Some vector valued Siegel modular forms of genus 2

Eberhard Freitag
Mathematisches Institut
Im Neuenheimer Feld 288
D69120 Heidelberg
freitag@mathi.uni-heidelberg.de

Riccardo Salvati Manni
Dipartimento di Matematica,
Piazzale Aldo Moro, 2
I-00185 Roma, Italy.
salvati@mat.uniroma1.it

2013

Introduction

We consider Igusa's group $\Gamma_2[4,8]$ (see Sect. 2) which is a subgroup of the Siegel modular group of degree 2. We want to study vector valued modular forms of the transformation type

$$f(MZ) = v_\vartheta(M)^r \det(CZ+D)^{r/2}(CZ+D)f(Z)\,{}^t(CZ+D) \quad (M \in \Gamma_2[4,8]).$$

Here f should be a symmetric 2×2-matrix of holomorphic functions. v_ϑ is the theta-multiplier system. Its square is trivial on $\Gamma_2[4,8]$. We denote by $\mathcal{M}(r)$ the vector space of all forms of the above transformation type.

The direct sum

$$\mathcal{M} := \bigoplus_{r\in\mathbb{Z}} \mathcal{M}(r)$$

is a module over the ring of all modular forms with respect to the group $\Gamma_2[4,8]$. We are interested in its structure. By Igusa, the ring of modular forms is generated by the ten classical theta constants $\vartheta[m]$. The module $\mathcal{M}$ contains a submodule $\mathcal{N}$ which is generated by 45 Cohen-Rankin brackets $\{\vartheta[m],\vartheta[n]\}$. We determine defining relations for this submodule and compute its Hilbert function (Theorem 2.4), i.e. the dimension formula for the spaces $\mathcal{N}(r)$. We prove that $\mathcal{M}$ is the intersection of the localizations of $\mathcal{N}$ by 60 elements (Theorem 5.4). This is a complete algebraic description of $\mathcal{M}$ and to get a finite system of generators of $\mathcal{M}$ is a computational problem. At the moment we cannot solve this problem. Examples of elements of $\mathcal{M}$ which are not contained in $\mathcal{N}$ can be given.

Our method is a further development of Wieber's geometric method [Wi] which he used to solve the $\Gamma_2[2,4]$-case. Meanwhile there appeared several papers with similar results using different methods, [Ao, CG, Do, Ib, Sa, Sat, Wi]. We want to thank Wieber for fruitful discussion and for his help with quite involved computer calculations.

1. A differential module over graded algebras

Let $A = \bigoplus_{d=0}^{\infty} A_d$ be a finitely generated graded algebra over a field K of characteristic 0. We assume that A is an integral domain and denote its field of fractions by $Q(A)$. We consider the Kähler differential module

$$\Omega = \Omega(Q(A)/K).$$

Recall that this is a $Q(A)$-vector space together with a K-linear derivation $d : Q(A) \to \Omega$. The dimension of Ω equals the transcendental degree of $Q(A)$ and Ω is generated by the image of d. In the following, we denote by $\deg(f)$ the degree of a non-zero homogeneous element of A. For two non-zero homogenous elements of positive degree $f, g \in A$ we define

$$\{f, g\} := \deg(g) g df - \deg(f) f dg.$$

Another way to write this is

$$\{f, g\} = \frac{g^{\deg(f)+1}}{f^{\deg(g)-1}}\, d\Big(\frac{f^{\deg(g)}}{g^{\deg(f)}}\Big).$$

This is a skew-symmetric K-bilinear pairing and it satisfies the following rule

$$\deg(h) h \{f, g\} = \deg(g) g \{f, h\} + \deg(f) f \{h, g\}.$$

1.1 Definition. *We denote by $\mathcal{N}$ the A-module that is generated by all $\{f, g\}$ where f, g are homogenous elements of positive degree of A.*

We are interested in a finite presentation of $\mathcal{N}$. There is no difficulty to get a finite system of generators, Let $A = K[f_1, \dots, f_m]$, (f_i homogenous). Then $\{f_i, f_j\}$ are generators of $\mathcal{N}$. It is more involved to get defining relations.

We use the notation $d_i = \deg(f_i)$. A polynomial $P \in K[X_1, \dots, X_m]$ is called isobaric of weight k (with respect to $(d_1, \dots, d_m)$) if it is of the form

$$P = \sum_{d_1\nu_1+\cdots+d_m\nu_m=k} a_{\nu_1,\dots,\nu_m} X_1^{\nu_1} \cdots X_m^{\nu_m}.$$

Then the Euler relation

$$\sum_{\nu=1}^{m} d_\nu \frac{\partial P}{\partial X_\nu} X_\nu = kP$$

holds.

The ideal of relations between $f_1, \dots, f_m$ is generated by isobaric polynomials. Let $R(f_1, \dots, f_m) = 0$ be an isobaric relation. Differentiation gives

$$\sum_{\nu=1}^{m} (\partial_\nu R) df_\nu = 0 \quad \text{where} \quad \partial_\nu R := \frac{\partial R}{\partial X_\nu}(f_1, \dots, f_m).$$

From this relation and the Euler relation we derive

$$\sum_{\nu=1}^{m} (\partial_\nu R)\{f_\nu, f_\mu\} = 0 \quad (\mu \text{ arbitrary}).$$

We want to formalize this and introduce a module $\mathcal{N}'$ which is defined by the so far known relations.

1.2 Definition. *We denote by $\mathcal{N}'$ the A-module that is generated by symbols $[f_i, f_j]$ with the following defining relations:*

$$(1) \qquad d_k f_k [f_i, f_j] = d_j f_j [f_i, f_k] + d_i f_i [f_k, f_j], \quad [f_i, f_j] + [f_j, f_i] = 0.$$

For each isobaric relation R between the $f_1, \dots, f_m$ one has

$$(2) \qquad \sum_{\nu=1}^{m} (\partial_\nu R)[f_\nu, f_\mu] = 0 \quad (\mu \textit{ arbitrary}).$$

It is of course enough to take for R a system of generators of the ideal of all relations.

There is a natural surjective homomorphism

$$\mathcal{N}' \longrightarrow \mathcal{N}, \quad [f_i, f_j] \longmapsto \{f_i, f_j\}.$$

We notice that $\mathcal{N}$ is torsion free for trivial reason but for $\mathcal{N}'$ this is not clear.

Under certain circumstances, $\mathcal{N}' \to \mathcal{N}$ is an isomorphism. To work this out we consider an arbitrary relation in $\mathcal{N}$

$$\sum_{i<j} P_{ij}\{f_i, f_j\} = 0, \quad P_{ij} \in A.$$

We multiply this relation by $d_1 f_1$ and insert

$$d_1 f_1 \{f_i, f_j\} = d_i f_i \{f_1, f_j\} - d_j f_j \{f_1, f_i\}.$$

Then we obtain the relation

$$\sum_j P_j \{f_1, f_j\} = 0,$$

where the elements $P_j \in A$ are defined as

$$P_j = \sum_{i<j} d_i f_i P_{ij} - \sum_{i>j} d_i f_i P_{ji}.$$

Let n be the transcendental degree of $Q(A)$. We can assume that $f_1,\dots,f_n$ are independent. Then each f_k, $k>n$, satisfies an algebraic relation

$$R_k(f_1,\dots,f_n,f_k)=0.$$

Here R_k is an irreducible polynomial in the variables $X_1,\dots,X_n,X_k$. Now we make use of the relation

$$(\partial_k R_k)\{f_1,f_k\}+\sum_{\nu=1}^{n}(\partial_\nu R_k)\{f_1,f_\nu\}=0.$$

We have to use the elements (from the ring A)

$$\Pi:=\prod_{k=n+1}^{m}\partial_k R_k,\quad \Pi^{(k)}:=\frac{\Pi}{\partial_k P_k}.$$

We multiply the original relation also by Π:

$$\Pi\sum_j P_j\{f_1,f_j\}=0.$$

For $k>n$ we have the formula

$$\Pi\{f_1,f_k\}=\Pi^{(k)}(\partial_k R_k)\{f_1,f_k\}=-\Pi^{(k)}\sum_{j=1}^{n}(\partial_j R_k)\{f_1,f_j\}.$$

Now we can eliminate the $\{f_1,f_k\}$ for $k>n$ to produce a relation between the $\{f_1,f_i\}$, $2\le i\le n$. But these elements are independent. Hence the coefficients of the relation must vanish. A simple calculation now gives the following lemma.

1.3 Lemma. *Let*

$$\sum_{i<j}P_{ij}\{f_i,f_j\}=0,\quad P_{ij}\in A.$$

Then the elements

$$P_j=\sum_{i<j}d_if_iP_{ij}-\sum_{i>j}d_if_iP_{ji}$$

satisfy the following system of relations.

$$P_j\Pi=\sum_{k=n+1}^{m}(\partial_j R_k)P_k\Pi^{(k)}\quad(1\le j\le n).$$

Supplement. *Conversely these relations imply in $\mathcal{N}'$ the relation*

$$f_1\Pi\sum_{i<j}P_{ij}[f_i,f_j]=0.$$

For the proof of the supplement we just have to notice that the calculations above only use the defining relations of $\mathcal{N}'$. □

Let us assume that multiplication by $f_1\Pi$ is injective on $\mathcal{N}'$. Then we see that $\sum P_{ij}\{f_i,f_j\}=0$ implies $\sum P_{ij}[f_i,f_j]=0$. Hence $\mathcal{N}'\to\mathcal{N}$ is an isomorphism and $\mathcal{N}'$ must be torsion free. This gives the following result.

1.4 Proposition. *Assume that the $f_1, \dots, f_n$ is a transcendental basis such that each f_k, $n < k \le m$, satisfies an irreducible algebraic relation*

$$R_k(f_1, \dots, f_n, f_k) = 0.$$

The homomorphism $\mathcal{N}' \to \mathcal{N}$ is an isomorphism if and only if $\mathcal{N}'$ is torsion free. For this it suffices that multiplication by f_1 and $\partial_k R_k$ $(n < k \le m)$ are injective on $\mathcal{N}'$.

2. Siegel modular forms and theta relations

We consider the Siegel modular group $\Gamma_n = \mathrm{Sp}(n, \mathbb{Z})$, its principal congruence subgroup

$$\Gamma_n[q] = \mathrm{kernel}(\mathrm{Sp}(n, \mathbb{Z}) \longrightarrow \mathrm{Sp}(n, \mathbb{Z}/q\mathbb{Z})$$

and Igusa's group

$$\Gamma_n[q, 2q] := \left\{ M \in \Gamma_n[q], \quad (C\,{}^tD)_0 \equiv (A\,{}^tB)_0 \equiv 0 \mod 2q \right\}.$$

Here S_0 denotes the column built of the diagonal of a square matrix S. We are particularly interested in the cases $n = 2$ and $q = 4$.

A scalar valued modular form f of weight $r/2$, $r \in \mathbb{Z}$, for a subgroup $\Gamma \subset \mathrm{Sp}(n, \mathbb{Z})$ is a holomorphic function f on $\mathbb{H}_n$ with the transformation property

$$f(MZ) = v(M)\sqrt{\det(CZ + D)}^{\,r} f(Z)$$

for all $M \in \Gamma$. In the case $n = 1$ a regularity condition at the cusps has to be added. Here $v(M)$ is system of complex numbers of absolute valued one, called the multiplier system. It depends on the choice of the holomorphic square root. It has to fulfil an obvious cocycle condition. We denote this space by $[\Gamma, r/2, v]$. Fixing some starting weight r_0 and a multiplier system v for it, we define the ring

$$A(\Gamma) = A(\Gamma, (r_0, v)) := \bigoplus_{r \in \mathbb{Z}} [\Gamma, rr_0/2, v^r].$$

This turns out to be a finitely generated graded algebra and its associated projective variety $\mathrm{proj}\, A(\Gamma)$ can be identified with the Satake compactification of $\mathbb{H}_n/\Gamma$. The ring depends on the starting weight and the multiplier system but the associated projective variety does not.

Basic examples of modular forms are given be theta series with characteristics. By definition, a theta characteristic is an element $m = \binom{a}{b}$ from $(\mathbb{Z}/2\mathbb{Z})^{2n}$. Here $a, b \in (\mathbb{Z}/2\mathbb{Z})^n$ are column vectors. The characteristic is called even if

${}^t ab = 0$ and odd otherwise. The group $\mathrm{Sp}(n, \mathbb{Z}/2\mathbb{Z})$ acts on the set of characteristics by

$$M\{m\} := {}^t M^{-1} m + \begin{pmatrix} (C\,{}^tD)_0 \\ (A\,{}^tB)_0 \end{pmatrix}.$$

It is well-known that $\mathrm{Sp}(n, \mathbb{Z}/2\mathbb{Z})$ acts transitively on the subsets of even and odd characteristics. Recall that for any characteristic the theta function

$$\vartheta[m] = \sum_{g \in \mathbb{Z}^n} e^{\pi \mathrm{i}(Z[g+a/2] + {}^t b(g+a/2))} \qquad (Z[g] = {}^t g Z g)$$

can be defined. Here we use the identification of $\mathbb{Z}/2\mathbb{Z}$ with the subset $\{0, 1\} \subset \mathbb{Z}$. It vanishes if and only if m is odd. Recall also that the formula

$$\vartheta[M\{m\}](MZ) = v(M, m)\sqrt{\det(CZ + D)}\vartheta[m](Z)$$

holds for $M \in \Gamma_n$ where $v(M, m)$ is a 8^{th} root of unity. From these formulas one can derive that the theta series are all modular form on the group $\Gamma_n[4, 8]$ and all with the same muliplier system v_ϑ. The square of this multiplier is trivial on $\Gamma_n[4, 8]$. We call this the *theta multiplier system.*

From now on we assume $n = 2$. In this case we will use the notation

$$\vartheta[m] = \vartheta\begin{bmatrix} a_1 a_2 \\ b_1 b_2 \end{bmatrix} \quad \text{for} \quad m = \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}.$$

There are ten even characteristics. We will order them as follows:

$$(m_1, \ldots, m_{10}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

The associated theta series are

$$\begin{aligned} &\vartheta_1 = \vartheta\begin{bmatrix} 00 \\ 00 \end{bmatrix}, \quad \vartheta_2 = \vartheta\begin{bmatrix} 00 \\ 01 \end{bmatrix}, \quad \vartheta_3 = \vartheta\begin{bmatrix} 00 \\ 10 \end{bmatrix}, \quad \vartheta_4 = \vartheta\begin{bmatrix} 00 \\ 11 \end{bmatrix}, \quad \vartheta_5 = \vartheta\begin{bmatrix} 01 \\ 00 \end{bmatrix}, \\ &\vartheta_6 = \vartheta\begin{bmatrix} 01 \\ 10 \end{bmatrix}, \quad \vartheta_7 = \vartheta\begin{bmatrix} 10 \\ 00 \end{bmatrix}, \quad \vartheta_8 = \vartheta\begin{bmatrix} 10 \\ 01 \end{bmatrix}, \quad \vartheta_9 = \vartheta\begin{bmatrix} 11 \\ 00 \end{bmatrix}, \quad \vartheta_{10} = \vartheta\begin{bmatrix} 11 \\ 11 \end{bmatrix}. \end{aligned}$$

They satisfy the quartic Riemann relations:

$$\vartheta_6^2\vartheta_8^2 - \vartheta_4^2\vartheta_9^2 + \vartheta_1^2\vartheta_{10}^2 = 0, \qquad \vartheta_4^2\vartheta_5^2 - \vartheta_2^2\vartheta_6^2 - \vartheta_7^2\vartheta_{10}^2 = 0,$$
$$\vartheta_5^2\vartheta_8^2 - \vartheta_2^2\vartheta_9^2 + \vartheta_3^2\vartheta_{10}^2 = 0, \qquad \vartheta_3^2\vartheta_5^2 - \vartheta_1^2\vartheta_6^2 - \vartheta_8^2\vartheta_{10}^2 = 0,$$
$$\vartheta_7^4 - \vartheta_8^4 - \vartheta_9^4 + \vartheta_{10}^4 = 0, \qquad \vartheta_2^2\vartheta_5^2 - \vartheta_4^2\vartheta_6^2 - \vartheta_8^2\vartheta_9^2 = 0,$$
$$\vartheta_6^2\vartheta_7^2 - \vartheta_3^2\vartheta_9^2 + \vartheta_2^2\vartheta_{10}^2 = 0, \qquad \vartheta_1^2\vartheta_5^2 - \vartheta_3^2\vartheta_6^2 - \vartheta_7^2\vartheta_9^2 = 0,$$
$$\vartheta_5^2\vartheta_7^2 - \vartheta_1^2\vartheta_9^2 + \vartheta_4^2\vartheta_{10}^2 = 0, \qquad \vartheta_3^4 - \vartheta_4^4 - \vartheta_6^4 + \vartheta_{10}^4 = 0,$$
$$\vartheta_4^2\vartheta_7^2 - \vartheta_3^2\vartheta_8^2 - \vartheta_5^2\vartheta_{10}^2 = 0, \qquad \vartheta_2^2\vartheta_3^2 - \vartheta_1^2\vartheta_4^2 + \vartheta_9^2\vartheta_{10}^2 = 0,$$
$$\vartheta_3^2\vartheta_7^2 - \vartheta_4^2\vartheta_8^2 - \vartheta_6^2\vartheta_9^2 = 0, \qquad \vartheta_1^2\vartheta_3^2 - \vartheta_2^2\vartheta_4^2 - \vartheta_5^2\vartheta_6^2 = 0,$$
$$\vartheta_2^2\vartheta_7^2 - \vartheta_1^2\vartheta_8^2 - \vartheta_6^2\vartheta_{10}^2 = 0, \qquad \vartheta_2^4 - \vartheta_4^4 - \vartheta_8^4 + \vartheta_{10}^4 = 0,$$
$$\vartheta_1^2\vartheta_7^2 - \vartheta_2^2\vartheta_8^2 - \vartheta_5^2\vartheta_9^2 = 0, \qquad \vartheta_1^2\vartheta_2^2 - \vartheta_3^2\vartheta_4^2 - \vartheta_7^2\vartheta_8^2 = 0,$$
$$\vartheta_5^4 - \vartheta_6^4 - \vartheta_9^4 + \vartheta_{10}^4 = 0, \qquad \vartheta_1^4 - \vartheta_2^4 - \vartheta_6^4 - \vartheta_9^4 = 0.$$

2.1 Theorem (Igusa). *The algebra*

$$A(\Gamma_2[4,8]) = \bigoplus [\Gamma_2[4,8], r/2, v_\vartheta^r]$$

is generated by the 10 *theta series. Defining relations are the Riemann quartic relations.*

A set $\mathfrak{m}$ of three even characteristics is called *syzygetic* if their sum is even. Otherwise it is called *azygetic.* A system of more than three is called syzygetic if every subsystem of three has this property.

2.2 Lemma. *Let* $\mathfrak{m} = \{m_1, m_2, m_3, m_4\}$ *be a syzygetic system. Then the* 4 *theta series* $\vartheta[m_i]$ *are algebraically independent and they have no joint zero. There are* 15 *syzygetic quadruples. The full modular group acts transitively on them.*

An example of an syzygetic system is

$$\begin{bmatrix}00\\00\end{bmatrix}, \quad \begin{bmatrix}01\\00\end{bmatrix}, \quad \begin{bmatrix}10\\00\end{bmatrix}, \quad \begin{bmatrix}11\\00\end{bmatrix}.$$

We call it the *standard syzygetic quadruple* and denote it by $\mathfrak{m}_{\text{standard}}$. In our numbering the corresponding theta series are

$$\vartheta_1, \quad \vartheta_5, \quad \vartheta_7, \quad \vartheta_9.$$

The algebraic relations R_k (relations between $\vartheta_1, \vartheta_5, \vartheta_7, \vartheta_9$ and one of the 6 remaining theta series ϑ_k) can be computed from the Riemann relation as follows.

$$R_2 = X_1^4X_7^4 - X_1^4X_2^4 - 2X_1^2X_5^2X_7^2X_9^2 + X_5^4X_9^4 + X_5^4X_2^4 - X_7^4X_2^4 + X_9^4X_2^4 + X_2^8,$$
$$R_3 = X_1^4X_5^4 - X_1^4X_3^4 - 2X_1^2X_5^2X_7^2X_9^2 - X_5^4X_3^4 + X_7^4X_9^4 + X_7^4X_3^4 + X_9^4X_3^4 + X_3^8,$$
$$R_4 = X_1^4X_9^4 - X_1^4X_4^4 - 2X_1^2X_5^2X_7^2X_9^2 + X_5^4X_7^4 + X_5^4X_4^4 + X_7^4X_4^4 - X_9^4X_4^4 + X_4^8,$$
$$R_6 = X_1^4X_5^4 - X_1^4X_6^4 - 2X_1^2X_5^2X_7^2X_9^2 - X_5^4X_6^4 + X_7^4X_9^4 + X_7^4X_6^4 + X_9^4X_6^4 + X_6^8,$$
$$R_8 = X_1^4X_7^4 - X_1^4X_8^4 - 2X_1^2X_5^2X_7^2X_9^2 + X_5^4X_9^4 + X_5^4X_8^4 - X_7^4X_8^4 + X_9^4X_8^4 + X_8^8,$$
$$R_{10} = X_1^4X_9^4 - X_1^4X_{10}^4 - 2X_1^2X_5^2X_7^2X_9^2 + X_5^4X_7^4 + X_5^4X_{10}^4 + X_7^4X_{10}^4 - X_9^4X_{10}^4 + X_{10}^8.$$

It is easy to check with the help of a computer that multiplication by ϑ_1 and by the $\partial_k R_k$ are injective on the module $\mathcal{N}'$. (The MAGMA computation is performed over the field of rational numbers. Since tensoring with $\mathbb{C}$ is exact it follows then over $\mathbb{C}$.) This gives the following result.

2.3 Proposition. *The following relations are defining relations for the $A(\Gamma_2[4,8])$-module*

$$\mathcal{N} := \sum_{1\le i<j\le 10} A(\Gamma[4,8])\{\vartheta_i,\vartheta_j\}.$$

(1) $\qquad \vartheta_k\{\vartheta_i,\vartheta_j\} = \vartheta_j\{\vartheta_k,\vartheta_i\} - \vartheta_i\{\vartheta_k,\vartheta_j\}, \quad \{\vartheta_i,\vartheta_j\} + \{\vartheta_j,\vartheta_i\} = 0.$

For each Riemann relation R one has

(2) $$\sum_{\nu=1}^{10} (\partial_\nu R)\{\vartheta_\nu,\vartheta_\mu\} = 0 \quad (1\le \mu\le m).$$

As an application, we can compute the Hilbert series of $\mathcal{N}$.

2.4 Theorem. *The Hilbert function of $\mathcal{N}$ is*

$$H(t) = \frac{t^2P(t)}{t^4-4t^3+6t^2-4t+1}$$

where

$$\begin{aligned} P(t) :=& -6t^{14} - 36t^{13} - 66t^{12} + 24t^{11} + 224t^{10} + 178t^9 - 297t^8 \\ & - 692t^7 - 427t^6 + 328t^5 + 868t^4 + 808t^3 + 435t^2 + 150t + 45. \end{aligned}$$

The first terms are given by

$$H(t) = 45t^2 + 330t^3 + 1485t^4 + 4948t^5 + 13025t^6 + 28350t^7 + 53130t^8 + \cdots.$$

3. Vector valued modular forms

As mentioned already in the introduction, the space $\mathcal{M}(r)$ consists of all modular forms of the type

$$f(MZ) = v_\vartheta(M)^r \det(CZ+D)^{r/2}(CZ+D)f(Z)\,{}^t(CZ+D) \quad (M \in \Gamma_2[4,8]).$$

Here f should be a symmetric 2×2-matrix of holomorphic functions. We take the direct sum

$$\mathcal{M} := \bigoplus_{r\in\mathbb{Z}} \mathcal{M}(r).$$

These is a module over the ring $A(\Gamma_2[4,8]) = \mathbb{C}[\vartheta_1,\dots,\vartheta_{10}]$.

Examples of elements of this module come from $\Gamma_2[4,8]$-invariant holomorphic tensors

$$T = \operatorname{tr}(f dZ) \otimes (dz_{11}\wedge dz_{12}\wedge dz_{22})^{\otimes k}, \quad dZ = \begin{pmatrix} dz_{11} & dz_{12} \\ dz_{12} & dz_{22}\end{pmatrix},$$

where f is a symmetric matrix of holomorphic functions. This tensor is invariant if and only if f has the transformation property

$$f(MZ) = \det(CZ+D)^{3k}(CZ+D)f(Z)\,{}^t(CZ+D) \qquad (M\in\Gamma_n[2,4]).$$

This is an element of $\mathcal{M}(6k)$.

The elements of the quotient field $Q(A(\Gamma_2[4,8])$ can be considered as meromorphic functions on the half plane $\mathbb{H}_2$. For such a meromorphic function φ, we can consider the meromorphic differential $d\varphi$ on $\mathbb{H}_2$. These differentials generate a 4-dimensional vector space Ω over $Q(A(\Gamma_2[4,8])$ which is a realization of the Kähler differential module. The following constructions are constructions inside Ω.

The easiest way to get vector valued modular forms is to consider brackets

$$\{f,g\} = g^2 d(f/g).$$

We write the three components of $\{f,g\}$ into a symmetric 2×2-matrix with the entries

$$\{f,g\}_{ij} = e_{ij} g^2 \frac{\partial(f/g)}{\partial z_{ij}}, \qquad e_{ij} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{else.}\end{cases}$$

If f,g are from $[\Gamma_2[4,8], v_\vartheta, 1/2]$, then $\{f,g\}$ can be considered as element of $\mathcal{M}(2)$. Hence we can consider

$$\mathcal{N} = \sum_{1\le i<j\le 10} \mathbb{C}[\vartheta_1,\dots,\vartheta_{10}]\,\{\vartheta_i,\vartheta_j\}$$

as sub-module of $\mathcal{M}$.

4. Holomorphic tensors

We study invariant tensors

$$T = \operatorname{tr}(f dZ) \otimes (dz_{11} \wedge dz_{12} \wedge dz_{22})^{\otimes k}$$

in more detail. For general reasons they are rational on the algebraic variety $\mathbb{H}_2/\Gamma_2[4,8]$. By means of the transcendental basis

$$a = \vartheta_5/\vartheta_1, \quad b = \vartheta_7/\vartheta_1, \quad c = \vartheta_9/\vartheta_1.$$

they can be expressed in the form

$$T = (T_1 da + T_2 db + T_3 dc) \otimes (da \wedge db \wedge dc)^{\otimes k}.$$

Here T_i are modular functions (i.e. rational functions on $\mathbb{H}_2/\Gamma_2[4,8]$). We have to work out what it means that T is holomorphic on $\mathbb{H}_2$, equivalently on $\mathbb{H}_2/\Gamma_2[4,8]$. (We use the known fact that $\Gamma_2[4,8]$ acts fixed point free.)

Assume that $Z \in \mathbb{H}_2$ is a point such that the a, b, c are holomorphic at a and such that they define a local coordinate system there. The holomorphicity of T at Z then means that the modular functions T_i are holomorphic there. Hence it is of interests to determine this locus.

We have to make use of Igusa's modular form χ_5 which, up to a constant factor, can be defined as the product of the 10 theta series.

4.1 Proposition. *Let $Z \in \mathbb{H}_2$ be a point such that the modular forms*

$$\vartheta\begin{bmatrix}00\\00\end{bmatrix} \quad \textit{and} \quad \chi_5(Z/2)$$

are different from 0. Then the functions a, b, c are holomorphic at Z and the Jacobian determinant is different from zero there.

Proof. The first of the two forms gives the denominator of a, b, c. They are holomorphic outside the zero locus of it. The second form gives the functional determinant in its projective form. Recall that for forms $f = (f_0, f_1, f_2, f_3)$ this version is the determinant of the matrix whose rows are f and the 3 derivatives of f. This determinant is known for the theta series of second kind

$$\vartheta\begin{bmatrix}00\\00\end{bmatrix}(2Z), \quad \vartheta\begin{bmatrix}01\\00\end{bmatrix}(2Z), \quad \vartheta\begin{bmatrix}10\\00\end{bmatrix}(2Z), \quad \vartheta\begin{bmatrix}11\\00\end{bmatrix}(2Z).$$

We remind that the determinant is a modular form of weight 5 with respect to the full modular group. The reason is that the full modular group acts on the space generated by the 4 theta series of second kind. Hence the determinant is a modular form of weight 5 with respect to the full modular group an therefore a multiple of χ_5. Replacing the variable Z by $Z/2$, we complete the proof of Proposition 4.1. $\square$

4.2 Lemma. *The modular form $\chi_5(Z/2)$ is contained in the algebra $A(\Gamma_2[4,8])$. Up to to a constant factor it equals*

$$\vartheta_2\vartheta_3\vartheta_4\vartheta_6\vartheta_8\vartheta_{10}(\vartheta_4^4-\vartheta_{10}^4).$$

Proof. Classical doubling formulae gives expressions of $\vartheta[m](Z)^2$ as polynomials in $\vartheta[n](2Z)$. Using them, $\chi_5(Z/2)^2$ can be expressed explicitly as polynomial in $\vartheta[n](Z)$. We omit this calculation. It is easy to check that the square of the expression in Lemma 4.2 is contained in the ideal generated by $\chi_5(Z/2)^2$ and by the Riemann relations. □

We can express now the forms $d(a),\dots$ by brackets $\{\vartheta_1,\vartheta_2\},\dots$. Then we can express the tensor T in the form

$$\sum_{i<j} f_{ij}\{f_i,f_j\}$$

with meromorphic modular forms f_{ij}. Their poles are located at the zeros of $\vartheta_1(Z)\chi_5(Z/2)$. They disappear if we multiply f_{ij} by a suitable power of this modular form. We obtain the following lemma.

4.3 Lemma. *Any $\Gamma_2[4,8]$-invariant tensor*

$$T=\operatorname{tr}(f\,dZ)\otimes(dz_{11}\wedge dz_{12}\wedge dz_{22})^{\otimes k}$$

is contained in the localization of the module $\sum_{1\le i<j\le 10}\mathbb{C}[\vartheta_1,\dots,\vartheta_{10}]\ \{\vartheta_i,\vartheta_j\}$ by the element

$$\vartheta\begin{bmatrix}00\\00\end{bmatrix}\chi_5(Z/2).$$

5. Change of the syzygetic system

The Hecke group $\Gamma_2^0[2]$ is defined by the condition $B\equiv 0 \bmod 2$. It is the stabilizer of the standard syzygetic quadruple in the full modular group there. Hence it has index 15. It is also easy to check that $\chi_5(Z/2)$ is a modular form of weight 5 on $\Gamma_2^0[2]$ with respect to some character. Let now $\mathfrak{m}$ be an arbitrary syzygetic quadruple. There exists an M in the full modular group such that $\mathfrak{m}$ agrees with $M\{\mathfrak{m}_{\text{standard}}\}$. Then we can use M to transform $\chi_5(Z/2)$:

$$X_{\mathfrak{m}}(Z):=\det(CZ+D)^{-5}\chi_5((MZ)/2).$$

This is a modular form of weight 5 with respect to some conjugate group of $\Gamma_2^0[2]$. Up to a constant factor it is independent of the choice of M.

We need an explicit formula for $X_{\mathfrak{m}}$.

5.1 Definition. *A set $\{\{m_1,n_1\},\{m_2,n_2\},\{m_3,n_3\}\}$ of three unordered pairs of even characteristics is called **compatible** if the involved 6 characteristics are pairwise different, if the complement of the 6 is a syzygetic quadruple and if the union of any two pairs is an azygetic quadruple.*

An example of a compatible triple is

$$\left\{\left\{\begin{bmatrix}00\\11\end{bmatrix},\begin{bmatrix}11\\11\end{bmatrix}\right\},\ \left\{\begin{bmatrix}00\\01\end{bmatrix},\begin{bmatrix}10\\01\end{bmatrix}\right\},\ \left\{\begin{bmatrix}00\\10\end{bmatrix},\begin{bmatrix}01\\10\end{bmatrix}\right\}\right\}.$$

It corresponds to the standard syzygetic quadruple.

5.2 Lemma. *There is a one to one correspondence between syzygetic quadruples and compatible triples. Let $\mathfrak{m}$ be a syzygetic quadruple and $\{m,n\}$ on of the three members of the associated compatible triple. Then up to a constant factor $X_{\mathfrak{m}}$ agrees with*

$$(\vartheta[m]^4 \pm \vartheta[n]^4)\prod_{x\notin\mathfrak{m}}\vartheta[x]$$

where the sign has to be chosen properly.

Here is a complete list with the correct signs.

$$\begin{array}{ll}
\vartheta_5\vartheta_6\vartheta_7\vartheta_8\vartheta_9\vartheta_{10}(\vartheta_7^4-\vartheta_8^4), & \vartheta_1\vartheta_3\vartheta_4\vartheta_6\vartheta_7\vartheta_{10}(\vartheta_1^4-\vartheta_7^4),\\
\vartheta_3\vartheta_4\vartheta_5\vartheta_6\vartheta_9\vartheta_{10}(\vartheta_3^4-\vartheta_4^4), & \vartheta_1\vartheta_3\vartheta_4\vartheta_5\vartheta_8\vartheta_9(\vartheta_4+\vartheta_5^4),\\
\vartheta_2\vartheta_4\vartheta_7\vartheta_8\vartheta_9\vartheta_{10}(\vartheta_8^4-\vartheta_{10}^4), & \vartheta_1\vartheta_2\vartheta_5\vartheta_6\vartheta_9\vartheta_{10}(\vartheta_1^4-\vartheta_2^4),\\
\vartheta_2\vartheta_3\vartheta_5\vartheta_6\vartheta_7\vartheta_8(\vartheta_5^4-\vartheta_7^4), & \vartheta_1\vartheta_2\vartheta_4\vartheta_6\vartheta_7\vartheta_9(\vartheta_4+\vartheta_7^4),\\
\vartheta_2\vartheta_3\vartheta_4\vartheta_6\vartheta_8\vartheta_{10}(\vartheta_2^4-\vartheta_8^4), & \vartheta_1\vartheta_2\vartheta_4\vartheta_5\vartheta_8\vartheta_{10}(\vartheta_1^4-\vartheta_5^4),\\
\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_7\vartheta_9(\vartheta_4^4-\vartheta_9^4), & \vartheta_1\vartheta_2\vartheta_3\vartheta_6\vartheta_8\vartheta_9(\vartheta_3^4+\vartheta_8^4),\\
\vartheta_1\vartheta_4\vartheta_5\vartheta_6\vartheta_7\vartheta_8(\vartheta_6^4+\vartheta_7^4), & \vartheta_1\vartheta_2\vartheta_3\vartheta_5\vartheta_7\vartheta_{10}(\vartheta_2^4+\vartheta_5^4).\\
\vartheta_1\vartheta_3\vartheta_7\vartheta_8\vartheta_9\vartheta_{10}(\vartheta_7^4+\vartheta_{10}^4), &
\end{array}$$

5.3 Lemma. *Any $\Gamma_2[4,8]$-invariant tensor*

$$T=\mathrm{tr}(f\,dZ)\otimes(dz_{11}\wedge dz_{12}\wedge dz_{22})^{\otimes k}$$

is contained in the intersection of the localizations of the modules

$$\sum_{1\le i<j\le 10}\mathbb{C}[\vartheta_1,\dots,\vartheta_{10}]\ \{\vartheta_i,\vartheta_j\}$$

by the 60 elements $\vartheta[m]X_{\mathfrak{m}}$ where $\mathfrak{m}$ runs through the 15 syzygetic quadruples and m runs through the 4 members of $\mathfrak{m}$.

This Lemma carries over to an arbitrary modular form of $\mathcal{M}(r)$ since we can multiply such a form by a suitable power of ϑ_i to get a form of tensorial type $(r=6k)$. Hence we obtain as an immediate consequence the following result.

5.4 Theorem. *Lemma 5.3 implies*

$$\mathcal{M} = \bigcap_{\mathfrak{m}} \bigcap_{m\in\mathfrak{m}} \Big(\sum_{1\le i<j\le 10} \mathbb{C}[\vartheta_1,\dots,\vartheta_{10}]\,\{\vartheta_i,\vartheta_j\}\Big)_{X_{\mathfrak{m}}\vartheta[m]}.$$

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