 0$ we obtain

$$\log\zeta(s) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k} p^{-ks} \le \sum_p p^{-s} + \sum_p \sum_{k=2}^{\infty} p^{-ks}.$$

By means of the geometric series we get

$$\log\zeta(s) \le \sum_p p^{-s} + \zeta(2).$$

2.3 Theorem. *The series* SumP

$$\sum_p \frac{1}{p}$$

diverges.

Proof. Otherwise the function $\log\zeta(s)$ would be bounded for $s \to 1$ which is not possible, since $\zeta(s)$ has a pole at $s = 1$. $\square$

Later we will derive much better prime number theorems from the Riemann zeta function.

We finish this section with a first comment for the zeros of the zeta function. From the functional equation we see that $\zeta(s)$ has zeros at the even negative integers. These zeros are called the trivial zeros. From the Euler product one sees that there are no zeros in the half plane $\operatorname{Re} s > 1$. Again using the functional equation we see that in the half plane $\operatorname{Re} s < 0$ there are only the trivial zeros. Hence the non-trivial zeros are all in the strip $0 \le \sigma \le 1$. The famous Riemann conjecture says that all non-trivial zeros lie on the middle of this strip ($\operatorname{Re} s = 1/2$). This conjecture is still unsolved.

3. Hecke's inversion theorem

The following generalization of the functional equation of the Riemann zeta function is due to Hecke. First we give a generalization of $\vartheta(z)$.

3.1 Definition. *Let λ, r be positive real numbers and let $\varepsilon = \pm 1$. The space* EckDef

$$[\lambda, r, \varepsilon]$$

consists of all Fourier series

$$f(z) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i n z}{\lambda}}$$

with the following properties:

1) *The sequence (a_n) has at most polynomial growth. In particular, $f(z)$ gives a holomorphic function $f : \mathcal{H} \to \mathbb{C}$.*
2) *The functional equation*

$$f\Big(-\frac{1}{z}\Big) = \varepsilon\Big(\frac{z}{\mathrm{i}}\Big)^r f(z)$$

is satisfied where the root is defined to have positive imaginary part.

We notice that

$$\vartheta(z) \in \Big[2, \frac{1}{2}, 1\Big].$$

Other examples are

$$G_r(z) \in [1, r, (-1)^{r/2}] \quad \text{for even } r > 2.$$

Now we define a similar space of Dirichlet series. Let R be a meromorphic function in the complex plane. It is called decaying in a vertical strip $a \le \sigma \le b$ if there exists for every $\varepsilon > 0$ a constant C such that

$$|R(s)| \le \varepsilon \quad \text{for} \quad a \le \sigma \le b,\ |t| \ge C.$$

(In particular there is no pole in this region.)

3.2 Definition. *Let λ, r be positive real numbers and let $\varepsilon = \pm 1$. The space* GeschD

$$\{\lambda, r, \varepsilon\}$$

consists of all Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

wit the following properties:

1) *The Dirichlet series is convergent.*
2) *The function that is represented in the convergence half plane by $D(s)$ can be extended to a meromorphic function on the whole plane. It is analytic outside $s = r$ and has in $s = r$ at most a pole of order one.*
3 *It satisfies the functional equation*

$$R(s) = \varepsilon R(r - s) \quad \textit{where} \quad R(s) = \Big(\frac{2\pi}{\lambda}\Big)^{-s}\Gamma(s)D(s).$$

4) *The meromorphic function $R(s)$ decays in any vertical strip.*

An example is

$$\zeta(2s) \in \Big\{2, \frac{1}{2}, 1\Big\}.$$

(The condition 4) will be proved a little later.)

3.3 Theorem. *The assignment* HecTh

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda} \longmapsto D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

defines an isomorphism

$$[\lambda, r, \varepsilon] \xrightarrow{\sim} \{\lambda, r, \varepsilon\}.$$

The residue of D at $s = r$ is

$$Res(D, r) = a_0 \Big(\frac{2\pi}{\lambda}\Big)^r \Gamma(r)^{-1}.$$

In particular, D is holomorphic if and only if $a_0 = 0$.

Proof. The first part of the proof is very similar to the proof of the functional equation of the Riemann zeta function, hence we can keep short. As in the case of the Riemann zeta function one proves

$$R(s) = \int_0^\infty t^s (f(\mathrm{i}t) - a_0) \frac{dt}{t}.$$

Then we decompose

$$R(s) = R_1(s) + R_2(s) = \int_1^\infty + \int_0^1 .$$

The first integral converges in the whole s-plane and defines a holomorphic function there. In the second integral we transform t by $1/t$ and apply the involution formula for f. This gives

$$R(s) = R_1(s) + \varepsilon R_1(r - s) - a_0 \Big(\frac{\varepsilon}{r - s} + \frac{1}{s}\Big).$$

Meromorphic continuation and functional equation are obvious.

It remains to prove that $R(s)$ decays in arbitrary vertical strips. It is enough to show this for $R_1(s)$. Since $|t^s| = t^\sigma$, it is clear that $R(s)$ is bounded in vertical strips. Partial integration $u(t) = f(\mathrm{i}t) - a_0$, $v(t) = t^{s-1}$ shows that $R_1(s)$ decays in vertical strips.

We come to the second part of the proof, the reverse direction. We start with Mellin's inversion formula

$$e^{-z} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\sigma + \mathrm{i}t)}{z^{\sigma + \mathrm{i}t}} dt.$$

Here we have to assume $\sigma > 0$ and $\operatorname{Re} z > 0$. The power

$$z^{\sigma + \mathrm{i}t} = e^{(\sigma + \mathrm{i}t) \log z}$$

is defined through the principal branch of the logarithm. We do not prove this formula here (s. [FB], Hilfssatz VII.3.5). We just mention that it rests on the fact that $\Gamma(s)$ is rapidly decreasing on vertical strips in the following precise sense.

Let ε be an arbitrary small positive number, $0 \le \varepsilon < \pi/2$. In every strip

$$a \le \sigma \le b, \quad |t| \ge \varepsilon,$$

there is an estimate

$$|\Gamma(s)| \le Ce^{-(\pi/2-\varepsilon)|t|}$$

with a suitable constant $C = C(a, b, \varepsilon)$.

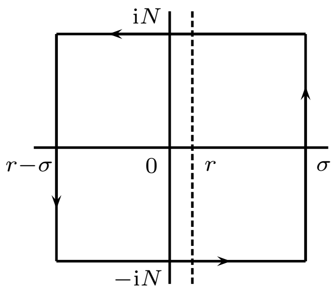
Now we define

$$f(z) = \sum_{n=0} a_n e^{\frac{2\pi i n z}{\lambda}},$$

where a_0 will be determined later. Now we insert the Mellin formula to obtain for big enough σ (bigger than the abscissa of convergence of $D(s)$)

$$f(\mathrm{i}y) - a_0 = \frac{1}{2\pi} \sum_{n=1}^{\infty} a_n \int_{-\infty}^{\infty} \frac{\Gamma(s)}{\left(\frac{2\pi}{\lambda} ny\right)^s} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R(s)}{y^s} dt.$$

Since $\Gamma(s)$ decreases fast enough, there is no problem with interchanging integration and summation. Now we want to shift the integration line from σ to the left (to $r - \sigma$).



To be precise, one has to integrate along the rectangle in the and then to take the limit $N \to \infty$. The contribution of the horizontal lines tends to zero, since $R(s)$ is decaying. So we get

$$f(\mathrm{i}y) - a_0 = \frac{1}{2\pi} \lim_{N\to\infty} \int_{-N}^{N} \frac{R(r-s)}{y^{r-s}}\, dt + \operatorname{Res}\left(\frac{R(s)}{y^s}, 0\right) + \operatorname{Res}\left(\frac{R(s)}{y^s}, r\right).$$

The functional equation $R(r - s) = \varepsilon R(s)$ shows that $\lim_{-N}^{N}$ is actually an absolutely convergent integral $\int_{-\infty}^{\infty}$ No we depose of the constant

$$a_0 := -\operatorname{Res}\left(\frac{R(s)}{y^s}\,;\ s = 0\right) = -\operatorname{Res}(R(s); s = 0).$$

The functional equation for $R(s)$ now gives

$$f\left(\frac{\mathrm{i}}{y}\right) = \varepsilon y^r f(\mathrm{i}y)$$

and we get by analytic continuation

$$f\left(-\frac{1}{z}\right) = \varepsilon\left(\frac{z}{\mathrm{i}}\right)^r f(z).$$

This finishes the proof of Hecke's theorem 3.3. □

A generalization

Instead of a number $\epsilon = \pm 1$ we introduce a matrix

$$\varepsilon \in \mathrm{GL}(N, \mathbb{C}), \quad \varepsilon^2 = E \quad \text{unit matrix.}$$

A function $f : \mathcal{H} \to \mathbb{C}^N$ is called holomorphic if each of its components is holomorphic.

3.4 Definition. *Let λ, r be positive real numbers and let $\varepsilon =\in \mathrm{GL}(N, \mathbb{C})$, $\varepsilon^2 = E$. The space* EckDefz

$$[\lambda, r, \varepsilon]$$

consists of all Fourier series

$$f(z) = \sum_n a_n e^{\frac{2\pi \mathrm{i} z}{\lambda}}, \quad a_n \in \mathbb{C}^N$$

with the following properties:

1) *The sequence a_n (i.e. its components) has at most polynomial growth. In particular, $f(z)$ gives a holomorphic function $f : \mathcal{H} \to \mathbb{C}^N$.*
2) *The functional equation*

$$f(z) = \varepsilon\left(\frac{z}{\mathrm{i}}\right)^r f(z)$$

is satisfied where the root is defined to have positive imaginary part.

Hecke's inversion theorem can be carried over literally to the general case.

3.5 Theorem. *Theorem 3.3 carries over to the case of a general $\varepsilon \in \mathrm{GL}(N, \mathbb{C})$, $\varepsilon^2 = E$.* HecThz

Proof. From linear algebra one knows that every matrix of finite order can be diagonalized. This means that there exists $A \in \mathrm{GL}(N, \mathbb{C})$ such that $A\varepsilon A^{-1}$ is a diagonal matrix. The entries must be ± 1. Hence the proof the theorem can be reduced to the special case. □

3.6 Theorem. *Let $f : \mathbb{H} \to \mathbb{C}$ be an analytic function. Assume that $f(z)$ as well as* VaHec

$$g(z) := \left(\frac{z}{\mathrm{i}}\right)^{-r} f\left(-\frac{1}{z}\right)$$

can be expanded into a Fourier series of the kind

$$f(z) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i n z}{\lambda}}, \quad g(z) = \sum_{n=0}^{\infty} b_n e^{\frac{2\pi i n z}{\lambda}}$$

where the coefficients have at most polynomial growth Then the two Dirichlet series

$$D_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad D_g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

can be expanded into the full plane as meromorphic functions. The satisfy the relation

$$R_f(s) = R_g(r-s) \quad \textit{where } R_f(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) D_f(s) \quad (\textit{analogously } R_g).$$

The functions $(s-r)D_f(s)$ und $(s-r)D_g(s)$ are holomorphic and we have

$$\operatorname{Res}(D_f; r) = a_0 \left(\frac{\lambda}{2\pi}\right)^r \Gamma(r)^{-1}, \quad \operatorname{Res}(D_g; r) = b_0 \left(\frac{\lambda}{2\pi}\right)^r \Gamma(r)^{-1}.$$

Proof. Apply the previous theorem to the pair (f, g) (and the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). □

4. More Euler products

We study in more detail the space $[1, r, (-1)^r/2]$ for even r. Its elements have the transformation formula

$$f(z+1) = f(z), \quad f(-1/z) = z^r f(z).$$

This are called modular forms. The have the more general transformation property

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^r f(z), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

To prove this, we introduce the Petersson notation

$$(f|M)(z) = (cz+d)^{-r} f(Mz)$$

for $M \in \mathrm{GL}(2, \mathbb{R})$, $\det M > 0$. An easy computation shows

$$f|MN = (f|M)|N.$$

So we see that the general transformation formula follows for MN if it is true for M, N. Since it is true for the generators T, S, it must be true in general.

5. Characters

A character of a group G is a homomorphism $\chi : G \to S^1$ into the group S^1 of complex numbers of absolute value 1. The product of two characters is a character, the group $\hat{G}$ of all characters is a group as well. If a is a real number, then

$$\chi_a : \mathbb{Z} \longrightarrow S^1, \quad \chi_a(x) = e^{2\pi \mathrm{i} a x},$$

is a character of $(\mathbb{Z}, +)$. The real number is determined mod 1. In other words χ_a depends only on the image of a in $\mathbb{R}/\mathbb{Z}$. An elementary result shows that

$$\mathbb{R}/\mathbb{Z} \xrightarrow{\sim} \hat{\mathbb{Z}}, \quad a \longmapsto \chi_a,$$

is an isomorphism. Next we treat the group $\mathbb{Z}/\ell\mathbb{Z}$, $\ell > 0$. This is a finite group. There is a natural surjective homomorphism $\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z}$. If we compose an arbitrary function $f : \mathbb{Z}/\ell\mathbb{Z} \to \mathbb{C}$ with this homomorphism, we get a function on $\mathbb{Z}$ with period ℓ and every function with period ℓ arises in this way. For sake of convenience we denote this function on $\mathbb{Z}$ by the same letter f. Hence $f(a)$ makes sense for $a \in \mathbb{Z}/\ell\mathbb{Z}$ and for $a \in \mathbb{Z}$. For any integer a we can define

$$\chi_a^{(\ell)}(x) = e^{2\pi \mathrm{i} a x / l}.$$

This does not change if we replace x by $x + n\ell$. Hence it defines a function

$$\chi_a^{(\ell)} : \mathbb{Z}/\ell\mathbb{Z} \longrightarrow S^1.$$

It is obviously a character. This character does not change if an replaces a by $a + n\ell$. Hence on can define $\chi_a^{(\ell)}$ for $a \in \mathbb{Z}/\ell\mathbb{Z}$. We will use the notation

$$\chi_a^{(\ell)}(x) = e^{2\pi \mathrm{i} a x / l}$$

also for $a, x \in \mathbb{Z}/\ell\mathbb{Z}$ in the obvious way.

5.1 Lemma. *The map* ChIs

$$\mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\sim} \widehat{\mathbb{Z}/\ell\mathbb{Z}}$$

is an isomorphism.

Proof. Just use the character is determined by the image of 1. □

Let G_1, G_2 be two groups and $G = G_1 \times G_2$ their cartesian product (componentwise multiplication). Let χ be a character on G. Then $\chi_1(g) = \chi(g, e)$ is a character of G_1 and similarly for G_2. Conversely, let χ_i be characters of G_i then $\chi(g_1), \chi(g_2)$ is a character of G. Hence we obtain $\widehat{G_1 \times G_2} \cong \hat{G}_1 \times \hat{G}_2$.

5.2 Proposition. *Let A be a finite abelian group. Then the groups $\hat{A}$ of characters is isomorphic to A. For every $a \neq 0$ there exists a character χ with the property $\chi(a) \neq 0$.* AbCh

Proof. This follows from the main theorem for finite abelian groups. It states that any finite abelian group is isomorphic to a direct product of finitely many groups of the form $\mathbb{Z}/\ell\mathbb{Z}$. □

5.3 Lemma. *Let A be a finite abelian group. Then* ChaSu

$$\sum_{\chi \in \hat{A}} \chi(a) = \begin{cases} \#A & \text{if } a = 0, \\ 0 & \text{if } a \neq 0 \end{cases}$$

$$\sum_{a \in A} \chi(a) = \begin{cases} \#A & \text{if } \chi = e, \\ 0 & \text{if } \chi \neq e \end{cases}.$$

Proof. We treat the first part. We can assume $a \neq 0$. Then there exists a character χ' with the property $\chi'(a) \neq 1$. We have

$$\sum_{\chi \in \hat{A}} \chi(a) = \sum_{\chi \in \hat{A}} \chi\chi'(a) = \chi'(a) \sum_{\chi \in \hat{A}} \chi(a).$$

This shows that the sum is zero. The second part is similar. □

A special case is the formula

$$\sum_{\nu \bmod \ell} e^{2\pi i \nu a/\ell} = \begin{cases} \ell & \text{if } a = 0 \text{ in } \mathbb{Z}/\ell\mathbb{Z} \\ 0 & \text{else} \end{cases},$$

where "$\nu \bmod \ell$" means that we have to sum over $\mathbb{Z}/\ell\mathbb{Z}$ which is the same as the sum over a system of representatives of $\mathbb{Z}/\ell\mathbb{Z}$ in $\mathbb{Z}$. This formula can be proved also by means of the geometric sum formula.

A slight generalization of the first formula in the Lemma 5.3 states

5.4 Lemma. *Let A be an abelian group and a, b be two elements of A. Then* ChaSuz

$$\sum_{\chi \in \hat{A}} \bar{\chi}(a)\chi(b) = \begin{cases} \#A & \text{if } b = a, \\ 0 & \text{else.} \end{cases}.$$

Proof. One uses $\bar{\chi}(a) = \chi(-a)$ to reduce the statement to the previous Lemma. □

Let $f : A \to \mathbb{C}$ be a complex valued function on a finite abelian group. Then one can define $\hat{A} \to \mathbb{C}$ by

$$\hat{f}(\chi) = \frac{1}{\#A} \sum_{a \in A} f(a)\chi(a)^{-1}.$$

5.5 Proposition. *Let $f : A \to \mathbb{C}$ be a function on a finite abelian group. Then* FouF

$$f(a) = \sum_{\chi \in \hat{A}} \hat{f}(\chi)\chi(a).$$

Proof. The right hand side is

$$\frac{1}{\#A}\sum_{\chi\in\hat{A}}\sum_{x\in A} f(x)\chi(x)^{-1}\chi(a).$$

We first sum over χ and apply Lemma 5.4. (Use $\bar{\chi}(x)=\chi(x)^{-1}$.) □

If we specialize to $A=\mathbb{Z}/\ell\mathbb{Z}$ and if we use the identification of the dual group of A with itself, we obtain the following formula for a function $f:\mathbb{Z}/\ell\mathbb{Z}\to\mathbb{Z}$.

$$f(a)=\sum_{n \bmod \ell}\hat{f}(n)e^{2\pi \mathrm{i} na/\ell}\quad\text{where } \hat{f}(n)=\frac{1}{\ell}\sum_{\nu \bmod \ell} f(\nu)e^{-2\pi\mathrm{i}\nu n/\ell}.$$

So far we used that $\mathbb{Z}/\ell\mathbb{Z}$ is an abelian group. This is not the complete truth. It is also a ring. As in every ring we can consider the group of invertible elements. We denote this group by $(\mathbb{Z}/\ell\mathbb{Z})^*$. The coset of an integer n is invertible if and only of n is prime to ℓ. This follows from a result in elementary number theory (a consequence of the euclidian algorithm) that for two coprime integers n,ℓ there exist integers x,y with the property $xn+y\ell=1$. Then x defined an inverse of n mod ℓ. Hence representatives of $(\mathbb{Z}/\ell\mathbb{Z})^*$ are the natural numbers in the interval $[0,\ell-1]$ that are coprime to ℓ. By the main theorem for finite abelian groups this group is isomorphic to a product of finitely many cyclic groups. We will make use of this (but we don't need the precise structure that can be determined, as is done usually in a course on elementary number theory).

5.6 Definition. *A Dirichlet character is a character on the group* $(\mathbb{Z}/\ell\mathbb{Z})^*$. DefFi

It turns out to be useful to extend a Dirichlet character $\chi:\mathbb{Z}/\ell\mathbb{Z})^*\to S^1$ to a function on the whole $\mathbb{Z}/\ell\mathbb{Z}$ by means of the definition

$$\chi(a)=\begin{cases}\chi(a) & \text{for } a\in(\mathbb{Z}/\ell\mathbb{Z})^*,\\ 0 & \text{else.}\end{cases}$$

As we have explained this can considered also as ℓ-periodic function on $\mathbb{Z}$. We use the same letter χ for it. A function

$$\chi:\mathbb{Z}\to S^1\cup\{0\}$$

comes from a Dirichlet character mod ℓ if and only if it has the following properties

$$\chi(n)\chi(m)=\chi(mn),\quad \chi(n+\ell)=\chi(n),\quad \chi(n)=0 \text{ if and only if } (n,\ell)=1.$$

Assume that ℓ' is a divisor of ℓ. Then there are natural homomorphisms

$$\mathbb{Z}/\ell\mathbb{Z}\longrightarrow\mathbb{Z}/\ell'\mathbb{Z},\quad(\mathbb{Z}/\ell\mathbb{Z})^*\longrightarrow(\mathbb{Z}/\ell'\mathbb{Z})^*.$$

It may happen that a Dirichlet character χ on $(\mathbb{Z}/\ell\mathbb{Z})^*$ comes from a Dirichlet character on $(\mathbb{Z}/\ell\mathbb{Z})^*$. This is obviously the case if and only if χ (considered as function on $\mathbb{Z}$ has period ℓ'.

5.7 Lemma. *Let ℓ' be a divisor of ℓ. Then the natural homomorphism* lisSur

$$(\mathbb{Z}/\ell\mathbb{Z})^* \longrightarrow (\mathbb{Z}/\ell'\mathbb{Z})^*$$

is surjective.

Proof. Let a be an integer that is coprime to ℓ'. We have to find an integer x such that $b = a + x\ell'$ such that b is coprime to ℓ. Just take x such that it is coprime to to the greatest common divisor of a and l. □

5.8 Definition. *A Dirichlet character $\chi : (\mathbb{Z}/\ell\mathbb{Z})^* \to S^1$ is called* ***primitive*** *if it is not identically one and if there exists no proper divisor $l'|l$ such that it comes from a Dirichlet character mod l'.* PriDC

Let χ be a Dirichlet character mod ℓ. Consider the smallest divisor ℓ' of ℓ such that χ comes from a Dirichlet character mod ℓ'. Then this character on $\mathbb{Z}/\ell'\mathbb{Z}$ is trivial or primitive. Hence every non-trivial Dirichlet character is induced by a primitive one. Hence we can restrict often to *primitive* Dirichlet characters.

5.9 Definition. *The* ***Gauss sum*** *of a Dirichlet character χ is the following number* DefGS

$$G(\chi) = \sum_{a \bmod \ell} \chi(a) e^{2\pi \mathrm{i} a/\ell}.$$

5.10 Lemma. *We have $G(\bar\chi) = \chi(-1)\overline{G(\chi)}$.* rPi

The proof is easy and can be omitted. □

Recall that we extended a Dirichlet character χ (by zero) to a function on $\mathbb{Z}/\ell\mathbb{Z}$. We can consider the "Fourier transform" $\hat\chi$ of this function. This also a function on $\mathbb{Z}/\ell\mathbb{Z}$.

5.11 Proposition. *For a primitive Dirichlet character χ mod ℓ we have* AdMul

$$\ell\hat\chi(-n) = G(\chi)\bar\chi(n) \quad \textit{for all } n \in \mathbb{Z}.$$

Proof. First we treat the case where n and ℓ are coprime. We denote by $\bar n$ an integer such that $n\bar n \equiv 1 \bmod \ell$. Then $\chi(\bar n) = \bar\chi(n)$. We have

$$\bar\chi(n)G(\chi) = \sum_{a \bmod \ell} \bar\chi(n)\chi(a)e^{2\pi \mathrm{i} a/\ell}.$$

We now replace a by na. This gives the claim.

So far we didn't use that χ is primitive. But now, in the case that n and ℓ are not coprime, we will have to use it. Since $\bar\chi(n) = 0$ it is sufficient to show

$\hat\chi(n)=0$. We write $n/\ell=n'/\ell'$ with coprime n',ℓ'. Since χ is primitive, it can't come from a character on $(\mathbb{Z}/\ell'\mathbb{Z})^*$ (via $(\mathbb{Z}/\ell\mathbb{Z})^*\to(\mathbb{Z}/\ell\mathbb{Z})^*$). Hence χ can not be trivial on the kernel of this map. Therefore there exists an integer b, coprime to n such that $b\equiv 1$ mod ℓ' but $\chi(b)\neq 1$. We obtain $bn\equiv n$ mod $n\ell'$. Since $n\ell'=n'\ell$ we get

$$bn\equiv n \mod \ell.$$

Now we get

$$\hat\chi(n)=\sum_{a \bmod \ell}\chi(a)e^{-2\pi\mathrm{i}an/\ell}=\sum_{a \bmod \ell}\chi(ab)e^{-2\pi\mathrm{i}abn/\ell}=\chi(b)\hat\chi(n).$$

This shows $\hat\chi(n)=0$. □

5.12 Proposition. *For a primitive character we have* GauS

$$|G(\chi)|=\sqrt{\ell}.$$

Proof. From the previous proposition we get

$$\ell^2|G(\bar\chi)|^2=\hat\chi(n)\overline{\hat\chi(n)}.$$

We sum over n.

$$|G(\bar\chi)|^2\,\frac{1}{\ell^2}\,\ell\#(\mathbb{Z}/\ell\mathbb{Z})^*=\#(\mathbb{Z}/\ell\mathbb{Z})^*.$$

The claim follows immediately. □

6. Dirichlet L-functions

Dirichlet L-functions are generalizations of the Riemann zeta function. They are defined for a Dirichlet character $\chi:(\mathbb{Z}/\ell\mathbb{Z})^*\to S^1$ through

$$L(s,\chi)=\sum_{n=1^\infty}\chi(n)n^{-s}$$

For $\ell=1$ (and the trivial character) this function agrees with the Riemann zeta function.

6.1 Proposition. *The Dirichlet series* DirPr

$$L(s,\chi)=\sum_{n=1^\infty}\chi(n)n^{-s}$$

converges absolutely for $\sigma>1$ an represents a holomorphic function there. It admits an absolutely convergent product expansion.

$$L(s,\chi)=\prod(1-\chi(p)p^{-s})^{-1}\quad(\sigma>1).$$

In particular, it has no zero for $\sigma>1$.

For a Dirichlet character $\chi(-1)^2=1$, hence $\chi(-1)=\pm1$. We call χ even if $\chi(1)=1$ and odd if $\chi(-1)=-1$. For an even character we have $\chi(n)=\chi(-n)$ and for an odd one we have $\chi(n)=-\chi(n)$.

We set

$$\delta=\begin{cases}0 & \text{if } \chi \text{ is even,}\\ 1 & \text{if } \chi \text{ is odd.}\end{cases}$$

6.2 Theorem. *Let $\chi:(\mathbb{Z}/\ell\mathbb{Z})^*\to S^1$ be a primitive Dirichlet character mod ℓ. Then the L-series $L(s,\chi)$ extends to a holomorphic function in the whole complex plane. It satisfies the functional equation* FunLF

$$\Big(\frac{\pi}{\ell}\Big)^{-\frac{s+\delta}{2}}\Gamma\Big(\frac{s+\delta}{2}\Big)L(s,\chi)=\frac{G(\chi)}{\mathrm{i}^\delta\sqrt{\ell}}\Big(\frac{\pi}{\ell}\Big)^{-\frac{1-s+\delta}{2}}\Gamma\Big(\frac{1-s+\delta}{2}\Big)L(1-s,\bar\chi).$$

Proof. The proof is similar to the proof of the functional equation of the Riemann zeta function. It uses a generalization of the theta inversion formula. To formulate it, we introduce for an arbitrary function

$$f:\mathbb{Z}/\ell\mathbb{Z}\longrightarrow\mathbb{C}$$

the theta function

$$\vartheta_f(z)=\sum_{n=-\infty}^{\infty}f(n)e^{2\pi\mathrm{i}zn^2/\ell}.$$

6.3 Proposition. *We have* ThChaF

$$\vartheta_f\Big(-\frac{1}{z}\Big)=\sqrt{\frac{z}{\mathrm{i}\ell}}\,\vartheta_{\hat f}(z).$$

Proof. It is sufficient to prove this formula for functions f of the type

$$f(n) = e^{2\pi i n a/\ell},$$

since every f can be written as linear combination of them. Then

$$\hat{f}(n) = \sum_{\nu \bmod \ell} e^{2\pi i (n-\nu)a)/\ell} = \begin{cases} \ell & \text{for } n = 0, \\ 0 & \text{else} \end{cases}.$$

Now the stated formula follows from

$$\sum_{n=-\infty}^{\infty} e^{\pi i(-n^2/z+2na)} = \sqrt{\frac{z}{i}} \sum_{n=-\infty}^{\infty} e^{\pi i(n+a)^2 z}.$$

(Just replace z by z/ℓ and a by a/ℓ.) The last formula is a special case of the Jacobi theta transformation formula. □

The formula in Proposition 6.3 is useless in the case that f is odd in the sense $f(n) = -f(-n)$ because both sides then are 0 for trivial reason. There is a modification which gives also a reasonable result of odd f.

6.4 Proposition. *Let* ThChaFz

$$\hat{\vartheta}_f(z) = \sum_{n=-\infty}^{\infty} n f(n) e^{2\pi i n z/\ell}.$$

Then

$$\hat{\vartheta}_f\Big(-\frac{1}{z}\Big) = i \sqrt{\frac{z}{i\ell}}^{\,3}\, \hat{\vartheta}_{\hat{f}}(z).$$

Proof. The trick is to differentiate Jacobi's theta transformation formula by w to obtain

$$z\sqrt{\frac{z}{i}} \sum_{n=-\infty}^{\infty} (n+w) e^{\pi i (n+w)^2 z} = \sum_{n=-\infty}^{\infty} n e^{\pi i n^2(-1/z)+2\pi i n w}$$

and then to apply the same method as in the proof of Proposition I.5.2. □

We now apply this to $f(n) = \chi(n)$ where $\chi(n)$ is a primitive Dirichlet character (to be precise, its extension by 0 to $\mathbb{Z}/\ell\mathbb{Z}$). We write

$$\vartheta(z,\chi) = \vartheta_\chi(z) \quad \text{and} \quad \hat{\vartheta}(z,\chi) = \hat{\vartheta}_\chi(z).$$

Using the formula for the Fourier transform of χ, we obtain the following result.

6.5 Proposition. *Let χ be an even primitive Dirichlet character. Then* ChiTra

$$\vartheta\Big(-\frac{1}{z},\chi\Big)=\varepsilon\sqrt{\frac{z}{\mathrm{i}}}\vartheta(z,\bar\chi)\quad \textit{where } \varepsilon=\frac{G(\chi)}{\sqrt{\ell}}.$$

Let χ be an odd primitive Dirichlet character. Then

$$\hat\vartheta\Big(-\frac{1}{z},\chi\Big)=\varepsilon\sqrt{\frac{z}{\mathrm{i}}}^{\,3}\hat\vartheta(z,\bar\chi)\quad \textit{where } \varepsilon=\frac{G(\chi)}{\mathrm{i}\sqrt{\ell}}.$$

Now we treat $L(s,\chi)$ for a character which is not necessarily primitive. We know that there exists a divisor $\ell'|\ell$ and a primitive character χ' mod ℓ' such that $\chi(n)=\chi'(n)$ for $(n,l)=1$. Hence we have

$$L(s,\chi')=L(s,\chi)\prod_{p\nmid l',\ p|l}(1-p^{-s})^{-1}.$$

The product is a finite product of meromorphic functions which are holomorphic for $\operatorname{Re}s>0$. Hence we obtain

6.6 Theorem. *Let $\chi:(\mathbb{Z}/\ell\mathbb{Z})^*\to S^1$ be a (not necessarily primitive) Dirichlet character. Then $L(s,\chi)$ extends to a meromorphic function on the whole plane. When χ is non-trivial then $L(s,\chi)$ is holomorphic for $\operatorname{Re}s>0$. In the trivial case it has in $\operatorname{Re}s>0$ one pole, namely at $s=1$ and this pole ha order one.* AnFN

If one is only interested in the result that $L(s,\chi)$ is holomorphic for $\operatorname{Re}s$ in the case of a non-trivial character, then there is a far easier proof which we want to explain now. Due to Theorem 1.4 it suffices to show that the sum $A(x)=\sum_{n\le x}\chi(n)$ remains bounded. From the Lemma 5.3 follows that $A(x)$ is periodic. Hence we obtain the following result.

6.7 Theorem. *Let χ be a non-trivial Dirichlet character. The Dirichlet L-series $L(s,\chi)$ converges for $\operatorname{Re}s>0$ and represents there a holomorphic function.* AnFoz

Chapter IV. Prime number theorems

1. The Dirichlet prime number theorem

Let a, b be two coprime natural numbers. The Dirichlet prime number theorem states that there are infinitely man primes $p \equiv a \bmod b$. We will prove more.

1.1 Theorem. *Let a, b be coprime natural numbers. The series* DPrT

$$\sum_{p \equiv a \bmod b} \frac{1}{p}$$

diverges.

The basic fact for the Dirichlet prime number theorem is the following result due to Dirichlet.

1.2 Theorem. *For a non-trivial Dirichlet character χ one has* LenZ

$$L(1, \chi) \neq 0.$$

Proof. There are two cases.
Case I. We assume that $\chi \neq \bar{\chi}$ (i.e. χ is not real. For real $\sigma > 1$ we get

$$\log L(\sigma, \chi) = \sum_p \log\Bigl(1 - \frac{\chi(p)}{p^\sigma}\Bigr)^{-1} = \sum_p \sum_{n=1}^{\infty} \frac{\chi(p)^m}{p^{m\sigma}}.$$

Now we introduce the function

$$P(s) = P_\ell(s) = \prod_{\chi \bmod \ell} L(s, \chi).$$

A simple calculation gives

$$\log P(\sigma) = \sum_p \sum_{n=1}^{\infty} \frac{1}{p^{n\sigma}} \sum_{\chi \bmod \ell} \chi(p)^n.$$

The second relation in Lemma III.5.3 shows that this expression is non-neagtive. Hence $P(s) \geq 1$ for $\sigma > 1$. One the other side we know $\lim P_{\sigma \to 1^+} = 0$, since in the product a pole of order one (prinicipal) and two zeros (belonging to $\chi, \bar{\chi}$) produce a zero of $P(s)$ at $s = 1$.

Case II. χ is real. Then χ can take only values $0, \pm 1$. We consider

$$f(s) = \zeta(s)L(s,\chi) = \prod f_p(s) \quad \text{where} \quad f_p(s) = \frac{1}{(1-p^{-s})(1-\chi(p)p^{-s})}.$$

We get

$$\begin{aligned} f_p(s) &= 1 + p^{-s} + p^{-2s} + \ldots && \text{for } \chi(p) = 0, \\ f_p(s) &= 1 + 2p^{-s} + 3p^{-2s} + \ldots && \text{for } \chi(p) = 1, \\ f_p(s) &= 1 + p^{-2s} + p^{-4s} + \ldots && \text{for } \chi(p) = -1. \end{aligned}$$

The point is that only non negative real coefficients occur. Hence the Dirichlet series $f(s) = \sum a_n n^{-s}$ has only non-negative real coefficients, even more, in the case of a square we see $a_{n^2} \geq 1$. Now assume that $L(1,\chi) = 0$. The $f(s)$ extends holomorphically to the whole plane. Now we apply Theorem III.1.5 and obtain that this series converges for all s. In particular, it converges for $s = 1/2$. But

$$\sum_{n=1}^{\infty} a_n n^{-1/2} \geq \sum_{n=1}^{\infty} a_{n^2} n^{-1} \geq \sum_{n=1}^{\infty} n^{-1}$$

which gives a contradiction. □

We have collected the tools for the proof of the Dirichlet prime number theorem. Similar to the proof of Theorem III.2.3 we start with

$$\log L(s,\chi) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k} \chi(p^k) p^{-ks}$$

for real $s > 1$. We obtain that

$$\operatorname{Log} L(s,\chi) = \sum_p \chi(p) p^{-s} + O(1),$$

where $O(1)$ stand symbolically for a function that remains bounden for $s \to 1^+$ (Landau O-symbol). We known that for non-trivial χ we have $L(1,\chi) \neq 0$, hence $\log L(s,\chi)$ remains bounded for $s \to 1^+$. So we get

$$\sum_p \chi(p) p^{-s} = O(1) \quad \text{for } s \to 1^+ \qquad (\chi \text{ non trivial}).$$

Now we use the character identity III.5.4 to obtain

$$\sum_{p \equiv a \bmod \ell} p^{-s} = \sum_p \frac{1}{\varphi(\ell)} \sum_\chi \bar{\chi}(a) \chi(p) p^{-s} \qquad (\varphi(\ell) := \#(\mathbb{Z}/\ell\mathbb{Z})^*).$$

We seperate the trivial and the non-trivial characters to show

$$\sum_{p\equiv a \bmod \ell} p^{-s} = \frac{1}{\varphi(l)} \sum_{p\nmid \ell} p^{-s} + O(1) \qquad (s \to 1^+).$$

We know

$$\sum_{p\nmid \ell} p^{-s} = \sum_{p} p^{-s} + O(1) = \zeta(s) + O(1).$$

Hence

$$\sum_{p\equiv a \bmod \ell} p^{-s}$$

remains unbounded for $s \to 1^+$. But then

$$\sum_{p\equiv a \bmod \ell} p^{-1}$$

cannot converge. □

2. A Tauber theorem

2.1 Theorem. *Let $a_1, a_2, a_3, \dots$ be a sequence of non-negative real numbers such that* Taub

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad (a_n \geq 0)$$

converges for $\sigma > 1$. We assume:

I. *The function $(s-1)D(s)$ admits an analytic continuation to an open subset that contains the closed half plane $\sigma \geq 1$. $D(s)$ has a pole of order 1 with residue*

$$\varrho = \operatorname{Res}(D, 1).$$

II. *There are estimates*

$$|D(s)| \leq C|t|^{\kappa} \quad \textit{and} \quad |D'(s)| \leq C|t|^{\kappa} \quad \textit{for } \sigma > 1,\ |t| \geq 1,$$

where C, κ are suitable constants.

Then

$$\sum_{n\leq x} a_n = \varrho(1 + r(x)) \quad \textit{where} \quad r(x) = O\Big(\frac{1}{\sqrt[N]{\log x}}\Big)$$

with a suitable constant N.

Proof. Besides $A(x) = \sum_{n\le x} a_n$ we will study the higher summatory functions

$$A_k(x) = \frac{1}{k!}\sum_{n\le x} a_n(x-n)^k.$$

So we have $A_0(x) = A(x)$ and

$$A'_{k+1}(x) = A_k(x), \quad A_{k+1}(x) = \int_1^x A_k(x)dt.$$

We will determine the asymptotic behaviour for all k, not only for $k = 0$. In fact, we show now that the asymptotic behaviour for A_k implies that of smaller k.

2.2 Lemma. *Define $r_k(x)$ by* AbSt

$$A_k(x) = \varrho\frac{x^{k+1}}{(k+1)!}(1 + r_k(x)).$$

Then

$$r_{k+1} = O(1/\sqrt[N]{\log x}) \Longrightarrow r_k = O(1/\sqrt[2N]{\log x}).$$

The Lemma shows that we have it suffices to prove

$$\boxed{r_k(x) = O(1/\log x)}$$

for big enough k. Later we will prove this for $k > \kappa + 1$.

Proof of Lemma 2.2. The function $A_k(x)$ is *monotonically increasing.* Hence for any positive $c > 0$ we have

$$cA_k(x) \le \int_x^{x+c} A_k(t)\,dt \quad (c > 0).$$

We apply this inequality for some $x \ge 1$ and for $c = hx$ where $h = h(x)$, $0 < h < 1$ will be determined later. The right hand side equals

$$A_{k+1}(x + hx) - A_{k+1}(x) = \\ \frac{\varrho}{(k+2)!}\big[(x + hx)^{k+2}(1 + r_{k+1}(x + hx)) - x^{k+2}(1 + r_{k+1}(x))\big].$$

We obtain

$$1 + r_k(x) \le \frac{(1+h)^{k+2}\big(1 + r_{k+1}(x + hx)\big) - \big(1 + r_{k+1}(x)\big)}{h(k+2)}.$$

Now we set

$$\varepsilon(x) := \sup_{0\le\xi\le 1} |r_{k+1}(x+\xi x)|$$

to obtain

$$\begin{aligned} r_k(x) &\le \frac{(1+h)^{k+2}\big(1+\varepsilon(x)\big) - \big(1-\varepsilon(x)\big)}{h(k+2)} - 1 \\ &= \frac{\big[(1+h)^{k+2}+1\big]\varepsilon(x)}{h(k+2)} + \frac{(1+h)^{k+2} - \big[1+(k+2)h\big]}{h(k+2)}. \end{aligned}$$

Now we choose $h = h(x) = \sqrt{\varepsilon(x)}$. For sufficiently big x this smaller than 1 Obviously h and hence also $(1+h)^{k+2}+1$ is bounded from above. The first term in the estimate of r_k is bounded from above by a constant multiple of $\varepsilon(x)/h = \sqrt{\varepsilon(x)}$. The second term is a polynomial in h whose constant term vanishes Up to a constant factor it can be estimated by $h = \sqrt{\varepsilon(x)}$. Obviously

$$\varepsilon(x) = O\left(1/\sqrt[N]{\log x}\right).$$

Hence we have with some bound K which is independent on k

$$r_k(x) \le K\sqrt{\varepsilon(x)}.$$

For an estimate of $|r_k(x)|$ we need also an estimate of r_k below. By means of the estimate

$$cA_k(x) \ge \int_{x-c}^{x} A_k(t)\,dt = A_{k+1}(x) - A_{k+1}(x-c) \text{ for } 0<c<x$$

we obtain along the same lines

$$r_k(x) \ge -K\sqrt{\varepsilon(x)}$$

with some new bound K. We obtain

$$r_k(x) = O\big(\sqrt{\varepsilon(x)}\big), \text{ also } r_k(x) = O\left(1/\sqrt[2N]{\log x}\right). \qquad \square$$

We need the convergence of the following integral

2.3 Remark. *Let k be a natural number, $x>0$ und $\sigma>1$. Then the integral* KonIn

$$\int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} \frac{|x^{s+k}|}{|s(s+1)\cdots(s+k)|}\,ds.$$

converges.

Here the improper integral along the line $\mathrm{Re}(s) = \sigma$ is defined through

$$\int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} f(s)\,ds := \mathrm{i}\int_{-\infty}^{\infty} f(\sigma+\mathrm{i}t)\,dt.$$

The proof of Remark 2.3 is trivial, since the integrand can be estimated by a constant multiple of $1/\sigma^2$. □

On the verical line $\mathrm{Re}\, s = \sigma$ the series $D(s)$ durch is bounded by the series

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma}.$$

Now the Lebesgue limit theorem implies the following result.

2.4 Corollary. *The integral* KonAb

$$\int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\,ds \quad (k\in\mathbb{N},\ x>0)$$

converges absolutely for $\sigma > 1$. We can interchange summation and integration

$$\sum_{n=1}^{\infty} a_n x^k \int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} \frac{(x/n)^s}{s(s+1)\cdots(s+k)}\,ds.$$

We want to prove the integral from Corollary 2.4.

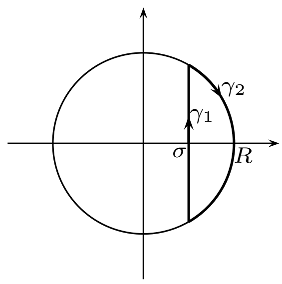
2.5 Lemma. *For $k\in\mathbb{N}$ and $\sigma > 0$ we have* BerInt

$$\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} \frac{a^s}{s(s+1)\cdots(s+k)}\,ds = \begin{cases} 0 & \text{for } 0 < a \le 1,\\ \frac{1}{k!}(1-1/a)^k & \text{for } a \ge 1.\end{cases}$$

Proof. Let

$$f(s) = \frac{a^s}{s(s+1)\cdots(s+k)}\,.$$

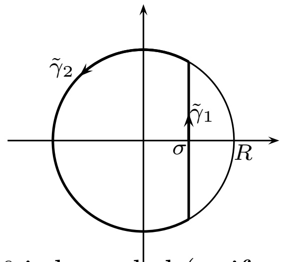
1) $(0 < a \le 1)$ The integral of $f(s)$ along the integration path $\gamma := \gamma_1 \oplus \gamma_2$



vanishes by the Cauchy integral theorem. From $0 < a \leq 1$ follows that the function a^s is bounded on the integration contour uniformly in R Take the limt $R \to \infty$ shows

$$\int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} f(s)\,ds = 0.$$

2) $(a \geq 1)$. Here we have to use the integration contour $\tilde\gamma = \tilde\gamma_1 \oplus \tilde\gamma_2$



since on this contour a^s is bounded (uniformly in R). Now the residue theorem shows

$$\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} = \sum_{\nu=0}^{k} \operatorname{Res}(f; -\nu) = \sum_{\nu=0}^{k} \frac{(-1)^\nu a^{-\nu}}{\nu!(k-\nu)!} = \frac{1}{k!}(1 - 1/a)^k. \qquad \square$$

We now obtain a formula for the generalized summatory function.

2.6 Lemma. *In the case $k \geq 1$ and $\sigma > 1$ we have* SumFu

$$\boxed{A_k(x) = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\,ds.}$$

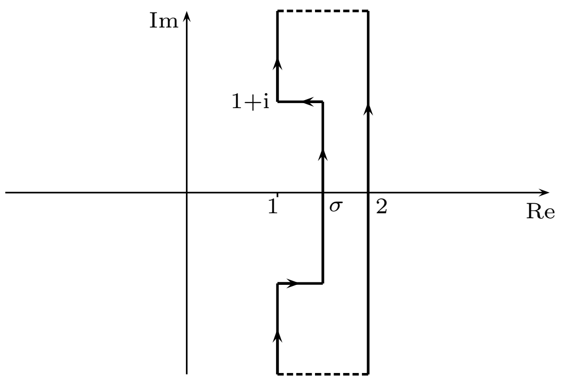
We apply this lemma for a fixed σ. We take $\sigma = 2$. The estimate

$$|D(s)| \leq C|t^\kappa| \qquad (|t| \geq 1\,,\ 1 < \sigma \leq 2)$$

holds also for $\sigma = 1$. For fixed x we obtain

$$\begin{aligned}\frac{|D(s)x^{s+k}\,ds}{s(s+1)\cdots(s+k)|} &\leq \mathrm{Const}\,|t|^{\kappa-k-1} \quad (|t| \geq 1\,,\ 1 \leq \sigma \leq 2),\\ &\leq \mathrm{Const}\,|t|^{-2},\ \text{if } k > \kappa + 1\end{aligned}$$

The Cauchy integral theorem allows to move the integration contour from $(\mathrm{Re}(s) = 2)$ to $\mathrm{Re}(s) = 1$ if we detour the singularity at $s = 1$. So we consider the integration path L



We obtain

2.7 Lemma. *In the case $k > \kappa + 1$ we have* Waru

$$A_k(x) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\, ds$$
$$\left(\int_L := \int_{1-\mathrm{i}\infty}^{1-\mathrm{i}} + \int_{1-\mathrm{i}}^{\sigma-\mathrm{i}} + \int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} + \int_{\sigma+\mathrm{i}}^{1+\mathrm{i}} + \int_{1+\mathrm{i}}^{1+\mathrm{i}} \right).$$

Now we estimate the two improper integrals along the 5 lines separately. We start with the two lines from $1 - \mathrm{i}$ to $1 - \mathrm{i}$ and from $1 + \mathrm{i}$ to $1 + \mathrm{i}$. For this purpose we need the following Lemma.

2.8 Lemma (B. Riemann, H. Lebesgue). *Let* LebLem

$$I = (a, b), \quad -\infty \le a < b \le \infty,$$

be some (not necessarily finite) interval and $f : I \to \mathbb{C}$ a function with the following properties

a) *f is bounded.*
b) *f is differentiable with continuous derivative.*
c) *f und f' are absolutely integrable (from a to b).*

Then the function $t \mapsto f(t)x^{\mathrm{i}t}$ $(x > 0)$ is absolutely integrable too and we have

$$\int_a^b f(t)x^{\mathrm{i}t}\, dt = O(1/\log x).$$

Proof. We choose sequences

$$a_n \to a, \; b_n \to b, \quad a < a_n < b_n < b.$$

We have

$$\int_a^b f(t)x^{\mathrm{i}t}\,dt = \lim_{n\to\infty}\int_{a_n}^{b_n} f(t)x^{\mathrm{i}t}\,dt$$
$$= \frac{1}{\mathrm{i}\log x}\lim_{n\to\infty}\left(\Big[f(t)x^{\mathrm{i}t}\Big]_{a_n}^{b_n} - \int_{a_n}^{b_n} f'(t)x^{\mathrm{i}t}\,dt\right).$$

By assumption, $f(t)$ is bounded and $|f'(t)x^{\mathrm{i}t}| = |f'(t)|$ is integrable. We obtain

$$\Big|\int_a^b f(t)x^{\mathrm{i}t}\,dt\Big| \le \mathrm{Const}\,\Big|\frac{1}{\log x}\Big|. \qquad \square$$

We now have prepared the tools to estimate $A_k(x)$ foe big k.

2.9 Lemma. *In the case $k > \kappa + 1$ we have* rk0

$$r_k(x) = O(1/\log x).$$

Now Lemma 2.8 gives

$$\frac{1}{2\pi\mathrm{i}}\int_{1+\mathrm{i}}^{1+\mathrm{i}} \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\,ds = O(x^{k+1}/\log x),$$

and similarly for the integral from $1-\mathrm{i}$ to $1-\mathrm{i}$. Hence both integrals only contribute the remainder term of $r_k(x)$!

Now we look at the integral along the vertical line from $\sigma-\mathrm{i}$ to $\sigma+\mathrm{i}$. At the moment we have $\sigma > 1$.

Lemma 2.8 implies

$$\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\,ds = O\left(x^{k+1}\frac{x^{\sigma-1}}{\log x}\right).$$

Unfortunately $x^{\sigma-1}/\log x$ is in the case $\sigma > 1$ not of magnitude $O(1/\log)$. But we have

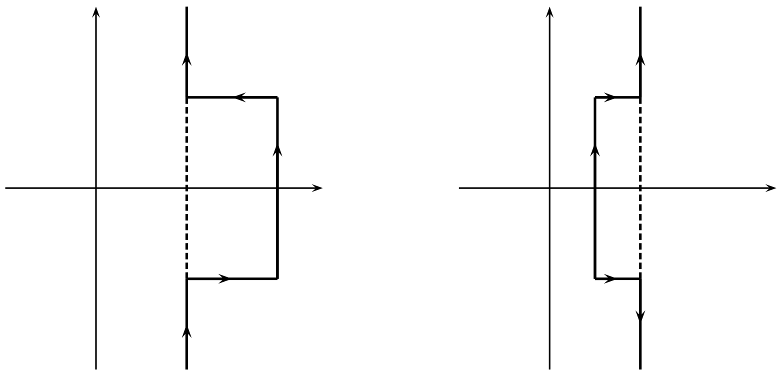
$$\boxed{x^{\sigma-1}/\log x = O\left(\frac{1}{\log x}\right) \quad \text{if } \sigma \le 1.}$$

So it seems to be natural to push the integration contour more to the left. Actually we know that $(s-1)D(s)$ extends analytically to an open subset which contains the half plane $\{s\in\mathbb{C},\ \mathrm{Re}(s)\ge 1\}$. There exists σ, $0<\sigma<1$, such that the closed rectangle with vertices $\sigma-\mathrm{i}$, $2-\mathrm{i}$, $2+\mathrm{i}$ and $\sigma+\mathrm{i}$

is contained in this open subset. The residue theorem shows

$$\int_E * = \int_F * + \operatorname{Res}\left(\frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\,;\ s=1\right).$$

Here E resp. F are the integration contours



Since $D(s)$ has a pole of order 1 with residue ϱ at $s = 1$t, the residue can be computed as

$$\frac{\varrho}{(k+1)!}\,x^{k+1}.$$

This is the main term in the asymptotic formula $A_k(x)$ in Lemma 2.9. All other terms have to disappear in the remainder. We showed this already for the integral from $\sigma - \mathrm{i}$ to $\sigma + \mathrm{i}$ (using $\sigma \le 1$). It remains to treat the two integrals along the horizontal line from $\sigma + \mathrm{i}$ to $1 + \mathrm{i}$ and from $\sigma - \mathrm{i}$ bis $1 - \mathrm{i}$. For example we show

$$\int_{\sigma+\mathrm{i}}^{1+\mathrm{i}} \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\,ds = O(x^{k+1}/\log x).$$

The integral can be estimated as follows.

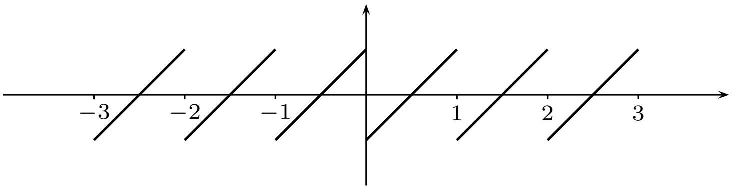
$$O\big(x^k \int_\sigma^1 x^t\,dt\big) = O(x^{k+1}/\log x).$$

This finishes the proof of the Tauber theorem. □

3. The prime number theorem

We need some bounds for the Riemann ζ function. For their proof we use another formula that gives the analytic continuation into the domain $\operatorname{Re} s > 0$. The proof rests on the function

$$\beta(t) = t - [t] - 1/2 \quad ([t] = \max\{n \in \mathbb{Z},\ n \le x\}).$$



Now we consider the integral

$$F(s) = \int_1^\infty t^{-s}\beta(t)\frac{dt}{t}.$$

It converges absolutely for $\operatorname{Re} s > 0$ and represents an analytic function there. Partial integration shows

$$\int_n^{n+1} \beta(t)\frac{d}{dt}t^{-s} = \frac{1/}{2}\big((n+1)^{-s} + n^{-s}\big) - \int_n^{n+1} t^{-s}dt.$$

Summing up, we get

$$-s\int_1^\infty \beta(t)t^{-s-1}dt = \zeta(s) + 1/2 - \int_0^\infty t^{-s}.$$

In the domain Res > 1 we have proved

$$\boxed{\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - sF(s).}$$

This gives an analytic continuation of $\zeta(s)$ into the half plane $\operatorname{Re} s > 0$ (with the pole of order 1 at $a = 1$. The residue is 1). We use this formula to give an estimate for $\zeta(s)$ and its derivatives from above.

3.1 Proposition. *For each integer $m \geq 0$ there exists a constant C_m such that* AbOb

$$|\zeta^{(m)}(t)| \leq C_m|t| \quad \textit{for} \quad |t| \geq 1 \textit{ and } \sigma > 1.$$

Proof. We can assume that $\sigma < 2$, since in the domain $\zeta^{(m)}(s =$ is bounded by $\zeta^{(m)}(2)$. Hence it is sufficient to prove that $F^{(m)}$ remains bounded in the domain $|t| \geq 1$, $1 < \sigma < 2$. In this domain we have

$$|F^{(m)}(s)| \leq \int_1^\infty (\log t)^m t^{-2}dt.$$

Since $\log t$ has smaller growth than any power of t we can estimate this by $\int_1^\infty t^{-3/2}dt$ up to a constant. □

The heart of the proof of the prime number theorem is an estimate of the zeta function from below.

3.2 Theorem. *There exists a constant $\delta > 0$ with* AbUn

$$|\zeta(s)| \geq \delta |t|^{-4} \quad \textit{for} \quad |t| \geq 1 \textit{ and } \sigma > 1.$$

Corollary. *The zeta function has no zero for $\sigma = 1$.*

Proof. We need the following elementary inequaltiy

$$\mathrm{Re}(a^4) + 4\,\mathrm{Re}(a^2) + 3 \geq 0 \quad \text{for} \quad |a| = 1.$$

If we apply it to $a = n^{-\mathrm{i}t/2}$ we get

$$\mathrm{Re}(n^{-2\mathrm{i}t}) + 4\,\mathrm{Re}(n^{-\mathrm{i}t}) + 3 \geq 0.$$

Let now

$$D(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

a Dirichlet series with non-negative real coefficients b_n which converges for $\sigma > 1$. Then we get

$$\mathrm{Re}\, D(\sigma + 2\mathrm{i}t) + 4\,\mathrm{Re}\, D(\sigma + \mathrm{i}t) + 3D(\sigma) \geq 0.$$

We apply this to

$$b_n = \begin{cases} 1/\nu & \text{if } n = p^\nu,\ p \text{ prime}, \\ 0 & \text{else}. \end{cases}$$

Then

$$D(s) = \sum_p \sum_\nu \frac{1}{\nu} p^{-\nu s} = -\sum_p \log(1 - p^{-s}).$$

So we get

$$e^{D(s)} = \zeta(s).$$

The above inequality now gives

$$|\zeta(\sigma + 2\mathrm{i}t)||\zeta(\sigma + \mathrm{i}t)|^4 \zeta(\sigma)^3 \geq 1.$$

It can be rewritten as

$$\boxed{\left|\frac{\zeta(\sigma + \mathrm{i}t)}{\sigma - 1}\right|^4 |\zeta(\sigma + 2\mathrm{i}t)|[\zeta(\sigma)(\sigma - 1)]^3 \geq (\sigma - 1)^{-1} \qquad (\sigma > 1).}$$

This inequality implies already that the zeta function has no zero for $\sigma = 1$. But it is not yet the estimate of Theorem 3.2. It is enough to prove this inequality for $\sigma \leq 2$. To get it, we use the already proved estimate

$$|\zeta(t)| \leq C_0 |t| \quad \text{for} \quad |t| \geq 1 \text{ and } \sigma > 1.$$

Together with the above estimate and using the fact that $(\sigma-1)\zeta(\sigma)$ is bounded from below by a positive constant on $[1,2]$, we obtain

$$|\zeta(s)| \ge A(\sigma-1)^{3/4}|t|^{-1/4} \qquad (1<\sigma\le 2).$$

Now we consider

$$\sigma(t) = 1+\varepsilon|t|^{-5}$$

where $0<\varepsilon<1$ is a small positive number which will be determined later. We consider two cases separately.

Case 1). $\sigma \ge \sigma(t)$. This case is trivial. Here we get

$$|\zeta(\sigma+\mathrm{i}t)| \ge A\varepsilon^{3/4}|t|^{-4}.$$

Case 2). here we use

$$\zeta(\sigma(t)+\mathrm{i}t)-\zeta(\sigma+\mathrm{i}t) = \int_\sigma^{\sigma(t)} \zeta'(x+\mathrm{i}t)dx.$$

Using the proved estimate for $|\zeta'(t)|$ from above, we obtain with some positive constant B ab estimat

$$|\zeta(\sigma+\mathrm{i}t)| ge |\zeta(\sigma(t)+\mathrm{i}t)| - B(\sigma(t)-1)|t| \ge (A\varepsilon^{3/4}-B\varepsilon)|t|^{-4}.$$

Taking ε so small that $\delta = Ae^{3/4}-B\varepsilon>0$, we get the claimed estimate.

□

We prove the prime number theorem in the form.

3.3 Theorem. *We define $r(x)$ for real $x>0$ by* PNT

$$\sum_{p\le x}\log p = x(1+r(x)).$$

Then there exists a natural number N such that

$$r(x) = O\Bigl(\frac{1}{\sqrt[N]{\log x}}\Bigr),$$

in particular,

$$\lim_{x\to\infty}\frac{\sum_{p\le x}\log p}{x} = 1.$$

One can derive in an elementary way the prime number theorem in its usual form

$$\lim_{x\to\infty}\frac{\#\{p,\ p\le x\}}{x/\log x} = 1.$$

Proof of Theorem 3.3. The negative of the logarithmic derivative of the Euler product of the zeta function is

$$D(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^\infty \Lambda(n)n^{-s}, \quad \Lambda(n) = \begin{cases}\log p & \text{if } n=p^\nu,\\ 0 & \text{else.}\end{cases}$$

It satisfies the assumptions of the Tauber theorem. This gives the proof of the prime number theorem. □

Index