On the variety associated to the ring of theta constants in genus 3

Eberhard Freitag
Mathematisches Institut
Im Neuenheimer Feld 288
D69120 Heidelberg
freitag@mathi.uni-heidelberg.de

Riccardo Salvati Manni
Dipartimento di Matematica,
Piazzale Aldo Moro, 2
I-00185 Roma, Italy.
salvati@mat.uniroma1.it

Abstract

Due to fundamental results of Igusa [Ig1] and Mumford [Mu] the $N = 2^{g-1}(2^g+1)$ theta constants of first kind

$$\sum_{n \text{ integral}} \exp \pi \mathrm{i}(Z[n+a/2] + 2b'(n+a/2)), \quad a, b \text{ integral.}$$

define for each genus g an injective holomorphic map of the Satake compactification $X_g(4,8) = \overline{\mathcal{H}_g/\Gamma_g[4,8]}$ into the projective space P^{N-1}. Moreover, this map is biholomorphic onto the image outside the Satake boundary. It is not biholomorphic on the whole in the cases $g \geq 6$ [Ig3]. Igusa also proved that in the cases $g \leq 2$ this map biholomorphic onto the image [Ig2]. In this paper we extend this result to the case $g = 3$. So we show that the theta map

$$X_3(4,8) \longrightarrow \mathbb{P}^{35}$$

is biholomorphic onto the image. This is equivalent to the statement that the image is a normal subvariety of $\mathbb{P}^{35}$.

Introduction

The algebra $R(g,q)$ is generated by the theta constants of second kind

$$f_{a,q} = \sum_{n \in \mathbb{Z}^g} \exp \pi \mathrm{i} q Z[n + a/q].$$

Here Z varies in the Siegel upper half space of genus g and a is a vector (column) in $\mathbb{Z}^g$. We use the standard notation $Z[a] = a'Za$ where a' is the transposed of a. The series depends only on $\pm a$ mod q. We always assume that q is an *even* natural number. The functions $f_{a,q}$ are modular forms with respect to the Igusa group $\Gamma_g[q, 2q]$ (Sect. 1). In particular, the series $f_{a,q}/f_{b,q}$ are invariant under $\Gamma_g[q,2q]$. The space of modular forms $[\Gamma_g[q,2q],r/2]$, $r\in\mathbb{Z}$, consists of all holomorphic functions f on the Siegel half plane $\mathcal{H}_g$ such that $f/f_{a,q}^r$ are invariant, where in the case $g=1$ the usual regularity condition at the cusps has to be added. The algebra of modular forms is

$$A(\Gamma_g[q,2q]) = \bigoplus_{r\in\mathbb{Z}} [\Gamma_g[q,2q],r/2].$$

By a result of Baily, the projective variety of the graded algebra $A(\Gamma_g[q,2q])$ can be identified, as a complex space, with the Satake compactification of $\mathcal{H}_g/\Gamma_g[q,2q]$,

$$\operatorname{proj}(A(\Gamma_g[q,2q])) = X_g(q,2q) := \overline{\mathcal{H}_g/\Gamma_g[q,2q]}.$$

Due to basic theorems of Igusa [Ig1] and Mumford [Mu], we have an everywhere regular, birational map

$$\overline{\mathcal{H}_g/\Gamma_g[q,2q]} \longrightarrow \operatorname{proj}(R(g,q)).$$

This implies that $A(\Gamma_g[q,2q])$ is the normalization of $R(q,g)$. In the case $q=4$ this map is bijective and biholomorphic outside the boundary. The case $q=2$ is exceptional. Here one knows that the ring $R(g,2)$ is normal if $g\le 3$ [Ru]. The variety $\operatorname{proj}(R(g,2))$ is not normal when $g\ge 4$, [SM]. It is not known if the map is bijective if $g>3$. The ring $R(g,4)$ is normal if and only if $g\le 2$ [Ig2, Ig5]. Moreover the ideal of the relations is generated by the so called Riemann relations. We shall obtain the following main result.

Theorem. *The map*

$$\overline{\mathcal{H}_3/\Gamma_3[4,8]} \longrightarrow \operatorname{proj}(R(3,4))$$

is biholomorphic.

We mention that Igusa uses a slightly different setting. He uses the theta constants of the first kind

$$\sum_{n \text{ integral}} \exp \pi \mathrm{i}(Z[n+a/q] + 2b'(n+a/q)), \quad a,b \text{ integral}.$$

There is a close relation to the theta constants of second kind. For example, one can show that the ring $R(g,q^2)$ is generated the "theta constants of first kind". In the case $g=3$, $q=2$, these are 36 different (up to sign) non-zero theta constants of first kind.

In a forthcoming paper we shall consider the projective variety related to Riemann's relations in genus $g=3$.

1. Local rings of modular varieties and their completion

We denote by

$$\mathcal{H}_g = \{Z \in \mathbb{C}^{g\times g}; \quad Z = Z',\ \operatorname{Im} Z \text{ positive definite}\}$$

the Siegel upper half plane and by $\mathrm{Sp}(g,\mathbb{Z})$ the Siegel modular group acting on $\mathcal{H}_g$ through $Z \mapsto (AZ+B)(CZ+D)^{-1}$. Recall that the principal congruence subgroup is defined as

$$\Gamma_g[q] = \operatorname{kernel}\bigl(\mathrm{Sp}(g,\mathbb{Z}) \longrightarrow \mathrm{Sp}(g,\mathbb{Z}/q\mathbb{Z})\bigr)$$

and Igusa's subgroup as

$$\Gamma_g[q,2q] := \bigl\{\ M \in \Gamma_g[q], \quad (CD')_0 \equiv (AB')_0 \equiv 0 \mod 2q\ \bigr\}.$$

Here S_0 denotes the column built of the diagonal of a square matrix S. We generalize results from [FK] and [Kn]. We consider the Siegel modular variety $\mathcal{H}_g/\Gamma_g[q,2q]$ and the Satake compactification

$$X_g(q,2q) = \overline{\mathcal{H}_g/\Gamma_g[q,2q]}.$$

For a decomposition $g = g_1 + g_2$ we consider the map

$$\mathcal{H}_{g_1} \longrightarrow X_g(q,2q), \quad \tau \longmapsto \lim_{t\to\infty} \begin{pmatrix} \tau & 0 \\ 0 & \mathrm{i}t \end{pmatrix}.$$

We call the image of τ the standard boundary point related to τ. The full Siegel modular group $\mathrm{Sp}(g,\mathbb{Z})$ acts on $X_g(q,2q)$. Every boundary point is equivalent to a standard boundary point. Hence we can restrict to study the standard boundary points. We recall the description of the analytic local ring of $X_g(q,2q)$ at such a point [Ig4].

1.1 Definition. *Let $U \subset \mathcal{H}_{g_1}$ be an open subset and let T be a semipositive integral symmetric $g_2\times g_2$-matrix. The space $\mathcal{J}_T(U)$ consists of all holomorphic functions $f : U \times \mathbb{C}^{g_2\times g_1} \to \mathbb{C}$ with the transformation property*

$$\begin{aligned} &f(\tau, z+qh) = f(\tau,z),\\ &f(\tau, z+qh\tau) = \exp\{-\pi \mathrm{i}\operatorname{tr}(qT[h]\tau + 2h'Tz)\} f(\tau,z) \quad \textit{for } h \in \mathbb{Z}^{g_2\times g_1}. \end{aligned}$$

For a point $\tau_0 \in \mathcal{H}_{g_1}$ we define

$$\mathcal{J}_T(\tau_0) = \varinjlim \mathcal{J}_T(U),$$

where U runs through all open neighborhoods of τ_0.

In the case $T=0$ we have an everywhere holomorphic abelian function of z which must be constant. So we see

$$\mathcal{J}_0(\tau_0)=\mathcal{O}_{\tau_0},$$

where $\mathcal{O}_{\tau_0}$ denotes the local ring of the complex manifold $\mathcal{H}_{g_1}$ at τ_0. In the case $q\geq 4$ we can identify $\mathcal{O}_{\tau_0}$ with the local ring of $\mathcal{H}_{g_1}/\Gamma_{g_1}[q,2q]$ at the image of τ_0, and we can consider $\mathcal{O}_{\tau_0}$ as subring of the local ring of $X_g(q,2q)$ at the cusp related to τ_0. The spaces $\mathcal{J}_T(\tau_0)$ are modules over $\mathcal{O}_{\tau_0}$, moreover multiplication gives a map

$$\mathcal{J}_{T_1}(\tau_0)\otimes_{\mathcal{O}_{\tau_0}}\mathcal{J}_{T_2}(\tau_0)\longrightarrow\mathcal{J}_{T_1+T_2}(\tau_0).$$

If we evaluate elements of the space $\mathcal{J}_T(\tau_0)$ at the point τ_0, we get spaces of theta functions:

1.2 Definition. *The space $J_T(\tau_0)$ consists of all holomorphic functions $f:\mathbb{C}^{g_2\times g_1}\to\mathbb{C}$ with the transformation property*

$$\begin{aligned} f(z+qh)&=f(z),\\ f(z+qh\tau_0)&=\exp\{-\pi\mathrm{itr}\,(q(T[h]\tau_0+2h'Tz)\}f(z)\quad \textit{for integral } h. \end{aligned}$$

We have the evaluation map

$$\mathcal{J}_T(\tau_0)\longrightarrow J_T(\tau_0).$$

1.3 Lemma. *The $\mathcal{O}_{\tau_0}$ modules $\mathcal{J}_T(\tau_0)$ are finitely generated and free.*

Proof. Since the elements of $\mathcal{J}_T(\tau_0)$ are periodic in z, they admit a Fourier expansion

$$f(\tau,z)=\sum_{k\text{ integral}}c_k\exp 2\pi\mathrm{itr}(k'z)/q.$$

The Fourier coefficients are in $\mathcal{O}_{\tau_0}$. The second equation in Definition 1.1 gives

$$c_{k+qTh}=\exp(\pi\mathrm{itr}\,(qT[h]\tau+2k'h\tau))c_k.$$

In the case that T is invertible, one can prescribe the Fourier coefficients c_k for a system of representatives mod $qT\mathbb{Z}^{g_2\times g_1}$ and then reconstruct f as a linear combination of theta functions. This shows that $\mathcal{J}_T(\tau_0)$ is free of finite rank. The case of a singular T can be reduced to the previous case in a standard way (taking a quotient by the nullspace of T). □

The same argument gives generators of the vector space $J_T(\tau_0)$ and hence the following result.

1.4 Lemma. *The evaluation map*

$$\mathcal{J}_T(\tau_0) \longrightarrow J_T(\tau_0)$$

is surjective.

Another way to express this is

$$J_T(\tau_0) = \mathcal{J}_T(\tau_0) \otimes_{\mathcal{O}_{\tau_0}} \mathbb{C}.$$

Now we assume $q \geq 4$. Then the groups $\Gamma_g[q, 2q]$ contain no element of finite order besides the unit matrix. In this case the analytic local ring $S^{\mathrm{an}}(\tau_0)$ of $X_g(q, 2q)$ at the image of τ_0 can be described as the set of series

$$\sum_T a_T \exp(\pi \mathrm{i}\, \mathrm{tr}(TW)/q), \quad a_T \in \mathcal{J}_T(\tau_0),$$
$$a_{T[U]}(\tau, z) = a_T(\tau, zU') \quad \text{for } U \in \mathrm{GL}(g_2, \mathbb{Z})[q],$$

where T runs through all symmetric integral semipositive $g_2 \times g_2$-matrices and such that a certain convergence condition is satisfied [Ig4]. In the paper [Ig4] it has been shown that the "Poincaré series"

$$H_{T,f}(\tau, z, W) = \sum f(\tau, WU') \exp(\pi \mathrm{i}\, \mathrm{tr}(WT[U])/q), \qquad f \in \mathcal{J}_T(\tau_0),$$

have this convergence property. The sum is taken over distinct $T[U]$ for $U \in \mathrm{GL}(g_2, \mathbb{Z})[q]$. From the Supplement of Theorem 1 in [Ig4], also the following result follows.

1.5 Proposition. *The maximal ideal of the ring $S^{\mathrm{an}}(\tau_0)$ is generated by the Poincaré series $H_{T,f}$ for non-zero T and by the maximal ideal of the local ring $\mathcal{O}_{\tau_0}$.*

We introduce a filtration $\mathfrak{m}_n$ on $S^{\mathrm{an}}(\tau_0)$. For a semipositive integral T we denote by $\lambda(T)$ the biggest number k such that T can be written as $T = T_1 + \dots + T_k$ with non-zero integral and semipositive T_i. In the case $T = 0$ this is understood as $\lambda(T) = 0$. The associated filtration is

$$\mathfrak{n}_n = \{P \in S^{\mathrm{an}}(\tau_0); \quad a_T = 0 \quad \text{for } \lambda(T) < n\}.$$

Then we define $\mathfrak{m}_n$ to be the ideal generated by

$$\mathfrak{m}(\mathcal{O}_{\tau_0})^\mu\, \mathfrak{n}_\nu, \quad \mu + \nu \geq n,$$

where $\mathfrak{m}(\mathcal{O}_{\tau_0})$ denotes the maximal ideal of $\mathcal{O}_{\tau_0}$. The ideal $\mathfrak{m} = \mathfrak{m}_1$ is the maximal ideal of $S^{\mathrm{an}}(\tau_0)$ and we have

$$\mathfrak{m}_1 \supset \mathfrak{m}_2 \supset \dots \quad \text{and} \quad \mathfrak{m}_\mu \mathfrak{m}_\nu \subset \mathfrak{m}_{\mu+\nu}.$$

The Poincaré series $H_{T,f}$ is contained in $\mathfrak{m}_n$ if either $\lambda(T) \geq n$ or if $\lambda(T) < n$ and

$$f \in \mathfrak{m}(\mathcal{O}_{\tau_0})^{n-\lambda(T)} \mathcal{J}_T(\tau_0).$$

So we have

$$S^{\mathrm{an}}(\tau_0)/\mathfrak{m}_n \cong \bigoplus_{\lambda(T)<n} \mathcal{J}_T(\tau_0)/\mathfrak{m}(\mathcal{O}_{\tau_0})^{n-\lambda(T)} \mathcal{J}_T(\tau_0).$$

We want to get rid of convergence conditions and therefore introduce a formal variant. First we define

$$\hat{\mathcal{J}}_T(\tau_0) = \mathcal{J}_T(\tau_0) \otimes_{\mathcal{O}_{\tau_0}} \hat{\mathcal{O}}_{\tau_0}$$

where $\hat{\mathcal{O}}_{\tau_0}$ denotes the completion of $\mathcal{O}_{\tau_0}$. Then we introduce the formal ring $\hat{S}(\tau_0)$ that consists of all formal series

$$\sum_T a_T \exp(\pi \mathrm{i}\, \mathrm{tr}(TW)/q), \quad a_T \in \hat{J}_T(\tau_0),$$
$$a_{T[U]}(\tau, z) = a_T(\tau, zU') \quad \text{for } U \in \mathrm{GL}(g_2, \mathbb{Z})[q].$$

The matrices T run through all integral semipositive $g_2 \times g_2$-matrices.

The ring $\hat{S}(\tau_0)$ is just the completion of $S^{\mathrm{an}}(\tau_0)$ with respect to the filtration $(\mathfrak{m}_n)$. We denote by $\bar{S}(\tau_0)$ the usual completion (by the powers of the maximal ideal $\mathfrak{m}$). From $\mathfrak{m}^n \subset \mathfrak{m}_n$ we obtain a natural homomorphism

$$\bar{S}(\tau_0) \longrightarrow \hat{S}(\tau_0).$$

1.6 Theorem. *The natural homomorphism*

$$\bar{S}(\tau_0) \longrightarrow \hat{S}(\tau_0)$$

is an isomorphism.

The case of the zero-dimensional boundary components has been treated (in the more general context of arbitrary tube domains) by Knöller [Kn] who refers to [FK] where the special case of the Hilbert modular group has been treated.

Proof of Theorem 1.6. First we prove that the homomorphism is surjective. Since $S^{\mathrm{an}}(\tau_0)/\mathfrak{m}^k$ is a finite dimensional vector space we find for each k an r such that

$$\mathfrak{m}^k \cap \mathfrak{m}_r = \mathfrak{m}^k \cap \mathfrak{m}_{r+1} = \cdots .$$

Therefore we can construct inductively a sequence of natural numbers $r_1 < r_2 < \cdots$ such that

$$\mathfrak{m}_{r_1} \subset \mathfrak{m}^2 + \mathfrak{m}_{r_2}, \quad \mathfrak{m}_{r_2} \subset \mathfrak{m}^3 + \mathfrak{m}_{r_3}, \quad \mathfrak{m}_{r_3} \subset \mathfrak{m}^4 + \mathfrak{m}_{r_4}, \quad \ldots .$$

An arbitrary element $f \in \hat{S}(\tau_0)$ can be written in the form

$$f = f_1 + f_2 + \cdots, \quad f_i \in \mathfrak{m}_{r_i}.$$

We construct inductively elements $g_j \in \mathfrak{m}^j$, $a_j \in \mathfrak{m}_{r_j}$ such that

$$f_1 = g_1 + a_2, \quad f_2 + a_2 = g_3 + a_3, \quad f_3 + a_3 = g_4 + a_4, \ldots .$$

Then

$$f_1 + f_2 + \cdots f_{k-1} = g_1 + g_2 + \cdots g_k + a_k.$$

The series $g_1 + g_2 + \cdots$ converges in $\bar{S}(\tau_0)$. Its image in $\hat{S}(\tau_0)$ is f. This shows the surjectivity. We now know that $\hat{S}(\tau_0)$ is noetherian too. To show injectivity it is enough that the dimension of $\hat{S}(\tau_0)$ is greater or equal than $\dim \bar{S}(\tau_0) = g(g+1)/2$. Here we use the well-known result of commutative algebra that for every ideal $\mathfrak{a}$ in a noetherian ring R that contains a non-zero divisor we have $\dim R > \dim R/\mathfrak{a}$. The dimension of a local noetherian ring can be computed as the highest coefficient of the Hilbert Samuel polynomial. Hence we must show that

$$\frac{\dim \hat{S}(\tau_0)/\hat{\mathfrak{m}}^k}{k^{g(g+1)/2-1}}, \qquad (\hat{\mathfrak{m}} \text{ maximal ideal of } \hat{S}(\tau_0),$$

is unbounded. We define the ideals $\hat{\mathfrak{m}}_k$ in $\hat{S}(\tau_0)$ in the same way as the ideals $\mathfrak{m}_k$ in $S^{\mathrm{an}}(\tau_0)$. This means that we set

$$\hat{\mathfrak{n}}_n = \{P \in \hat{S}(\tau_0); \quad a_T = 0 \quad \text{for } \lambda(T) < n\}$$

and $\hat{\mathfrak{m}}_n$ to be the ideal generated by

$$\mathfrak{m}(\hat{\mathcal{O}}_{\tau_0})^\mu \, \hat{\mathfrak{n}}_\nu, \quad \mu + \nu \geq n,$$

where $\mathfrak{m}(\hat{\mathcal{O}}_{\tau_0})$ denotes the maximal ideal of $\hat{\mathcal{O}}_{\tau_0}$. It is sufficient to show that

$$\frac{\dim \hat{S}(\tau_0)/\hat{\mathfrak{m}}_k}{k^{g(g+1)/2-1}}$$

remains unbounded. The description above by means of Poincaré series shows

$$\dim \hat{S}(\tau_0)/\hat{\mathfrak{m}}_k = \dim S^{\mathrm{an}}(\tau_0)/\mathfrak{m}_k.$$

During the following estimates, T always runs through a system of semipositive integral matrices mod $\mathrm{GL}(g_2, \mathbb{Z})[q])$ and $C_1, C_2 \ldots$ will denote suitable constants. We have

$$\begin{aligned}\dim \hat{S}(\tau_0)/\hat{\mathfrak{m}}_k &= \dim S^{\mathrm{an}}(\tau_0)/\mathfrak{m}_k \\ &= \sum_{\nu+\lambda(T)=k} \dim \mathcal{J}_T(\tau_0)/\mathfrak{m}(\mathcal{O}_{\tau_0})^\nu \mathcal{J}_T(\tau_0) \\ &\geq C_1 \sum_{\nu+\lambda(T)=k} \dim J_T(\tau_0) \nu^{g_1(g_1+1)/2}.\end{aligned}$$

We only keep T which are invertible. Then the dimension of $J_T(\tau_0)$ is $\det(T)^{g_1}$ up to a constant factor [Ig4]. We obtain

$$\geq C_2 \sum_{\nu+\lambda(T)=k} (\det T)^{g_1} \nu^{g_1(g_1+1)/2}.$$

A trivial estimate states $\mathrm{tr}(T) \geq \lambda(T)$. We claim that also $(\det T)^{1/g_2}$ is greater or equal than $\lambda(T)$ up to a constant factor. Since this statement is invariant under unimodular transformation, it is sufficient to prove this for Minkowski reduced matrices. It follows from the standard inequalities for Minkowski reduced matrices (cf. [Fr] page 33). Therefore we get

$$\geq C_3 \sum_{\nu+\lambda(T)=k} \lambda(T)^{g_1 g_2} \nu^{g_1(g_1+1)/2}.$$

Now we restrict the summation to the range $k/2 \leq \lambda(T) \leq 3k/4$. Then $\nu \geq k/4$. Hence we get

$$\geq C_4 k^{g_1 g_2 + g_1/(g_1+1)/2} \, \#\{T; T \mod \mathrm{GL}(g_2, \mathbb{Z})[q],\ k/4 \leq \lambda(T) \leq 3k/4\}.$$

The asymptotic behaviour of the number of all T with an upper bound for λ has been determined by Knöller [Kn], Satz 2.3.1. This gives

$$\geq C_5 k^{g_1 g_2 + g_1/(g_1+1)/2} \cdot k^{g_2(g_2+1)/2} = C_5 k^{g(g+1)/2}.$$

This finishes the proof of Theorem 1.6. $\square$

2. Optimal decompositions

We use the notation

$$\mathcal{T}_g = \{\text{integral semipositive } g \times g\text{-matrices}\}.$$

The group $\mathrm{GL}(g, \mathbb{Z})$ acts on $\mathcal{T}_g$ through $T \mapsto T[U] = U'TU$ from the right. In our context, matrices $T \in \mathcal{T}_g$ of rank one are important.

We call a non-zero element of $\mathcal{T}_g$ *irreducible* if it cannot be written as sum of two non zero elements of $\mathcal{T}_g$. We recall that for a semipositive $T \in \mathcal{T}_g$ we denote by $\lambda(T)$ the biggest number k such that T can be written as $T = T_1 + \cdots T_k$ with non-zero $T_i \in \mathcal{T}_g$. Notice that the irreducible elements T are characterized by $\lambda(T) = 1$ and that $\lambda(T)$ is invariant under unimodular transformations.

2.1 Definition. *Let T be a semipositive definite integral matrix. A decomposition into irreducible integral matrices*

$$T = T_1 + \cdots + T_k, \quad \lambda(T) = k,$$

*is called **q-optimal** if all T_i are of rank 1 and if for arbitrary $U_1, \dots, U_k$ in $\mathrm{GL}(g,\mathbb{Z})[q]$ one of the following two conditions holds.*

a) $\lambda(T_1[U_1] + \cdots + T_k[U_k]) > k$.

b) $T_1[U_1] + \cdots + T_k[U_k] \sim T \mod \mathrm{GL}(g,\mathbb{Z})[q]$.

We will make use of the following two simple facts.

1) If $T = T_1 + \cdots + T_k$ is optimal then $T[U] = T_1[U] + \cdots + T_k[U]$ is optimal for all $U \in \mathrm{GL}(g,\mathbb{Z})$.
2) If $T = T_1 + \cdots + T_k$ is optimal then

$$\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} T_k & 0 \\ 0 & 0 \end{pmatrix}$$

is optimal too.

An integral matrix is called *primitive* if its entries are coprime. Primitive semipositive matrices of rank 1 can be written as products

$$T = aa', \quad a \text{ primitive column},$$

where a is unique up to the sign. The group $\mathrm{GL}(g,\mathbb{Z})$ acts transitively on the set of all primitive columns. Hence it acts transitively on the set of all primitive integral matrices of rank 1.

2.2 Lemma. *Let $q = 2$ or $q = 4$. Two primitive semipositive integral matrices T, S of rank one are equivalent mod $\mathrm{GL}(g,\mathbb{Z})[q]$ if and only if they are congruent mod q (i.e. $T \equiv S \mod q$).*

Proof. We write T, S in the form $T = aa'$, $S = bb'$. There must be an index i such that a_i is odd. From $a_i^2 \equiv b_i^2$ and $q = 2, 4$ we conclude $a_i \equiv \pm b_i \mod 4$. Since we can replace b by $-b$ we can assume $a_i \equiv b_i \mod q$. Then $a_i a_j \equiv b_i b_j$ implies $a \equiv b \mod q$. Hence there exists a matrix $U \in \mathrm{GL}(g,\mathbb{Z})[q]$ such that $b = Ua$. This shows $S = T[U]$. □

2.3 Lemma. *Let*

$$T = \begin{pmatrix} t_0 & t_1 \\ t_1 & t_2 \end{pmatrix}, \quad 0 \le t_1 \le t_0, t_2,$$

be an integral semi positive matrix. Then

$$\lambda(T) = t_0 + t_2 - t_1.$$

Proof. The equality

$$T = t_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (t_0 - t_1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (t_2 - t_1) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

shows $\lambda(T) \geq t_0 + t_2 - t_1$. We have to show the reverse inequality. Let $T = T_1 + \cdots + T_k$, $k = \lambda(T)$, where the T_i are integral, positive semidefinite and different from 0. Consider the matrix

$$S = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}.$$

It is positive definite. Obviously

$$t_0 + t_2 - t_1 = \operatorname{tr}(TS) = \operatorname{tr}(ST_1) + \cdots + \operatorname{tr}(ST_k) \geq k = \lambda(T).$$

This implies $\lambda(T) \leq t_0 + t_2 - t_1$. □

2.4 Lemma. *Assume that T is a positive definite integral 2×2-matrix. Then*

$$\lambda(T) \geq \frac{3}{2}\sqrt{\det T}.$$

Proof. Since λ and det are unimodular invariant, we can assume that T is Minkowski reduced $(0 \leq 2t_1 \leq t_0 \leq t_2)$. Then

$$\lambda(T) = t_0 + t_2 - t_1 \geq \frac{3}{4}(t_0 + t_2) \geq \frac{3}{2}\sqrt{t_0 t_2} \geq \frac{3}{2}\sqrt{t_0 t_2 - t_1^2}. \qquad \square$$

2.5 Lemma. *Let*

$$T = \begin{pmatrix} t_1 & t' \\ t & T_2 \end{pmatrix}, \quad T_2 \in \mathcal{T}_{g-1},$$

be an integral primitive semipositive $g \times g$-matrix of rank 1. Assume

$$\lambda \begin{pmatrix} t_1 + 1 & t' \\ t & T_2 \end{pmatrix} = 2.$$

Then T_2 is primitive or zero.

Proof. After a suitable unimodular transformation with a matrix of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & U \end{smallmatrix}\right)$ we can assume that

$$T_2 = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

We have to show $d \leq 1$. We have

$$T = \begin{pmatrix} t_1 & s & 0 \\ s & d & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

hence

$$\det \begin{pmatrix} t_1 + 1 & s \\ s & d \end{pmatrix} = d$$

and the claim follows from Lemma 2.4 □

2.6 Lemma. *Let T be a semipositive $g\times g$-matrix of rank one with coprime entries and let $U\in\mathrm{GL}(g,\mathbb{Z})[2]$ such that $\lambda(T+T[U])=2$. Then $T[U]=T$.*

Proof. This statement is invariant under $T\mapsto T[V]$ where $V\in\mathrm{GL}(g,\mathbb{Z})$. Hence we can assume that T is the matrix with $t_{11}=1$ and zeros elsewhere. Let

$$H=T[U]=\begin{pmatrix} h_1 & h' \\ h & H_2\end{pmatrix}\equiv\begin{pmatrix}1&0\\0&0\end{pmatrix}\bmod 2$$

The entries of H_2 are even and hence not coprime. From Lemma 2.5 follows that they are zero. Since H is semipositive this implies $h=0$ and $h_1=1$. This shows $H=T[U]=T$. □

3. Degree two

We prove the existence of optimal decompositions in the case $g=2$.

3.1 Proposition. *In the cases $g=2$, q arbitrary (even), every semipositive integral matrix T admits an optimal decomposition.*

This proposition is invariant under unimodular transformation. Hence it is enough to prove Proposition 3.1 for invertible Minkowski-reduced T (i.e. $0\le 2t_{12}\le t_{11}\le t_{22}$).

Proof of Proposition 3.1. We can assume that T is invertible and Minkowski reduced. Then we claim that

$$T=(t_0-t_1)E_1+(t_2-t_1)E_2+t_1E_3\quad (k=\lambda(T)=t_0+t_2-t_1)$$

where

$$E_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad E_2=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad E_3=\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

and

$$r_1=t_0-t_1,\quad r_2=t_2-t_1,\quad r_3=t_1.$$

is optimal (in both cases $q=2$ and $q=4$). We write the decomposition of T in the form $T_1+\cdots+T_k$ where T_i belong to $\{E_1,E_2,E_3\}$. We have to consider

$$\tilde T=T_1[U_1]+\cdots+T_k[U_k],\quad U_i\in\mathrm{GL}(2,\mathbb{Z})[q].$$

We can assume that $\lambda(\tilde T)=\lambda(T)$. Then we have to show that T and $\tilde T$ are equivalent under the group $\mathrm{GL}(2,\mathbb{Z})[q]$. From Lemma 2.6 we can assume that

$$T_i=T_k\Longrightarrow U_i=U_k.$$

Hence we can write

$$\tilde{T} = r_1 E_1[U_1] + r_2 E_2[U_2] + r_3 E_3[U_3], \quad U_i \in \mathrm{GL}(2,\mathbb{Z})[q].$$

We can assume that $U_1 = E$ is the unit matrix. Since T is Minkowski reduced, $r_1 = t_0 - t_1$ and $r_2 = t_2 - t_1$ both are positive. From Lemma 2.5 we see that $(E_2[U_2])_{11} \le 1$. But since this expression is even, it must be zero. Then necessarily $E_2[U_2] = E_2$. So we can assume $U_1 = U_2 = E$. In the case $r_3 = 0$ we are finished. Otherwise, we can apply Lemma 2.5 again to see that the diagonal elements of $E_3[U_3]$ are ≤ 1. They are odd, hence both are 1. So we get

$$E_3[U_3] = \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.$$

In case of the plus sign we are done. The minus sign only can occur of $q \le 2$. Then we can transform with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ which is a matrix in $\mathrm{GL}(2,\mathbb{Z})[2]$. This completes the proof of Proposition 3.1. □

4. Degree three

We prove the existence of optimal decompositions in the case $g = 3$.

4.1 Proposition. *In the cases $g = 3$, $q = 2, 4$, every semipositive integral matrix T admits an optimal decomposition.*

We recall from [CS], Chapt. 15, Sect. 10 that a 3×3-matrix symmetric positive definite real matrix T is reduced in the sense of Minkowski if

$$\begin{aligned} &t_{11} \le t_{22} \le t_{33}, \\ &0 \le 2t_{12} \le t_{11}, \quad 0 \le 2t_{23} \le t_{22}, \quad 2|t_{13}| \le t_{11}, \\ &2(t_{12} + t_{23} + |t_{13}|) \le t_{11} + t_{22}. \end{aligned}$$

4.2 Lemma. *Let T be a positive definite reduced integral 3×3-matrix. Then*

$$\lambda(T) = \begin{cases} t_{11} + t_{22} + t_{33} - t_{12} - t_{23} + t_{13} & \text{if } t_{13} \le 0, \\ t_{11} + t_{22} + t_{33} - t_{12} - t_{23} - t_{13} + \min(t_{12}, t_{13}, t_{23}) & \text{if } t_{13} > 0. \end{cases}$$

Proof. The following system of matrices will play a basic role.

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ E_4 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ E_6 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},\ E_7 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

It is easy to check that it is a system of representatives of integral semipositive matrices of rank one with respect to the action $T \mapsto T[U]$ of the group $\mathrm{GL}(2,\mathbb{Z})[3]$. We also introduce the modified matrix

$$E_6^- = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

In the case $t_{13} \le 0$ we use the decomposition

$$\begin{aligned} T =&(t_{11} - t_{12} + t_{13})E_1 + (t_{22} - t_{12} - t_{23})E_2 + (t_{33} + t_{13} - t_{23})E_3 \\ &+ t_{12}E_4 + t_{23}E_5 - t_{13}E_6^- . \end{aligned}$$

We notice that E_7 does not occur in this decomposition. Since the coefficients are nonnegative, we get $\lambda(T) \ge t_{11} + t_{22} + t_{33} - t_{12} - t_{23} + t_{13}$. For the reverse inequality we use the positive matrix

$$S = \frac{1}{2}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

The product of a positive and a non-zero semipositive matrix has positive trace, hence $\mathrm{tr}(SH) > 0$ is a positive integer. This implies $\mathrm{tr}(SH) \ge \lambda(H)$, in fact if $H = H_1 + \dots H_k$, then $SH = SH_1 + \dots SH_k$. In our case we get $\lambda(T) \le \mathrm{tr}(ST) = t_{11} + t_{22} + t_{33} - t_{12} - t_{23} + t_{13}$. This completes the proof in the first case $t_{13} \le 0$.

In the second case, $t_{13} > 0$, we use a similar decomposition. Setting $m = \min\{t_{12}, t_{23}, t_{13}\}$, we take

$$\begin{aligned} &T = \\ &(t_{11} - t_{12} - t_{13} + m)E_1 + (t_{22} - t_{12} - t_{23} + m)E_2 + (t_{33} - t_{23} - t_{13} + m)E_3 \\ &+ (t_{12} - m)E_4 + (t_{23} - m)E_5 + (t_{13} - m)E_6 \\ &+ mE_7 \end{aligned}$$

which shows $\lambda(T) \ge t_{11} + t_{22} + t_{33} - t_{12} - t_{13} - t_{23} + m$. We observe that at least one of the coefficients of E_4, E_5, E_6 is 0. To prove the reverse inequality one uses $\mathrm{tr}(ST) \ge \lambda(T)$ for one of the following three matrices

$$S = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

depending on whether m is t_{12} or t_{23}, or t_{13}. □

4.3 Lemma. *Let T be an integral positive definite 3×3-matrix. Then*
$$\lambda(T)^3 \geq 8\det T.$$

Proof. We can assume that T is reduced. From the inequality
$$2(t_{12}+t_{23}+|t_{13})| \leq t_{11}+t_{22} \leq t_{11}+t_{33} \leq t_{22}+t_{33}$$
together with the trivial inequality
$$\lambda(T) \geq (t_{11}+t_{22}+t_{33}-t_{12}-t_{23}-|t_{13}|)$$
we get
$$\lambda(T) \geq 2\frac{t_{11}+t_{22}+t_{33}}{3} \geq 2(t_{11}t_{22}t_{33})^{1/3}.$$
The statement of the Lemma now follows from Hadamard's inequality
$$t_{11}t_{22}t_{33} \geq \det T.$$
□

4.4 Lemma. *Let T be a matrix of rank ≤ 1. Then*
$$\det\left(E_1+E_2+T\right) = t_{33}.$$
In addition, let T be semipositive and integral. Then one of the following two inequalities hold.

a) $\lambda\left(E_1+E_2+T\right) > 3$.

b) $t_{33} \leq 1$.

Proof. The computation of the determinant is easy. Hence we have to prove only the second statement. We assume that $t_{33} > 1$. Since it is a square, we obtain $t_{33} \geq 4$. But t_{33} is the determinant of the matrix. Hence we get from Lemma 4.3 that $\lambda^3 \geq 32$. This gives $\lambda(T) > 3$. □

We will apply several times not only Lemma 4.4 but also an obvious generalization. Let $U \in \mathrm{GL}(3,\mathbb{Z})$. Then one has for $\operatorname{rank} T \leq 1$
$$\det\left(\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}[U]+T\right) = T[U^{-1}]_{33}.$$

4.5 Corollary of Lemma 4.4. *Assume that T is a semipositive integral matrix of rank one. Then*
$$\begin{aligned}
&\lambda(E_1+E_2+T)=3 \Longrightarrow t_{33}\leq 1,\\
&\lambda(E_1+E_3+T)=3 \Longrightarrow t_{22}\leq 1,\\
&\lambda(E_1+E_4+T)=3 \Longrightarrow t_{33}\leq 1,\\
&\lambda(E_1+E_5+T)=3 \Longrightarrow t_{22}-2t_{23}+t_{33}\leq 1,\\
&\lambda(E_2+E_3+T)=3 \Longrightarrow t_{11}\leq 1,\\
&\lambda(E_2+E_4+T)=3 \Longrightarrow t_{33}\leq 1,\\
&\lambda(E_2+E_5+T)=3 \Longrightarrow t_{11}\leq 1,\\
&\lambda(E_3+E_4+T)=3 \Longrightarrow t_{11}-2t_{12}+t_{22}\leq 1,\\
&\lambda(E_3+E_5+T)=3 \Longrightarrow t_{11}\leq 1,\\
&\lambda(E_4+E_5+T)=3 \Longrightarrow t_{11}+t_{22}+t_{33}+2t_{12}-2t_{23}+2t_{13}\leq 1.
\end{aligned}$$

For the proof of Proposition 4.1 we can assume that T is positive definite. The proposition is invariant under arbitrary unimodular transformation. Hence we can assume that T is reduced. We have to differ between the two cases:

Case A. $t_{13} \le 0$.
Case B. $t_{13} > 0$.

We start with case A. We use the decomposition

$$\begin{aligned} T =& r_1E_1 + r_2E_2 + r_3E_3 + r_4E_4 + r_5E_5 + r_6E_6^-, \\ & r_1 = t_{11} + t_{13} - t_{12}, \quad r_2 = t_{22} - t_{12} - t_{23}, \quad r_3 = t_{33} + t_{13} - t_{23}, \\ & r_4 = t_{12}, \quad r_5 = t_{23}, \quad r_6 = -t_{13}. \end{aligned}$$

We will show that it is q-optimal in both cases $q = 2$ and $q = 4$. The reduction inequalities read as

$$\begin{aligned} &r_4 \le r_1 + r_6, \quad r_1 + r_6 \le r_2 + r_5, \\ &r_6 \le r_1 + r_4, \quad r_2 + r_4 \le r_3 + r_6, \\ &r_5 \le r_2 + r_4, \quad r_5 + r_6 \le r_1 + r_2. \end{aligned}$$

Since T is positive definite we have also that the diagonal elements are positive, in particular

$$r_1 + r_4 + r_6 > 0.$$

We also mention that at least two of the coefficients r_1, r_2, r_3 do not vanish. More precisely we state.

Only the following 4 cases are possible.

1) $r_1 > 0$, $r_2 > 0$, $r_3 > 0$.
2) $r_1 > 0$, $r_2 > 0$, $r_3 = 0$ and $r_6 > 0$, $r_4 = 0$, $r_1 = r_2 = r_5 = r_6$.
3) $r_1 > 0$, $r_2 = 0$, $r_3 > 0$ and $r_6 = 0$, $r_1 = r_4 = r_5$.
4) $r_1 = 0$, $r_2 > 0$, $r_3 > 0$ and $r_6 > 0$, $r_4 = r_6$.

For the proof one has to discuss the three cases $r_i = 0$ separately. We start with

Case 1) There is nothing to prove.

Case 2) $r_3 = 0$. Thus $t_{33} - t_{23} + t_{13} = 0$. Hence $t_{23} = -t_{13} = t_{33}/2$, thus $r_6 > 0$ and by the basic inequalities

$$2(t_{12} + t_{33}) \le t_{11} + t_{22} \le 2t_{33}.$$

Hence

$$t_{12} = 0; \; t_{33} = t_{11} = t_{22}.$$

This implies $r_4 = 0, r_1 = r_2 = r_5 = r_6$.

We observe that in this case the matrix T has the form

$$T = \begin{pmatrix} 2a & 0 & -a \\ 0 & 2a & a \\ -a & a & 2a \end{pmatrix}$$

Case 3) and Case 4) can be proved in similar way, we just observe that the corresponding matrices T have the forms

$$T = \begin{pmatrix} 2a & a & 0 \\ a & 2a & a \\ 0 & a & c \end{pmatrix} \quad T = \begin{pmatrix} 2a & a & -a \\ a & b & h \\ -a & h & c \end{pmatrix}$$

Now we will prove that the described decomposition

$$T = r_1 E_1 + \cdots r_5 E_5 + r_6 E_6^-$$

is q-optimal in each of the 4 cases. As in the case $g = 2$ we can apply Lemma 2.6 to formulate Proposition 4.1 as follows. Consider matrices $U_1, \ldots, U_6 \in \mathrm{GL}(3, \mathbb{Z})[q]$ and

$$\tilde{T} = r_1 E_1[U_1] + \cdots r_5 E_5[U_5] + r_6 E_6^-[U_6].$$

Assume $\lambda(T) = \lambda(\tilde{T})$. Then $T \sim \tilde{T} \bmod \mathrm{GL}(3, \mathbb{Z})[q]$.

Proof of Proposition 4.1 in the case A1.

Without loss of generality we can assume that in the decomposition of $\tilde{T}$ we have $U_1 = E$. Then we have $\lambda(E_1 + E_2[U_2]) = 2$ since this sum is a partial sum of $\tilde{T}$. Now Lemma 2.5 shows that

$$E_2[U_2] = H = \begin{pmatrix} * & * \\ * & H_2 \end{pmatrix} \quad \text{where } H_2 \text{ is primitive.}$$

(The other case in Lemma 2.5, $H_2 = 0$, cannot arise since the first diagonal element of H_2 is odd.) We have the freedom to act on H with a matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix}$ where $V \in \mathrm{GL}(2, \mathbb{Z})[q]$ since this does not change E_1. Thanks to Lemma 2.2 we can replace H_2 by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since H has rank one, h_{11} must be zero. The semidefinitness now implies $H = E_2$. Hence we can assume now $U_1 = U_2 = E$. Now we use $\lambda(E_1 + E_2 + E_3[U_3]) = 3$. Lemma 4.4 shows $E_3[U_3]_{33} = 1$. (Zero is not possible since this element is odd.) We still can apply transformations with matrices of $\mathrm{GL}(3, \mathbb{Z})[2]$ if they fix E_1 and E_2. Hence we can multiply simultaneously the third row (column) by a multiple of q and add it to another row (column). This allows us to assume

$$E_3[U_3] = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the rank is one we get $E_3[U_3] = E_3$. Hence we can assume $U_1 = U_2 = U_3 = E$. Next we apply Corollary 4.5 to show that all diagonal elements of the matrices $E_i[U_i]$, $i > 3$, are 0 or 1. This shows that

$$E_4[U_4] = E_4[D_4], \quad E_5[U_5] = E_5[D_5], \quad E_6^-[U_6] = E_6^-[D_6]$$

where D_i are diagonal matrices in $\mathrm{GL}(3,\mathbb{Z})$. In the case $q = 4$ we are finished since then the congruence mod 4 shows $D_i = E$. So we can assume $q = 2$, The diagonal matrices fix E_1, E_2, E_3. Hence we can assume first $D_4 = E$ and then $D_5 = E$. There remain two possibilities $E_6^-[D_i] = E_6$ or $E_6^-[D_i] = E_6^-$. The second is what we want, hence it remains to discuss $E_6^-[D_i] = E_6$. In this case we claim that one of the r_4, r_5 is zero. Otherwise $E_4 + E_5 + E_6 = E_1 + E_2 + E_3 + E_7$ would be a partial sum of $\tilde{T}$ which is not possible. So assume $r_4 = 0$. Then there is a diagonal matrix D with the property $E_6[D] = E_6^-$ which does not change anything in the first five summands. This finishes the proof of A1.

Proof of Proposition 4.1 in the case A2.

The decomposition of T reads as

$$T = r_1(E_1[U_1] + E_2[U_2] + E_5[U_5] + E_6^-[U_6]).$$

As in the case A1 it is no loss of generality to assume $U_1 = U_2 = U_5 = E$. Let $H = E_6^-[U_6]$. From $\lambda(E_1 + E_2 + H) = 3$ and Corollary 4.5 follows $h_{33} = 1$ and similarly from $\lambda(E_2 + E_5 + H) = 3$ follows $h_{11} = 1$. Since $h_{11}h_{33} = h_{13}^2$ we have $h_{13} = \pm 1$. But $t_{13} \leq 0$, hence $h_{13} = -1$. The matrix H is semipositive of rank 1. Hence it is of the form

$$H = \begin{pmatrix} 1 & a & -1 \\ a & a^2 & -a \\ -1 & -a & 1 \end{pmatrix}.$$

Now we use $\lambda(E_1 + E_5 + H) = 3$. Lemma 4.4 shows $(a+1)^2 \leq 1$. Since a is even, we get $a = 0$ or $a = -2$. In the case $a = 0$ we are done. The case $a = -2$ occurs only if $q = 2$. Then we can apply the transformation

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{pmatrix} \in \mathrm{GL}(3,\mathbb{Z})[2].$$

It fixes E_1, E_2, E_5 and sends H to E_6^-. This finishes the proof of A2.

Proof of Proposition 4.1 in the case A3.

We have

$$T = r_1(E_1 + E_4 + E_5) + r_3E_3 \text{ and } \tilde{T} = r_1(E_1[U_1] + E_4[U_4] + E_5[U_5]) + r_3E_3[U_3].$$

Again we can assume $U_1 = E$. Considering the partial sum $T_1 + T_3[U_3]$ we can reduce to $U_3 = E$. Then, considering $E_1 + E_3 + E_4[U_4]$, we get $E_4[U_4]_{22} \leq 1$. It must be 1 since it is odd. Now, applying to $E_4[U_4]$ a unimodular substitution from $\mathrm{GL}(3,\mathbb{Z})[q]$ that fixes E_1 and E_3, we can get $U_4 = E$. So we can assume

$$\tilde{T} = r_1(E_1 + E_4 + E_5[U_5]) + r_3 E_3.$$

Now we apply Corollary 4.5 to

$$(E_1 + E_3) + E_5[U_5], \quad (E_1 + E_4) + E_5[U_5]$$

to obtain that $E_5[U_5]_{22} = 1$ and $E_5[U_5]_{33} = 0$. This means

$$E_5[U_5] = \begin{pmatrix} a^2 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 1 \end{pmatrix}.$$

We have $a \equiv 0 \bmod q$. We transform with the matrix from $\mathrm{GL}(2,\mathbb{Z})[q]$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-a & 0 & 1 \end{pmatrix}.$$

This transforms $\tilde{T}$ to T. This completes the proof of A3.

Proof of Proposition 4.1 in the case A4.

We have

$$T = r_2 E_2 + r_3 E_3 + r_4 E_4 + r_5 E_5 + r_4 E_6^-, \quad r_2, r_3, r_4 > 0,$$
$$\tilde{T} = r_2 E_2[U_2] + r_3 E_3[U_3] + r_4 E_4[U_4] + r_5 E_5[U_5] + r_4 E_6^-[U_6].$$

Similar to the previous cases we can assume $U_2 = U_3 = E$. Since $E_2 + E_3 + E_4[U_4]$ is optimal, we get $E_4[U_4]_{11} = 1$. We can transform $\tilde{T}$ by a matrix from $\mathrm{GL}(3,\mathbb{Z})[q]$ that fixes E_2, E_3. This means that we can multiply the first row (resp. column) of $E_4[U_4]$ by a factor which is a multiple of q and add it to the second (or third row). In this way we can get

$$E_4[U_4] = \begin{pmatrix} 1 & 1 & 0 \\ 1 & * & * \\ 0 & * & * \end{pmatrix}.$$

Since it is matrix of rank one, we then have $E_4[U_4] = E_4$. So we can assume $U_4 = E$. Now we assume $r_5 > 0$. Then we can apply Lemma 4.4 to $E_2 + E_3 + E_5[U_5]$ to obtain $E_5[U_5]_{11} = 0$ (it is even) which implies

$$E_5[U_5] = \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}.$$

The 2×2-matrix H is primitive, semidefinit of rank one and its entries are $\equiv 1 \bmod q$. Now Lemma 2.2 implies that it is of the form

$$H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} [U], \quad U \in \mathrm{GL}(2, \mathbb{Z})[q].$$

So we can assume $r_5 = 0$ or $U_5 = E$. It remains to treat $E_6^-[U_6]$. Since $E_2 + E_3 + E_6^-[U_6]$ are optimal, we get $E_6[U_6]_{11} = 1$. Since $E_2 + E_4 + E_6^-[U_6]$ is optimal, we get $E_6[U_6]_{33} = 1$. Since this matrix is symmetric and of rank 1, it is of the form

$$E_6^-[U_6] = \begin{pmatrix} 1 & a & \pm 1 \\ a & a^2 & \pm a \\ \pm 1 & \pm a & 1 \end{pmatrix}, \quad a \equiv 0 \bmod q.$$

In the case $q = 0$ the minus sign must be there. The case $r_5 = 0$ can be transformed to the case A2. (Interchange the first and the third row and column). Hence we can assume $r_5 > 0$. Then we can consider $E_4 + E_5 + E_6^-[U_6]$ which is optimal. Corollary 4.5 gives $a^2 \leq 1$ if the minus sign holds and $(a-2)^2 \leq 1$ if the plus sign holds. In the first case we get $a = 0$ which finishes the proof. So as only possibility $q = 2$ and

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

remains. One can transform this matrix by the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

to E_6^-. The other occurring matrices E_2, E_3, E_4, E_5 are fixed under this transformation. This finishes the proof in the case A4. So case A is settled.

It remains to treat the case B. This case is very similar to the case A. Hence we can keep short. Recall that the case B we consider the decomposition

$$T = r_1 E_1 + r_2 E_2 + r_3 E_3 + r_4 E_4 + r_5 E_5 + r_6 E_6 + r_7 E_7$$

where

$$r_1 = t_{11} - t_{12} - t_{13} + m, \ r_2 = t_{22} - t_{12} - t_{23} + m, \ r_3 = t_{33} - t_{13} - t_{23} + m,$$
$$r_4 = t_{12} - m, \quad r_5 = t_{23} - m, \quad r_6 = t_{13} - m, \quad r_7 = m.$$

The reduction conditions for T imply that all r_i are nonnegative. At least one of the r_4, r_5, r_6 is zero. The remaining reduction inequalities are

$$\begin{aligned}
&r_1 + r_6 \leq r_2 + r_5, \quad r_2 + r_4 \leq r_3 + r_6, \quad r_4 + r_7 \leq r_1 + r_6,\\
&r_5 + r_7 \leq r_2 + r_4, \quad r_6 + r_7 \leq r_1 + r_4,\\
&r_5 + r_6 + 4r_7 \leq r_1 + r_2.
\end{aligned}$$

Since the diagonal elements of T are positive, we also have

$$r_1 + r_4 + r_6 + r_7 > 0.$$

Again we distinguish between 4 cases where either all r_1, r_2, r_3 are positive or one of them is zero. We claim that only the following 4 cases are possible,

1) $r_1 > 0$, $r_2 > 0$, $r_3 > 0$.
2) $r_1 > 0$, $r_2 > 0$, $r_3 = 0$ and $r_4 = r_7 = 0$, $r_1 = r_2 = r_5 = r_6$.
3) $r_1 > 0$, $r_2 = 0$, $r_3 > 0$ and $r_6 = r_7 = 0$, $r_1 = r_4 = r_5$.
4) $r_1 = 0$, $r_2 > 0$, $r_3 > 0$ and $r_6 = r_4 > 0$, $r_5 = r_7 = 0$.

If m is positive, then we are in the first case. Hence we can assume for the rest that $m = 0$.

As in the case A) we list the corresponding matrices T. They have the forms

$$T = \begin{pmatrix} 2a & 0 & a \\ 0 & 2a & a \\ a & a & 2a \end{pmatrix}, \quad T = \begin{pmatrix} 2a & a & 0 \\ a & 2a & a \\ 0 & a & c \end{pmatrix}, \quad T = \begin{pmatrix} 2a & a & a \\ a & b & 0 \\ a & 0 & c \end{pmatrix}$$

Really the case $B3$) does not occur, since it contradicts $t_{13} > 0$.

Proof of Proposition 4.1 in the case B1.

As in the proof of A1 we can assume that $U_1 = U_2 = U_3 = E$ and $U_i = D_i$ is diagonal for $i > 3$ if $r_i \neq 0$ In the case $q = 4$ the congruence $E_i[D_i] \equiv E_i$ mod 4 implies $E_i[D_i] = E_i$. Hence we can assume $q = 2$. As we have shown during the proof of A1, we have $\lambda(E_4 + E_5 + E_6) > 3$. Hence one of the r_4, r_5, r_6 must be zero. The case $r_7 = 0$ is similar to the case A1 and can be omitted. Hence we can assume $r_7 > 0$. There are three possibilities for $E_7[D_i]$ which behave similarly. We restrict to treat the case

$$E_7[D_i] = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Since $\lambda(E_1 + E_5 + E_7) = \lambda(E4 + E_6^- + E_2 + E_3) > 3$ we must have $r_5 = 0$. Similarly $\lambda(E_2 + E_6 + E_7) > 3$ shows E_6. Now we can apply the diagonal matrix with entries $1, 1, -1$. It transforms $E_7[D_7]$ to E_7 and keeps the other non-zero terms fixed.

In the cases B2) and B4) the coefficient r_7 is zero. Hence we are in nearly the same situation as in the cases A2) and A4). This finishes the proof of Proposition 4.1. □

5. Localizations of rings of theta series

The algebra $R(g,q)$ of theta constants is generated by the theta constants which we introduced in the introduction

$$f_{a,q} = \sum_{n\in\mathbb{Z}^g} \exp \pi \mathrm{i} q Z[n + a/q].$$

We consider a decomposition $g = g_1 + g_2$ and

$$Z = \begin{pmatrix} \tau & z' \\ z & W \end{pmatrix}, \quad \tau \in \mathcal{H}_{g_1}, \quad W \in \mathcal{H}_{g_2}.$$

According to [Ig4], the Fourier-Jacobi expansion with respect to W as variable can be written in the form

$$f_{a,q} = \sum_T f^T_{a,q}(\tau, z) \exp \frac{\pi \mathrm{i}}{q} \operatorname{tr}(TW).$$

The coefficients $f^T_{a,q}$ can be considered as elements of $\mathcal{J}_T(\tau_0)$. They can be different from 0 only if the rank of T is ≤ 1.

5.1 Proposition. *Assume that T is an integral semipositive $g_2 \times g_2$-matrix of rank one and with coprime entries, i.e. $T = a_2 a_2'$, $a_2 \in \mathbb{Z}^{g_2}$ a primitive vector. The $\mathcal{O}_{\tau_0}$-module $\mathcal{J}_T(\tau_0)$ is generated by all*

$$f^T_{a,q}(\tau, z) = \sum_{n_1\in\mathbb{Z}^{g_1}} \exp \frac{\pi \mathrm{i}}{q} (\tau[qn_1 + a_1] + 2a_2' z(qn_1 + a_1)), \quad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

where a_1 runs through a system of representatives of $\mathbb{Z}^{g_1}/q\mathbb{Z}^{g_1}$.

Proof. The formula for the $f^T_{a,q}$ is obtained by a simple calculation. By Nakayama's Lemma, it is sufficient to show that the vector space $J_T(\tau_0)$ is generated by the $f^T_{a,q}(\tau_0, z)$. They span a space of dimension q^{g_1}. But this is also the dimension of this space (see [Ig4] for some explanations about the dimensions of the spaces $J_T(\tau_0)$. Also the proof of Lemma 1.3 can be extended to the computation of the dimensions.). □

The image of the point in the standard component $\tau_0 \in \mathcal{H}_{g_1}$, $0 \le g_1 \le g$, in $\operatorname{proj} R(g,q)$ corresponds to the homogenous maximal ideal $\mathfrak{m} \subset R(g,q)$ consisting of all elements f with the property

$$\lim_{t\to\infty} f \begin{pmatrix} \tau_0 & 0 \\ 0 & \mathrm{i}tE \end{pmatrix} = 0.$$

We consider its homogenous localization $R(g,q)_{(\mathfrak{m})}$. It consists of quotients f/g, $g \notin \mathfrak{m}$, where f, g are homogenous and of the same degree. We are interested in cases where this ring is normal.

5.2 Lemma. *The ring $R(g,q)_{(\mathfrak{m})}$ is normal if and only if it is analytically irreducible and if the ideal $\mathfrak{m}$ generates the maximal ideal of $\hat{S}(\tau_0)$.*

Proof. We recall that a local noetherian integral domain is analytically irreducible if its completion is an integral domain. By Zarisk's main theorem $R(g,q)_{(\mathfrak{m})}$ is analytically irreducible if it is normal. Hence we have to prove the converse. We denote by $\hat{R}(g,q)_{(\mathfrak{m})}$ the completion of $R(g,q)_{(\mathfrak{m})}$. The natural homomorphism

$$R(g,q)_{(\mathfrak{m})} \longrightarrow S^{\mathrm{an}}(\tau_0)$$

among the algebraic and the analytic local ring, induces a homomorphism among the completions

$$\hat{R}(g,q)_{(\mathfrak{m})} \longrightarrow \hat{S}(\tau_0).$$

It is surjective since $\mathfrak{m}$ generates the maximal ideal of $\hat{S}(\tau_0)$. Since the left hand side is an integral domain by assumption and since both sides have the same dimension, the map is also injective, hence it is an isomorphism. Hence $\hat{R}(g,q)_{(\mathfrak{m})}$ is a normal integral domain. This implies that $R(g,q)_{(\mathfrak{m})}$ is normal by faithfully flatness of the map $R(g,q)_{(\mathfrak{m})} \longrightarrow \hat{R}(g,q)_{(\mathfrak{m})}$ ([Ma] p. 156).

$\square$

Igusa proved that in the case $q=4$ that the map $\overline{\mathcal{H}_q/\Gamma_g[4,8]} \to \operatorname{proj} R(g,4)$ is bijective. Therefore the local rings of the right hand side are analytically irreducible in this case.

5.3 Proposition. *Assume that each $T \in \mathcal{T}_{g_2}$ admits a q-optimal decomposition $T = T_1 + \cdots + T_k$ such that the multiplication map*

$$J_{T_1}(\tau_0) \otimes_{\mathbb{C}} \ldots, \otimes_{\mathbb{C}} J_{T_k}(\tau_0) \longrightarrow J_T(\tau_0)$$

is surjective. Then $\hat{R}(g,q)_{(\mathfrak{m})} \longrightarrow \hat{S}(\tau_0)$ is surjective.

Proof. We have to show the following. Let P be an element of the maximal ideal of $\hat{S}(\tau_0)$. For each k there exists an element Q in the maximal ideal of $R(g,q)_{(\mathfrak{m})}$ such that $P - Q \in \hat{\mathfrak{m}}_k$. It is sufficient to show that for each $P \in \hat{\mathfrak{m}}_k$ there exists Q in the maximal ideal of $R(g,q)_{(\mathfrak{m})}$ such that $P - Q \in \hat{\mathfrak{m}}_{k+1}$. By definition of of $\hat{\mathfrak{m}}_k$ we can write P as a sum of products AB where A is in $\mathfrak{m}(\mathcal{O}_{\tau_0})^\mu$ and where the coefficients of B are zero for $\lambda(T) < \nu$ and where $\mu + \nu = k$. We can prove the statement separately for A (with μ instead of k) and for B (with ν instead of k). So it is sufficient to assume that $P = A$ or $P = B$.

Case 1. $P \in \mathfrak{m}(\mathcal{O}_{\tau_0})^\mu$. In this case we can use the result that the ring $R(g_1,q)$ gives a biholomorphic embedding of $\mathcal{H}_{g_1}/\Gamma_{g_1}[q,2q]$ into a projective space. Since the natural projection $R(g,q) \to R(g_1,q)$ is surjective, this implies that the maximal ideal of $\mathcal{O}_{\tau_0}$ can be generated by (images of) linear combinations of $f_{a,q} \in R(g,q)$ which vanish at τ_0 divided by a suitable $f_{b,q}$ that does not vanish at τ_0.

Case 2. The coefficients of P are zero for $\lambda(T) < k$. Then we choose an admissible decomposition $T = T_1 + \cdots + T_k$ and use the assumption in Proposition 5.3. This finishes the proof of this proposition. □

It remains to check whether the assumption of Proposition 5.3 is fulfilled. We restrict now to $g = 3$ and $q = 4$. Then admissible decompositions exist, cf. Proposition 4.1. We have to differ between three cases.

The case of a genus-zero boundary component. This case is trivial, since in this case the spaces J_T all are of dimension 1.

The case of a genus-two boundary component. In this case $T = m$ is an integer. The statement is that

$$J_1(\tau_0)^{\otimes m} \longrightarrow J_m(\tau_0)$$

is surjective. Since $J_1(\tau_0)$ is the space of sections of an ample line bundle of the form $\mathcal{L}^4$, the statement follows from the well-known result that

$$H^0(\mathcal{L})^{\otimes m} \longrightarrow H^0(\mathcal{L}^{\otimes m})$$

is surjective for $m \geq 3$.

The case of a genus-one boundary component. The elements of $L_T(\tau_0)$ can be identified with the sections of a line bundle on $E \times E$, where $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_0)$. In the case $T = E_1 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ the space is spanned by the 4 theta series

$$\sum_{n\in\mathbb{Z}} \exp 4\pi \mathrm{i}\{\tau(n + a_1/4)^2 + 2(n + a_1/4)z_1\}.$$

They can be considered as sections of a line bundle $\mathcal{L}$ on the first component E of $E \times E$, i.e. the line bundle on $E \times E$ is the inverse image $\mathcal{L}_1 := p^*\mathcal{L}$ with respect to the first projection. Similarly in the case $T = E_2 = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ we have to consider the line bundle $\mathcal{L}_2 := q^*\mathcal{L}$ where q is the projection on the second E. Finally in the case $T = E_3 = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ the line bundle $\mathcal{L}_3 = (p + q)^*\mathcal{L}$ has to be considered.

We have to consider optimal decompositions of 2×2-matrices. We can restrict to reduced matrices, then the optimal decompositions are of the form

$$T = aE_1 + bE_2 + cE_3.$$

What we have to show is that the multiplication map

$$H^0(\mathcal{L}_1)^{\otimes a} \otimes_{\mathbb{C}} H^0(\mathcal{L}_2)^{\otimes b} \otimes_{\mathbb{C}} H^0(\mathcal{L}_3)^{\otimes c} \longrightarrow H^0(\mathcal{L}_1^{\otimes a}\mathcal{L}_2^{\otimes b}\mathcal{L}_3^{\otimes c})$$

is surjective. This is the problem for the cartesian square of an elliptic curve. All what we must know is that $\mathcal{L}$ is a line bundle (= divisor class) on E of degree 4. But this follows from $\dim H^0(\mathcal{L}) = 4$. Any divisor of degree 4 is equivalent to a translate of $4[0]$. Since we are free to change the origin we can assume that $\mathcal{L}$ is the line bundle associated to the divisor $4[0]$. So we can reformulate the problem as follows.

5.4 Proposition. *We denote by $L(a,b,c)$ the space of all meromorphic functions on $E \times E$ which are regular or have poles of order $\le a$ on $\{0\} \times E$, of order $\le b$ on $E \times \{0\}$ and of order $\le c$ on the diagonal. The multiplication map*
$$L(4,0,0)^{\otimes a} \otimes_{\mathbb{C}} L(0,4,0)^{\otimes b} \otimes_{\mathbb{C}} L(0,0,4)^{\otimes c} \longrightarrow L(4a,4b,4c)$$
is surjective.

We shall prove this proposition in the next section. It will imply our main result.

5.5 Main-Theorem. *In the case $g = 3$ the theta functions f_a, $a \in (\mathbb{Z}/4\mathbb{Z})^6/\pm$, define a biholomorphic embedding of the Satake compactification $\overline{\mathcal{H}_3/\Gamma_3[4,8]}$ into the projective space.*

As we mentioned already one can replace the f_a by the standard 36 theta constants of first kind.

6. Cartesian square of an elliptic curve

In this section we give the proof of Proposition 5.4 (and hence of Main-Theorem 5.5). We consider the elliptic curve $E = \mathbb{C}(\mathbb{Z} + \mathbb{Z}\tau)$, $\operatorname{Im}\tau > 0$. We will construct the spaces $L(a,b,c)$ (see Proposition 5.4) by means of the Weierstrass $\wp$-function. We will use the basic fact that every elliptic function (meromorphic function on E) that is holomorphic outside the origin can be written as unique linear combination of the (higher) derivatives of $\wp$ including $\wp$ and the constant function 1. We consider the list
$$\begin{array}{llll} 1, & \wp(z), & \wp'(z), & \wp''(z), \\ 1, & \wp(w), & \wp'(w), & \wp''(w), \\ 1, & \wp(z-w), & \wp'(z-w), & \wp''(z-w). \end{array}$$
If we take from the first line a elements, from the second line b elements and from the third line c elements and multiply them, we get an element of $L(4a,4b,4c)$. We denote the subspace of $L(4a,4b,4c)$ generated by them by $M(4a,4b,4c)$. So the statement of Proposition 5.4 is $L(4a,4b,4c) = M(4a,4b,4c)$. We notice that
$$\wp^{(k)}(z) \in M(4a,0,0), \quad \text{if } k+2 \le 4a.$$

6.1 Lemma. *The function*
$$\varphi(z,w) = \frac{\wp'(z) + \wp'(w)}{\wp(z) - \wp(w)}$$
has poles of first order at the 3 special divisors ($E \times \{0\}$, $\{0\} \times E$ and diagonal) and has no other pole. Hence it is contained in $L(1,1,1)$.

Proof. This follows form the addition formula for the $\wp$-function,

$$\varphi(z,w)^2 = \wp(z-w) + \wp(z) + \wp(w). \qquad \square$$

We consider the matrix group G generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

It is isomorphic to S_3. It acts on $E \times E$ through

$$\begin{pmatrix} z \\ w \end{pmatrix} \longmapsto g \begin{pmatrix} z \\ w \end{pmatrix}$$

from the left and hence on functions in z, w from the right. It permutes the three special divisors.

6.2 Lemma. *The function $\varphi(z,w)$ has the property*

$$-\varphi(z,w) = \varphi(-z,-w).$$

Moreover, it is invariant under G up to the character $\varepsilon(g) = \det(g)$. The formula

$$\lim_{z\to 0} z\varphi(z,w) = -2$$

holds.

We want to exhibit all functions from the space $M(4,4,4)$ that have the same transformation formula as φ. The 64 generating functions all are symmetric or skew symmetric under $(z,w) \mapsto (-z,-w)$. We are only interested in the skew symmetric ones. The group G acts on them. A system of representatives is given by the functions

$$\wp'(z), \quad \wp(z)\wp'(w), \quad \wp''(z)\wp'(w), \quad \wp(z)\wp(w)\wp'(z-w),$$
$$\wp(z)\wp''(w)\wp'(z-w), \quad \wp''(z)\wp''(w)\wp'(z-w), \quad \wp'(z)\wp'(w)\wp'(z-w).$$

We symmetrize them with respect to the Character ε.

$$\begin{aligned}
f_1 =\ & \wp'(z) - \wp'(w) - \wp'(z-w), \\
f_2 =\ & \wp(z)\wp'(w) - \wp'(z)\wp(w) + \wp'(w)\wp(z-w) + \wp(w)\wp'(z-w) - \\
& \wp'(z)\wp(z-w) + \wp(z)\wp'(z-w), \\
f_3 =\ & \wp''(z)\wp'(w) - \wp'(z)\wp''(w) + \wp'(w)\wp''(z-w) + \wp''(w)\wp'(z-w) - \\
& \wp'(z)\wp''(z-w) + \wp''(z)\wp'(z-w), \\
f_4 =\ & \wp(z)\wp(w)\wp'(z-w) - \wp'(z)\wp(w)\wp(z-w) + \wp(z)\wp'(w)\wp(z-w), \\
f_5 =\ & \wp(z)\wp''(w)\wp'(z-w) + \wp''(z)\wp(w)\wp'(z-w) - \wp'(z)\wp''(w)\wp(z-w) - \\
& \wp'(z)\wp(w)\wp''(z-w) + \wp''(z)\wp'(w)\wp(z-w) + \wp(z)\wp'(w)\wp''(z-w), \\
f_6 =\ & \wp''(z)\wp''(w)\wp'(z-w) - \wp'(z)\wp''(w)\wp''(z-w) + \wp''(z)\wp'(w)\wp''(z-w), \\
f_7 =\ & \wp'(z)\wp'(w)\wp'(z-w).
\end{aligned}$$

We have to investigate the pole behavior of these functions. The only poles are along the three special divisors. The symmetry properties show that the behavior at each of the three special divisors is the same. Hence it is sufficient to concentrate on the divisor $z=0$. We compute some Laurent coefficients for fixed $w=a\neq 0$. What we need is

$$\wp(z)=\frac{1}{z^2}+O(1),\quad \wp'(z)=-\frac{2}{z^3}+O(1),\quad \wp''(z)=\frac{6}{z^4}+O(1).$$

Here $O(1)$ stands for a bounded function in a small neighborhood of the origin. We also need

$$\begin{aligned}
\wp(z-a)&=\wp(a)-\wp'(a)z+\frac{\wp''(a)}{2}z^2-\frac{\wp^{(3)}(a)}{6}z^3+\cdots,\\
\wp'(z-a)&=-\wp'(a)+\wp''(a)z-\frac{\wp^{(3)}(a)}{2}z^2+\frac{\wp^{(4)}}{6}z^3+\cdots,\\
\wp''(z-a)&=\wp''(a)-\wp^{(3)}(a)z+\frac{\wp^{(4)}}{2}z^2-\frac{\wp^{(5)}}{6}z^3+\cdots,
\end{aligned}$$

By means of these formualae we are able to compute the Laurent coefficients. The differential equation of the $\wp$-function allows the express the higher derivatives of $\wp$ explicitly in terms of $\wp$ and $\wp'$. Now a somewhat tedious but straightforward computation gives the following result.

6.3 Proposition. *The functions*

$$\begin{aligned}
F_1&=f_3-30f_4-(5/2)g_2f_1\\
F_2&=2f_5-9f_7-5g_2f_2+15f_1
\end{aligned}$$

have poles of order 1 *along the three divisors* D_i. *They are contained in* $L(1,1,1)\cap M(4,4,4)$. *We have*

$$\begin{aligned}
\lim_{z\to 0} zF_1(z,w)&=-36g_2\wp(w)-54g_3,\\
\lim_{z\to 0} zF_2(z,w)&=108g_3\wp(w)+6g_2^2.
\end{aligned}$$

The functions 1, F_1 *and* F_2 *are linearly independent. They span the space* $L(1,1,1)$.

In particular, φ must be a linear combination of F_1 and F_2. Here is it.

6.4 Proposition. *We have*

$$3(g_2^3-27g_3^2)\varphi=-g_2F_2+3g_3F_1.$$

Notice that the discriminant $g_2^3-27g_3^2$ is different from 0.

6.5 Proposition. *The spaces $L(4a, 4b, 4c)$ and $M(4a, 4b, 4c)$ agree.*

Proof. Let $f(z, w) \in L(4a, 4b, 4c)$. We assume that the order of f along one of the three components is zero. Without loss of generality we can assume that the order at the diagonal is zero. Then, for fixed $w \neq 0$ the function $z \mapsto f(z, w)$ has only a pole at $z = 0$. Hence it can be written as linear combination in the derivatives of the $\wp$-function (including the constant function),

$$f(z, w) = a_0 + \sum_{\nu \geq 0} a_\nu \wp^{(\nu)}(z).$$

The coefficients a_ν are elliptic functions in w with poles only at $w = 0$. Hence they can be expressed by derivatives of $\wp(w)$ (including the function constant 1 and $\wp(w)$).

Now f has no poles along the three special divisors, so we can assume that its order along $z = 0$ is $m > 0$. We first treat the case where $m > 1$. Again we fix $w \neq 0$. Then $f(z, w)$ has a pole of order > 1 at $z = 0$ (and may be a pole at $z = w$). We subtract from $f(z, w)$ a constant multiple of $\wp^{(m-2)}(z)$ such that the difference $f(z, w) - a\wp^{(k-2)}(z)$ has smaller pole order at $z = 0$. Again the coefficient $a = a(w)$ is an elliptic function with poles only at $w = 0$. It can be expressed by derivatives of $\wp(w)$.

In the remaining case $m = 1$ we consider for fixed w the difference

$$f(z, w) - a\varphi(z, w).$$

The pole at $z = 0$ can be cancelled. The coefficient a is an elliptic function in w with no poles outside $w = 0$. Hence it can be expressed by derivatives of $\wp(w)$. This finishes the proof of Proposition 5.4 and hence of our main result. □

We mention that in Proposition 6.5 the factor 4 is essential. For example $L(1, 1, 1)$ is not contained in $M(3, 3, 3)$.

References

[CS] Conway, J.H., Sloane, N.J.A.: *Sphere Packings, Lattices and Groups,* Grundlehren der math. Wiss. Vol. **290**, Springer Verlag New York Berlin Heidelberg London Paris Tokyo (1988)

[Fr] Freitag, E., : *Siegelsche Modulfunktionen,* Grundlehren der math. Wiss. Vol. **254**, Springer Verlag Berlin Heidelberg New York (1983)

[FK] Freitag, E., Kiehl, R.: *Algebraische Eigenschaften der lokalen Ringe in den Spitzen der Hilbertschen Modulgruppe,* Invent. Math. Vol. **24**, 121-148 (1974)

[Ig1] Igusa, J.I.: *On the Graded Ring of Theta-Constants,* Am. J. of Math. Vol. **86**, No. 1, pp. 219-246 (1964)

[Ig2] Igusa, J.I.: *On the Graded Ring of Theta-Constants II,* Am. J. of Math. Vol. **88**, No. 1, pp. 221-236 (1966)

[Ig3] Igusa, J.I.: *On the Variety Associated with the Ring of Thetanullwerte,* Am. J. of Math. Vol. **103**, No. 2, pp. 377–398, (1981)

[Ig4] Igusa, J.I.: *A desingularization problem in the theory of Siegel modular varieties,* Math. Ann. Vol. **168**, 228–260 (1967)

[Ig5] Igusa, J.I.: *On the nullwerte of Jacobians of odd theta functions,* Symposia Mathematica, Vol. XXIV (Sympos., INDAM, Rome, 1979), pp. 83-95, Academic Press, London-New York (1981)

[Kn] Knöller, F.W.: *Multiplizitäten "unendlich ferner" Spitzen,* Monatsh. Math. Vol. **88**, 7–26 (1979)

[Ma] Matsumura, H. :*Commutative algebra*, second ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co. (1980).

[Mu] Mumford, D., with Nori, M. and Norman, P.: *Tata Lectures on Theta III,* Modern Birkhäuser Classics (2010)

[Ru] Runge, B. :*On Siegel modular forms. I,* J. Reine Angew. Math. Vol. **436**, 57-85 (1993)

[SM] Salvati Manni, R. : *On the projective varieties associated with some subrings of the ring of Thetanullwerte,* Nagoya Math. J. Vol. **133**, 71-83 (1994)