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THE TRANSFORMATION FORMALISM OF VECTOR VALUED THETA FUNCTIONS WITH RESPECT TO THE SIEGEL MODULAR GROUP

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(Dedicated to the memory of Srinivasa Ramanujan)

I The results

We denote by $\mathbf{H}_n$ the Siegel upper half plane which consists of all symmetric $n \times n$-matrices with positive definite imaginary part by $Sp_{2n}(\mathbf{R})$ the real symplectic group which acts on $\mathbf{H}_n$ by the usual formula

$$Z \longmapsto MZ = (AZ+B)(CZ+D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1}$$

The Eichler imbedding

The Kronecker product of two matrices

$$A = A^{(m,n)}, \ B = B^{(r,s)}$$

is the (mr, ns)-matrix defined by

$$A \otimes B = \begin{pmatrix} Ab_{11} & \dots & Ab_{1s} \\ \vdots & & \vdots \\ Ab_{r1} & \dots & Ab_{rs} \end{pmatrix}. \tag{2}$$

This product is associative and bilinear. Furthermore the following formulae are easily verified (under obvious assumptions on the size of the matrices)

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

$$\det(A \otimes B) = (\det A)^n \cdot (\det B)^m \quad (A = A^{(m)},\ B = B^{(n)})$$

$$\sigma(A \otimes B) = \sigma(A)\,\sigma(B) \qquad (\sigma \text{ denotes the trace})$$

$$(A \otimes B)' = A' \otimes B'.$$

If $A = A^{(m)}$ and $B = B^{(n)}$ are symmetric matrices, one has

$$(A \otimes B)[g] = \sigma(A[G]B) \tag{3}$$

where

$$G = G^{(m.n)} = (g_1, \ldots, g_n)$$

denotes an $m \times n$-matrix and g the column vector

$$g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}.$$

We use the usual notation

$$A[G] = G' AG. \tag{4}$$

If $S = S^{(r)}$, $Y = Y^{(n)}$ are real symmetric positive definite matrices then $S \otimes Y$ is also symmetric and positive definite. We especially obtain an imbedding of Siegel half planes

$$\begin{aligned} \mathbf{H}_n &\to \mathbf{H}_{nr} \\ Z &\longmapsto S \otimes Z \end{aligned} \tag{5}$$

which is compatible with the action of the symplectic groups in the following sense. The mapping

$$\begin{aligned} S_{p_{2n}}(\mathbf{R}) &\to S_{p_{2nr}}(\mathbf{R}) \\ M &\longmapsto M^S \\ \| \quad & \quad\quad \| \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \quad \begin{pmatrix} E \otimes A & S \otimes B \\ S^{-1} \otimes C & E \otimes D \end{pmatrix} \end{aligned} \tag{6}$$

is an injective homomorphism with the property

$$M^S(S \otimes Z) = S \otimes (MZ). \tag{7}$$

We are now going to define certain important subgroups of the symplectic group.

1) The Siegel modular group

$$\Gamma_n = S_{p_{2n}}(\mathbf{Z}).$$

2) The theta group

$$\Gamma_{n,\vartheta} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n,\ AB' \text{ and } CD' \text{ have even diagonal elements} \right\}. \tag{8}$$

3) The main congruence group

$$\Gamma_n[q] = \ker\left(S_{p_{2n}}(\mathbb{Z}) \to S_{p_{2n}}(\mathbb{Z}/q\mathbb{Z})\right). \tag{9}$$

4) Igusa's group [6]

$$\Gamma_n[q, 2q] = \{M \in \Gamma_n[q], \text{ the diagonals of } AB'/q \text{ and } CD'/q \text{ are even}\}. \tag{10}$$

Obviously

$$\Gamma_n = \Gamma_n[1],\ \Gamma_{n,\vartheta} = \Gamma_n[1, 2].$$

5) The generalized Hecke groups

$$\Gamma_{n,0}[q] = \{M \in \Gamma_n,\ C \equiv 0 \bmod q\}. \tag{11}$$

6) For a symmetric positive definite rational matrix $S = S^{(r)}$ we define

$$\Gamma_n(S) = \{M \in Sp_{2n}(\mathbb{R}),\ M^S \in \Gamma_{n,\vartheta}\}. \tag{12}$$

Because S is rational this group is obviously a congruence group, i.e. it contains some main congruence group $\Gamma_n[q]$ as subgroup of finite index.

We recall that S is called even, if S is integral and if the elements in the diagonal are even, equivalently

$$S[g] \equiv 0 \bmod 2 \text{ for } g \in \mathbb{Z}^r.$$

1.1 *Remark:* Assume that S is even and that q is a natural number such that qS^{-1} is even too. Then $\Gamma_n(S)$ contains the group $\Gamma_{n,0}[q]$ (11).

1.2 *Remark:* Let q be a natural number such that

$$qS \text{ and } qS^{-1}$$

both are integral. Then $\Gamma_n(S)$ contains Igusa's group $\Gamma_n[q, 2q]$ (10).

The proofs are trivial. □

Theta series

We consider theta series of the type

$$\theta_{S,P}(Z;U,V) = \sum_{G=G^{(r,n)} \text{ integral}} PS^{1/2}(G+U/2))\exp(\pi i\sigma\{S[G+U/2]Z+V'G\}). \qquad (13)$$

Hereby r and n are natural numbers and

1) $S=S^{(r)}$ is a positive definite real matrix,
2) $U=U^{(r,n)}$, $V=V^{(r,n)}$ are (complex) matrices, the so-called **characteristics**,
3) $P:\mathbb{C}^{(r,n)}\to \mathbf{Z}$, $\dim_{\mathbb{C}} \mathbf{Z}<\infty$, is a polynomial on the space of $r\times n$-matrices with values in a finite-dimensional complex vector space,
4) Z varies in the Siegel half-plane $\mathbf{H}_n$.

We are mainly interested in the case where P is a harmonic form with respect to a given rational representation

$$\rho_0: Gl(n,\mathbb{C})\to Gl(\mathbf{Z}).$$

1.3 Definition: *A harmonic form with respect to a rational representation*

$$\rho_0: Gl(n,\mathbb{C})\to Gl(\mathbf{Z})$$

is a polynomial

$$P:\mathbb{C}^{(n,m)}\to \mathbf{Z}$$

with the following two properties

a) $$P(XA)=\rho_0(A')P(X),\ A\in Gl(n,\mathbb{C}).$$

b) $$\Delta P = 0 \qquad \left(\Delta=\sum_{\substack{1\leqslant i\leqslant r\\ 1\leqslant k\leqslant n}} \frac{\partial^2}{(\partial x_{ik})^2}\right).$$

Remark: If P is a harmonic form, then all the functions

$$X\longmapsto P(XA),\ A=A^{(n)}$$

are harmonic. This implies that P is in fact **pluriharmonic**, i.e. it satisfies the system of differential equations

$$\sum_{j=1}^{r}\frac{\partial}{\partial x_{ji}}\frac{\partial}{\partial x_{jk}}P=0\ (1\leqslant i,k\leqslant n). \qquad (14)$$

Theta multiplier systems

Because $\mathbf{H}_n$ is convex there exists a unique holomorphic function

$$h:\mathbf{H}_n \to \mathbb{C}$$

with the properties

a) $$h(Z)^2 = \det(Z/\mathrm{i}),$$

b) $$h(\mathrm{i}Y) = +\sqrt{\det Y}\cdot$$

One usually writes

$$h(Z) = \det(Z/\mathrm{i})^{1/2} = \sqrt{\det(Z/\mathrm{i})}.$$

(But this notation is not quite correct, because it might happen that there are points Z, W in $\mathbf{H}_n$ with the same determinant but different $h(Z)$, $h(W)$. Let M be a real symplectic matrix with invertible D. We define

$$I(M, Z) = \sqrt{\det D}\; h(Z)\, h(-Z^{-1} - D^{-1}C) \tag{15}$$

where the square root is taken on the positive real or imaginary axis. Of course

$$I(M, Z)^2 = \det(CZ + D)$$

and one may write (not quite correctly)

$$I(M, Z) = \det(CZ + D)^{1/2}. \tag{16}$$

If r is an even number, then

$$I(M, Z)^r = \det(CZ + D)^{r/2}$$

is of course defined without any ambiguity and also without the restriction $\det D \neq 0$. Let now $S = S^{(r)}$ be a positive (real) matrix. We define for $M \in \Gamma_n(S)$, $\det D \neq 0$, the multiplier

$$\epsilon_S(M) = \sqrt{\det D^{-r}} \sum_{G \in \mathbb{Z}^{(r,\,n)}/\mathbb{Z}^{(r,\,n)}D'} \exp\left(\pi i \sigma\left(BD^{-1}S[G]\right)\right) \tag{17}$$

where again the square root of $\det D$ has to be taken on the positive real or imaginary axis. It is easy to verify that the terms of the above sum remain unchanged if one replaces

$$G \longmapsto G + XD', \quad X \in \mathbb{Z}^{(r,\,n)}.$$

Hence the sum is well-defined.

The main formula

1.4 THEOREM. *Let*

$$P: \mathbb{C}^{(r,n)} \to \mathbb{Z} \quad (\dim_{\mathbb{C}} \mathbb{Z} < \infty)$$

be a harmonic form with respect to the rational representation

$$\rho_0: Gl(n, \mathbb{C}) \to Gl(\mathbb{Z}).$$

The theta series (13)

$$\tilde{\theta}_{S,P}(Z: U, V) := \exp(\pi i \sigma(U'V)/4)\, \theta_{S,P}(Z; U, V) \tag{18}$$

satisfies for all $M \in \Gamma_n(S)$ (9), (12) *the transformation formula*

$$\tilde{\theta}_{S,P}(MZ; U, V) = \epsilon_S(M) \det(CZ+D)^{r/2} \rho_0(CZ+D)\, \tilde{\theta}_{S,P}(Z; \tilde{U}, \tilde{V}) \tag{19}$$

where

$$\tilde{U} = UA + S^{-1}VC, \; \tilde{V} = SUB + VD. \tag{20}$$

$\epsilon_S(M)$ *is a system of complex numbers of absolute value* 1 *which depends on the choice of a holomorphic square root of* $\det(CZ+D)$. *If* $\det D \neq 0$, *they are defined by the formulae* (15), (16), (17).

Assume that the quadratic form S and the characteristics U, V are rational. Then obviously there exists a rational number m such that the theta-function (18) remains unchanged under a substitution

$$U \longmapsto U + mX, \; V \longmapsto V + mY; \; X, Y \in \mathbb{Z}^{(r,n)}.$$

Formula (20) shows

$$\tilde{U} \equiv U, \; \tilde{V} \equiv V \bmod m$$

if M is contained in a suitable main congruence group. One hence obtains from 1.4:

1.5 COROLLARY. *If the quadratic form S and the characteristics U, V are rational, there exists a main congruence subgroup*

$$\Gamma_n[q] \subset \Gamma_n(S)$$

such that

$$f(Z) := \theta_{S,P}(Z; U, V)$$

satisfies

$$f(MZ) = \epsilon_S(M) \det (CZ + D)^{r/2} \rho_0(CZ + D) f(Z) \tag{21}$$

for all $M \in \Gamma[q]$.

Hence $f(Z)$ is a modular form of level q. Of course Theorem 1.4 gives precise information about possible levels q. For example:

1.6 COROLLARY. *Assume that S is even (that means integral with even diagonal) and that q is a natural number such that qS^{-1} is even too. Then the function*

$$f(Z) = \theta_{S,P}(Z; 0, 0)$$

satisfies for all $M \in \Gamma_{n,0}[q]$ (11)

$$f(MZ) = \epsilon_S(M) \det (CZ + D)^{r/2} \rho_0(CZ + D) f(Z).$$

The proof follows from 1.1 and 1.4. □

1.7 COROLLARY. *Assume that S is integral and that q is a natural number such that $q^2 S^{-1}$ is integral too. Let furthermore $V = V^{(r,n)}$ be an integral matrix such that $qS^{-1}V$* and *$S^{-1}[V]$ are both integral. Then the function*

$$f(Z) = \sum_{G \in \mathbb{Z}^{(r,n)}} P(S^{1/2}G) \exp\left(\frac{\pi i}{q} \sigma\{S[G]Z + 2V'G\}\right)$$

satisfies

$$f(MZ) = \epsilon_{S/q}(M) \det (CZ + D)^{r/2} \rho_0(CZ + D) f(Z) \tag{22}$$

for all $M \in \Gamma_n[q, 2q]$.

Proof. From 1.2 we obtain

$$\Gamma_n(S/q) \supset \Gamma_n[q, 2q].$$

Because of 1.4 we only have to show

$$\theta_{S/q,P}(Z; 0, 2V/q) = \theta_{S/q,P}(Z; 2S^{-1}VC, 2VD/q).$$

In the exponent of the general term of the second series occurs

$$S[G + S^{-1}VC].$$

By assumption $qS^{-1}V$ and C/q hence $S^{-1}VC$ are integral. The transformation of the summation variable

$$G \longmapsto G - S^{-1}VC$$

gives the desired identity, if one makes use of the fact that

$$(D - E)/q \text{ and } S^{-1}[V]$$

are integral. □

Some results about the theta multiplier systems

It is well-known and easy to prove that

$$\epsilon_S(M) = 1 \text{ for all } M \in \Gamma_n$$

if S is an even unimodular matrix [5].

Andrianov and Maloletkin proved in [1] the following result.

1.8 Proposition. *Assume that $S = S^{(r)}$ is a positive even matrix and that $q > 1$ is a natural number such that qS^{-1} is even, too. Assume furthermore*

$$r \equiv 0 \bmod 2.$$

Then

$$\epsilon_S(M) = \left(\frac{(-1)^{r/2} \det(S)}{\det D}\right) \tag{23}$$

for all $M \in \Gamma_{n,0}[q]$.

(Here $(\div)$ denotes the generalized Legendre symbol).

Stark gives in [9] a method to reduce the case of an odd r to the easier case of an even r. All secret of the multiplier systems $\epsilon_S(M)$ is contained in one multiplier system $v_\theta[M]$ which can be defined by

$$\theta(MZ) = v_\theta(M) \det(CZ + D)^{1/2}\theta(Z) \tag{24}$$

for all $M \in \Gamma_n[\theta]$ where $\theta(Z)$ is the simplest theta series, namely

$$\theta[Z] = \sum_{g \in \mathbb{Z}^n} e^{\pi i z[g]} \; (= \theta_{(1),1}(Z; 0, 0)). \tag{25}$$

Of course $v_\theta(M)$ depends on the choice of the root of $\det(CZ + D)$. In case of an invertible D we may use the agreements (15), (16) and obtain

$$v_\theta(M) = \sqrt{\det D} \sum_{g \in \mathbb{Z}^n/D\mathbb{Z}^n} \exp(\pi i \sigma(BD^{-1}[g])) \qquad (M \in \Gamma_{n,\theta}, \det D \neq 0) \tag{26}$$

In general one has

$$\epsilon_S(M) = v_\theta(M^S) \tag{27}$$

using the notations (6), (17), (26).

The computation of $v_\theta(M)$, $M \in \Gamma_{n,\theta}$, is very difficult and the results are still incomplete. Partial results about $v_\theta(M)$ can be found in Igusa's book [6] and in Styer's papers [10].

Historical note: The formulae are classical in case $n = 1$. In case $n > 1$ the main formula 1.4 and several consequences have been proved in the already mentioned paper of Andrianov and Maloletkin [1] for special harmonic forms, namely finite sums

$$P(X) = \sum_A \det(A'X)^k, \ A'A = 0.$$

$$(X = X^{(r,n)}, \ A = A^{(r,n)})$$

Maass observed that in case $k = 1$ the condition $A'A = 0$ is not necessary. But it is not clear how to generalize their method to the case of general (vector-valued) P. If one is only interested in the fact that theta-functions $\theta_{S,P}(Z; A, B)$ are for rational S, A, B modular forms of some level q (Corollary 1.5) and not in precise informations about possible $q - s$ and multiplier systems one can avoid the precise formula in 1.4. A direct simple proof for 1.5 can be found for example in Mumford's book [8].

II General coefficient functions

Assume that

$$P: \mathbb{C}^{(r,n)} \to \mathbf{Z} \quad (\dim_{\mathbb{C}} \mathbf{Z} < \infty)$$

is a harmonic form with respect to some rational representation

$$\rho_0: Gl(n, \mathbb{C}) \to Gl(\mathbf{Z}).$$

We define a polynomial

$$P_0: \mathbb{C}^{rn} \to \mathbf{Z}$$

by

$$P_0(g) = P(S^{1/2} G), \tag{28}$$

where g is the corresponding big column vector coming from G, i.e.

$$G = (g_1, \ldots, g_n), \ g' = (g_1', \ldots, g_n'). \tag{29}$$

We may consider the theta series on $\mathbf{H}_{rn}$

$$f(Z) := \sum P_0(g)\, e^{\pi i Z[g]} = \theta_{(1),\, Q_0}(Z; 0, 0) \tag{30}$$

where

$$Q_0(g) = P_0(g'),\ g \in \mathbb{C}^{(1,\, rn)}.$$

One obviously has

$$f(S \otimes Z) = \theta_{S,\, P}(Z; 0, 0). \tag{31}$$

The idea of the proof of transformation-formulae of $\theta_{S,P}$ under $\Gamma_n(S)$ is to reduce it to that of f under the big group $\Gamma_{rn,\vartheta}$. In case of a trivial coefficient $P \equiv 1$ this method has been used by Eichler [2] (in case $n = 1$) and by Andrianov-Maloletkin in the already mentioned paper [1]. In the case of general P a certain difficulty arises, namely: Even if P is a harmonic form, the function P_0 needs not to be a form (with respect to some representation of $Gl(rn, \mathbb{C})$. We hence are forced to consider also more general coefficient functions. A good class of coefficient functions which is stable under the transformations which we need is given by the following

2.1 DEFINITION. *A coefficient function is a mapping*

$$P: \mathbb{C}^n \times \mathbf{H}_n \to \mathbb{C},$$

which can be written as

$$P(g, Z) = \sum_{j=1}^{k} P_j(g)\, A_j(Z)$$

where $P_1, \ldots, P_k$ are polynomials on $\mathbb{C}^n$ and $A_1, \ldots, A_k$ holomorphic functions on $\mathbf{H}_n$.

We now consider theta series of the type

$$\theta_P[m](Z) = \sum_{g \in \mathbb{Z}^n} P(g + a/2, Z) \exp(\pi i \{Z[g + a/2] + b'g\}) \tag{32}$$

where

$$m = \begin{bmatrix} a \\ b \end{bmatrix};\ a, b \in \mathbb{C}^n \quad \text{(column vector)}$$

is the so-called characteristic and

$$P(g) = P(g, Z)$$

a coefficient function in the sense of 2.1.

It will be clear very soon why we are forced to admit a Z-dependency in the coefficient function and why we are forced to admit coefficient functions which are not harmonic (as functions of g).

2.2 DEFINITION. *The Gauss-transform of a polynomial P on $\mathbf{C}^n$ is*

$$P^*(x) = \int_{\mathbf{R}^n} P(x+u)\, e^{-\pi u'u}\, du.$$

Obviously P^* again is a polynomial.

2.3 LEMMA *The Gauss-transform of a polynomial P is*

$$P^*(x) = e^{\Delta/4\pi} P(x) = \sum \frac{1}{j!}\left(\frac{\Delta}{4\pi}\right)^j P(x) \tag{33}$$

(This sum is finite). Here

$$\Delta = \sum \partial^2/\partial^2 x_\nu$$

is the usual Laplacian.

COROLLARY. *The Gauss-transformation is invertible,*

$$P = e^{-\Delta/4\pi} P^*. \tag{34}$$

A proof of this well-known lemma can be found in [3], III, 3.2. □

Before we may formulate the general theta involution formula we need another ingredient, namely a holomorphic matrix valued square root on the Siegel upper half-plane.

2.4 LEMMA *There exists a unique holomorphic mapping*

$$l: \mathbf{H}_n \to \mathbf{Z}_n$$

($\mathbf{Z}_n$ = vectorspace of all symmetric $n \times n$-matrices) with the properties

a) $e^{l(Z)} = Z/\mathrm{i}$,

b) $l(\mathrm{i}Y)$ *is real if* $Y > 0$.

Notations

$$\log(Z/i) = l(Z),$$
$$(Z/i)^{1/2} = \exp\left(\tfrac{1}{2}\log(Z/i)\right) \tag{35}$$

Proof. *Existence of l*: Let $Z \in \mathbf{H}_n$ be a fixed point. All points on the line

$$\alpha(t) := E + t(Z/i - E), \quad 0 \leqslant t \leqslant 1,$$

are invertible matrices because $\mathbf{H}_n$ is convex. We hence may define

$$l(x) = l(x, Z) = \int_0^x \dot{\alpha}(t)/\alpha(t)\, dt, \quad 0 \leqslant x \leqslant 1.$$

One has

$$\dot{l}(t) = \dot{\alpha}(t)/\alpha(t)$$

hence

$$[e^{l(t)}/\alpha(t)]^{\cdot} = 0$$

because the matrices $\alpha(t)$, $\dot{\alpha}(t)$ generate a commutative algebra. Hence the matrix in the bracket is constant and we obtain

$$e^{l(t)} = \alpha(t)$$

or, in the special case $t = 1$

$$e^{l(1,Z)} = Z/i.$$

Uniqueness of l: It has to be shown that for each $Y > 0$ there exists only one symmetric real matrix A with the property

$$e^A = Y.$$

This well-known fact can easily be proved by means of an orthogonal transformation of Y into a diagonal matrix. One has to make use of the fact that each matrix which commutes with A commutes with Y, too. □

We now are able to formulate the main-result of this section.

2.5 Proposition. *We have*

$$\theta_P\begin{bmatrix} a \\ b \end{bmatrix}(-Z^{-1}) = \exp(\pi i a' b/2) \det (Z/i)^{1/2}\, \theta_Q \begin{bmatrix} -b \\ a \end{bmatrix}(Z) \tag{36}$$

where $Q(g, Z)$ *is the Gauss-transform of the polynomial*

$$u \longmapsto P((Z/\mathrm{i})^{1/2} u, -Z^{-1})$$

at

$$-\mathrm{i}(Z/\mathrm{i})^{1/2} g.$$

Remark. Even if $P(g) = P(g, Z)$ is a harmonic polynomial independent of Z the transformed polynomial $Q(g, Z)$ in general depends on Z and is no longer harmonic in g.

Proof. We proceed as usual and notice that the function

$$\exp(\pi i a' b/2)\, \theta_P[m](Z)$$

is periodic as function of a and admits a Fourier expansion

$$\sum_{g \in \mathbb{Z}^n} \alpha(g) \exp(\pi i g' a).$$

The Fourier coefficients can be computed by means of the Fourier integral. If $Z = \mathrm{i}Y$ is purely imaginary one obtains

$$\alpha(g) = \int_0^1 \cdots \int_0^1 \sum_{h \text{ integral}} P(h + a, \mathrm{i}Y)$$

$$\exp(-\pi\{Y[h + a] - \mathrm{i}b'(h + a) - \mathrm{i}2g'a\})\, da$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(x, \mathrm{i}Y) \exp(-\pi\{Y[x] - \mathrm{i}(b - 2g)' x\})\, dx \quad (37)$$

$$= \exp(-\pi Y^{-1}[g - b/2]) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(x, \mathrm{i}Y)$$

$$\exp(-\pi\{Y[x + \mathrm{i}Y^{-1}(g - b/2)]\})\, dx$$

$$= \exp(-\pi Y^{-1}[g - b/2]) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(x - \mathrm{i}Y^{-1}(g - b/2), \mathrm{i}Y)$$

$$\exp(-\pi Y[x])\, dx.$$

By means of the substitution

$$x = Y^{-1/2} u, \quad dx = \det Y^{-1/2}\, du$$

we obtain

$$\alpha(g) = \det Y^{-1/2} \exp(-\pi Y^{-1}[g - b/2]) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(Y^{-1/2} u -$$

$$- iY^{-1}(g - b/2), iY) \exp(-\pi u'u)\, du.$$

We now have proved 2.5 in the special case $Z = iY$. The general case follows by analytic continuation. □

We now want to determine the action of arbitrary modular substitutions on the theta-series $\theta_P[m]$ and recall firstly the (affine) action of the modular group on characteristics. One defines

$$M\{m\} = \begin{bmatrix} D & -C \\ -D & A \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} (CD')_0 \\ (AB')_0 \end{bmatrix}; \tag{38}$$

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_n; \quad m \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{C}^{2n}.$$

Hereby S_0 denotes in general the column-vector made from the diagonal-elements of S. One has

$$MN\{m\} \equiv M\{N\{m\}\} \bmod 2. \tag{39}$$

2.6 PROPOSITION *There exists a unique action of the modular group Γ_n on the space of all coefficient functions*

$$(P, M) \longmapsto P_M$$

such that $P_M = P$, if P is constant and such that the transformation formula

$$\theta_P[m](MZ) = w(M, m) \det(CZ + D)^{1/2} \theta_{P_M}[M^{-1}\{m\}](Z) \tag{40}$$

holds. Here $w(M, m)$ is a system of complex numbers which is independent of Z and of P.

Of course $w(M, m)$ depends on the choice of a holomorphic root of $\det(CZ + D)$.

Proof of 2.6. *Uniqueness of P_M*: The system of numbers $w(M, m)$ is determined by the demand $P_M = P$ for constant P. The coefficient-function of a theta-series is uniquely determined by the theta function which is immediately clear if one considers b as variable.

Existence of P_M: If the formula is true for M and N it is also true for $M \cdot N$. This follows from

$$(M \cdot N)\{m\} \equiv M\{N\{m\}\} \bmod 2$$

and from the fact that a change of the characteristic mod 2 can be absorbed by the coefficient-function:

$$\theta_{\tilde{P}}[\tilde{m}] = \theta_P[m] \text{ if } \tilde{m} \equiv m \bmod 2$$

where

$$\tilde{P}(g, Z) = \exp[\pi i b'(a - \tilde{a})/2]\, P(g + (a - \tilde{a})/2, Z).$$

It is hence sufficient to prove the transformation formula for generators of the modular group, hence for

a) $Z \longmapsto Z + S; \quad S = S'$ integral,

b) $Z \longmapsto -Z^{-1}$.

The case of a translation is trival. The case of the involution has been treated in 2.5. □

The determination of $w(M, m)$ is difficult. A simplification is obtained if one restricts to the case where M is contained in the theta group $\Gamma_{n,\theta}$ which is characterized by

$$(CD')_0 \equiv (AB')_0 \equiv 0 \bmod 2$$

equivalently

$$M^{-1}\{m\} \equiv M' m \bmod 2$$

(The last formula in connection with (39) shows that $\Gamma_{n,\theta}$ is a group). In this connection it is convenient to use the modified theta-series

$$\tilde{\theta}_P[m](Z) = \exp(\pi i a' b/4)\, \theta_P[m](Z) \tag{41}$$

From 2.6 we obtain a transformation-formula

$$\tilde{\theta}_P[m](MZ) = v(M, m) \det(CZ + D)^{1/2}\, \tilde{\theta}_{P^M}[M' m](Z)$$

for $$M \in \Gamma_{n,\theta}. \tag{42}$$

Again $v(M, m)$ is normalized by the demand

$$P^M = P \text{ if } P \text{ is constant.}$$

Of course P^M can be expressed by P_M, but we do not need that.

Actually $\nu(M, m)$ is independent of m! A proof of this remarkable fact can be found in [6]. We hence obtain.

2.7 PROPOSITION. *There exists a unique action of the theta group $\Gamma_{n,\vartheta}$ on the space of all coefficient functions*

$$(P, M) \longmapsto P^M$$

such that $P^M = P$ if P is constant and such that the formula

$$\tilde{\vartheta}_P[m](MZ) = v_\vartheta(M) \det (CZ + D)^{1/2} \tilde{\vartheta}_{P^M}[M' m](Z) \tag{43}$$

holds for all $M \in \Gamma_{n,\vartheta}$. Here $v_\vartheta(M)$ is a system of complex numbers of absolute value 1 which is independent of Z, m and P.

In the next section we will determine an explicit formula for the action $(P, M) \longmapsto P^M$ and for $v_\vartheta(M)$.

III The action of the theta group on coefficient functions

We describe the action (2.7)

$$(P, M) \longmapsto P^M \quad (M \in \Gamma_{n,\vartheta})$$

in case of an invertible D.

If the determinant of D is different from 0 one has

$$MZ = W + R \tag{44}$$

where

$$R = BD^{-1}, \quad W = D'^{-1} Z (CZ + D)^{-1}. \tag{45}$$

The matrix R is rational and symmetric. Hence W is contained in the half-plane $\mathbf{H}_n$. One has

$$\begin{aligned} \vartheta_P[m](MZ) &= \sum P(g + a/2, MZ) \\ &\qquad \exp(\pi i \{W[g + a/2] + R[g + a/2] + b' g\}) \\ &\exp(\pi i R[a]/4) \sum P(g + a/2, MZ) \\ &\qquad \exp(\pi i \{W[g + a/2] + R[g] + (Ra + b)' g\}). \end{aligned} \tag{46}$$

We want to simplify the formula and assume $b = -Ra$ which is sufficient for our purpose. We notice that $R[g]$ remains unchanged if one replaces

$$g \text{ by } g + Dh, \quad h \text{ integral}.$$

If g_0 runs through a system of representatives of $\mathbb{Z}^n/D\mathbb{Z}^n$ one obtains

$$\exp(\pi i R[a]/4) \sum_{g_0 \bmod D} \exp(\pi i R[g_0]) \sum_{g \text{ integral}} P(g_0 + Dg + a/2, MZ).$$
$$\cdot \exp(\pi i W[g_0 + Dg + a/2]). \tag{47}$$

We put

$$\tilde{a} = D^{-1}(a + 2g_0), \quad \tilde{b} = 0$$

$$\tilde{P}(g, Z) = P(Dg, Z[D^{-1}] + R), \tag{48}$$

hence

$$\tilde{P}(g, W[D]) = P(Dg, MZ) \tag{49}$$

and obtain

$$\exp(\pi i R[a]/4) \sum_{g_0 \bmod D} \exp(\pi i R[g_0]) \cdot \theta_{\tilde{P}}[\tilde{m}](W[D]). \tag{50}$$

Now we apply the inversion formula 2.5

$$\theta_P[m](MZ) = \exp(\pi i R[a]/4) \det(-W[D]^{-1}/i)^{1/2}$$

$$\sum_{g_0 \bmod D} \exp(\pi i R[g_0]\, \theta_{\tilde{Q}} \begin{bmatrix} -\tilde{b} \\ \tilde{a} \end{bmatrix} (-W[D]^{-1}) \tag{51}$$

Here $\tilde{Q}(g, Z)$ is the Gauss-transform of the polynomial

$$u \longmapsto \tilde{P}((Z/i)^{1/2} u, -Z^{-1}) = P(D \cdot (Z/i)^{1/2} u, -Z^{-1}[D^{-1}] + R)$$

at

$$-i(Z/i)^{1/2} g.$$

We have now replaced in the transformation-formula (43) MZ on the left hand side by

$$-W[D]^{-1} = -Z^{-1} - D^{-1}C. \tag{52}$$

The same will be done on the right-hand side. One has

$$n = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} := M'm = \begin{bmatrix} A' & C' \\ B' & D' \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}, \tag{53}$$

and hence because of $b = -Ra$

$$\alpha = D^{-1}a, \quad \beta = 0.$$

From the inversion formula (2.5) follows

$$\theta_{PM}[M'm](Z) = \det(Z/i)^{-1/2}\,\theta_{QM}\begin{bmatrix}-\beta\\ \alpha\end{bmatrix}(-Z^{-1}). \tag{54}$$

Here $Q^M(g, Z)$ is the Gauss-transform of the polynomial

$$u \longmapsto P^M((Z/i)^{1/2}u, -Z^{-1})$$

at

$$-i(Z/i)^{1/2}g.$$

If one makes use of

$$\tilde{\theta}_P[m] = \exp(-\pi i R[a]/4)\,\theta_P[m]$$

$$\tilde{\theta}_{PM}[M'm] = \theta_{PM}[M'm] \tag{55}$$

because of $b = -Ra$, the transformation formula (43) equals

$$\det(-(Z^{-1}+D^{-1}C)/i)^{1/2}\cdot \sum_{g_0 \bmod D} \exp(\pi i R[g_0])$$

$$\cdot \sum_{g \text{ integral}} \tilde{Q}(g, -Z^{-1}-D^{-1}C)$$

$$\exp(\pi i\{-Z^{-1}[g] - D^{-1}C[g] + g'D^{-1}(a+2g_0)\})$$

$$= v_\theta(M)\det(CZ+D)^{1/2}\det(Z/i)^{1/2}$$

$$\cdot \sum_{g \text{ integral}} Q^M(g, -Z^{-1})\exp(\pi i\{-Z^{-1}[g] + g'D^{-1}a\}). \tag{56}$$

This is an identity between Fourier series with respect to the variable a. Comparison of the Fourier-coefficients gives for each fixed g

$$\det(-(Z^{-1}+D^{-1}C)/i)^{1/2}\cdot \sum_{g_0 \bmod D} \exp(\pi i R[g_0])$$

$$\tilde{Q}(g, -Z^{-1}-D^{-1}C)\exp(\pi i\{-D^{-1}C[g] + 2g'D^{-1}g_0\})$$

$$= v_\theta(M)\det(CZ+D)^{1/2}\det(Z/i)^{-1/2}\cdot Q^M(g, -Z^{-1}). \tag{57}$$

or

$$\tilde{Q}(g, -Z^{-1}-D^{-1}C)\cdot(\det D)^{-1/2}$$

$$\sum_{g_0 \bmod D} \exp(\pi i\{R[g_0] - D^{-1}C[g] + 2g'D^{-1}g_0\})$$

$$= v_\theta(M)\cdot Q^M(g, -Z^{-1}). \tag{58}$$

In the special case $P(g, Z) = 1$ one has

$$\tilde{P}(g, Z) = \tilde{Q}(g, Z) = 1$$

and

$$P^M(g, Z) = Q^M(g, Z) = 1.$$

We hence obtain the well-known formula for the multiplier system

$$v_\theta(M) = (\det D)^{-1/2} \cdot \sum_{g_0 \bmod D} \exp(\pi i \{R[g_0] - D^{-1}C[g] + 2g' D^{-1} g_0\}). \quad (59)$$

This sum is especially independent of g and hence has to be computed only for special g, for example $g = 0$. A second application of (56) now gives us

$$\tilde{Q}(g, -Z^{-1} - D^{-1}C) = Q^M(g, -Z^{-1}). \quad (60)$$

3.1 PROPOSITION. *The action*

$$(P, M) \longmapsto P^M \qquad (\det D \neq 0)$$

of the theta-group on the coefficient functions can be described as follows. Let $\tilde{Q}(g, Z)$ *be the Gauss-transform of*

$$u \longmapsto P(D \cdot (Z/i)^{1/2} u, -Z^{-1}[D^{-1}] + R)$$

at

$$-i(Z/i)^{1/2} g$$

and let $Q^M(g, Z)$ *be the Gauss-transform of*

$$u \longmapsto P^M((Z/i)^{1/2} u, -Z^{-1})$$

at

$$-i(Z/i)^{1/2} g.$$

Then

$$Q^M(g, -Z^{-1}) = \tilde{Q}(g, -Z^{-1} - D^{-1}C). \quad (61)$$

This proposition gives in fact an explicit formula (which we do not write down) because the Gauss-transformation is invertible (2.3).

The action described in 3.1 can be simplified if P satisfies certain conditions of harmonicity.

3.2 *Remark.* Assume that $P = P(g, Z)$ is a coefficient function. Let

$$(Z, M), \; Z \in \mathbf{H}_n, \; M \in \Gamma_{n,\theta}$$

be a pair, such that the two polynomials

$$u \longmapsto P(D((-Z^{-1} - D^{-1}C)/i)^{1/2} u)$$

and

$$u \longmapsto P((DZ^{-1} + C)(Z/i)^{1/2} u)$$

are harmonic. Then

$$P^M(g, Z) = P((CZ + D) g) \tag{62}$$

Proof. The first condition of harmonicity implies

$$Q^M(g, -Z^{-1}) = \tilde{Q}(g, -Z^{-1} - D^{-1}C) = P((DZ^{-1} + C) g).$$

From the second one follows, that

$$u \longmapsto Q^M((Z/i)^{1/2} u, -Z^{-1})$$

is harmonic. Consequently

$$Q^M(g, -Z^{-1}) = P^M(g, Z). \quad \square$$

IV Eichler's imbedding trick

We now consider theta series with respect to a positive (symmetric and real) matrix $S = S^{(r)}$ of the type

$$\theta_{S,P}(Z; U, V) = \sum_{G \in \mathbb{Z}^{(m,n)}} P(S^{1/2}(G + U/2), Z)$$

$$\exp(\pi i \sigma \{S[G + U/2] Z + VG'\}) \tag{63}$$

where $U = U^{(r,n)}$, $V = V^{(r,n)}$ are arbitrary complex matrices (the characteristics) and where $P(G, Z)$ is a coefficient-function analogous to 2.1 (i.e. P is a polynomial of bounded degree in G whose coefficients depend holomorphically on Z). We denote by g, a, b the big column-vector (in $\mathbb{C}^{rn}$) which corresponds to the matrices G, U, V (29). If we define

$$P_0(g, Z) := P(G, Z) \tag{64}$$

we have (with notation (32))

$$\theta_{S,P}(Z; U, V) = \theta_{P_0}[m](S \otimes Z) \left(m = \begin{pmatrix} a \\ b \end{pmatrix}\right). \tag{65}$$

From the inversion formula 2.5 we obtain

4.1 Proposition. *We have*

$$\theta_{S,P}(-Z^{-1}; U, V) = \exp[\pi i \sigma (U'V)/2] (\det S)^{-n/2}$$

$$\det (Z/i)^{r/2} \theta_{S^{-1},Q}(Z; -V, U) \tag{66}$$

$$\tilde{U} = UA + S^{-1} VC, \quad \tilde{V} = SUB + VD$$

$$\epsilon_S(M) = v_\theta(M^S)$$

for all $M \in \Gamma_n(S)$ with invertible D.

It is easy to prove that an arbitrary congruence group $\Gamma \subset Sp_{2n}(\mathbb{R})$ is generated by the set of all

$$M \in \Gamma, \det D \neq 0$$

Hence formula (67) is valid for all $M \in \Gamma_n(S)$ with certain numbers $\epsilon_S(M)$ of absolute value 1 which depends on the choice of the square-root of det $(CZ + D)$.

All results stated in section I now have been proved.

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