AUTOMORPHY FACTORS OF HILBERT’S MODULAR GROUP

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INTRODUCTION. Let $\Gamma$ be a group of analytic automorphisms of a domain $D \subset \mathbb{C}^n$. By an automorphy factor of $\Gamma$ we understand a family of functions $I(z, \gamma)$, $z \in D$, $\gamma \in \Gamma$ holomorphic on $D$ and without zeros, which satisfy the condition

$$I(z, \gamma \gamma') = I(z, \gamma) I(\gamma z, \gamma').$$

The most-occurring factors are the following ones

1) \textit{The trivial factors}

$$I(z, \gamma) = \frac{h(\gamma z)}{h(z)}.$$

Here $h$ is a holomorphic function on $D$ without zeros.

2) \textit{The powers of the complex functional determinant} (Jacobian).

3) \textit{The abelian characters} $v$ of $\Gamma$

$$I(z, \gamma) = v(\gamma).$$

The determination of all automorphy factors belonging to a \textit{discontinuous} group is a difficult problem in general. It is roughly equivalent to the calculation of

$$\text{Pic } D/\Gamma = \text{group of analytic line bundles on } D/\Gamma.$$ 

More precisely, if $\Gamma$ operates without fixed points, we have

$$\text{Pic } D/\Gamma = \frac{\text{group of automorphy factors}}{\text{subgroup of trivial factors}}.$$

There is a well-known isomorphism

$$\text{Pic } D/\Gamma = H^1(D/\Gamma, \mathcal{O}^*)$$

($\mathcal{O}$ = sheaf of automorphic functions, $\mathcal{O}^*$ = sheaf of invertible automorphic functions).
By means of the exact sequence
\[ 0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0, \]
we reduce the original problem to the calculation of

a) the singular cohomology of \( D/\Gamma \)

b) the analytical cohomology \( H^*(D/\Gamma, \mathcal{O}) \).

This program could be carried out almost completely for the domain
\[ D = H^* = H \times \ldots \times H, \]
\( H \) the usual upper half-plane.

Matsushima and Shimura succeeded in calculating those groups in case of a compact quotient by means of the Hodge theory [3]. As for the non-compact quotients \( D/\Gamma \) (Hilbert's modular groups) similar complete results have been found.

a) The singular cohomology was investigated by G. Harder [2].

Let us give a very brief indication of the specific problems arising in the non-compact case.

By "cutting off cusps" of \( D/\Gamma \) one gets a manifold with boundary.

There is a natural mapping from the cohomology of the whole space \( D/\Gamma \) to the (well-known) cohomology of the boundary. In the mentioned paper, Harder determined the image and the kernel of this map. His detailed study of this problem leads into the theory of non-analytic modular forms, especially into the theory of Eisenstein series.

b) The analytical cohomology was determined in [1].

To overcome the discrepancy between the standard compactification of \( D/\Gamma \) and a non-singular model, we had to carry out a thorough investigation of the algebraic nature of the cusps. But it was not necessary to get a concrete resolution of the cusps.

1. The main result. In the following let \( \Gamma \) be a group of simultaneously fractional linear substitutions

\[ M(z_1, \ldots, z_n) = \left( \frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n} \right) \]
of the half space

\[ H^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; \ \text{Im} \ z_v > 0 \ \text{for} \ 1 \leq v \leq n\}. \]

We are only interested in the case in which \( \Gamma \) is commensurable with Hilbert's modular group of a totally real number field. We define the complex power

\[ a^b = e^{b \log a}, \quad a \neq 0 \]

by the principal branch of the logarithm.

**Theorem 1.** In case of \( n > 3 \)† the only automorphy factors of \( \Gamma \) are:

\[ I(z, M) = v(M) \prod_{v=1}^{n} \left( c_v z_v + d_v \right)^{2r_v} \frac{h(Mz)}{h(z)} \]

where

a) \( r = (r_1, \ldots, r_n) \) is a vector of rational numbers;

b) \( \{v(M)\}_{M \in \Gamma} \) is a system of complex numbers of absolute value one;

c) \( h \) is a holomorphic invertible function on \( H^n \).

This factorisation of \( I \) is unique.

By a system of multipliers of weight \( r = (r_1, \ldots, r_n) \) we understand a family \( \{v(M)\}_{M \in \Gamma} \) of complex numbers of absolute value one, such that

\[ I(z, M) = v(M) \prod_{v=1}^{n} \left( c_v z_v + d_v \right)^{2r_v} \]

is a factor of automorphy.

**Amendment to Theorem 1.**

1) The group of abelian characters of \( \Gamma \) is finite.‡

2) If \( r = (r_1, \ldots, r_n) \) is the weight of a multiplier system, the components \( r_v \) have to be rational and their denominators are bounded (by a number which may depend on the group).

†The method used for the proof is valid also in case of \( n < 3 \). But one has to carry out some separate investigations because in this case the first cohomology groups are not trivial.

‡A more general result has been proved by Serre [4].
We now discuss an application of the main theorem.

A meromorphic modular form with respect to $\Gamma$ is a meromorphic function on $H^*$ satisfying the functional equations:

$$f(Mz) = v(M) \prod_{r=1}^{n} (c(z_r + d_z)^{2r} f(z) \text{ for } M \in \Gamma.$$ 

We call $r = (r_1, \ldots, r_n)$ the weight and $v(M)$ the multiplier system of $f$.

We are interested in the zeros and poles of $f$ which we describe by a divisor $(f)$ as usual.

By a divisor we understand a formal and locally finite sum

$$D = \sum_{Y} n_Y Y, \quad n_Y \in \mathbb{Z}$$

the summation being taken over irreducible closed analytic sub-varieties of codimension one.

**Theorem 2.** Let $D$ be a $\Gamma$-invariant divisor on $H^*, n > 3$. There exists a meromorphic modular form $f$ with the property

$$D = (f).$$

**Proof.** The space $H^*$ is a topologically trivial Stein-space. Therefore we can find a meromorphic function $g$ on $H^*$ with

$$D = (g).$$

The function

$$I(z, M) = \frac{g(Mz)}{g(z)}, \quad M \in \Gamma$$

is without poles and zeros because $D$ is $\Gamma$-invariant. We therefore can apply Theorem 1. Put

$$f = \frac{g}{h}.$$

2. Sketch of proof. The group $\Gamma$ operates in a natural way on the multiplicative group $H^0(D, \mathcal{O}^*)$ of holomorphic invertible functions on $D = H^*$. 
The automorphy factors are nothing else but the 1-cocycles with regard to the standard complex and the trivial factors $h(\gamma z)/h(z)$ are the 1-cochain factors, i.e.

$$H^1(\Gamma, H^0(D, \mathcal{O}^*)) = \frac{\text{group of automorphy factors}}{\text{subgroup of trivial factors}}$$

Theorem 1 may thus be formulated as follows

**Theorem 3.** The group $H^1(\Gamma, H^0(D, \mathcal{O}^*))$ is finitely generated and of free rank $n$.

We now want to pass on a subgroup $\Gamma_0 \subset \Gamma$ of finite index in order to eliminate the elements of finite order. Let $\Gamma_0 \subset \Gamma$ be a normal subgroup of finite index. Putting

$$A = H^0(D, \mathcal{O}^*)$$

we obtain by means of the Hochschild-Serre sequence

$$0 \to H^1(\Gamma/\Gamma_0, C^*) \to H^1(\Gamma, A) \to H^1(\Gamma_0, A)^{\Gamma/\Gamma_0} \to H^2(\Gamma/\Gamma_0, C^*).$$

(Since every holomorphic modular function is constant, we have $A^{\Gamma} = C^*$.)

The groups $H^*(\Gamma/\Gamma_0, C^*)$ are finite. This is proved by means of the sequence

$$0 \to Z \to C \to C^* \to 0.$$

In general, the cohomology groups of a finite group which acts trivially on $Z$, are finite.

We therefore can assume without loss of generality:

The group $\Gamma$ is a congruence-subgroup of Hilbert's modular group without torsion.

In the case at hand it is easy to be seen

$$H^1(\Gamma, H^0(D, \mathcal{O}^*)) = H^1(D/\Gamma, \mathcal{O}^*),$$

i.e. there is a one-to-one correspondence between the factor classes and the classes of isomorphic analytical line bundles on $X_0 = D/\Gamma$. 

We now treat the group

\[ \text{Pic } X_\circ = H^1(X_\circ, \mathcal{O}^*), \quad X_\circ = D/\Gamma \]

by means of the sequence

\[ 0 \to \mathbb{Z} \to \mathcal{O} \to \exp \mathcal{O}^* \to 0. \]

Hereby \( \mathcal{O} \) is the sheaf of holomorphic functions on \( X_\circ \). From the long cohomology sequence results

\[ H^1(X_\circ, \mathcal{O}) \to \text{Pic } X_\circ \to H^0(X_\circ, \mathcal{O}). \]

We thus have to calculate the groups \( H^1(X_\circ, \mathcal{O}) \) and \( H^0(X_\circ, \mathcal{O}) \).

**Theorem 4.** The groups \( H^r(X_\circ, \mathcal{O}) \) vanish for \( 1 \leq r < n - 2 \).

**Proof.** Let \( S \) be the finite set of cusp classes of \( \Gamma \) and

\[ X = X_\circ \cup S \]

the standard compactification of \( X_\circ = D/\Gamma \). There is a long exact sequence, which combines the cohomology with supports in \( S \) with the usual cohomology of sheaves

\[ H^r_S(X, \mathcal{O}) \to H^r(X, \mathcal{O}) \to H^r(X_\circ, \mathcal{O}) \to H^{r+1}_S(X, \mathcal{O}) \to H^{r+1}(X, \mathcal{O}). \]

From my paper [1] the result (Theorem 7.1)

\[ H^r_S(X, \mathcal{O}) \cong H^r(X, \mathcal{O}) \quad \text{for } 1 \leq r < n \]

is taken.

An analysis of the proof shows that this isomorphism is induced by the natural mapping

\[ H^r_S(X, \mathcal{O}) \to H^r(X, \mathcal{O}). \]

In case of \( n > 3 \) we now obtain the exact sequence

\[ 0 \to \text{Pic } X_\circ \to H^1(X_\circ, \mathcal{O}). \]

Obviously the free rank of \( \text{Pic } X_\circ \) is not smaller than \( n \) because the automorphy factors

\[ I_r(z, M) = (c, z_r + d_r)^n \quad (1 < r < n) \]

are independent of each other.
Therefore Theorem 3 has been proved if one knows that $H^2(X_\varnothing, \mathbb{Z})$ is of free rank $n$. That means

**Theorem 5.** In case of $n > 3$ we have

$$\dim_C H^2(X_\varnothing, C) = n, \ X_\varnothing = D/\Gamma.$$ 

**Proof.** We derive the calculation of the 2nd Betti number of $X_\varnothing$ from Harder's investigations on the singular cohomology of $X_\varnothing = D/\Gamma$ [2].

This will be explained briefly in the following.

By cutting off cusps we obtain a bounded manifold $X^*$ which is homotopically equivalent to $X_\varnothing$. (The boundary component at the cusp is given by

$$\prod_{r=1}^n \text{Im } z_r = C; \ C \gg 0.$$ 

In the paper quoted above, Harder gives a decomposition of the singular cohomology of $X_\varnothing$:

$$H^*(X_\varnothing, C) = H^*(X^*, C)$$

$$= H_{\text{inf}}(X^*, C) \oplus H_{\text{univ}}(X^*, C) \oplus H_{\text{cusp}}(X^*, C).$$

This decomposition has the following properties:

1. The canonical mapping

$$\zeta^*: H^*(X^*, C) \to H^*(\partial X^*, C)$$

defines an isomorphism of $H_{\text{inf}}(X_\varnothing, C)$ onto the image of $\zeta^*$.

2. $H_{\text{univ}}(X_\varnothing, C)$ is a subring, generated by the cohomology-classes attached to the universal harmonic forms

$$\frac{dx_r \wedge dy_r}{y_r^2}, \ 1 < r < n.$$ 

3. The cohomology classes in $H^*_{\text{cusp}}(X_\varnothing, C)$ can be represented by harmonic cusp-forms (which are rapidly decreasing at infinity).

The image of $\zeta^*$ can be represented by means of the theory of Eisenstein series. One has
For the subspace of cusp forms $H^\cdot_{\text{cusp}}(X_0, \mathbb{C})$ one has a Hodge decomposition

$$H^\cdot_{\text{cusp}} = \bigoplus_{p+q=r} H^p_{\text{cusp}}.$$ 

This part of the theory coincides with the investigations of Matsushima and Shimura [3] who treated the case of a compact quotient $D/\Gamma$. One has

$$H^\cdot_{\text{cusp}}(X_0, \mathbb{C}) = 0 \text{ for } \nu \neq n \text{ and therefore }$$

$$H^\bullet(D/\Gamma, \mathbb{C}) = H^{2\nu}_{\text{uni}}(X_0, \mathbb{C})(n > 3).$$

A basis of this vector space is represented by the harmonic forms

$$\frac{dx_\nu \wedge dy_\nu}{y^3}, \quad 1 < \nu < n.$$ 

**Remark.** The dimension of $H^p_{\text{cusp}}$ can be calculated explicitly using the methods of [1]. One obtains an expression in terms of Shimizu's rank polynomials.

3. **Line bundles on the standard compactification $D/\Gamma$.** Every $\Gamma$-invariant divisor $\mathcal{D}$ on $H^\bullet(n > 3)$ can be represented by a modular form of a certain weight $r = (r_1, \ldots, r_n)$ according to Theorem 2.

Finally we investigate the problem in which case the weight $r$ satisfies the condition

$$r_1 = \ldots = r_n.$$ 

Firstly, $\mathcal{D}$ defines a divisor on $X_0 = \mathcal{H}^n/\Gamma$ which can be continued to a divisor on the standard compactification $X = D/\Gamma$ due to a well-known theorem of Remmert.

Then, we call $\mathcal{D}$ a Cartier divisor, if the associated divisorial sheaf on $X$ is a line bundle, i.e. for each point $x \in X$ (even if it is a cusp) exists an open neighborhood $U$ and a meromorphic function $f: U \to \mathbb{C}$ which represents $\mathcal{D}$:

$$(f) = \mathcal{D}/U.$$ 

At the regular points, this is automatically the case.
Theorem 6. In the case of \( n > 3 \) the following two conditions are equivalent for a \( \Gamma \)-invariant divisor:

i) The divisor \( D \) is defined by a meromorphic modular form of the type

\[
f(Mz) = v(M) \left[ \prod_{r=1}^{n} (c_r z + d_r)^{2r} \right] f(z) \quad \text{for } M \in \Gamma\]

\((|v(M)| = 1; r \in \mathbb{Q})\).

ii) A suitable multiple \( kD \), \( k \in \mathbb{Z} \), of \( D \) is a Cartier divisor.

Proof. The condition ii) is only relevant in the case of cusps. One has to observe that in \( H^n/\Gamma \) only a finite number of quotient singularities occur.

Firstly we show i) \( \Rightarrow \) ii).

We may assume that \( r \) is integer because we can replace \( f \) by a suitable power. In that case \( v \) has to be an abelian character.

This character has — owing to amendment 2 of Theorem 1 — finite order. Therefore we may assume \( v = 1 \).

Let \( \alpha \) be a cusp of \( \Gamma \).

The form \( f \) is invariant by the affine substitutions of \( \Gamma \)

\[ z \to cz + \alpha \]

and therefore induces a meromorphic function in a neighbourhood of the cusp, which obviously represents the divisor \( D \).

ii) \( \Rightarrow \) i):

We may assume that \( D \) is a Cartier divisor. This means for the cusp: There exists a meromorphic function

\[ g : U_0 \to \mathbb{C}; U_0 = \{ z \in H^n; N(\text{Im} z) > C \} \]

with the properties

a) \( g \) is \( \Gamma \)-invariant,

b) \( (g) = D \) in \( U_0 \).
The function $h = \frac{f}{g}$ is holomorphic and without zeros. The transformation law

$$h(\epsilon z + \alpha) = v \left[ \begin{array}{cc} \epsilon^{1/2} & \epsilon^{-1/2} \\ 0 & \epsilon^{-1/2} \end{array} \right] \prod_{\nu=1}^{n} \epsilon_{\nu}^{-r} h(z)$$

is valid. By means of a variant of Koecher's principle one sees, that the limit value

$$\lim_{N(\Im z) \to \infty} h(z) = C$$

exists and is finite.

The same argument also holds for the function $\frac{1}{h}$ instead of $h$.

Therefore $C$ has to be different from zero. It follows that

$$v \left[ \begin{array}{cc} \epsilon^{1/2} & \epsilon^{-1/2} \\ 0 & \epsilon^{-1/2} \end{array} \right] \prod_{\nu=1}^{n} \epsilon_{\nu}^{-r} = 1$$

and therefore

$$r_1 = ... = r_n.$$

Finally we mention an interesting application without supplying the proof.

**Conclusion to Theorem 4.** In case of $n > 3$ we have

$$\text{Pic } X = Z, \quad X = H^n/\Gamma \text{ (standard-compactification)}.$$

We also have got some information about a generator of $\text{Pic } X$. Choose a natural number $\epsilon$, such that

$$\gamma^\epsilon = \text{id} \quad \text{for } \gamma \in \Gamma.$$

Then $X^\epsilon (X = \text{canonical divisor})$ defines a line-bundle on $X$.

This bundle generates a subgroup of finite index in $\text{Pic } X$

$$N = [\text{Pic } X : \{ X^\nu, \nu \equiv 0 \mod \epsilon \}].$$

It is possible to calculate the Chern-class of a generator of $\text{Pic } X$
\[ c(X^{\sigma/N}) = \frac{1}{2\pi i} \frac{1}{N} \sum_{r=1}^{k} \frac{dz_r \wedge d\bar{z}_r}{y_r^2}. \]

This defines in fact a cohomology class on \( X \).

We now make use of the fact that Chern-classes are always integral, i.e.

\[ \int_{\gamma} c(X^{\sigma/N}) \in \mathbb{Z}, \]

where \( \gamma \) is a two-dimensional cycle on \( X \). Such cycles can be constructed by means of certain specializations. The simplest case is

\[ z_1 = \ldots = z_n. \]

If one carries out the integration, one obtains conditions for \( N \).

**Example.** If \( \Gamma \) is the full Hilbert-modular group, then

\[ \int_{z_1 = \ldots = z_n} c(X^{\sigma/N}) = \frac{en}{3N} \in \mathbb{Z}. \]

**References**

2. G. Harder: On the cohomology of \( \text{Sl} (2, \mathbb{O}) \) (preprint).