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AUTOMORPHY FACTORS OF HILBERT'S MODULAR GROUP

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INTRODUCTION. Let Γ be a group of analytic automorphisms of a domain $D \subset \mathbf{C}^n$. By an automorphy factor of Γ we understand a family of functions $I(z, \gamma)$, $z \in D$, $\gamma \in \Gamma$ holomorphic on D and without zeros, which satisfy the condition

$$I(z, \gamma'\gamma) = I(z, \gamma)\, I(\gamma z, \gamma').$$

The most-occurring factors are the following ones

1) *The trivial factors*

$$I(z, \gamma) = \frac{h(\gamma z)}{h(z)}.$$

Here h is a holomorphic function on D without zeros.

2) *The powers of the complex functional determinant* (Jacobian).

3) *The abelian characters v of* Γ

$$I(z, \gamma) = v(\gamma).$$

The determination of all automorphy factors belonging to a *discontinuous* group is a difficult problem in general. It is roughly equivalent to the calculation of

$$\text{Pic } D/\Gamma = \text{group of analytic line bundles on } D/\Gamma.$$

More precisely, if Γ operates without fixed points, we have

$$\text{Pic } D/\Gamma = \frac{\text{group of automorphy factors}}{\text{subgroup of trivial factors}}.$$

There is a well-known isomorphism

$$\text{Pic } D/\Gamma = H^1(D/\Gamma, \mathcal{O}^*)$$

($\mathcal{O}$ = sheaf of automorphic functions,

$\mathcal{O}^*$ = sheaf of invertible automorphic functions).

By means of the exact sequence

$$0 \to \mathbf{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0,$$

we reduce the original problem to the calculation of

a) the singular cohomology of D/Γ

b) the analytical cohomology $H^\bullet(D/\Gamma, \mathcal{O})$.

This program could be carried out almost completely for the domain

$D = H^n = H \times \ldots \times H$, H the usual upper half-plane.

Matsushima and Shimura succeeded in calculating those groups in case of a compact quotient by means of the Hodge theory [3]. As for the non-compact quotients D/Γ (Hilbert's modular groups) similar complete results have been found.

a) The singular cohomology was investigated by G. Harder [2].

Let us give a very brief indication of the specific problems arising in the non-compact case.

By "cutting off cusps" of D/Γ one gets a manifold with boundary.

There is a natural mapping from the cohomology of the whole space D/Γ to the (well-known) cohomology of the boundary. In the mentioned paper, Harder determined the image and the kernel of this map. His detailed study of this problem leads into the theory of non-analytic modular forms, especially into the theory of Eisenstein series.

b) The analytical cohomology was determined in [1].

To overcome the discrepancy between the standard compactification of D/Γ and a non-singular model, we had to carry out a thorough investigation of the algebraic nature of the cusps. But it was not necessary to get a concrete resolution of the cusps.

1. The main result. In the following let Γ be a group of simultaneously fractional linear substitutions

$$M(z_1, \ldots, z_n) = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right)$$

of the half space

$$H^n = \{(z_1, \dots, z_n) \in \mathbf{C}^n \,;\, \operatorname{Im} z_\nu > 0 \text{ for } 1 \leqslant \nu \leqslant n\}.$$

We are only interested in the case in which Γ is commensurable with Hilbert's modular group of a totally real number field. We define the complex power

$$a^b = e^{b \log a}, \quad a \neq 0$$

by the principal branch of the logarithm.

THEOREM 1. *In case of $n \geqslant 3$*† *the only automorphy factors of Γ are:*

$$I(z, M) = v(M) \prod_{\nu=1}^{n} (c_\nu z_\nu + d_\nu)^{2r_\nu} \frac{h(Mz)}{h(z)}$$

where

a) $r = (r_1, \dots, r_n)$ *is a vector of rational numbers;*

b) $\{v(M)\}_{M \in \Gamma}$ *is a system of complex numbers of absolute value one;*

c) h *is a holomorphic invertible function on H^n.*

This factorisation of I is unique.

By a system of multipliers of weight $r = (r_1, \dots, r_n)$ we understand a family $\{v(M)\}_{M \in \Gamma}$ of complex numbers of absolute value one, such that

$$I(z, M) = v(M) \prod_{\nu=1}^{n} (c_\nu z_\nu + d_\nu)^{2r_\nu}$$

is a factor of automorphy.

AMENDMENT TO THEOREM 1.

1) *The group of abelian characters of Γ is finite.*‡

2) *If $r = (r_1, \dots, r_n)$ is the weight of a multiplier system, the components r_ν have to be rational and their denominators are bounded (by a number which may depend on the group).*

† The method used for the proof is valid also in case of $n < 3$. But one has to carry out some separate investigations because in this case the first cohomology groups are not trivial.

‡ A more general result has been proved by Serre [4].

We now discuss an application of the main theorem.

A meromorphic modular form with respect to Γ is a meromorphic function on H^n satisfying the functional equations:

$$f(Mz) = v(M) \prod_{\nu=1}^{n} (c_\nu z_\nu + d_\nu)^{2r_\nu} f(z) \text{ for } M \in \Gamma.$$

We call $r = (r_1, \dots, r_n)$ the weight and $v(M)$ the multiplier system of f.

We are interested in the zeros and poles of f which we describe by a divisor (f) as usual.

By a divisor we understand a formal and locally finite sum

$$D = \sum_Y n_Y Y, \quad n_Y \in \mathbf{Z}$$

the summation being taken over irreducible closed analytic subvarieties of codimension one.

Theorem 2. *Let $\mathcal{D}$ be a Γ-invariant divisor on H^n, $n > 3$. There exists a meromorphic modular form f with the property*

$$\mathcal{D} = (f).$$

Proof. The space H^n is a topologically trivial Stein-space. Therefore we can find a meromorphic function g on H^n with

$$\mathcal{D} = (g).$$

The function

$$I(z, M) = \frac{g(Mz)}{g(z)}, \quad M \in \Gamma$$

is without poles and zeros because $\mathcal{D}$ is Γ-invariant. We therefore can apply Theorem 1. Put

$$f = \frac{g}{h}.$$

2. Sketch of proof. The group Γ operates in a natural way on the multiplicative group $H^0(D, \mathcal{O}^*)$ of holomorphic invertible functions on $D = H^n$.

The automorphy factors are nothing else but the 1-cocycles with regard to the standard complex and the trivial factors $h(\gamma z)/h(z)$ are the 1-coboundaries, i.e.

$$H^1(\Gamma, H^0(D, \mathcal{O}^*)) = \frac{\text{group of automorphy factors}}{\text{subgroup of trivial factors}}$$

Theorem 1 may thus be formulated as follows

THEOREM 3. *The group $H^1(\Gamma, H^0(D, \mathcal{O}^*))$ is finitely generated and of free rank n.*

We now want to pass on a subgroup $\Gamma_\bullet \subset \Gamma$ of finite index in order to eliminate the elements of finite order. Let $\Gamma_\bullet \subset \Gamma$ be a normal subgroup of finite index. Putting

$$A = H^0(D, \mathcal{O}^*)$$

we obtain by means of the Hochschild-Serre sequence

$$0 \to H^1(\Gamma/\Gamma_\bullet, \mathbf{C}^*) \to H^1(\Gamma, A) \to H^1(\Gamma_\bullet, A)^{\Gamma/\Gamma_\bullet} \to H^2(\Gamma/\Gamma_\bullet, \mathbf{C}^*).$$

(Since every holomorphic modular function is constant, we have

$$A^{\Gamma_\bullet} = \mathbf{C}^*.)$$

The groups $H^r(\Gamma/\Gamma_\bullet, \mathbf{C}^*)$ are finite. This is proved by means of the sequence

$$0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}^* \to 0.$$

In general, the cohomology groups of a finite group which acts trivially on $\mathbf{Z}$, are finite.

We therefore can assume without loss of generality:

The group Γ is a congruence-subgroup of Hilbert's modular group without torsion.

In the case at hand it is easy to be seen

$$H^1(\Gamma, H^0(D, \mathcal{O}^*)) = H^1(D/\Gamma, \mathcal{O}^*),$$

i.e. there is a one-to-one correspondence between the factor classes and the classes of isomorphic analytical line bundles on $X_\bullet = D/\Gamma$.

We now treat the group

$$\operatorname{Pic} X_0 = H^1(X_0, \mathcal{O}^*),\ X_0 = D/\Gamma$$

by means of the sequence

$$0 \to \mathbf{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0.$$

Hereby $\mathcal{O}$ is the sheaf of holomorphic functions on X_0. From the long cohomology sequence results

$$H^1(X_0, \mathcal{O}) \to \operatorname{Pic} X_0 \to H^2(X_0, \mathbf{Z}).$$

We thus have to calculate the groups $H^1(X_0, \mathcal{O})$ and $H^2(X_0, \mathbf{Z})$.

THEOREM 4. *The groups $H^\nu(X_0, \mathcal{O})$ vanish for $1 < \nu < n-2$.*

PROOF. Let S be the finite set of cusp classes of Γ and

$$X = X_0 \cup S$$

the standard compactification of $X_0 = D/\Gamma$. There is a long exact sequence, which combines the cohomology with supports in S with the usual cohomology of sheafs

$$H_S^\nu(X, \mathcal{O}) \to H^\nu(X, \mathcal{O}) \to H^\nu(X_0, \mathcal{O}) \to H_S^{\nu+1}(X, \mathcal{O}) \to H^{\nu+1}(X, \mathcal{O}).$$

From my paper [1] the result (Theorem 7.1)

$$H_S^\nu(X, \mathcal{O}) \simeq H^\nu(X, \mathcal{O}) \quad \text{for } 1 < \nu < n$$

is taken.

An analysis of the proof shows that this isomorphism is induced by the natural mapping

$$H_S^\nu(X, \mathcal{O}) \to H^\nu(X, \mathcal{O}).$$

In case of $n > 3$ we now obtain the exact sequence

$$0 \to \operatorname{Pic} X_0 \to H^1(X_0, \mathbf{Z}).$$

Obviously the free rank of Pic X_0 is not smaller than n because the automorphy factors

$$I_\nu(z, M) = (c_\nu z_\nu + d_\nu)^2 \qquad (1 < \nu < n)$$

are independent of each other.

Therefore Theorem 3 has been proved if one knows that $H^2(X_0, \mathbf{Z})$ is of free rank n. That means

THEOREM 5. *In case of $n > 3$ we have*

$$\dim_{\mathbf{C}} H^2(X_0, \mathbf{C}) = n, \ \ X_0 = D/\Gamma.$$

PROOF. We derive the calculation of the 2nd Betti number of X_0 from Harder's investigations on the singular cohomology of $X_0 = D/\Gamma$ [2].

This will be explained briefly in the following.

By cutting off cusps we obtain a bounded manifold X^* which is homotopically equivalent to X_0. (The boundary component at the cusp is given by

$$\prod_{\nu=1}^{n} \operatorname{Im} z_\nu = C; \quad C \gg 0.)$$

In the paper quoted above, Harder gives a decomposition of the singular cohomology of X_0

$$H^\cdot(X_0, \mathbf{C}) = H^\cdot(X^*, \mathbf{C})$$
$$= H^\cdot_{\mathrm{inf}}(X^*, \mathbf{C}) \oplus H^\cdot_{\mathrm{univ}}(X^*, \mathbf{C}) \oplus H^\cdot_{\mathrm{cusp}}(X^*, \mathbf{C}).$$

This decomposition has the following properties:

(1) The canonical mapping

$$\zeta^* : H^\cdot(X^*, \mathbf{C}) \to H^\cdot(\partial X^*, \mathbf{C})$$

defines an isomorphism of $H_{\mathrm{inf}}(X_0, \mathbf{C})$ onto the image of ζ^*.

(2) $H_{\mathrm{univ}}(X_0, \mathbf{C})$ is a subring, generated by the cohomology-classes attached to the universal harmonic forms

$$\frac{dx_\nu \wedge dy_\nu}{y_\nu^2}, \ 1 < \nu < n.$$

(3) The cohomology classes in $H^\cdot_{\mathrm{cusp}}(X_0, \mathbf{C})$ can be represented by harmonic cusp-forms (which are rapidly decreasing at infinity).

The image of ζ^* can be represented by means of the theory of Eisenstein series. One has

$$H^{\nu}_{\rm inf}(X_0, \mathbf{C}) = 0 \quad \text{for } 1 < \nu < n-1.$$

For the subspace of cusp forms $H^{\bullet}_{\rm cusp}(X_0, \mathbf{C})$ one has a Hodge decomposition

$$H^{r}_{\rm cusp} = \bigoplus_{p+q=r} H^{p,q}_{\rm cusp}.$$

This part of the theory coincides with the investigations of Matsushima and Shimura [3] who treated the case of a compact quotient D/Γ. One has

$$H^{\nu}_{\rm cusp}(X_0, \mathbf{C}) = 0 \text{ for } \nu \neq n \text{ and therefore}$$

$$H^{2}(D/\Gamma, \mathbf{C}) = H^{2}_{\rm univ}(X_0, \mathbf{C}) \; (n > 3).$$

A basis of this vector space is represented by the harmonic forms

$$\frac{dx_\nu \wedge dy_\nu}{y_\nu^2} \quad 1 < \nu < n.$$

Remark. The dimension of $H^{p,q}_{\rm cusp}$ can be calculated explicitly using the methods of [1]. One obtains an expression in terms of Shimizu's rank polynomials.

3. Line bundles on the standard compactification D/Γ. Every Γ-invariant divisor $\mathscr{D}$ on $H^n (n > 3)$ can be represented by a modular form of a certain weight $r = (r_1, \ldots, r_n)$ according to Theorem 2.

Finally we investigate the problem in which case the weight r satisfies the condition

$$r_1 = \ldots = r_n.$$

Firstly, $\mathscr{D}$ defines a divisor on $X_0 = H^n/\Gamma$ which can be continued to a divisor on the standard compactification $X = D/\Gamma$ due to a well-known theorem of Remmert.

Then, we call $\mathscr{D}$ a Cartier divisor, if the associated divisorial sheaf on X is a line bundle, i.e. for each point $x \in X$ (even if it is a cusp) exists an open neighbourhood U and a meromorphic function $f: U \to \mathbf{C}$ which represents $\mathscr{D}$:

$$(f) = \mathscr{D}/U.$$

At the regular points, this is automatically the case.

THEOREM 6. *In the case of $n > 3$ the following two conditions are equivalent for a Γ-invariant divisor:*

i) *The divisor $\mathcal{D}$ is defined by a meromorphic modular form of the type*

$$f(Mz) = v(M)\left[\prod_{\nu=1}^{n}(c_\nu z_\nu + d_\nu)^2\right]^r f(z) \quad \text{for } M \in \Gamma$$
$$(|v(M)| = 1;\ r \in \mathbf{Q}).$$

ii) *A suitable multiple $k\mathcal{D}$, $k \in \mathbf{Z}$, of $\mathcal{D}$ is a Cartier divisor.*

PROOF. The condition ii) is only relevant in the case of cusps. One has to observe that in H^n/Γ only a finite number of quotient singularities occur.

Firstly we show i) $\Rightarrow$ ii).

We may assume that r is integer because we can replace f by a suitable power. In that case v has to be an abelian character.

This character has — owing to amendment 2 of Theorem 1 — finite order. Therefore we may assume $v = 1$.

Let ∞ be a cusp of Γ.

The form f is invariant by the affine substitutions of Γ

$$z \to \epsilon z + \alpha$$

and therefore induces a meromorphic function in a neighbourhood of the cusp, which obviously represents the divisor $\mathcal{D}$.

ii) $\Rightarrow$ i):

We may assume that $\mathcal{D}$ is a Cartier divisor. This means for the cusp: There exists a meromorphic function

$$g: U_C \to \mathbf{C};\ U_C = \{z \in H^n;\ N(\operatorname{Im} z) > C\}$$

with the properties

a) g is Γ-invariant,

b) $(g) = \mathcal{D}$ in U_C.

The function $h = \frac{f}{g}$ is holomorphic and without zeros. The transformation law

$$h(\epsilon z + \alpha) = v \begin{bmatrix} \epsilon^{1/2} & \epsilon^{-1/2}\alpha \\ 0 & \epsilon^{-1/2} \end{bmatrix} \prod_{\nu=1}^{n} \epsilon_\nu^{r_\nu} h(z)$$

is valid. By means of a variant of Koecher's principle one sees, that the limit value

$$\lim_{N(\operatorname{Im} z) \to \infty} h(z) = C$$

exists and is finite.

The same argument also holds for the function $\frac{1}{h}$ instead of h. Therefore C has to be different from zero. It follows that

$$v \begin{bmatrix} \epsilon^{1/2} & \epsilon^{-1/2}\alpha \\ 0 & \epsilon^{-1/2} \end{bmatrix} \prod_{\nu=1}^{n} \epsilon_\nu^{r_\nu} = 1$$

and therefore

$$r_1 = \dots = r_n.$$

Finally we mention an interesting application without supplying the proof.

CONCLUSION TO THEOREM 4. *In case of $n > 3$ we have*

$$\operatorname{Pic} X = \mathbf{Z},\ X = \overline{H^n/\Gamma}\ (\textit{standard-compactification}).$$

We also have got some information about a generator of Pic X. Choose a natural number e, such that

$$\gamma^e = id \qquad \text{for } \gamma \in \Gamma.$$

Then $\mathscr{K}^e(\mathscr{K} =$ canonical divisor) defines a line-bundle on X.

This bundle generates a subgroup of finite index in Pic X

$$N = [\operatorname{Pic} X : \{\mathscr{K}^\nu, \nu \equiv 0 \bmod e\}].$$

It is possible to calculate the Chern-class of a generator of Pic X

$$c(\mathscr{K}^{e/N}) = \frac{1}{2\pi i}\,\frac{e}{N}\sum_{\nu=1}^{n}\frac{dz_\nu \wedge d\bar{z}_\nu}{y_\nu^2}.$$

This defines in fact a cohomology class on X.

We now make use of the fact that Chern-classes are always integral, *i.e.*

$$\int_\gamma c(\mathscr{K}^{e/N}) \in \mathbf{Z},$$

where γ is a two-dimensional cycle on X. Such cycles can be constructed by means of certain specializations. The simplest case is

$$z_1 = \dots = z_n.$$

If one carries out the integration, one obtains conditions for N.

Example. If Γ is the full Hilbert-modular group, then

$$\int_{z_1=\dots=z_n} c(\mathscr{K}^{e/N}) = \frac{en}{3N} \in \mathbf{Z}.$$

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