Vector valued Siegel modular forms of level [2,4,8]

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Introduction

Bert van Geemen and Duco van Straten introduced the subgroup $\Gamma_n[2q,4q,8q]$ of the Siegel modular group. It is a subgroup of index two of Igusa's group $\Gamma[2q,4q]$ where

$$\Gamma_n[q,2q] := \left\{ M \in \Gamma_n[q], \quad (C\,{}^tD)_0 \equiv (A\,{}^tB)_0 \equiv 0 \mod 2q \ \right\}$$

which is defined by the additional condition

$$\operatorname{tr}(A) \equiv E \mod 4q \qquad (E \text{ unit matrix}).$$

The group $\Gamma_n[2,4,8]$ can also be defined by the property that the theta series of first kind

$$\vartheta[m] = \sum_{g\in\mathbb{Z}^n} e^{\pi \mathrm{i}(Z[g+a/2]+{}^tb(g+a/2))}, \qquad g = \begin{pmatrix} a \\ b \end{pmatrix} \in \{0,1\}^{2n}, \quad {}^tab \equiv 0 \mod 2,$$

and those of second kind

$$f_a(Z) := \vartheta\begin{bmatrix} a \\ 0 \end{bmatrix}(2Z)$$

are modular forms on this group and all have the same multiplier system v_ϑ. We recall the classical relations

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}^2 = \sum_x (-1)^{{}^txb} f_a f_{a+x}.$$

Due to a result of van Geemen and van Straten [GS] a beautiful thing happens in the case $n = q = 2$. The ring of all modular forms is generated by these 14 theta series and the 10 relations above are defining ones. Geometrically this means that the Satake compactification of $\mathbb{H}_2/\Gamma_2[2,4,8]$ is a complete intersection of ten quadrics.

In this paper we study the space $\mathcal{M}(r)$, $r \in \mathbb{Z}$, of all vector valued modular forms of the transformation type

$$f(MZ) = v_\vartheta(M)^r \det(CZ+D)^{r/2}(CZ+D)f(Z)\,{}^t(CZ+D) \quad (M \in \Gamma_2[2,4,8]).$$

Here f is a symmetric 2×2-matrix of holomorphic functions on the Siegel upper half plane $\mathbb{H}_2$. We collect these spaces to

$$\mathcal{M} = \bigoplus_{r\in\mathbb{Z}} \mathcal{M}(r)$$

which is a module over the ring

$$\mathbb{C}[\ldots,\vartheta[m],\ldots,f_a,\ldots].$$

This module contains a submodule $\mathcal{N}$ which is generated by the 91 Cohen-Rankin brackets $\{f,g\}$ between the 14 generators of this ring. We will determine generating relations betweem the 91 generators which enables us to compute the Hilbert function of $\mathcal{N}$ (Theorem 1.4). Similar as in the $\Gamma[4,8]$-case the module $\mathcal{M}$ is the intersection of the localizations of $\mathcal{N}$ by 60 explicitly described elements (Theorem 2.3). In principle this is a complete algebraic description of $\mathcal{M}$ and the determination of a finite system of generators and relations remains a computational problem. So far we could not solve this problem.

Analogous results have been proved in [Wi] for the group $\Gamma_2[2,4]$ and the multiplier system v_Θ that belongs to the thetas of second kind and in [FS1] for the group $\Gamma_2[4,8]$ and the multiplier system v_ϑ which belongs to the thetas of first kind. Since both multipliers agree on $\Gamma_2[2,4,8]$, the results of this paper can be considered as a common roof of the results in [Wi], [FS1].

As in [Wi] and [FS1] we use a geometric method. If r divides 6 the elements of $\mathcal{M}(r)$ can be identified with certain holomorphic tensors on the regular locus of the modular variety whose structure is simple enough to describe them completely.

1. Defining relations

The space of scalar valued forms $[\Gamma[2,4,8],r/2,v_\vartheta^r]$ consists of holomorphic function f on the Siegel half plane with the property

$$f(MZ)=v_\vartheta(M)^r\det(CZ+D)^{r/2}f(Z)\quad(M\in\Gamma_2[2,4,8]).$$

The ring of modular forms is

$$A(\Gamma[2,4,8]):=\bigoplus_{r\in\mathbb{Z}}[\Gamma[2,4,8],r/2,v_\vartheta^r].$$

The following result is essentially contained in [GS].

1.1 Theorem. *The $\mathbb{C}$-algebra $A(\Gamma[2,4,8])$ is generated by the 10 theta series $\vartheta[m]$ of first kind and the 4 theta series f_a of second kind. Defining relations are the quadratic relations which we described in the introduction.*

Proof. In [GS] actually has been proved that the ring generated by the 14 theta series gives an embedding of the Satake compactification of $\mathbb{H}_2/\Gamma[2,4,8]$ into the projective space. This implies that the ring of all modular forms is the normalization of this subring. It is easy to show that this ring is normal.

□

The easiest way to get vector valued modular forms is to consider brackets

$$\{f,g\} = g^2 d(f/g).$$

We write the three components of $\{f,g\}$ into a symmetric 2×2-matrix with the entries

$$\{f,g\}_{ij} = e_{ij} g^2 \frac{\partial(f/g)}{\partial z_{ij}}, \qquad e_{ij} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{else.} \end{cases}$$

If f,g are from $[\Gamma_2[4,8], v_\vartheta, 1/2]$, then $\{f,g\}$ can be considered as element of $\mathcal{M}(2)$.

We number the 10 theta series of first kind as follows

$$\begin{aligned}
&\vartheta_1 = \vartheta\begin{bmatrix}00\\00\end{bmatrix}, \quad \vartheta_2 = \vartheta\begin{bmatrix}00\\01\end{bmatrix}, \quad \vartheta_3 = \vartheta\begin{bmatrix}00\\10\end{bmatrix}, \quad \vartheta_4 = \vartheta\begin{bmatrix}00\\11\end{bmatrix}, \quad \vartheta_5 = \vartheta\begin{bmatrix}01\\00\end{bmatrix}, \\
&\vartheta_6 = \vartheta\begin{bmatrix}01\\10\end{bmatrix}, \quad \vartheta_7 = \vartheta\begin{bmatrix}10\\00\end{bmatrix}, \quad \vartheta_8 = \vartheta\begin{bmatrix}10\\01\end{bmatrix}, \quad \vartheta_9 = \vartheta\begin{bmatrix}11\\00\end{bmatrix}, \quad \vartheta_{10} = \vartheta\begin{bmatrix}11\\11\end{bmatrix}.
\end{aligned}$$

and the theta series of second kind as

$$\begin{aligned}
f_0(Z) &= \vartheta\begin{bmatrix}00\\00\end{bmatrix}(2Z), \quad f_1(Z) = \vartheta\begin{bmatrix}01\\00\end{bmatrix}(2Z), \\
f_2(Z) &= \vartheta\begin{bmatrix}10\\00\end{bmatrix}(2Z), \quad f_3(Z) = \vartheta\begin{bmatrix}11\\00\end{bmatrix}(2Z).
\end{aligned}$$

The explicit form of the quadratic relations is

$$\begin{aligned}
\vartheta_1^2 &= f_0^2 + f_1^2 + f_2^2 + f_3^2, & \vartheta_6^2 &= 2f_1f_0 - 2f_3f_2, \\
\vartheta_2^2 &= f_0^2 - f_1^2 + f_2^2 - f_3^2, & \vartheta_7^2 &= 2f_2f_0 + 2f_3f_1, \\
\vartheta_3^2 &= f_0^2 + f_1^2 - f_2^2 - f_3^2, & \vartheta_8^2 &= 2f_2f_0 - 2f_3f_1, \\
\vartheta_4^2 &= f_0^2 - f_1^2 - f_2^2 + f_3^2, & \vartheta_9^2 &= 2f_3f_0 + 2f_2f_1, \\
\vartheta_5^2 &= 2f_1f_0 + 2f_3f_2, & \vartheta_{10}^2 &= 2f_3f_0 - 2f_2f_1,
\end{aligned}$$

We use the notation

$$\vartheta_{11} = f_0, \quad \vartheta_{12} = f_1, \quad \vartheta_{13} = f_2, \quad \vartheta_{14} = f_3.$$

So we have

$$A(\Gamma[2,4,8]) = \mathbb{C}[\vartheta_1,\dots,\vartheta_{14}].$$

1.2 Definition. *We denote by $\mathcal{N}$ the submodule of $\mathcal{M}$ that is generated by the brackets $\{\vartheta_i, \vartheta_j\}$, $1 \le i, j \le 14$.*

We write the ten relations above in the form $R(\vartheta_1, \dots, \vartheta_{14}) = 0$ where R is a quadratic polynomial. An example is

$$R = X_1^2 - (X_{11}^2 + X_{12}^2 + X_{13}^2 + X_{14}^2).$$

Then we set

$$\partial_i R = \frac{\partial R}{\partial X_i}(\vartheta_1, \dots, \vartheta_{14}).$$

A basic result is that defining relations for the module $\mathcal{N}$ are known.

1.3 Proposition. *Defining relations of the module $\mathcal{N}$ are*

$$\vartheta_k\{\vartheta_i, \vartheta_j\} = \vartheta_j\{\vartheta_k, \vartheta_i\} - \vartheta_i\{\vartheta_k, \vartheta_j\}, \quad \{\vartheta_i, \vartheta_j\} + \{\vartheta_j, \vartheta_i\} = 0. \tag{1}$$

For each of the ten quadratic relations R one has

$$\sum_{\nu=1}^{14} (\partial_\nu R)\{\vartheta_\nu, \vartheta_\mu\} = 0 \quad (1 \le \mu \le m). \tag{2}$$

Proof. We have to use a general criterion [FS1], Proposition 1.4. This proposition says that it is enough to prove that the module $\mathcal{N}'$, that is defined through these relations, is torsion free. Even more, it is enough to prove that multiplication by some special elements is injective on $\mathcal{N}'$. These special elements depend on the choice of a transcendental basis. Here we take $f_0, \dots, f_3$. Then this special elements turn out to be the generators ϑ_i, $1 \le i \le 14$. It is easy to implement this module in a computer algebra system as SINGULAR and to verify this injectivity. □

The knowledge of the defining relations enables us to compute the Hilbert function.

1.4 Theorem. *The Hilbert function of the $A(\Gamma[2,4,8])$-module $\mathcal{N}$ which is generated by the Cohen-Rankin brackets $\{f, g\}$ where f, g is*

$$H(t) = \frac{t^2 P(t)}{t^4 - 4t^3 + 6t^2 - 4t + 1}$$

where

$$\begin{aligned} P(t) := &- 10t^{10} - 86t^9 - 311t^8 - 580t^7 - 465t^6 + 300t^5 \\ &+ 1218t^4 + 1488t^3 + 1021t^2 + 406t + 91. \end{aligned}$$

The first terms are given by

$$H(t) = 91t^2 + 770t^3 + 3555t^4 + 11452t^5 + 28685t^6 + 59778t^7 + 108790t^8 + \cdots.$$

2. Holomorphic tensors

Elements of the module $\mathcal{M}$ (see introduction) can be obtained by invariant holomorphic tensors

$$T = \operatorname{tr}(f dZ) \otimes (dz_{11} \wedge dz_{12} \wedge dz_{22})^{\otimes k}, \quad dZ = \begin{pmatrix} dz_{11} & dz_{12} \\ dz_{12} & dz_{22} \end{pmatrix},$$

where f is a symmetric matrix of holomorphic functions. The invariance means that $f \in \mathcal{M}(6k)$. For general reasons these tensors are rational on the algebraic variety $\mathbb{H}_2/\Gamma_2[2,4,8]$. By means of the transcendental basis a, b, c of the field of modular functions they can be expressed in the form

$$T = (T_1 da + T_2 db + T_3 dc) \otimes (da \wedge db \wedge dc)^{\otimes k}.$$

Here T_i are modular functions (i.e. rational functions on $\mathbb{H}_2/\Gamma_2[2,4,8]$). We have to work out what it means that T is holomorphic on $\mathbb{H}_2$, equivalently on $\mathbb{H}_2/\Gamma_2[2,4,8]$.

Assume that $Z \in \mathbb{H}_2$ is a point such that the a, b, c are holomorphic at a and such that they define a local coordinate system there. The holomorphicity of T at Z then means that the modular functions T_i are holomorphic there. Hence it is of interests to find transcendental bases a, b, c where this locus can be determined.

We can use here the same transcendental bases as in the paper [FS1]. Recall that a characteristic $n = \binom{a}{b}$ is called even if ${}^t ab \equiv 0 \bmod 2$. A set of 4 even characteristics $\mathfrak{m} = \{m_1, m_2, m_3, m_4\}$ is called syzygetic if the sum of three of them is even. There are 15 syzygetic quadruples. For a syzygetic quadruple we consider

$$a = \frac{\vartheta[m_2]}{\vartheta[m_1]}, \quad b = \frac{\vartheta[m_3]}{\vartheta[m_1]}, \quad b = \frac{\vartheta[m_4]}{\vartheta[m_1]}.$$

This is a transcendental basis of the field of modular functions.

$$\begin{array}{ll}
\vartheta_5\vartheta_6\vartheta_7\vartheta_8\vartheta_9\vartheta_{10}(\vartheta_7^4 - \vartheta_8^4), & \vartheta_1\vartheta_3\vartheta_4\vartheta_6\vartheta_7\vartheta_{10}(\vartheta_1^4 - \vartheta_7^4), \\
\vartheta_3\vartheta_4\vartheta_5\vartheta_6\vartheta_9\vartheta_{10}(\vartheta_3^4 - \vartheta_4^4), & \vartheta_1\vartheta_3\vartheta_4\vartheta_5\vartheta_8\vartheta_9(\vartheta_4^4 + \vartheta_5^4), \\
\vartheta_2\vartheta_4\vartheta_7\vartheta_8\vartheta_9\vartheta_{10}(\vartheta_8^4 - \vartheta_{10}^4), & \vartheta_1\vartheta_2\vartheta_5\vartheta_6\vartheta_9\vartheta_{10}(\vartheta_1^4 - \vartheta_2^4), \\
\vartheta_2\vartheta_3\vartheta_5\vartheta_6\vartheta_7\vartheta_8(\vartheta_5^4 - \vartheta_7^4), & \vartheta_1\vartheta_2\vartheta_4\vartheta_6\vartheta_7\vartheta_9(\vartheta_4^4 + \vartheta_7^4), \\
\vartheta_2\vartheta_3\vartheta_4\vartheta_6\vartheta_8\vartheta_{10}(\vartheta_2^4 - \vartheta_8^4), & \vartheta_1\vartheta_2\vartheta_4\vartheta_5\vartheta_8\vartheta_{10}(\vartheta_1^4 - \vartheta_5^4), \\
\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_7\vartheta_9(\vartheta_4^4 - \vartheta_9^4), & \vartheta_1\vartheta_2\vartheta_3\vartheta_6\vartheta_8\vartheta_9(\vartheta_3^4 + \vartheta_8^4), \\
\vartheta_1\vartheta_4\vartheta_5\vartheta_6\vartheta_7\vartheta_8(\vartheta_6^4 + \vartheta_7^4), & \vartheta_1\vartheta_2\vartheta_3\vartheta_5\vartheta_7\vartheta_{10}(\vartheta_2^4 + \vartheta_5^4). \\
\vartheta_1\vartheta_3\vartheta_7\vartheta_8\vartheta_9\vartheta_{10}(\vartheta_7^4 + \vartheta_{10}^4), &
\end{array}$$

It seems to be worth while to point out the following by-product.

2.1 Remark. *The function* $\vartheta_7^4 - \vartheta_8^4$ *has (up to constant factors* 15 *transformed functions of the form* $\vartheta_i^4 \pm \vartheta_j^4$. *The product of these forms is a constant multiple of Igusa's modular form* χ_{30}.

Recall that χ_{30} is a modular form of weight 30 which generates the space of all modular forms of this weight which belong to the non-trivial character of the full modular group. In [FS2] a similar result has been proved. The function $\vartheta_1(Z/2)$ has 60 transformed function (the forms f_a belong to them) and their product gives χ_{30}. So there are two different ways to write χ_{30} as product of 60 forms.

2.2 Lemma. *There is one-to-one correspondence between the* 15 *syzygetic quadruples and the* 15 *modular forms above. We denote by* $X_{\mathfrak{m}}$ *the modular form in this list that corresponds to* $\mathfrak{m}$. *Then the transcendental basis*

$$a = \frac{\vartheta[m_2]}{\vartheta[m_1]}, \quad b = \frac{\vartheta[m_3]}{\vartheta[m_1]}, \quad b = \frac{\vartheta[m_4]}{\vartheta[m_1]}$$

defines a local analytic chart outside the zero locus of $\vartheta[m_1]X_{\mathfrak{m}}$.

This means that the components T_i of a holomorphic tensor

$$T = (T_1 da + T_2 db + T_3 dc) \otimes (da \wedge db \wedge dc)^{\otimes k}.$$

are holomorphic outside this zero locus. If one multiplies them with a suitable power of $\vartheta[m_1]X_{\mathfrak{m}}$ one gets holomorphic functions on the upper half-plane. This implies that T is contained in

$$\bigcap_{\mathfrak{m}} \bigcap_{m \in \mathfrak{m}} \mathcal{N}_{X_{\mathfrak{m}}\vartheta[m]}.$$

But since we can multiply an arbitrary element from $\mathcal{M}(r)$ by a suitable power of any of the $\vartheta[m]$ to obtain a tensorial form, this result extends to the full module $\mathcal{M}$.

2.3 Theorem. *We have*

$$\mathcal{M} = \bigcap_{\mathfrak{m}} \bigcap_{m \in \mathfrak{m}} \Big(\sum_{1 \le i < j \le 10} \mathbb{C}[\vartheta_1, \dots, \vartheta_{14}] \, \{\vartheta_i, \vartheta_j\} \Big)_{X_{\mathfrak{m}}\vartheta[m]}.$$

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