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HOLOMORPHIC TENSORS ON SUBVARIETIES OF THE SIEGEL MODULAR VARIETY

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Introduction.

The Siegel modular variety

$$A_n = H_n/\Gamma_n \quad , \quad \Gamma_n = Sp(n,\mathbb{Z}) \ ,$$

$$H_n = \text{Siegel upper half plane of genus } n$$

is for sufficiently high genus n of general type. This important result is due to Y. Tai [8]. His bound ($n \geq 9$) has been improved (see [3] for $n \geq 8$, [6] for $n \geq 7$).

It may be expected that similar structure theorems are true for subvarieties of A_n if they are not in too special position (in a sense which has to be made precise).

One promising method to investigate a subvariety $Y \subset A_n$ is to construct holomorphic tensors

$$T \in \Omega^{\otimes d} \ (\tilde{A}_n)$$

on a desingularization $\tilde{A}_n$ of a compactification of A_n. The restriction of such a tensor to Y extends to a holomorphic tensor on any nonsingular variety $\tilde{Y}$ which is birationally equivalent to Y. If the tensor T is a multicanonical form

$$T \in (\Lambda^N\Omega)^{\otimes r} \ (\tilde{A}_n) \quad , \quad N = n(n+1)/2 \ ,$$

the restriction to any proper subvariety vanishes.

For this reason we have to consider more general types of tensors. In this connection the following weakened form of the notion "general type" seems to be natural.

Definition: A nonsingular compact irreducible algebraic variety X is of type G ("general") if there exist $n = \dim X$ algebraically independent rational functions $f_1, \ldots, f_n$ and a holomorphic tensor

$$T \in \Omega^{\otimes d}(X) \quad , \quad d > 0 \quad , \quad T \neq 0 \, ,$$

such that the tensors $f_1 T, \ldots, f_n T$ are holomorphic on X.

This notion is of course birational invariant. We call an arbitrary irreducible variety of type G if it is birational equivalent to a nonsingular compact variety with this property.

Of course varieties of type G are far away from being unirational.

Our main result is the following

Theorem: There is a certain bound n_o such that for $n \geq n_o$ each irreducible subvariety

$$Y \subset A_n = H_n / \Gamma_n$$

of codimension 1 is of type G.

Let

$$\tilde{A}_n \xrightarrow{\pi} \bar{A}_n \supset A_n$$

be a desingularization of the Satake-compactification. By means of the explicit construction of such an $\tilde{A}_n$ [1] we may deduce:

In contrast to the above theorem no irreducible subvariety Y of codimension 1 which is contained in the inverse image of the boundary

$$\pi(Y) \subset \bar{A}_n \setminus A_n$$

is of type G.

A remarkable consequence of this observation and the above theorem is the following (compare [5], Satz 7).

Corollary 1. There is no birational automorphism of A_n, $n \geq n_o$, besides the identy.

Equivalently: Each automorphism of the field $K(\Gamma_n)$ of modular functions which fixes $\mathbb{C}$ is trivial.

Another consequence of the theorem is the following minimality property of A_n (compare [5], S. 33, Folgerung).

Corollary 2. Let $(A_n)_{reg}$ $(n \geq n_o)$ be the regular locus of A_n and $\tilde{A}_n$ a nonsingular compactification

$$\tilde{A}_n \supset (A_n)_{reg} .$$

Let

$$\pi : \tilde{A}_n \to X$$

be any birational everywhere holomorphic map. The restriction

$$\pi | (A_n)_{reg} \to X$$

is an open embedding.

The tensors we are considering are of the form

(1) $\quad T = fT_o .$

Here f is a usual scalar-valued Siegel modular form and

(2) $\quad T_o \in Z_n'^{\otimes d}$

where Z_n' is the dual of the tangent space of H_n, i.e. the space of symmetric n-rowed matrices,

$$Z_n \cong \mathrm{Symm}^2(\mathbb{C}^n) .$$

The "constant tensor" T_o has to be invariant under the natural action of $Sl(n,\mathbb{C})$. For a suitable chosen weight of f the tensor T will then be invariant under the modular group Γ_n.

In the paper [5] I used scalar-valued Hilbert-modular forms to prove similar results for Hilbert-modular varieties of high level. During my stay at Harvard 1981 I had the chance to discuss with D. Mumford the possibility of constructing holomorphic tensors on a nonsingular compact model $\tilde{M}_n$ of the variety M_n (the moduli variety of curves of genus n).

D. Mumford gave an example of a constant $Sl(n,\mathbb{C})$-invariant tensor T_o whose restriction to M_n does not vanish. It is possible to investigate the conditions which a modular form f must satisfy so that

$$fT_o^{\otimes m}$$

defines a holomorphic tensor on $\tilde{M}_n$. It is further possible to prove the existence of such f by means of the Mumford-Hirzebruch-proportionality theorem. (This method has been used by Tai [8] to prove the existence of many multicanonical tensors on $\tilde{A}_n$.) But it seems to be very hard to get concrete examples of such modular forms f which do not vanish identically on M_n.

I still believe that the indicated method is good enough to prove structure theorems for M_n and for many subvarieties of M_n.

The present paper is of course highly influenced by many conversations with D. Mumford.

§ 1. Γ_n-invariant tensors.

The symplectic group $Sp(n,\mathbb{R})$ acts on the Siegel upper half plane H_n by means of the well known formula

(3) $$Z \to MZ = (AZ + B)(CZ + D)^{-1} , \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

The derivative of M at a point $Z_o \in H_n$ is given by

(4) $$(dM)(Z_o) : \mathcal{Z}_n \to \mathcal{Z}_n$$
$$W \to (CZ_o + C)'^{-1} W (CZ_o + D)^{-1} .$$

Hereby $\mathcal{Z}_n$ denotes the tangent space of H_n,

$$\mathcal{Z}_n = \{Z = Z' = Z^{(n)}\} .$$

There is a natural action of $Gl(n,\mathbb{C})$ on $\mathcal{Z}_n$, namely

$$W \to \rho(A)W = A'WA \quad , \quad A \in Gl(n,\mathbb{C}) .$$

The representation $(\mathcal{Z}_n,\rho)$ is isomorphic with the natural representation of $Gl(n,\mathbb{C})$ on $Symm^2(\mathbb{C}^n)$. We denote by $\mathcal{Z}_n' = Hom(\mathcal{Z}_n,\mathbb{C})$ the dual space of $\mathcal{Z}_n$ and by ρ' the contragredient representation.

A holomorphic tensor on H_n is a holomorphic map

$$T : H_n \to Z_n'^{\otimes d} = Z_n' \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} Z_n' .$$

The tensor T is invariant under a symplectic substitution M if and only if

$$(5) \qquad T(MZ) = \rho'^{\otimes d}(CZ + D)T(Z) .$$

We are interested in tensors T which are invariant under some subgroup $\Gamma \subset Sp(n,\mathbb{R})$, commensurable with the modular group $\Gamma_n = Sp(n,\mathbb{Z})$. We want to construct such tensors by means of usual (scalar-valued) Siegel modular forms. A modular form of weight $r \in \mathbb{Z}$ is a holomorphic function

$$f : H_n \to \mathbb{C}$$

with the transformation property

$$(6) \qquad f(MZ) = \det(CZ + D)^r f(Z) \quad , \quad M \in \Gamma .$$

If n = 1 a well known growth condition at infinity has to be added.

The Jacobian determinant of a symplectic substitution is

$$(7) \qquad \det(CZ + D)^{-(n+1)} \quad ,$$

i.e. modular forms of weight $r(n+1)$ correspond to multi-canonical tensors of degree r.

It is a remarkable fact that scalar-valued modular forms also can be used to construct tensors of other types. The reason is, that in the tensor space

$$Z_n'^{\otimes d} \quad , \quad d \text{ suitable} ,$$

there exist tensors $T_o \neq 0$ which are invariant under $Sl(n,\mathbb{C})$.

1.1 Lemma: *Let*

$$T_o \in Z_n'^{\otimes nr/2} = Symm^2(\mathbb{C}^n)^{\otimes nr/2} \qquad (nr \equiv 0 \bmod 2)$$

be a $Sl(n,\mathbb{C})$-invariant tensor and f a modular form of weight r with respect to Γ. The tensor

$$T = fT_o$$

is Γ-invariant.

<u>Proof</u>. The $Sl(n,\mathbb{C})$-invariance of T_o implies

$$T_o|A \; (:= \rho'^{\otimes nr/2}\, T_o) = (\det A)^m T_o \; .$$

The exponent m can be determined if one specializes $A = aE$ (E = unit matrix). One obtains $m = r$.

Some examples of tensors T_o.

1) <u>A symmetric tensor</u>.

Symmetric tensors in $Z_n'^{\otimes d}$ can be identified with polynomials on Z_n. Let T_o be the $Sl(n,\mathbb{C})$-invariant polynomial

$$Z_n \to \mathbb{C}$$
$$W \to \det W \; .$$

If f is any modular form of even weight 2r with respect to Γ then

$$f \cdot T_o^{\otimes r}$$

is a <u>holomorphic</u> Γ-<u>invariant symmetric tensor</u> on H_n.

2) <u>Multicanonical tensors</u>.

If f is modular form of weight $r(n+1)$, the tensor

$$(8) \qquad f \cdot T_o^r \; , \quad T_o = \bigwedge_{1 \le i \le k \le n} dz_{ik}$$

is Γ-invariant.

3) To get more complicated examples we consider polynomials $P(W_1,\ldots,W_k)$ on $Z_n \times \ldots \times Z_n$ with the property

a) (9) $P(t_1W_1,\ldots,t_kW_k) = t_1^{d_1} \ldots t_k^{d_k} P(W_1,\ldots,W_k)$.

There exists a unique multilinear form on $Z_n^{(d_1+\ldots+d_k)}$

$$M(W_1^{(1)},\ldots,W_1^{(d_1)},\ldots,W_k^{(1)},\ldots,W_k^{(d_k)})$$

with the property

i) $P(W_1,\ldots,W_k) = M(\overbrace{W_1,\ldots,W_1}^{d_1},\ldots,\overbrace{W_k,\ldots,W_k}^{d_k})$

ii) M is symmetric in each $(W_\nu^{(1)},\ldots,W_\nu^{(d_\nu)})$.

We identify P with the tensor M

$$P \in Z_n'^{\otimes(d_1+\ldots+d_k)} \; .$$

We impose on P a further condition:

b) $P(W_1,\dots,W_k)$ depends only on the Plücker-coordinates of $(W_1,\dots,W_k)$, i.e. on

$$W_1 \wedge \dots \wedge W_k \in \Lambda^k Z_n \ .$$

It is easy to be seen that this is equivalent with

$$P \in \mathrm{Symm}^d(\Lambda^k Z_n) = \mathrm{Symm}^d(\Lambda^k(\mathrm{Symm}^2(\mathbb{C}^n))) \ ,$$

$$d = d_1 = \dots = d_k \ .$$

As we already mentioned we are interested in $Sl(n,\mathbb{C})$-invariant polynomials P:

c) $P(A'W_1A,\dots,A'W_kA) = P(W_1,\dots,W_k)$, $A \in Sl(n,\mathbb{C})$

which implies

(10) c') $P(A'W_1A,\dots,A'W_kA) = (\det A)^{2kd/n} P(W_1,\dots,W_k)$.

Such polynomials arise in the theory of Chow forms [7]. The idea to use Chow forms and especially the following example is due to Mumford.

The set of matrices

(11) $$Z_n[h] = \{W \in Z_n,\ \mathrm{rank}(W) \leq h\} \ , \quad 0 \leq h \leq n$$

is an irreducible algebraic variety, invariant under the action of $Gl(n,\mathbb{C})$. Each $W \in Z_n[h]$ can be written in the form

$$W = A'A \ , \quad A = A^{(h,n)} \ ,$$

A is unique up to left multiplication with an orthogonal matrix. The dimension of the orthogonal group $O(h,\mathbb{C})$ is $h(h-1)/2$. We therefore obtain

(12) $$k := \dim Z_n[h] = hn - h(h-1)/2 \ .$$

To compute the Chow form of $Z_n[h]$ (more precisely of the corresponding projective variety in the projective space of Z_n) one has to consider intersections with hyperplanes. Any hyperplane can be written in the form

(13) $$H_S = \{W \in Z_n,\ \sigma(WS) = 0\},\ S \in Z_n \setminus \{0\} \quad (\sigma = \text{trace}).$$

The Chow form is - up to a constant factor - the unique irreducible polynomial $P(W_1,\ldots,W_k)$ such that

(14) $$H_{W_1} \cap \ldots \cap H_{W_k} \cap Z_n[h] \neq \{0\} \Leftrightarrow P(W_1,\ldots,W_k) = 0 .$$

1.2 Lemma: There exists (up to a constant factor) a unique irreducible polynomial

$$P(W_1,\ldots,W_k) \ , \ k = nh - h(h-1)/2 \quad (0 < h \leq n)$$

such that the following two conditions are equivalent:

a) $P(W_1,\ldots,W_k) = 0$.

b) There exists a matrix $S \in Z_n[h]$, $S \neq 0$, with the property

$$\sigma(W_1 S) = \ldots = \sigma(W_k S) = 0 .$$

This polynomial defines a tensor

$$P \in \mathrm{Symm}^d(\Lambda^k Z_n)$$

which is invariant under $Sl(n,\mathbb{C})$.

A dual construction yields

1.3 Lemma: There exists (up to a constant factor) a unique irreducible polynomial

$$P(W_1,\ldots,W_k) \ , \ k = n(n+1)/2 - nh + h(h-1)/2$$

such that the following two conditions are equivalent:

a) $P(W_1,\ldots,W_k) = 0$.

b) $(\mathbb{C}W_1 + \ldots + \mathbb{C}W_k) \cap Z_n[h] \neq \{0\}$.

This polynomial defines a tensor

$$P \in \mathrm{Symm}^d(\Lambda^k Z_n)$$

which is invariant under $Sl(n,\mathbb{C})$.

Let $H \subset Z_n$ be a linear subspace. By means of the natural map $Z_n' \to H'$ we obtain a restriction map

$$Z_n'^{\otimes d} \to H'^{\otimes d}$$

$$T \to T|H .$$

1.4 Lemma: Let $H \subset Z_n$ be any linear subspace of co-dimension 1. Let $P = P_h$ be the tensor defined in 1.3. Assume

$$h = 1 \quad , \quad n > 1 .$$

Then

$$P|H \neq 0 .$$

Proof. Let

$$H = \{W \in Z_n, \ \sigma(WS) = 0\} \quad , \quad S = S' \neq 0 .$$

The invariance of P under $Sl(n,\mathbb{C})$ allows us to replace

$$S \to S[A] = A'SA \quad , \quad A \in Gl(n,\mathbb{C}) .$$

Therefore we may assume

$$S = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & 0 \end{pmatrix} .$$

The space

$$\{W = W', \ w_{11} = \ldots = w_{nn} = 0\}$$

is contained in H. Its dimension is $n(n-1)/2$. Take any basis $W_1, \ldots, W_{n(n-1)/2}$ of this space. We claim

$$P(W_1, \ldots, W_{n(n-1)/2}) \neq 0 .$$

This means precisely that no non trivial linear combination of the W_ν-s has rank ≤ 1 and this follows from the simple fact that no symmetric matrix with zero-diagonal is of rank 1.

Let $Y \subset H_n/\Gamma$ be an algebraic subvariety, T a tensor on H_n. We say that T vanishes on Y if

(15) $$T|[p^{-1}(Y)]_{reg} = 0 .$$

Here $[p^{-1}(Y)]_{reg}$ is the regular locus of the inverse image of Y under the natural projection $p : H_n \to H_n/\Gamma$.

From 1.1 and 1.4 we obtain

1.5 Lemma: Let

a) $P = P_1 \in \mathrm{Symm}^d(\Lambda^{n(n-1)/2} Z_n')$

be the tensor defined in 1.3 ($h = 1$),

b) f a scalar-valued modular form of an arbitrary weight r,

c) $Y \subset H_n/\Gamma$ an irreducible subvariety of codimension 1,

d) m a natural number such that $m' = mr/d(n-1)$ is integral.

The Γ-invariant tensor $f^m \cdot P^{m'}$ vanishes on Y if and only if the modular form f vanishes (as a function) on Y (i.e. on $p^{-1}(Y)$).

§ 2. Extension of holomorphic tensors to smooth compactifications.

In this section we assume $n \geq 3$. The set of elliptic fixed points $\mathrm{Fix}(\Gamma_n)$ of the modular group $\Gamma_n = \mathrm{Sp}(n,\mathbb{Z})$ is of codimension ≥ 2. Therefore

$$H_n^o/\Gamma_n \quad , \quad H_n^o := H_n - \mathrm{Fix}(\Gamma_n) \tag{16}$$

is the regular locus of H_n/Γ_n. We denote by $\widetilde{H_n/\Gamma_n}$ a smooth compactification of H_n^o/Γ_n which lies over the Satake-compactification $\overline{H_n/\Gamma_n}$,

$$\begin{array}{ccc} H_n^o/\Gamma_n & \hookrightarrow & \widetilde{H_n/\Gamma_n} \\ & \searrow \quad \swarrow & \\ & \overline{H_n/\Gamma_n} & \end{array} \tag{17}$$

It is possible to investigate the conditions that a holomorphic tensor on H_n^o/Γ_n extends holomorphically to $\widetilde{H_n/\Gamma_n}$ without making use of an explicit construction of a smooth compactification. The method is described in detail in [3], [4] for the main-congruence-subgroup of level $l \geq 3$ (in this case no elliptic fixed-points occur) and rather sketchily for Γ_n and multicanonical tensors in [3], pp. 195.

We now give the calculations in some detail for tensors of the type

$$T = fP \quad , \quad P \in \mathrm{Symm}^d(\Lambda^k(Z_n')) \ .$$

We have to consider commutative diagrams of holomorphic maps

$$(18)\qquad \begin{array}{ccccc} (z,w) & H\times E^{N-1} & \overset{\Psi}{\to} & H_n^o \\ \downarrow & \downarrow & & \downarrow \\ (e^{2\pi i z},w) & E^{\bullet}\times E^{N-1} & \overset{\psi}{\to} & H_n^o/\Gamma_n \\ & \cap & & \cap \\ & E^N & \to & \overline{H_n/\Gamma_n} & \text{(Satake-compactification)} \end{array}$$

A holomorphic tensor T on H_n^o/Γ_n extends to $\widetilde{H_n/\Gamma_n}$ iff for each such diagram $\psi^*(T)$ extends to E^N. A suitable lift Ψ of ψ is of the form

$$(19)\qquad \Psi(z,w) = S_o z + \Psi_o(q_m,w)\ ,\ q_m = e^{2\pi i z/m}\ ,$$

$$S_o = \begin{pmatrix} 0 & 0 \\ 0 & S^{(n-j)} \end{pmatrix}$$

where m is a natural number and Ψ_o is holomorphic in $q_m = 0$ (compare [3], III 5.7, 5.8 and the remarks on p. 199).

We define $Z_1 = Z_1^{(j)} = Z_1(w)$ by

$$(20)\qquad \Psi_o(0,w) = \begin{pmatrix} Z_1 & * \\ * & * \end{pmatrix}\ .$$

The imaginary part of Z_1 is positive definite and not only semipositive! This follows from the fact that each holomorphic map $\varphi : E \to Z_n$ with the property

$$\varphi(E) \subset \overline{H_n}\quad ,\quad \varphi(E) \cap \partial H_n \neq \emptyset$$

is constant. The point $Z_1 \in H_j$ represents the limit point

$$\lim_{q\to 0} \psi(q,w)$$

in the Satake-compactification. We further know

$$(21)\qquad \Psi(z+1,w) = M\Psi(z,w)\ .$$

Hereby M is contained in the subgroup $\Gamma_{n,j} \subset \Gamma_n$, i.e.

$$(22)\quad A = \begin{pmatrix} A_1 & 0 \\ A_3 & A_4 \end{pmatrix}\ ,\ B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}\ ,\ C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}\ ,$$

$$D = \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}\ .$$

The image of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ under the natural homomorphism

(23) $$\Gamma_{n,j} \to \Gamma_j \quad , \quad M \to M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

fixes Z_1, $M_1 Z_1 = Z_1$ and therefore is of finite order. A suitable power of M is of the form

$$M^l = \begin{pmatrix} * & * \\ O & * \end{pmatrix} \quad , \quad l > 0 \; .$$

But then it follows easily [3], III 5.8

$$M^h = \begin{pmatrix} E & * \\ O & E \end{pmatrix} \quad , \quad h > 0 \text{ suitable multiple of } m \; .$$

From (19), (21) we obtain

(24) $$M^h = \begin{pmatrix} E & hS_o \\ O & E \end{pmatrix} \quad , \quad S_o = \begin{pmatrix} O & O \\ O & S \end{pmatrix} \; .$$

We now distinguish two cases:

Case I.

(25) $$M = \pm \begin{pmatrix} E & T \\ O & E \end{pmatrix} \; .$$

We obtain

$$S_o(z+1) + \Psi_o(e^{2\pi i/m} q_m) = S_o z + \Psi_o(q_m) + T$$

and therefore

$$S_o = T \quad \text{(integral!)}$$

(26) Ψ_o depends only on $q = q_m^m$.

So we may write (new notation)

$$\Psi(z,w) = S_o z + \Psi_o(q,w) \quad , \quad S_o \text{ integral} \; .$$

We now consider a Γ_n-invariant tensor of the form fP, where f is a modular form of weight r and

$$P \in \text{Symm}^d(\Lambda^k Z_n') \quad , \quad 2dk = nr$$

is a polynomial with the properties a) - c) (§ 1). We compute

$$\Psi^*(fP) = (f \circ \Psi)\cdot\Psi^*(P) \; ,$$

$$\Psi^*(P)(z^{(1)},\ldots,z^{(k)}) = P(d\Psi(z^{(1)}),\ldots,d\Psi(z^{(k)})) \; .$$

The differential $d\Psi$ of Ψ (in a point (z,w)) is a linear map

$$d\Psi : \mathbb{C}^N \to Z_n \; .$$

If we denote by $z = (\xi,\eta_2,\dots,\eta_N)$ the coordinates of $\mathbb{C}^N$ we have

$$(27) \qquad d\Psi(z) = \frac{\partial\psi}{\partial z}\,\xi + \sum_{\nu=2}^{N} \frac{\partial\psi}{\partial w_\nu}\,\eta_\nu \quad .$$

We obtain

$$(28) \quad \psi^*(P)(z^{(1)},\dots,z^{(k)}) =$$

$$P(S_o\xi^{(1)} + W_1(q,w),\dots,S_o\xi^{(k)} + W_k(q,w)) \ ,$$

$$W_j(q,w) = W_j(q,w,\xi,\eta) \ .$$

This tensor on $H \times E^{N-1}$ is invariant under $(z,w) \mapsto (z+1,w)$. It therefore defines a tensor on $\acute{E} \times E^{N-1}$ namely

$$(29) \quad P(\frac{1}{q}\,\tilde{S}_o + \tilde{W}_1(q,w),\dots,\frac{1}{q}\,\tilde{S}_o + \tilde{W}_k(q,w)) \ , \ \tilde{S}_o = \frac{1}{2\pi i}\,S_o \ .$$

This function has in $q = 0$ a pole of order $\leq d$. Hereby we make use of the fact that $P(W_1,\dots,W_k)$ depends only on the Plücker-coordinates of $W_1,\dots,W_k$. (Otherwise we would obtain only the estimate "$\leq kd$".) The tensor $\psi^*(fP)$ is holomorphic in $q = 0$ if either $S_o = 0$ or if the function

$$(30) \quad f(S_o z + \Psi_o(q,w)) \ , \ S_o \neq 0 \ , \ S_o \geq 0 \ , \ S_o \text{ integral} \ ,$$

vanishes in $q = 0$ of order $\geq d$.

We express this as a condition for the Fourier-coefficients of

$$(31) \qquad f(Z) = \sum_{H=H'\geq 0} a(H)e^{\pi i\sigma(HZ)}$$

(H even, i.e. H is integral with even diagonal). We obviously have to demand

$$a(H) \neq 0 \ \Rightarrow \ \frac{1}{2}\,\sigma\,(HS_o) \geq d$$

$$S_o = S_o' \geq 0 \text{ integral}, \neq 0 \ .$$

By a result of Barnes and Cohn [2]

$$\min \sigma(HS_o) = \min_{g\in\mathbb{Z}^n\setminus\{0\}} g'Hg =: \min(H) \ .$$

The minimum $\min(H)$ is invariant under $H \to U'HU$, $U \in Sl(n,\mathbb{Z})$ as well as $a(H)$. Each unimodular class $\{U'HU,\ U \in Sl(n,\mathbb{Z})\}$ contains a representative H with the property

$$\min H = h_{11} \ .$$

2.1 Definition: The vanishing order of a modular form (at infinity)

$$f(Z) = \sum_{H=H'\geq O,\text{integral}} a(H)e^{\pi i\sigma(HZ)}$$

relative to its weight r is greater or equal than a certain number α if

$$a(H) \neq O \rightarrow \frac{1}{2} h_{11} \geq \alpha r \ .$$

2.2 Résumé of the first case: If the vanishing order of the modular form f (relative to its weight) is greater or equal than

$$n/2k \ ,$$

the tensor $f \cdot P$ has the desired extension-property.

Case II.

(32) $$M \neq \pm\begin{pmatrix} E & * \\ O & E \end{pmatrix} \qquad (\text{s. } (21)) \ .$$

We transform the fixed point $Z_1 = Z_1(w)$ of M_1 (23) into the generalized unit disc

(33) $$E_j = \{Z = Z^{(j)} = Z',\ E - Z\overline{Z} > O\}$$

by means of a complex symplectic substitution

$$N_o : H_j \rightarrow E_j \ , \ N_o = \begin{pmatrix} A_o & B_o \\ C_o & D_o \end{pmatrix} \in Sp(j,\mathbb{C}) \ .$$

After a suitable choice of N_o we have

$$N_o(Z_1) = O$$

(34) $$N_o M_1 N_o^{-1} = \left(\begin{array}{ccc|ccc} \xi_1 & & & & & \\ & \ddots & & & & \\ & & \xi_j & & & \\ \hline & & & \overline{\xi}_1 & & \\ & & & & \ddots & \\ & & & & & \overline{\xi}_j \end{array}\right)$$

We now introduce a partial Cayley-transform of H_n. The (complex) symplectic substitution replace m by h and therefore assume m = h. We diagonalize U

(43) $$A'UA = \begin{pmatrix} e^{2\pi i a_1/m} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i a_n/m} \end{pmatrix}$$

$$0 \le a_\nu < m \quad (1 \le \nu \le n) .$$

The polynomial $P(W_1,\ldots,W_k)$ is - up to a constant factor - invariant under $W_\nu \to A'W_\nu A$. We therefore may assume

$$U = \begin{pmatrix} e^{2\pi i a_1/m} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i a_n/m} \end{pmatrix} .$$

The most general solution of (42) is of the form

$$W_\nu(q_m) = (w_{ij}^{(\nu)}(q_m))$$

(44) $$w_{ij}^{(\nu)}(q_m) = q^{[a_i+a_j]/m}\, \hat{w}_{ij}^{(\nu)}(q)$$

$$(\hat{w}_{ij}^{(\nu)}(q) \text{ holomorphic in } |q| < 1,\ q = q_m^m = e^{2\pi i z}).$$

Here $[a_i + a_j] = [a_i + a_j]_m$ is defined by

(45) $$0 \le [a_i + a_j] < m$$
$$[a_i + a_j] \equiv a_i + a_j \mod m .$$

<u>2.3 Résumé of the second case</u>: <u>If the vanishing order of</u>

$$q_m \to P(W_1(q_m),\ldots,W_k(q_m))$$
$$W_\nu(q_m) \text{ as in } (42)$$

<u>in</u> $q_m = 0$ <u>is of order</u> $\ge md$, <u>the tensor</u> $f \cdot P$ <u>has the desired extension-property</u>.

This vanishing order of course depends on the polynomial P. In the next section we give a very rough estimate which depends only on n and k and $a_1,\ldots,a_n$.

§ 3. Estimations for vanishing orders.

We have to consider certain roots of unity

$$\xi_1 = e^{2\pi i a_1/m}, \ldots, \xi_n = e^{2\pi i a_n/m}$$

$$0 \le a_\nu < m \;,\; (a_1,\ldots,a_n) \ne (0,\ldots,0) \;.$$

The only information we want to use is that $\xi_1,\ldots,\xi_n,\overline{\xi_1},\ldots,\overline{\xi_n}$ are the eigenvalues of an integral matrix (namely M (21)). This has the following consequence:

Denote by P_l the set of primitive roots of unity of order l. If $\{\xi_1,\ldots,\xi_n\}$ contains <u>one</u> element of P_l (especially $l|m$), then

$$P_l \subset \{\xi_1,\ldots,\xi_n,\overline{\xi_1},\ldots,\overline{\xi_n}\} \;.$$

A complete half-system of primitive roots of unity of order l is a set of representatives

$$\eta_1,\ldots,\eta_t$$

of P_l mod $\eta \to \overline{\eta}$. We have

$$(46) \qquad t = \begin{cases} 1 & \text{if} \quad l = 1,2 \\ \varphi(l)/2 & \text{if} \quad l > 2 \;. \end{cases}$$

With this notation we obtain: <u>The system</u> $\{e^{2\pi i a_1/m},\ldots,e^{2\pi i a_n/m}\}$ <u>is a disjoint union of complete half-systems</u>.

We now estimate the zero-order (in $q_m = 0$) of the function

$$q_m \to P(W_1(q_m),\ldots,W_k(q_m)) \;.$$

Here $P(W_1,\ldots,W_k)$ is a polynomial as in § 1, i.e.

$$\text{"}P \in \mathrm{Symm}^d(\Lambda^k Z_n')\text{"} \;,$$

the functions $W_\nu(q_m)$ are as described in (42), (44). Obviously such a function $W(q_m) = W_\nu(q_m)$ can be written in the following manner

$$(47) \qquad W(q_m) = \tilde{W}(q) \begin{bmatrix} q_m^{a_1} & & 0 \\ & \ddots & \\ 0 & & q_m^{a_n} \end{bmatrix}$$

$$(35) \qquad N = \left(\begin{array}{cc|cc} A_o & O & B_o & O \\ O & E & O & O \\ \hline C_o & O & D_o & O \\ O & O & O & E \end{array}\right) \in Sp(n,\mathbb{C})$$

maps H_n onto a domain $H_{n,j}$

$$(36) \qquad N : H_n \tilde{\to} H_{n,j} \subset Z_n$$

$$(H_{n,o} = H_n,\ H_{n,n} = E_n) .$$

We consider

$$(37) \qquad \tilde{\Psi} = N \bullet \Psi : H \times E^{N-1} \to H_{n,j}$$

instead of Ψ. We have

$$(38) \qquad \tilde{\Psi}(z+1,w) = \tilde{M}\tilde{\Psi}(z,w) .$$

Obviously $\tilde{M}$ is of the form

$$\tilde{M} = \begin{pmatrix} E & T \\ O & E \end{pmatrix}\begin{pmatrix} U' & O \\ O & U^{-1} \end{pmatrix} , \quad U \neq \pm E .$$

From (24) we deduce

$$(39) \qquad \tilde{M}^h = \begin{pmatrix} E & * \\ O & E \end{pmatrix} ,$$

i.e. U is of finite order! We have

$$\tilde{\Psi}(z,w) = S_o z + \tilde{\Psi}_o(q_m,w) , \quad S_o = \begin{pmatrix} O & O \\ O & S \end{pmatrix}$$

(the same S_o as in (24)!)

and

$$(40) \qquad d\tilde{\Psi}(z+1,w) = U' d\tilde{\Psi}(z,w) U .$$

We now consider the pullback $\Psi^*(P)(z^{(1)},\ldots,z^{(k)})$ $(z^{(\nu)} \in \mathbb{C}^N)$ as a function of z. A similar consideration as in the first case shows that this function is of the form

$$(41) \qquad q^{-d} P(W_1(q_m),\ldots,W_k(q_m)) .$$

Here $W_\nu(q_m)$ $(1 \leq \nu \leq k)$ are holomorphic functions in $|q_m| < 1$ with the property

$$(42) \qquad W_\nu(e^{\frac{2\pi i}{m}} q_m) = U' W_\nu(q_m) U .$$

If $U \neq \pm E$ the function $P(W_1(q_m),\ldots,W_k(q_m))$ will have a zero of a certain order at $q_m = 0$. We want to estimate this order. The matrix U is of finite order h, $m|h$. We may is also > 1/(n-1) if m is big enough!

This simple observation completes the proof of our theorem.

Construction of $f_1,\dots,f_t$.
We use the wellknown "Thetanullwerte"

$$\vartheta_{a,b}(Z) = \sum_{g\in\mathbb{Z}^n} e^{\pi i(Z[g+\frac{1}{2}a]+b'g)}$$

$$a,b \in \{0,1\}^n\ ,\ a'b \equiv 0 \bmod 2\ .$$

The functions

$$f_l = \sum_{(\alpha,\beta)} \prod_{(a,b)\neq(\alpha,\beta)} \vartheta_{a,b}(Z)^{8l}\ ,\ l = 1,2,\dots$$

are modular forms with respect to the full modular group. Their vanishing order (relative to the weight) has been computed [3], p. 204. One obtains the value

$$\geq \frac{(n+1)(2^{2(n-1)}-1)}{8[2^{(n-2)}(2^n+1)-\frac{1}{2}]} > \frac{1}{n-1} \quad \text{if} \quad n \geq 10\ .$$

So the vanishing order of f_l ($l = 1,2,\dots$) is sufficiently high. But we still have to show that the set of common zeros of all f_l, i.e. the union of the sets

$$\{Z;\ \vartheta_{a,b}(Z) = \vartheta_{\alpha,\beta}(Z) = 0;\ (a,b) \neq (\alpha,\beta)\}$$

is of codimension ≥ 2. It is easy to see that two Thetanullwerte are linearly independent if their characteristics are different. The assertion (and therefore our theorem) now follows from

3.3 Proposition: Let f be a modular form of weight 1/2 (and some multiplier system) with respect to some subgroup $\Gamma \subset \Gamma_n$ of finite index. Assume $n \geq 4$. The zero-divisor of f in H_n/Γ is irreducible.

Proof. If $n \geq 4$ each divisor in H_n/Γ is the zero divisor of a modular form. It is therefore sufficient to prove:

Assume $n \geq 2$. There is no nonvanishing modular form of weight

$$r\ ,\quad 0 < r < \frac{1}{2}\ .$$

Such a modular form has to be singular (this follows from

[3], A 4.1.2). But the weight of a nonvanishing singular modular form is always a multiple of $\frac{1}{2}$ (this shows the proof of A 4.1 in [3]). Hence proposition 3.3 and therefore our theorem has been proved.

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