Some vector valued Siegel modular forms of genus 3

Eberhard Freitag
Mathematisches Institut
Im Neuenheimer Feld 288
D69120 Heidelberg
`freitag@mathi.uni-heidelberg.de`

Riccardo Salvati Manni
Dipartimento di Matematica,
Piazzale Aldo Moro, 2
I-00185 Roma, Italy.
`salvati@mat.uniroma1.it`

2013

Introduction

There seem not to be many structure results about vector valued Siegel modular forms in the case of degree $n > 1$. In the case $n = 2$ there are several structure results [Do], [Sat] which rest on the dimension formulae of Tsushima [Ts]. Another geometric appraoch in the case $n = 2$ is due to T. Wieber. His method is closely related to the method which we use here. His work is still in progress. In this paper we treat an $n = 3$ case.

We consider the Siegel modular group $\Gamma_n = \mathrm{Sp}(n, \mathbb{Z})$, its principal congruence subgroup

$$\Gamma_n[q] = \mathrm{kernel}(\mathrm{Sp}(n, \mathbb{Z}) \longrightarrow \mathrm{Sp}(n, \mathbb{Z}(q\mathbb{Z})$$

and Igusa's group

$$\Gamma_n[q, 2q] := \left\{ M \in \Gamma_n[q], \quad \frac{1}{q}(C\,{}^tD) \equiv \frac{1}{q}(A\,{}^tB) \equiv 0 \bmod 2 \right\}.$$

We are particularly interested in the cases $n = 3$ and $q = 2$. We recall the theta constants of second kind

$$\Theta[a](Z) = \sum_{g\in\mathbb{Z}^n} e^{2\pi \mathrm{i} Z[g+a/2]}, \qquad a \in (\mathbb{Z}/2\mathbb{Z})^n.$$

These are functions on the Siegel upper half plane

$$\mathbb{H}_n = \{Z \in \mathbb{C}^{n\times n}; \quad Z = {}^tZ, \ \mathrm{Im}(Z) \text{ positive definit}\}.$$

They are modular forms for $\Gamma_n[2, 4]$ of weight $1/2$ which all have the same multiplier system v_Θ. One has $v_\Theta^4 = 1$. In the cases $n \le 3$ one has

$$A(\Gamma_n[2, 4]) = \bigoplus_{r\in\mathbb{Z}} [\Gamma_3[2, 4], v_\Theta^r, r/2] = \mathbb{C}[\Theta[\cdot]], \qquad n \le 3.$$

Here we use the standard notation $[\Gamma, v, r/2]$ for the space of modular forms of transformation type

$$f(MZ) = v(M)\det(CZ + D)^{r/2} f(Z).$$

(Since we include half integral weights, we have to make a choice of a holomorphic square root of $\det(CZ + D)$.) In the cases $n \le 2$, the forms $\Theta[a]$ are algebraically independent. The case $n = 3$ has been treated by Runge [Ru] (in a slightly modified form). In the following we will use the notation

$$\begin{aligned} f_0 &= \Theta[0,0,0], \quad f_1 = \Theta[0,0,1], \quad f_2 = \Theta[0,1,0], \quad f_3 = \Theta[0,1,1], \\ f_4 &= \Theta[1,0,0], \quad f_5 = \Theta[1,0,1], \quad f_6 = \Theta[1,1,0], \quad f_7 = \Theta[1,1,1]. \end{aligned}$$

Theorem (Runge). *There is – up to constant factor – a unique polynomial of degree 16 in 8 variables such that $P(f_0, \dots, f_7) = 0$. We have*

$$A(\Gamma_3[2,4]) \cong \mathbb{C}[X_0, .., X_7]/(P).$$

We call R the Runge polynomial. Recall that by Baily's theorem the projective hypersurface in $P^7\mathbb{C}$ defined by $R = 0$ can be identified with the Satake compactification of $\mathbb{H}_3/\Gamma_3[2,4]$.

In this paper we study vector valued modular forms of the transformation type

$$f(MZ) = v_\Theta(M)^r \det(CZ+D)^{r/2}(CZ+D)f(Z)\,{}^t(CZ+D) \qquad (M \in \Gamma_n[2,4])$$

(mainly in the case $n = 3$). Here f should be a symmetric $n \times n$-matrix of holomorphic functions. We denote by $\mathcal{M}_n^+(r)$ the vector space of all forms of the above transformation type and we consider

$$\mathcal{M}_n^+ = \bigoplus_k \mathcal{M}_n^+(r).$$

This is a module over the ring $\mathbb{C}[\Theta[\cdot]]$.

Similarly we introduce the space $\mathcal{M}_n^-(r)$ by the transformation formula

$$f(MZ) = v_\Theta^{r+2} \det(CZ + D)^{r/2}(CZ + D)f(Z)\,{}^t(CZ + D) \qquad (M \in \Gamma_n[2,4])$$

and we consider

$$\mathcal{M}_n^- = \bigoplus_k \mathcal{M}_n^-(r)$$

which is also a module over $\mathbb{C}[\Theta[\cdot]]$.

These functions are related to $\Gamma_n[2,4]$-invariant tensors in the following way. We consider the symmetric matrix dZ with the entries dz_{ik} and we denote by

$$\omega = dz_{11} \wedge dz_{12} \wedge \ldots \wedge dz_{nn}$$

the wedge product of all dz_{ij}, $i \leq j$, in lexicographical ordering. By dZ we understand the symmetric matrix with entries dz_{ij} The tensor

$$T = \operatorname{tr}(f dZ) \otimes \omega^{\otimes k}, \quad f \text{ a symmetric matrix of holomorphic functions},$$

is invariant under $\Gamma_n[2,4]$ if and only if f has the transformation property

$$f(MZ) = \det(CZ+D)^{k(n+1)}(CZ+D)f(Z)\,{}^t(CZ+D) \qquad (M \in \Gamma_n[2,4]).$$

This means

$$f \in \begin{cases} \mathcal{M}_n^+(2k(n+1)) & \text{if } k(n+1) \text{ is even,} \\ \mathcal{M}_n^-(2k(n+1)) & \text{if } k(n+1) \text{ is odd.} \end{cases}$$

In this paper we are mainly interested in the case $n = 3$. In this case the holomorphic tensors belong to $\mathcal{M}_n^+$.

The easiest way to get such vector valued modular forms is to consider brackets of the form

$$\{f, g\} = g^2 d(f/g).$$

Here the three components of $d(f/g)$ are written into a symmetric $n \times n$-matrix with the entries

$$\{f,g\}_{ij} = e_{ij} g^2 \frac{\partial(f/g)}{\partial z_{ij}}, \qquad e_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

If f, g are from $[\Gamma_3[2,4], v_\Theta, 1/2]$, then $\{f,g\}$ is an element of $\mathcal{M}^+(1)$. We can consider the sub-module

$$\sum_{a,b} \mathbb{C}[\Theta[\cdot]]\ \{\Theta[a], \Theta[b]\} \subset \mathcal{M}_n^+.$$

Here is our main result.

Theorem. *Assume $n = 3$. Defining relations of the sub-module*

$$\sum_{a,b} \mathbb{C}[\Theta[\cdot]]\ \{\Theta[a], \Theta[b]\} = \sum_{0 \leq i,j \leq 7} \mathbb{C}[f_0, \ldots, f_7]\{f_i, f_j\} \subset \mathcal{M}_3^+$$

are

$$f_j\{f_i, f_k\} = f_i\{f_j, f_k\} + f_k\{f_i, f_j\}, \quad \{f_i, f_j\} + \{f_j, f_i\} = 0,$$

$$\sum_{j=0}^{7} R_j\{f_j, f_i\} = 0 \qquad \text{(for each } i\text{).}$$

Here R_j denote the derivatives of the Runge polynomial by the 8 variables. For the full module one has

$$\mathcal{M}_3^+ = \bigcap_{0 \leq k \leq 7} \frac{1}{f_k} \sum_{0 \leq i,j \leq 7} \mathbb{C}[f_0, \ldots, f_7]\{f_i, f_j\}.$$

This is in principle a complete algebraic description of $\mathcal{M}_3^+$. Using Gröbner algorithms, computer algebra systems can compute the intersection of modules which have been defined by relations. The computers which we could use couldn't do this job, since the Ringe polynomial and its derivatives are quite involved.

One might conjecture that the module and its sub-module agree. But this is not the case. It will turn out that the two modules are different. We will construct counter examples in Sect. 5 using a quite sophistic method.

1. Holomorphic tensors in genus three

From now on we restrict to the case $n = 3$. We want to study $\Gamma_3[2,4]$-invariant holomorphic tensors on the Siegel half plane of genus three of the type

$$T = \operatorname{tr}(f\,dZ) \otimes \omega^{\otimes k}, \quad \omega = dz_{11} \wedge dz_{12} \wedge \ldots \wedge dz_{33}.$$

From compactification theory follows that they can be considered as rational tensors on the hypersurface defined by the Runge polynomial R. let $\varphi_1, \ldots, \varphi_6$ by a transcendental basis of the field of modular functions (which is the field of rational functions on the hypersurface). Then there exist modular functions $T_1, \ldots, T_6$ such that

$$T = \Big(\sum_{\nu=1}^{6} T_\nu d\varphi_\nu\Big)\ (d\varphi_1 \wedge \ldots \wedge d\varphi_6)^{\otimes k}.$$

We have to work out what it means that they are holomorphic on $\mathbb{H}_3$. It is known that $\Gamma_3[2,4]$ has no fixed point sets of codimension 1. Hence the holomorphicity means that the tensor is holomorphic on the regular locus on the hyper surfaces.

We take the concrete transcendental basis

$$\varphi_\nu = \frac{f_\nu}{f_0}, \quad 1 \le \nu \le 6.$$

We also use the notations $\varphi = (\varphi_1, \ldots, \varphi_6)$ and

$$z_1 = z_{11}, \quad z_2 = z_{12}, \quad z_3 = z_{13}, \quad z_4 = z_{22}, \quad z_5 = z_{23}, \quad z_6 = z_{33}.$$

We denote by $J = J(\varphi, z)$ the Jacobian matrix with the entries

$$J_{ik} = \frac{\partial \varphi_i}{\partial z_k}.$$

The determinant of J can be determined.

1.1 Theorem. *There exists a constant C such that*

$$f_0^7 \det J = CR_7.$$

Corollary. *The formula*

$$d\varphi_1 \wedge \ldots \wedge d\varphi_6 = C\,\frac{R_7}{f_0^7}\,dz_1 \wedge \ldots \wedge dz_6$$

holds.

Recall that R_i are the derivatives of the Ringe polynomial.

Proof of Theorem 1.1. We have to make use of the theta functions in two variables

$$\Theta[a](Z,w) = \sum_{g\in\mathbb{Z}^n} e^{2\pi \mathrm{i}(Z[g+a/2]+2\,{}^t(g+a/2)w)}$$

So the functions $\Theta[a](Z)$ which have been introduced already are just their "nullwerte", $\Theta[a](Z,0)$. We have to study them for fixed Z as functions of w. At the moment the degree n can be arbitrary. These functions span for a fixed non-split Z a vector space of dimension 2^n. This vector space has an important subspace which is sometimes denoted by Γ_{00}. It consists of all functions that vanish at the origin $w=0$ of order at least 4. We will use the description of van Geemen and van der Geer for this space, [GG]. The dimension of this space is $2^n - n(n+1)/2 - 1$. Generators of this space can be constructed as follows. Let I be a homogenous polynomial in 2^n variables which describes a relation between the nullwerte,

$$I(\ldots,\Theta[a](Z,0),\ldots) = 0.$$

We denote by $\partial I/\partial\Theta[a]$ the derivatives of this polynomial. Then

$$f_I(w) = \sum_{a\in\mathbb{Z}^n} \partial I/\partial\Theta[a]\;\Theta[a](Z,w)$$

is contained in the space Γ_{00} and this space is generated by these functions.

We need the following important result of Sasaki [Sa] which characterizes the split points $Z \in \mathbb{H}_n$. A point is called a split-point if it is equivalent modulo the full modular group to a bloc matrix of the form

$$\begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix}$$

where Z_{11} and Z_{22} have degree $< n$. We have to use the following result of Sasaki [Sa].

Proposition (Sasaki). *Let $Z \in \mathbb{H}_n$ be a non-split point. Assume that $\epsilon(i)$ runs trough the $N = 2^n$ elements of $(\mathbb{Z}/2\mathbb{Z})^n$. Then the matrix*

$$\begin{pmatrix} \Theta[\epsilon_1] & \dots & \Theta[\epsilon_N] \\ \frac{\partial\Theta[\epsilon_1]}{z_{11}} & \dots & \frac{\partial\Theta[\epsilon_N]}{z_{11}} \\ \dots & \dots & \dots \\ \frac{\partial\Theta[\epsilon_1]}{z_{nn}} & \dots & \frac{\partial\Theta[\epsilon_N]}{z_{nn}} \end{pmatrix}$$

has maximal rank $n(n+1)/2+1$ at the point Z.

Proof of Theorem 1.1 continued. Another construction of elements of Γ_{00} has been described in [GS]. By Sasaki's result one can find for a non-split point Z elements $\epsilon_1, \dots, \epsilon_{n(n+1)/2+1}$ such the the nullwerte $\Theta[\epsilon_i]$ are analytically independent at Z. Then for any $\epsilon \in \mathbb{F}_2^n$ different from the ϵ_i the function

$$f(\tau_0,\, z) = \det \begin{pmatrix} \Theta[\epsilon_1](\tau_0, z) & \Theta[\epsilon_1](\tau_0, 0) & \frac{\partial\Theta[\epsilon_1](\tau_0,0)}{\partial\tau_{11}} & \dots & \frac{\partial\Theta[\epsilon_1](\tau_0,0)}{\partial\tau_{nn}} \\ \vdots & \vdots & \vdots & \vdots & \\ \Theta[\epsilon_N](\tau_0, z) & \Theta[\epsilon_N](\tau_0, 0) & \frac{\partial\Theta[\epsilon_N](\tau_0,0)}{\partial\tau_0\tau_{11}} & \dots & \frac{\partial\Theta[\epsilon_N](\tau_0,0)}{\partial\tau_{nn}} \\ \Theta[\epsilon](\tau_0, z) & \Theta[\epsilon](\tau_0, 0) & \frac{\partial\Theta[\epsilon](\tau_0,0)}{\partial\tau_{11}} & \dots & \frac{\partial\Theta[\epsilon](\tau_0,0)}{\partial\tau_{nn}} \end{pmatrix}$$

is contained in Γ_{00} and these functions define a basis of Γ_{00}.

Now we specialize to the case $n = 3$. Here Γ_{00} is one-dimensional. We can take for ϵ_i all but one characteristic ϵ. We use the notation

$$D(\epsilon) := \begin{pmatrix} \Theta[\epsilon_1] & \dots & \Theta[\epsilon_7] \\ \frac{\partial\Theta[\epsilon_1]}{z_{11}} & \dots & \frac{\partial\Theta[\epsilon_7]}{z_{11}} \\ \dots & \dots & \dots \\ \frac{\partial\Theta[\epsilon_1]}{z_{nn}} & \dots & \frac{\partial\Theta[\epsilon_7]}{z_{nn}} \end{pmatrix}$$

So we have with certain signs $\pm$ that

$$f(Z, w) = \sum_{\epsilon} \pm D(\epsilon)\Theta[\epsilon](Z, w).$$

We compare this with the expression of van Geemen and van der Geer using for the relation I Runge's relation R. Comparing coefficients we get

$$D(\varepsilon) = c\frac{\partial R}{\Theta[\epsilon]}.$$

The constant c might depend on Z. But it is holomorphic outside the split locus and it is invariant under $\Gamma_3[2, 4]$. Since the split-locus is of codimension ≥ 2 it is holomorphic everywhere and hence independent of Z. Now Theorem 1.1 is an immediate consequence. $\square$

We come back to the tensor

$$T = \Big(\sum_{\nu=1}^{6} T_\nu d\varphi_\nu\Big)\ (d\varphi_1 \wedge \ldots \wedge d\varphi_6)^{\otimes k}.$$

Its holomorphicity means that the row

$$\Big(\frac{R_7}{f_0^7}\Big)^k (T_1, \ldots, T_6) \cdot J$$

is holomorphic on $\mathbb{H}_3$. From construction we have that $f_0^2 J$ is holomorphic on the whole Siegel half plane.

We denote by J^* the adjoint matrix of J. It has the property

$$J \cdot J^* = \det J E = \frac{R_7}{f_0^7} E.$$

It is easy to check that $f_0^6 J^*$ is holomorphic. This shows that

$$F_\nu = R_7 f_0 \frac{1}{f_0^2}\Big(\frac{R_7}{f_0^7}\Big)^k T_\nu$$

are holomorphic. They are contained in $A(\Gamma_3[2,4])$ and hence expressible as polynomials in the f_a. We obtain

$$T = \Big(\sum \frac{F_\nu f_0}{R_7} d\varphi_\nu\Big)\ (dz_1 \wedge \cdots \wedge dz_6)^k.$$

This shows

$$T \in \frac{1}{f_0 R_7} \sum_{i \neq 0} \mathbb{C}[f_0, \ldots, f_7]\{f_0, f_i\}.$$

During the proof we marked the indices 0 and 7. We could have done this for any pair $\alpha \neq \beta$. Hence lets define

$$\mathcal{N}(\alpha,\beta) = \frac{1}{f_\alpha R_\beta} \sum_{i \neq \alpha} \mathbb{C}[f_0, \ldots, f_7]\{f_\alpha, f_i\} \qquad (0 \le \alpha, \beta \le 7,\ \alpha \neq \beta).$$

1.2 Lemma. *A holomorphic vector valued modular form f of the transformation type*

$$f(MZ) = \det(CZ+D)^{k(n+1)}(CZ+D)f(Z)^t(CZ+D) \qquad (M \in \Gamma_n[2,4])$$

is contained in

$$\bigcap_{\alpha \neq \beta} \mathcal{N}(\alpha,\beta).$$

2. Relations

The forms $\{f_i, f_j\}$ satisfy the obvious relations

$$f_j\{f_i, f_k\} = f_i\{f_j, f_k\} + f_k\{f_i, f_j\}, \quad \{f_i, f_j\} + \{f_j, f_i\} = 0.$$

From $R(f_0, \dots, f_7) = 0$ we can derive more relations as follows. This relations implies

$$R(1, \varphi_1, \dots, \varphi_7) = 0 \qquad (\varphi_i = f_i/f_0)$$

and hence

$$0 = \partial R/\partial z_i = \sum_{j=1}^{7} R_j(1, \varphi_1, \dots, \varphi_7)\frac{\partial \varphi_j}{\partial z_i}$$

or

$$\sum_{j=1}^{7} R_j\{f_j, f_0\} = 0.$$

We can do this for every index i instead of 0. So we get for each i the relation

$$\sum_{j=0}^{7} R_j\{f_j, f_i\} = 0 \qquad (i \text{ is fixed}).$$

We want to prove that the relations so far are defining relations. To get a precise formulation, we consider over $\mathbb{C}[f_0, \dots, f_7]$ the free module $\mathcal{F}$ that is generated by symbols $\{X_i, X_j\}$, $0 \le i, j \le 7$. There is a natural $\mathbb{C}[f_0, \dots, f_7]$-linear map

$$\mathcal{F} \longrightarrow \mathcal{M}_3^+, \quad \{X_i, X_j\} \longmapsto \{f_i, f_j\}.$$

2.1 Lemma. *The expression*

$$\sum_{i<j} P_{ij}\{X_i, X_j\}$$

is in the kernel of the natural map $\mathcal{F} \to \mathcal{M}_3^+$ if and only if the relations

$$P_j R_7 = P_7 R_j \qquad (1 \le l \le 6)$$

where

$$P_j = X_0 P_{0j} + \sum_{i>j} X_i P_{ji} - \sum_{0<i<j} X_i P_{ij}$$

are satisfied.

Proof. We consider a relation

$$\sum_{i<j} P_{ij}\{f_i, f_j\} = 0.$$

We multiply with f_i and insert in the case $i > 0$

$$f_0\{f_i, f_j\} = f_j\{f_0, f_i\} - f_i\{f_0, f_j\}.$$

This gives

$$\sum_{0<j} f_0 P_{0j}\{f_0, f_j\} + \sum_{0<i<j} f_j P_{ij}\{f_0, f_i\} - \sum_{0<i<j} f_i P_{ij}\{f_0, f_j\} = 0.$$

We set

$$P_j = f_0 P_{0j} + \sum_{i>j} f_i P_{ji} - \sum_{0<i<j} f_i P_{ij}$$

Then we have

$$\sum_{j=1}^{7} P_j\{f_0, f_j\} = 0.$$

We recall that we also have

$$\sum_{j=1}^{7} R_j\{f_0, f_j\} = 0.$$

We multiply the first by R_7 and the second by P_7 and subtract.

$$\sum_{j=1}^{6} (P_j R_7 - P_7 R_j)\{f_0, f_j\} = 0.$$

Since the six functions $\varphi_1, \ldots, \varphi_6$ are algebraically independent, we obtain

$$P_j R_7 - P_7 R_j = 0 \qquad (1 \le l \le 6).$$

3. An algebraic result

We consider the polynomial ring $\mathbb{C}[X_0,\dots,X_n]$ of arbitrarily many variables. Let R be a homogenous irreducible polynomial. We denote its partial derivative by $R_i=\partial R/\partial X_i$. We assume that the all are different from zero. We consider the ring

$$\mathcal{R}=\mathbb{C}[X_0,\dots,X_n]/(R).$$

As in the previous section we consider the free module

$$\mathcal{F}=\sum_{i<j}\mathcal{R}\{X_i,X_j\}$$

where $\{X_i,X_j\}$ are just symbols. It is convenient to define also $\{X_i,X_i\}=0$ and $\{X_i,X_j\}=-\{X_j,X_i\}$ for $i>j$. We consider the submodule which is generated by the elements

$$\begin{aligned}&X_k\{X_i,X_j\}-X_j\{X_k,X_i\}-X_i\{X_j,X_k\},\\ &\sum_{j\neq i}R_j\{X_j,X_i\}\quad (i \text{ is fixed})\end{aligned}$$

We denote the quotient of $\mathcal{F}$ by this submodule by $\mathcal{N}$. We can consider $\mathcal{N}$ as module over $\mathbb{C}[X_0,\dots,X_n]$ but also as module over $\mathcal{R}$. We can consider the natural projection

$$\mathcal{F}\longrightarrow\mathcal{N}.$$

3.1 Proposition. *We assume that the $n-1$ elements $\{X_0,X_i\}$, $0<i<n$, are independent in the sense that they generate a free submodule of the $\mathcal{R}$-module $\mathcal{N}$. Then the following three conditions are equivalent:*

1) *The module $\mathcal{N}$ is torsion free as $\mathcal{R}$-module.*
2) *Multiplication by X_0R_n defines an injective map $\mathcal{N}\to\mathcal{N}$.*
3) *An element*

$$\sum_{i<j}P_{ij}\{X_i,X_j\}$$

is a relation (i.e. in the kernel of $\mathcal{F}\to\mathcal{N}$) if and only if the relations

$$P_jR_n-P_nR_j=0 \quad \textit{in}\quad \mathcal{R}\qquad (1\le j\le n)$$

where

$$P_j=X_0P_{0j}+\sum_{i>j}X_iP_{ji}-\sum_{0<i<j}X_iP_{ij}$$

are satisfied.

Proof. 1)⇒2) is trivial. Also 3)⇒1) is clear. So it remains to show 2)⇒3). So let's assume that 2) is satisfied. One has to show 3). The proof is very similar to that of Proposition 4.3. So let's just explain the essential points. First lets assume that

$$\sum_{i<j} P_{ij}\{X_i, X_j\} \qquad (P_{ij} \in \mathcal{R})$$

is a relation. We can do the same calculation as in Proposition 4.3: We multiply by X_0 and insert in the case $i > 0$ the relation

$$X_0\{X_i, X_j\} - X_j\{X_0, X_i\} + X_i\{X_0, X_j\}.$$

This gives that

$$\sum_{0<j} X_0 P_{0j}\{X_0, X_j\} + \sum_{0<i<j} X_j P_{ij}\{X_0, X_i\} - \sum_{0<i<j} X_i P_{ij}\{X_0, X_j\}$$

is a relation. We set

$$P_j = X_0 P_{0j} + \sum_{i>j} X_i P_{ji} - \sum_{0<i<j} X_i P_{ij}$$

Then we have that

$$X_0 \sum_{i<j} P_j\{X_0, X_j\}$$

is a relation. We recall that we also have that

$$\sum_{j=1}^{n} R_j\{X_0, X_j\}$$

is a relation. We multiply the first by R_n and the second by P_n and subtract. Then we get that

$$\sum_{j=1}^{n-1} (P_j R_7 - P_7 R_j)\{X_0, X_j\}$$

is a relation. From the assumption in the proposition we get

$$P_j R_n - P_n R_j = 0 \qquad (1 \le l \le n-1)$$

where

$$P_j = X_0 P_{0j} + \sum_{i>j} X_i P_{ji} - \sum_{0<i<j} X_i P_{ij}.$$

We also have to show the converse, namely that the conditions $P_j R_n - P_n R_j = 0$ imply that

$$\sum_{i<j} P_{ij}\{X_i, X_j\}.$$

By the assumption in 2) it is enough to prove that

$$X_0 R_n \sum_{i<j} P_{ij}\{X_i, X_j\}.$$

But the same calculation as above shows that this is equivalent to the fact that

$$R_n \sum_{j=1}^{7} P_j\{X_0, X_j\}$$

is a relation. But this follows from the relation $R_n P_j = P_n R_j$. $\square$

A better result can be obtained if one has more assumptions for the polynomial R.

3.2 Proposition. *Assume that the polynomial R satisfies the following assumption:*

The codimension of the zero locus of (R, R_n, R_i) in $P^n(\mathbb{C})$ has codimension ≥ 3 for $i < n$.

Then, in Proposition 3.1 it is sufficient to replace 2) by the following weaker condition:

2) *Multiplication by X_0 defines an injective map $\mathcal{N} \to \mathcal{N}$.*

Proof. We have to show that the conditions $P_j R_n - P_n R_j = 0$ imply that

$$X_0 \sum_{i<j} P_{ij}\{X_i, X_j\}$$

is a relation. We write $P_j = A_j R_j$ with a rational function A_j. The assumption implies that A_j is regular outside a subset of codimension ≥ 2 in the zero locus of R. This implies that A_j is regular, i.e. contained in $\mathcal{R}$. We also obtain that A_j is independent of j,

$$P_j = A R_j.$$

During the proof of Proposition 3.1 we have seen that

$$X_0 \sum_{i<j} P_{ij}\{X_i, X_j\}$$

is a relation if and only if

$$\sum_{j} P_j\{X_0, X_j\}$$

is a relation. But this is one of the defining relations since P_j is a multiple of R_j. $\square$

4. A structure theorem

We apply the results of the previous section to the case $n = 8$ where R is the Runge polynomial R.

4.1 Lemma. *Let R be the Runge polynomial which describes the defining relation between the theta series f_i. For any two indices $i \neq j$ the dimension of the joint zero locus in $P^7(\mathbb{C})$ of R_i, R_j, R is 4.*

Proof. Let X be the joint zero locus of R_i, R_j, R. It is sufficient to find a 4-dimensional subspace $L \subset P^7$ such that $X \cap L$ has dimension zero. For example for $i = 0$ and $j = 7$ one can take of L the zero set of the linear forms

$$X_1 - X_2,\ X_2 - X_4,\ X_5 - X_7,\ X_0 - 3X_6.$$

The proof can be given with the help of a computer. We used the command `dim` of the computer algebra system SINGULAR. □

From the Lemma follows that the singular locus of the zero locus of R has codimension ≥ 2. This implies that the factor ring $\mathbb{C}[X_0, \dots X_7]/(R)$ is normal. This is the essential point in the proof of Runge's result that $A(\Gamma_3[2,4])$ is generated by the theta functions $\Theta[a]$ and that R gives the defining relation. So we get a new proof of this result.

Recall that we have to consider the ring

$$\mathcal{R} = \mathbb{C}[X_0, \dots, X_7]/(R)$$

where now R is the Runge polynomial, and we consider the free module

$$\mathcal{F} = \sum_{i<j} \mathcal{R}\{X_i, X_j\}$$

and its quotient $\mathcal{N}$ which is defined by the relations

$$\begin{aligned} &X_k\{X_i, X_j\} - X_j\{X_k, X_i\} - X_i\{X_j, X_k\}, \\ &\sum_{j\neq i} R_j\{X_j, X_i\} \quad (i \text{ is fixed}) \end{aligned}$$

4.2 Lemma. *Multiplication by X_0 defines an injective map $\mathcal{N} \to \mathcal{N}$.*

Proof. The proof can again be given by means of SINGULAR. □

Now we obtain from Lemma 2.1 and the Proposition 3.2 the following description of the relations between the vector valued modular forms $\{f_i, f_j\}$.

4.3 Proposition. *The kernel of the map $\mathcal{F} \to \mathcal{M}_3^+$ is generated as $\mathbb{C}[f_0,\dots,f_7]$-module by*

$$\begin{aligned}
&X_k\{X_i,X_j\} - X_j\{X_k,X_i\} - X_i\{X_j,X_k\},\\
&\{X_i,X_j\} - \{X_j,X_i\},\\
&\sum_{j\neq i} R_j\{X_j,X_i\} \quad \textit{(i is fixed).}
\end{aligned}$$

The next result can also be proved with the help of SINGULAR, since we know the relations of the occurring modules.

4.4 Lemma. *Let $\alpha \neq \beta$. Then*

$$\begin{aligned}
\frac{1}{f_\alpha}\sum_{i\neq\alpha}\mathbb{C}[f_0,\dots,f_7]\{f_\alpha,f_i\} \,\cap\, \frac{1}{f_\beta}\sum_{i\neq\beta}\mathbb{C}[f_0,\dots,f_7]\{f_\beta,f_i\} = \\
\sum_{i<j}\mathbb{C}[f_0,\dots,f_7]\{f_i,f_j\}.
\end{aligned}$$

From this lemma we obtain for the intersections of the modules $\mathcal{N}(\alpha,\beta)$ (see Lemma 1.2):

$$\bigcap_{\alpha\neq\beta}\mathcal{N}(\alpha,\beta) = \bigcap_{\alpha}\frac{1}{R_\alpha}\sum_{i<j}\mathbb{C}[f_0,\dots,f_7]\{f_i,f_j\}.$$

We know that each holomorphic tensor is contained in this module. But we can prove more.

4.5 Theorem. *We have*

$$\mathcal{M}_3^+ = \bigcap_{\alpha}\frac{1}{R_\alpha}\sum_{i<j}\mathbb{C}[f_0,\dots,f_7]\{f_i,f_j\}.$$

Proof. Let $F \in \mathcal{M}_3^+$ a homogenous element. We can multiply it by a monomial P in the forms f_a of a suitable weight to obtain a tensor ($r = 8k$). Then we know that PF is contained in the right hand side. This means that R_iPF is contained in $\sum_{i<j}\mathbb{C}[f_0,\dots,f_7]\{f_i,f_j\}$. From Lemma 4.4 we can deduce that R_iF is contained in $\sum_{i<j}\mathbb{C}[f_0,\dots,f_7]\{f_i,f_j\}$. □

Theorem 4.5 is a complete algebraic description of $\mathcal{M}_3^+$. One might think that the right hand side equals $\sum_{i<j}\mathbb{C}[f_0,\dots,f_7]\{f_i,f_j\}$. But this is not the case. In the next section we will give counter examples.

It is possible to compute the Hilbert function of the sub-module.

4.6 Proposition. *The Hilbert function of the sub-module*

$$\sum_{i<j} \mathbb{C}[f_0,\dots,f_7]\{f_i,f_j\}$$

is

$$\frac{28r^2-56r^3+70r^4-56r^5+28r^6-8r^7+r^8-8r^{17}+r^{32}}{(1-r)^8}=$$
$$28r^2+168r^3+630r^4+1848r^5+4620r^6+10296r^7+21021r^8+\cdots.$$

(Here $r/2$ is the weight that has been introduced in the introduction. Recall that f_i has weight $1/2$ and $\{f_i,f_j\}$ has weight 1.)

5. Theta series of the first kind

We consider the theta series of first kind

$$\vartheta[\mathfrak{m}] = \sum_{g\in\mathbb{Z}^n} e^{\pi \mathrm{i}(Z[g+a/2]+{}^tb(g+a/2))}.$$

Here

$$\mathfrak{m} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^{2n} \quad \text{(column)}$$

is the so-called characteristic. Up to sign the series depends only on $\mathfrak{m}$ mod 2. Usually we will take the entries in $\{0,1\}$. The characteristic is called even if ${}^tab \equiv 0$ mod 2 and odd else. The theta series vanishes identically if and only of the characteristic is odd. We also notice

$$\Theta[a](Z) = \vartheta\begin{bmatrix} a \\ 0 \end{bmatrix}(2Z).$$

The $\Theta[a]$ are called the theta series of second kind. A basic theta relation states

$$\vartheta[\mathfrak{m}]^2 = \sum_{x \bmod 2} (-1)^{{}^txb}\Theta[a+x]\Theta[x].$$

The theta constants are modular forms on $\Gamma[2,4]$ but with respect to rather delicate multipliers. We are interested in monomials which have the same multiplier as the monomials of the same weight in the forms $\Theta[a]$. A result of the second authors states.

5.1 Proposition. *Assume that $\mathfrak{m}_1, \dots, \mathfrak{m}_m$ are characteristics with the following properties.*

1) *m is even.*
2) *The matrix M with the columns $\mathfrak{m}_i$ satisfies*

$$M\ {}^tM \equiv 0 \bmod 2.$$

Then $\vartheta[\mathfrak{m}_1] \cdots \vartheta[\mathfrak{m}_m]$ is a modular form for $\Gamma_n[2,4]$ with respect to the multiplier system v_Θ^m. So in the case $n \le 3$ it is expressible as polynomial in the $\Theta[a]$.

We can use this construction to find new elements of $\mathcal{M}_n^+$.

5.2 Lemma. *Let $\mathfrak{m}_1, \dots, \mathfrak{m}_m$ be as in Proposition 5.1. Then*

$$\vartheta[\mathfrak{m}_1] \cdots \vartheta[\mathfrak{m}_{m-2}]\ \{\vartheta[\mathfrak{m}_{m-1}], \vartheta[\mathfrak{m}_m]\}$$

is contained in the module $\mathcal{M}_n^+$.

Proof. It is sufficient to treat the case $M = (\mathfrak{m}, \mathfrak{n}, \mathfrak{m}, \mathfrak{n})$ since in the general case we can write

$$\begin{aligned}&\vartheta[\mathfrak{m}_1] \cdots \vartheta[\mathfrak{m}_{m-2}]\ \{\vartheta[\mathfrak{m}_{m-1}], \vartheta[\mathfrak{m}_m]\} = \\ &\frac{\vartheta[\mathfrak{m}_1] \cdots \vartheta[\mathfrak{m}_m]}{\vartheta[\mathfrak{m}_{m-1}]^2 \vartheta[\mathfrak{m}_m]^2} \cdot \vartheta[\mathfrak{m}_{m-1}]\vartheta[\mathfrak{m}_m]\ \{\vartheta[\mathfrak{m}_{m-1}], \vartheta[\mathfrak{m}_m]\}.\end{aligned}$$

In the spacial case $M = (\mathfrak{m}, \mathfrak{n}, \mathfrak{m}, \mathfrak{n})$ we can say more, namely, that

$$\vartheta[\mathfrak{m}]\vartheta[\mathfrak{n}]\ \{\vartheta[\mathfrak{m}], \vartheta[\mathfrak{n}]\}$$

is contained in the module

$$\sum_{a,b} \mathbb{C}[\dots \Theta[\cdot] \dots]\ \{\Theta[a], \Theta[b]\}.$$

This follows form the formula

$$2\vartheta[\mathfrak{m}]\vartheta[\mathfrak{n}]\{\vartheta[\mathfrak{m}], \vartheta[\mathfrak{n}]\} = \{\vartheta[\mathfrak{m}]^2, \vartheta[\mathfrak{n}]^2\}.$$

We want to make this explicit. Using the relation

$$\{fg, hk\} = fh\{g,k\} + gk\{f,h\}$$

we get for

$$\mathfrak{m} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \mathfrak{n} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

the expression

$$2\vartheta[\mathfrak{m}]\vartheta[\mathfrak{n}]\,\{\vartheta[\mathfrak{m}],\vartheta[\mathfrak{n}]\} = \sum_{x,y \mod 2} (-1)^{\,{}^t x b + {}^t y \beta}$$
$$\big(\Theta[x]\Theta[y]\,\{\Theta[x+a],\Theta[y+\alpha]\} + \Theta[x+a]\Theta[y+\alpha]\big)\,\{\Theta[x],\Theta[y]\}).$$

We want to describe some of the extra forms explicitly. For this purpose we have to construct systems $M = (\mathfrak{m}_1, \dots, \mathfrak{m}_m)$ as considered in Proposition 5.1. For this purpose we consider the vector space $\mathbb{F}_2^{2n}$ over the field of two elements. We equip it with the symplectic form

$$\langle \mathfrak{m}, \mathfrak{n}\rangle = {}^t a\beta + {}^t \alpha b, \qquad \mathfrak{m} = \begin{pmatrix} a \\ b \end{pmatrix},\ \mathfrak{n} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

The symplectic group $\mathrm{Sp}(n, \mathbb{F}_2)$ is the group of all linear transformations which preserve this form. A subspace is called isotropic if the symplectic form is zero on it. The maximal dimension of an isotropic subspace is n. The symplectic group $\mathrm{Sp}(n, \mathbb{F}_2)$ acts transitively on the maximal isotropic subspaces. An example is given by the set of all $\mathfrak{m}$ with $b = 0$. We write the 2^n elements of a maximal isotropic subspace L in an arbitrary chosen ordering into a matrix $M = (\mathfrak{m}_1, \dots, \mathfrak{m}_{2^n})$. It is easy to check that in the case $n \geq 3$ this matrix has the property $M\,{}^tM = 0$. Instead of L we can take an arbitrary coset $\mathfrak{m} + L$ and write its elements into a matrix M. This matrix has also the property $M\,{}^tM = 0$. Of course we are only interested in matrices where all entries are even (now considered as characteristics, i.e. as elements of $\{0,1\}^{2n}$). It can be shown that each maximal isotropic subspace L has exactly one coset which consists of even elements. So, for each isotropic subspace L, we have constructed a matrix M whose columns are even characteristics and such $M\,{}^tM \equiv 0 \bmod 2$. This matrix is determined only up to the ordering of the columns.

We have introduced the symplectic group $\mathrm{Sp}(n, \mathbb{F}_2)$ as the subgroup of $\mathrm{GL}(2n, \mathbb{F}_2)$ which preserves the symplectic form. We denote the image of an element $\mathfrak{m}$ (similarly of a set of elements) by $M\mathfrak{m}$ or $M(\mathfrak{m})$. This is just the matrix product of the matrix M and the column $\mathfrak{m}$. Besides this linear action we also need the affine action which is defined by

$$M\{m\} := {}^tM^{-1}\mathfrak{m} + \begin{pmatrix} (C\,{}^tD)_0 \\ (A\,{}^tB)_0 \end{pmatrix}.$$

Here X_0 denotes the column built from the diagonal of the matrix X. Under the affine action even elements are mapped to even elements. Let $L \subset \mathbb{F}_2^{2n}$ be a maximal isotropic subspace and $\mathfrak{m} + L$ its even coset, then the even coset of $M(L)$ is ${}^tM^{-1}\{\mathfrak{m} + L\}$. This follows from the fact that $M\{\mathfrak{m} + L\}$ and ${}^tM^{-1}(\mathfrak{m} + L)$ have the same underlying vector space (namely $M(L)$) and that ${}^tM^{-1}\{\mathfrak{m} + L\}$ is even.

Now we restrict to the case $g = 3$. Let L be a 3-dimensional isotropic space. Denote be M the matrix whose columns are the 8 elements of the even coset of M. We use the standard notation

$$\vartheta[M] = \vartheta[\mathfrak{m}_1]\cdots\vartheta[\mathfrak{m}_8] \qquad (M = (\mathfrak{m}_1, \dots, \mathfrak{m}_8).$$

by Runge's theorem it must be possible to express this as polynomial in the $\Theta[a]$. We want to make this explicit. Fur this purpose we need certain Riemann relations.

5.3 Proposition. *Let $U \subset \mathbb{F}_2^6$ be a 2-dimensional isotropic subspace. It has precisely three even cosets U_1, U_2, U_3. A relation*

$$\vartheta[U_1] = \pm\vartheta[U_2] \pm \vartheta[U_3]$$

with certain signs holds.

As a consequence one gets

$$\vartheta[U_2]\vartheta[U_3] = \pm(\vartheta[U_1]^2 - \vartheta[U_2]^2 - \vartheta[U_3]^2).$$

Hence $\vartheta[U_2]\vartheta[U_3]$ is explicitly expressible by the $\Theta[a]$.

5.4 Lemma. *Let $U \subset \mathbb{F}_2^3$ be an isotropic 2-dimensional subspace. It is contained in three maximal isotropic subspaces L. The even coset of each L is the union of two of the even cosets of U.*

We give an example. We take for U the space defined by $m_1 = m_2 = m_3 = m_4 = 0$. Its three even cosets are

$$U,\ \mathfrak{m} + U,\ \mathfrak{n} + U, \quad \text{where} \quad {}^t\mathfrak{m} = (0,0,0,1,0,0),\ {}^t\mathfrak{n} = (1,0,0,0,0,0).$$

The union of the first two is the maximal isotropic space defined by $m_1 = m_2 = m_3 = 0$. The Riemann relations (with correct signs if one identifies the elements of $\mathbb{F}_2$ with the integers $0, 1$) is

$$\vartheta[U] = \vartheta[\mathfrak{m} + U] + \vartheta[\mathfrak{n} + U].$$

We obtain

$$\begin{aligned} 2\prod_{b\in\mathbb{F}_2^3}\vartheta\begin{bmatrix}0\\ b\end{bmatrix} = \quad & \vartheta\begin{bmatrix}000\\100\end{bmatrix}^2\vartheta\begin{bmatrix}000\\101\end{bmatrix}^2\vartheta\begin{bmatrix}000\\110\end{bmatrix}^2\vartheta\begin{bmatrix}000\\111\end{bmatrix}^2 \\ & +\vartheta\begin{bmatrix}000\\000\end{bmatrix}^2\vartheta\begin{bmatrix}000\\001\end{bmatrix}^2\vartheta\begin{bmatrix}000\\010\end{bmatrix}^2\vartheta\begin{bmatrix}000\\011\end{bmatrix}^2 \\ & -\vartheta\begin{bmatrix}100\\000\end{bmatrix}^2\vartheta\begin{bmatrix}100\\001\end{bmatrix}^2\vartheta\begin{bmatrix}100\\010\end{bmatrix}^2\vartheta\begin{bmatrix}100\\011\end{bmatrix}^2. \end{aligned}$$

Here we used the standard notation

$$\vartheta[\mathfrak{m}] = \vartheta\begin{bmatrix}a_1a_2a_3\\ b_1b_2\,b_3\end{bmatrix} \quad \text{for} \quad {}^t\mathfrak{m} = (a_1, a_2, a_3, b_1, b_2, b_3).$$

The number of maximal isotropic subspaces is 135. The symplectic group (linearly) transitively on them. Hence from the one relation above we can produce 135 relations by applying the full modular group.

5.5 Proposition. *Let $L \subset \mathbb{F}_2^6$ be a maximal isotropic subspace and let $M = \{\mathfrak{m}_1, \dots, \mathfrak{m}_8\}$ its even orbit (in an arbitrary ordering). Then*

$$T = \vartheta[\mathfrak{m}_1] \cdots \vartheta[\mathfrak{m}_6]\, \{\vartheta[\mathfrak{m}_7], \vartheta[\mathfrak{m}_8]\}$$

defines a holomorphic tensor in $\mathcal{M}_3^+$ which is not contained in the module $\sum_{i<j} \mathbb{C}[f_0, \dots, f_7]\{f_i, f_j\}$.

Proof. As we have seen,

$$\tilde{T} = \vartheta[\mathfrak{m}_7]^2 \vartheta[\mathfrak{m}_8]^2 \cdot T$$

is contained in $\sum_{i<j} \mathbb{C}[f_0, \dots, f_7]\{f_i, f_j\}$ and even more, one can get an explicit expression in this module. Now we argue indirect. If T were in $\sum_{i<j} \mathbb{C}[f_0, \dots, f_7]\{f_i, f_j\}$, then $\tilde{T}$ would be in

$$\vartheta[\mathfrak{m}_7]^2 \vartheta[\mathfrak{m}_8]^2 \sum_{i<j} \mathbb{C}[f_0, \dots, f_7]\{f_i, f_j\}.$$

Since $\vartheta[\mathfrak{m}]^2$ can be expressed by the f_a this would be a statement inside the module $\sum_{i<j} \mathbb{C}[f_0, \dots, f_7]\{f_i, f_j\}$ which can be falsified by means of a computer calculation, since we have determined generating relations. (We used the computer algebra system SINGULAR.)

6. The Jacobian ideal

In this section we study the Jacobian ideal $(R_0, \dots, R_7)$. Recall that the R_i are the derivatives of Runge's polynomial which is a homogenous polynomial of degree 16 in 8 variables X_i, $0 \le i < 8$. Usually the Jacobian ideal is considered as ideal in the polynomial ring $\mathbb{C}[X_0, \dots, X_7]$. From Euler's identity follows that R is contained in this ideal. Hence it makes no difference if we consider the Jacobian ideal as ideal in $\mathcal{R} = \mathbb{C}[X_0, \dots, X_7]((R)$. We prefer the second point of view. Set theoretically its zero locus coincides with the singular locus of the zero locus of R (which equals the Satake compactification of $\mathbb{H}_3/\Gamma_3[2,4]$. As one knows, this singular locus is the split locus. One of the components of the split locus is given by the closure of the image of matrices of the type

$$\begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix}, \quad Z \in \mathbb{H}_2,\ \tau \in \mathbb{H}_1.$$

We call this component of the split locus the standard component. There are six even characteristics $\mathfrak{m}$ such that $\mathfrak{m}_3 = \mathfrak{m}_6 = 1$. The corresponding thetas $\theta[\mathfrak{m}]$ vanish along the standard component. Recall that their squares can be expressed in a unique way as quadratic polynomials in the variables $X_0, \dots, X_7$.

6.1 Lemma. *The six theta squares $\theta[\mathfrak{m}]^2$ with the property $\mathfrak{m}_3 = \mathfrak{m}_6 = 1$, considered as quadratic polynomials in the variables X_i generate a prime ideal in $\mathbb{C}[X_0, \dots, X_7]$. This prime ideal contains R. Its zero locus is the standard split component. The orbit of this prime ideal with respect to Γ_3 consists of* 336 *prime ideals. The intersection of these* 336 *ideals is the vanishing ideal of the split (=singular) locus.*

We recall the notion of the saturation of an ideal $\mathfrak{a}$ in a finitely generated graded algebra $A = \sum_{d \geq 0} A_d$. Let $\mathfrak{m}$ be the ideal $\sum_{d>0} A_d$. The saturation $\mathfrak{a}^{\text{sat}}$ of $\mathfrak{a}$ consists of all elements $a \in A$ such that $a\mathfrak{m}^d$ is contained in A for sufficiently large d. The ideals $\mathfrak{a}$ and $\mathfrak{a}^{\text{sat}}$ define the same sub-scheme of $\text{proj}(A)$.

6.2 Proposition. *The saturation $(R_0, \dots R_7)^{\text{sat}}$ consists of all f such that $f\mathfrak{m}^2 \in (R_0, \dots, R_7)$. The Hilbert functions of the two ideals can be computed as follows:*

Hilbert function of $\mathcal{R}/(R_0, \dots, R_7)$:

$$\frac{1 - 8r^{15} + 315r^{24} - 1008r^{25} + 1512r^{26} - 1344r^{27} + 720r^{28} - 216r^{29} + 28r^{30}}{(1-r)^8}$$

$$=$$

$$1 + 8r + 36r^2 + 120r^3 + 330r^4 + 792r^5 + 1716r^6 + 3432r^7 + 6435r^8 +$$
$$11440r^9 + 19448r^{10} + 31824r^{11} + 50388r^{12} + 77520r^{13} + 116280r^{14} +$$
$$170536r^{15} + 245093r^{16} + 345816r^{17} + 479740r^{18} + 655160r^{19} +$$
$$881694r^{20} + 1170312r^{21} + 1533324r^{22} + \cdots.$$

Hilbert function of $\mathcal{R}/(R_0, \dots, R_7)^{\text{sat}}$:

$$\frac{1 - 8r^{15} - 8r^{21} + 36r^{22} - 21r^{24}}{(1-r)^8}.$$

Both Hilbert series coincide in degrees ≤ 20. The difference of the first and the second one is

$$8r^{21} + 28r^{22} + \cdots.$$

References

[Do] von Dorp, C.: *Vector-valued Siegel modular forms of genus 2,* MSc Thesis, Korteweg-de Vries Instituut voor Wiskunde, Universiteit van Amsterdam (2011)

[GG] van Geemen, B., van der Geer, G.: *Kummer varieties and the moduli spaces of abelian varieties,*, Amer. J. of Math. **108**, 615–642 (1986)

[GS] Grushevsky, S., Salvati Manni, R.: *The vanishing of two-point functions for three-loop superstring scattering amplitudes,* Comm.Math. Phys. **294**, 343–352 (2010).

[Ru] Runge, B.: *On Siegel modular forms, Part I,*, J. reine angew. Math. **436**, 57–85 (1993)

[Sa] Sasaki, R.: *Modular forms vanishing at the reducible points of the Siegel upper-half space,* J. Reine Angew. Math. **345**, 111–121 (1983).

[Sat] Satoh, T.: *On Certain Vector Valued Siegel Modular forms of Degree Two,* Mathematische Annalen **274**, 335-352 (1986)

[Ts] Tsushima, R.: *An Explicit Dimension Formula for the Spaces of Generalized Automorphic Forms with Respect to* $\mathrm{Sp}(2,\mathbb{Z})$*,* Proc. Japan Acad. **59** (4), 139–142 (1983).