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Siegel Eisenstein Series of Arbitrary Level and Theta Series

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Introduction. In this paper we consider Siegel modular forms of genus n and arbitrary level q, which do not vanish at all zero dimensional cusps. If such a form is an eigenform of some power $T(p)^m$, $m \geq 1$, of the Hecke operator $T(p)$ with respect to at least one prime $p \equiv \pm 1 \mod q$ and if the weight of f is big enough, $r > n+1$, then this form is uniquely determined by the values of f at the zero dimensional cusps. This is a generalization of an observation of Elstrodt [3], who proved this result for the full modular group.

The Siegel Eisenstein series

$$E_r(Z) = \sum_{M:\Gamma_n[q]_0\backslash\Gamma_n[q]} \det(CZ+D)^{-r}$$

is an example of such a form. More generally, the transformed $E_r|L$ for arbitrary $L \in \mathrm{Sp}(n, \mathbb{Z})$ and all linear combinations of them have this property. This means that each f can be expressed explicitly as a linear combination of those Eisenstein series. As Andrianov [1] pointed out, the Siegel mean value of the theta series of the genus of a positive definite integral quadratic form is an eigenform of many $T(p)^2$, for example if $p \equiv 1 \mod q$.

The above result allows us to write this mean value as an Eisenstein series. We recover the analytic form of Siegel's Hauptsatz.

(For the sake of simplicity we restrict to integral weights.) In the case of the full modular group and in some other special cases Andrianov obtained the same result using a similar method.

In this paper we also want to do the converse, namely to express linear combinations of Eisenstein series as linear combinations of theta series. We will see that this is not always possible. For example the above Eisenstein series of level $q = 5$ cannot be a linear combination of theta series of any type. The reason is that the values of theta series at 0-dimensional cusps satisfy certain relations. We prove that a linear combination E of Eisenstein series can be expressed as linear combination of theta series if and only if the values of E at the 0-dimensional cusps satisfy these relations.

Our method gives a bound for the level of the theta series which have to be used. In principle it is possible to work out an explicit formula for such representations. But this would demand more detailed investigations.

1 Hecke operators with respect to the principal congruence subgroup

Hecke operators with respect to the full Siegel modular group $\Gamma_n = \mathrm{Sp}(n,\mathbb{Z})$: Let

$$O_n(l) = \{M = M^{(2n)} \text{ integral}; \quad I[M] = lI\}, \quad I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

$$O_n = \bigcup_{l=1}^{\infty} O_n(l).$$

The Hecke algebra

$$\mathcal{H}_n = \mathcal{H}(\Gamma_n, O_n)$$

is commutative. It is the tensor product of the p-components

$$\mathcal{H}_n^{(p)} = \mathcal{H}\left(\Gamma_n, O_n^{(p)}\right); \quad O_n^{(p)} := \bigcup O_n(p^m).$$

For the *principal congruence subgroup of level* one has to consider similarly

$$O_n(l)[q] := \left\{ M \in O_n(l); \quad M \equiv \begin{pmatrix} E & 0 \\ 0 & lE \end{pmatrix} \pmod q \right\},$$

$$O_n[q] := \bigcup_{(l,q)=1} O_n(l)[q].$$

The application

$$\Gamma_n[q]M\Gamma_n[q] \longmapsto \Gamma_n M \Gamma_n$$

defines a homomorphism of algebras

$$\mathcal{H}(\Gamma_n[q], O_n[q]) \longrightarrow \mathcal{H}(\Gamma_n, O_n),$$

which is an isomorphism onto the part which is prime to q,

$$\mathcal{H}_n[q] := \mathcal{H}(\Gamma_n[q], O_n[q]) \xrightarrow{\sim} \bigotimes_{(p,q)=1} \mathcal{H}_n^{(p)}.$$

As a consequence this algebra is commutative too.

We also have to consider the bigger semigroup

$$\widetilde{O}_n[q] = \bigcup_{(l,q)=1} O_n(l).$$

The associated Hecke algebra

$$\widetilde{\mathcal{H}}[q] = \mathcal{H}(\Gamma_n[q], \widetilde{O}_n[q])$$

is noncommutative. It is composed by the Hecke algebra of $\Gamma_n[q]$ in Γ_n and of $\mathcal{H}(\Gamma_n[q], O_n[q])$. The Hecke algebra of a normal subgroup equals the group ring of the quotient group,

$$\mathcal{H}(\Gamma_n[q], \Gamma_n) = \mathbb{C}[\mathrm{Sp}(n, \mathbb{Z}/q\mathbb{Z})].$$

The natural homomorphism

$$\mathbb{C}[\mathrm{Sp}(n,\mathbb{Z}/q\mathbb{Z})] \otimes_{\mathbb{C}} \mathcal{H}_n[q] \longrightarrow \widetilde{\mathcal{H}}_n[q]$$

is an *isomorphism of vector spaces.* But it is not a homomorphism of rings if the left hand side is considered as tensor product of algebras. More precisely we have:

1.1 Remark. For each

$$M \in O_n(l), \quad (l,q) = 1,$$

there exists

$$L \in \Gamma_n \quad \text{with} ML \in O_n(l)[q].$$

Additional Remark. In $\widetilde{\mathscr{H}}_n[q]$ we have

$$[ML] = [M][L].$$

Here we use the notation

$$[M] = \Gamma_n[q]M\Gamma_n[q].$$

Proof. We solve the congruence $ll^* \equiv 1 \pmod q$. The matrix

$$M\begin{pmatrix} E & 0 \\ 0 & l^*E \end{pmatrix}$$

is symplectic mod q, hence the image of a matrix in Γ_n by the approximation theorem. □

This remark immediately implies the claimed isomorphism of vector spaces. For the description of the structure as a ring we consider the coset of an element $M \in O_n(l)$, $(l,q) = 1$ mod q. This is an element in $\mathrm{GL}(2n, \mathbb{Z}/q\mathbb{Z})$, more precisely a symplectic similarity. It acts on $\mathrm{Sp}(n, \mathbb{Z}/q\mathbb{Z})$ by conjugation. This means:

1.2 Remark. For each $M \in O_n(l)$ and for each $L \in \Gamma_n$ there exists $L^M \in \Gamma_n$ uniquely determined mod $\Gamma_n[q]$ by the property

$$ML^M \equiv LM \pmod q.$$

Additional Remark. In $\widetilde{\mathscr{H}}_n[q]$ we have

$$[M][L^M] = [L][M].$$

For an arbitrary congruence subgroup $\Gamma \subset \Gamma_n$ we denote by $[\Gamma, r]$ the space of all modular forms of integral weight r with respect to Γ. Their transformation law is $f(MZ) = \det(CZ + D)^r f(Z)$ for $M \in \Gamma$. The algebra $\widetilde{H}_n[q]$ acts on $[\Gamma_n[q], r]$. The operator T_M associated to a double coset is defined by means of the decomposition into right cosets,

$$\Gamma_n[q]M\Gamma_n[q] = \bigcup \Gamma_n[q]M_i,$$

by

$$f|T_M = \sum_i f|M_i,$$

where we use the Petersson notation

$$(f|M)(Z) = l^{nr-\frac{n(n+1)}{2}} \det(CZ+D)^{-r} f(MZ).$$

From 1.2 follows

$$f|L|T_M = f|T_M|L^M.$$

Combining this with the obvious rule

$$L^{MN} = \left(L^M\right)^N$$

we obtain

$$f|L|T_M^m = f|T_M^m|L^{M^m}.$$

The matrix M has finite order mod q. Hence we obtain:

1.3 Lemma. *There exists a natural number m, such that for every $M \in O_n(l)$ the commutation law*

$$f|L|T_M^m = f|T_M^m|L$$

holds. If f is an eigenform of T_M^m, all

$$f|L, \quad L \in \Gamma_n,$$

are eigenforms of T_M^m with respect to the same eigenvalue:

$$f|T_M^m = \lambda f \Longrightarrow (f|L)|T_M^m = \lambda f|L.$$

For m we can take the order of $\mathrm{GL}(2n, \mathbb{Z}/q\mathbb{Z})$.

2 Diagonalization of Hecke operators

For the sake of completeness we prove that Hecke operators can be diagonalized on the space of all modular forms (not only cusp forms) (compare [4, 7]). For this purpose one needs the commutation rule between the Hecke- and the Φ-operators.

Decomposition by characters. We consider in the modular group $\mathrm{Sp}(n, \mathbb{Z}/q\mathbb{Z})$ the abelian group of diagonal matrices

$$\begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}$$

and denote by $\Delta_n[q]$ the inverse image in Γ_n. The application

$$D \longmapsto (d_1, \dots, d_n)$$

induces an isomorphism

$$\Delta_n[q]/\Gamma_n[q] \longrightarrow (\mathbb{Z}/q\mathbb{Z})^{*n}.$$

We can decompose $[\Gamma_n[q], r]$ by characters of this group,

$$[\Gamma_n[q], r] = \sum [\Delta_n[q], r, \chi].$$

We identify the characters χ with n-tuples of Dirichlet characters $(\chi_1, \dots, \chi_n)$ via

$$\chi(D) = \chi_1(d_1) \cdots \chi_n(d_n).$$

The Hecke operators in $\mathscr{H}_n[q]$ commute with the projection operators

$$[\Gamma_n[q], r] \longrightarrow [\Delta_n[q], r, \chi].$$

This is a consequence of the fact that the elements of $\Delta_n[q]$ and $O_n[q]$ commute mod $\Gamma_n[q]$. Especially the spaces $[\Gamma_n[q], r, \chi]$ are invariant under $\mathscr{H}_n[q]$. Those eigenspaces are important because they allow a simple formulation of the commutation law between the Hecke operators and the Siegel Φ-operator

$$(f|\Phi)(Z) := \lim_{t\to\infty} f\begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix}.$$

This law is extremely simple for the operator

$$T(p) = \Gamma_n[q]\begin{pmatrix} E & 0 \\ 0 & pE \end{pmatrix}\Gamma_n[q], \quad p \text{ prime}, \quad (p,q) = 1.$$

It states for $f \in [\Gamma_n[q], r]$:

$$f\,|T(p)|\Phi = (1 + \overline{\chi_n(p)}p^{r-n})f\,|\Phi|T(p).$$

The proof of the general commutation law rests on the precise description of the p-component $\mathscr{H}_n^{(p)}$ of the Hecke algebra (s.[5], IV, §3):

One attaches to a coset

$$\Gamma_n M, \quad M = \begin{pmatrix} A & * \\ 0 & D \end{pmatrix}; \quad A = \begin{pmatrix} p^{k_1} & & * \\ & \ddots & \\ 0 & & p^{k_n} \end{pmatrix},$$

the polynomial

$$Q(\Gamma_n M) = X_0^{k_0}\prod_{\nu=1}^{n}\left(\frac{X_\nu}{p^\nu}\right)^{k_\nu}$$

and defines a similar map for double cosets by decomposing them into right cosets and summing up the Q-polynomials of the right cosets. After that we extend Q linearly to the whole Hecke algebra. We obtain an injective homomorphism of algebras

$$\mathscr{H}_n^{(p)} \hookrightarrow \mathbb{C}[X_0, X_0^{-1}, X_1, \dots, X_n].$$

For an arbitrary complex number $u \in \mathbb{C}$ one defines the homomorphism

$$\begin{aligned} \Psi(u) \;:\; \mathscr{H}_n^{(p)} &\longrightarrow \mathscr{H}_{n-1}^{(p)}, \\ X_0 &\longmapsto u^{-1}X_0, \\ X_\nu &\longmapsto X_\nu, \quad 1 \le \nu < n, \\ X_n &\longmapsto u. \end{aligned}$$

Then the commutation law between Hecke operator and Siegel Φ-operator reads:

2.1 Proposition. *Let $f \in [\Gamma_n[q], r, \chi]$. For $T \in \mathscr{H}_n^{(p)}$, $(p,q) = 1$, one has*

$$f|T|\Phi = f|\Phi|\Psi(\chi_n(p)p^{n-r})(T).$$

The commutation law for $T(p)$ mentioned above is a special case. The Q-polynomial of $T(p)$ is

$$Q(T(p)) = X_0^{-1}(E_0 + \cdots + E_n) = X_0 \prod_{\nu=1}^{n}(1 + X_\nu),$$

where E_ν denotes the ν-th elementary symmetric polynomial.

2.2 Proposition. *The vector space $[\Gamma_n[q], r]$ admits a basis of simultanous eigenforms of the Hecke algebra $\mathscr{H}_n[q]$.*

Proof. Of course it is sufficient to show that each individual Hecke operator is diagonalizable:

On attaches to each modular form $f \in [\Gamma_n[q], r]$ a finite system of modular forms of degree $(n-1)$: Firstly on applies the substitutions $L(s)$, which permute the cusps. After that one decomposes by characters χ. Finally one applies the Φ-operator.

$$\begin{aligned} \hat{\Phi} \,:\, [\Gamma_n[q], r] &\longrightarrow \prod_{s,\chi} [\Gamma_{n-1}[q], r, \chi] \\ f &\longmapsto \big(((f \,|L(s))^\chi)|\Phi\big)_{(s,\chi)}. \end{aligned}$$

The kernel of this application is the subspace of cusp forms. It is well known that on this space the Hecke operators are diagonalizable. The commutation law 2.1 between Φ-operator and Hecke operator T_M, $M \in O_n[q]$, in connection with 1.2 states, that the application $\hat{\Phi}$ is equivariant in the following sense:

$$\hat{\Phi} \circ T_M = \hat{T}_M \circ \hat{\Phi}.$$

Hereby

$$\hat{T}_M \,:\, \prod_{s,\chi} [\Gamma_{n-1}[q], r, \chi] \longrightarrow \prod_{s,\chi} [\Gamma_{n-1}[q], r, \chi]$$

denotes an operator which is the composition of a permutation of the components and certain Hecke operators of the single factors. We notice that the permutation operator and the diagonal Hecke operators commute. It is known that each permutation operator is diagonalizable. We want to argue by induction and hence can assume that the "diagonal" Hecke operators are diagonalizable. The composition of commuting diagonalizable operators is diagonalizable. Now it is easy to prove that $\hat{T}_M$ is diagonalizable. One uses a simple lemma of linear algebra:

Lemma. *Let $l : V \to W$ be a linear map between finite dimensional complex vector spaces and let $A : V \to V$ and $B : W \to W$ be linear operators which are equivariant under l. If B is diagonalizable and if A is diagonalizable on the kernel of l, then A is diagonalizable on V.*

□

3 Zero dimensional cusps

We now want to consider the values of modular forms at 0-dimensional cusp classes. By definition the set of 0-dimensional cusps is $\Gamma_n/\Gamma_{n,0}$, where $\Gamma_{n,0} := \{M \in \Gamma_n;\ C = 0\}$.

We would like to define the value of a modular form f at a cusp s represented by $L \in \Gamma_n$ by $\lim_{t\to\infty}(f|L)(t\,\mathrm{i}\,E)$. But if the weight of f is odd, this value is only well-defined by s up to a sign. Therefore it is better for our purpose to replace the set of cusps by a double covering

$$\mathfrak{M} \longleftrightarrow \Gamma_n/\Gamma_{n,0}^+,$$
$$\Gamma_{n,0}^+ := \{M \in \Gamma_{n,0};\ \det A = 1\}.$$

If $L \in \Gamma_n$ represents the element $s \in \mathfrak{M}$, we define

$$f(s) := \lim_{t\to\infty}(f|L)(t\,\mathrm{i}\,E).$$

This value is independent of the choice of L even if the weight is odd. The statements

A modular form vanishes at a certain cusp.

or

Two modular forms of the same weight agree at a certain cusp.

make sense on the set of (geometrical) cusps and not only on the double covering $\mathfrak{M}$.

Now let $\Gamma \subset \Gamma_n$ by an arbitrary congruence subgroup. The set of 0-dimensional cusp-classes of Γ is the set of double cosets $\Gamma\backslash\Gamma_n/\Gamma_{n,0}$. Because we want to include odd weights, we again prefer to consider instead of this set

$$\Gamma\backslash\mathfrak{M} = \Gamma\backslash\Gamma_n/\Gamma_{n,0}^+,$$

which is a covering of degree ≤ 2. The value $f(s)$ of a modular form $f \in [\Gamma, r]$ depends only on the class of s in this set.

3.1 Lemma. *Let*

$$f \in [\Gamma_n[q], r], \quad r > n+1,$$

be a non-vanishing eigenform of at least one Hecke operator which vanishes at all 0-dimensional cusps,

$$T(p)^m, \quad (p,q) = 1,\ m \in \mathbb{N}\ \textit{suitable}, \quad f|T(p)^m = \lambda f.$$

Then we have

$$\sqrt[m]{|\lambda|} \leq p^{\frac{r}{2}-1}(p+1)\prod_{\nu=2}^{n}(1+p^{r-\nu}).$$

Proof. In a first step one assumes that $f \in [\Gamma_n[q], r]$ is a cusp form. Here the standard Hecke method (compare [5], IV.4.8, [3]) gives the desired estimate. The general proof is given by induction on n. Let f be a non-cusp form. After suitable choice of m all conjugate forms $f|L$, $L \in \Gamma_n$, are eigenforms of $T(p)^m$ with respect to the same eigenvalue. Hence we may assume $f|\Phi \neq 0$. Now we decompose f by characters of the group $\Delta_n[q]/\Gamma_n[q]$. The forms which arise in this decomposition are eigenforms too and they vanish at the 0-dimensional cusps. At least one of them is not annihilated by Φ. Hence we may assume from advance $f \in [\Gamma_n[q], r, \chi]$, for suitable χ. Now we use the commutation law

$$f|T(p)|\Phi = (1 + \varepsilon p^{r-n}) f|\Phi|T(p), \quad |\varepsilon| = 1.$$

We may assume that the eigenvalue λ is different from 0. Then $1 + \varepsilon p^{r-n}$ must be different from 0 and $f|\Phi$ is an eigenform for $T(p)^m$ with eigenvalue

$$\lambda' = \frac{\lambda}{(1 + \varepsilon p^{r-n})^m}.$$

From the induction hypothesis we obtain

$$\sqrt[m]{|\lambda'|} \leq p^{-\frac{r}{2}}(p+1) \prod_{\nu=2}^{n-1} (1 + p^{r-\nu}).$$

Connecting this with $|1 + \varepsilon p^{r-n}| \leq 1 + p^{r-n}$, we obtain the claim. □

Let

$$T = \sum_{\nu} \Gamma_n[q] M_\nu$$

be the decomposition of a Hecke operator in $\mathscr{H}_n[q]$ into right cosets. We assume that the matrices M_ν are of the form

$$M_\nu = \begin{pmatrix} A_\nu & B_\nu \\ 0 & D_\nu \end{pmatrix}.$$

Then we have for a modular form $f \in [\Gamma_n[q], r]$

$$\lim_{t \to \infty} (f|T)(\mathrm{i}\, tE) = \sum_{\nu} C_\nu (\det D_\nu)^{-r} \lim_{t \to \infty} f(\mathrm{i}\, tE).$$

We want to apply this to $T(p)$. We know that

$$\Gamma_n \begin{pmatrix} E & 0 \\ 0 & pE \end{pmatrix} \Gamma_n = \bigcup \Gamma_n M_\nu$$

where

$$M_\nu = \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & D_\nu \end{pmatrix}, \quad \det A_\nu = p^\alpha, \ \det D_\nu = p^\delta.$$

Now we assume $p \equiv \pm 1 \pmod q$. Then we can approximate A_ν by an element in $\mathrm{GL}(n, \mathbb{Z})$. Hence we can assume $M_\nu \in O_n[q]$ and $C_\nu = 0$ and obtain

$$\lim_{t \to \infty} (f|T(p))(\mathrm{i}\, tE) = \prod_{\nu=1}^{n} (1 + p^{r-\nu}) \lim_{t \to \infty} f(\mathrm{i}\, tE).$$

3.2 Lemma. *Let $f \in [\Gamma_n[q], r]$ be an eigenform of a suitable power of a*

$$T(p), \quad p \equiv \pm 1 \pmod q,$$

which does not vanish at all 0-dimensional cusps. Then we have for suitable m

$$f|T(p)^m = \left[\prod_{\nu=1}^{n}(1+p^{r-\nu})\right]^m f.$$

Proof. We may assume that all conjugates $f|L$, $L \in \Gamma_n$, are eigenforms of $T(p)^m$. We also may assume that f doesn't vanish at the cusp "$\lim_{t\to\infty} \mathrm{i}\, tE$". Now we can apply the consideration above. □

Lemma 3.1 and 3.2 imply a characterization of eigenforms by their values in the 0-dimensional cusps, if not all of these values vanish.

3.3 Proposition. *Let*

$$f, g \in [\Gamma_n[q], r]$$

be two eigenforms of a suitable

$$T(p)^m, \quad p \equiv \pm 1 \pmod q,$$

which have the same values at all 0-dimensional cusps. We assume that not all of them vanish. Then

$$f = g.$$

Proof. By 3.2 the eigenvalues of f and g agree. Therefore $f-g$ is an eigenform with the same eigenvalue. If $f-g$ were different from 0, we could apply 3.1, which gives a contradiction. □

4 Eisenstein series

The Siegel Eisenstein series with respect to $\Gamma_n[q]$ is given by

$$E_r(Z) = \sum_{M:\Gamma_n[q]_0\backslash\Gamma_n[q]} \det(CZ+D)^{-r}.$$

Here $\Gamma_n[q]_0$ denotes the subgroup of $\Gamma_n[q]$ which is defined by the condition $C = 0$. The Eisenstein series is well-defined if

$$\det(D)^r = 1 \quad \text{for} \quad D \in \mathrm{GL}(n,\mathbb{Z}), \quad D \equiv E \pmod q.$$

This is the case if r is even or if $q > 2$. Hence we assume that r is even in the cases $q = 1$ and 2. The Eisenstein series converges for $r > n+1$ and represents a modular form,

$$E_r \in [\Gamma_n[q], r].$$

Its value at the cusp "$\lim_{t\to\infty} \mathrm{i}\, tE$" is one. To determine the values at the other cusps, one has to consider the conjugate Eisenstein series

$$(E_r|L)(Z) = \sum_{M:\Gamma_n[q]_0\backslash\Gamma_n[q]L} \det(CZ+D)^{-r} \quad (L \in \Gamma_n).$$

When L is not contained in the double coset $\Gamma_n[q]\Gamma_{n,0}$, the series $E_r|L$ vanishes at the cusp "$\lim_{t\to\infty} \mathrm{i}\, tE$".

This means that the Eisenstein series E_r vanishes at all (geometric) cusp classes with one exception. For an element $s \in \mathfrak{M}$ represented by $L \in \Gamma_n$, we define the Eisenstein series

$$E_r^s(Z) := (E_r|L^{-1})(Z).$$

This series depends only on the class of s in $\Gamma_n[q]\backslash\mathfrak{M}$.

4.1 Remark. Let $r > n+1$. In the cases $q = 1$ and 2 we assume r to be even. Let s be an element of $\Gamma_n[q]\backslash\mathfrak{M}$. We want to consider the value $E_r^s(t)$ of the Eisenstein series E_r^s at some other element $t \in \Gamma_n[q]\backslash\mathfrak{M}$. This value is 1 if $t = s$ and 0 if s and t are geometrically different, which means that they have different images in $\Gamma_n[q]\backslash\Gamma_n/\Gamma_{n,0}$.

The Eisenstein series are eigenforms of certain Hecke operators.

4.2 Lemma. *Let $p \equiv \pm 1 \pmod q$ be a prime. The Eisenstein series $E_r(Z)$ is an eigenform of $T(p)$. As a consequence all Eisenstein series $E_r^s(Z)$ and all linear combinations of them are eigenforms with respect to a suitable power of $T(p)$.*

Proof. We know that $T(p)$ admits a system of representatives M_ν of right cosets, which satisfy the condition $C_\nu = 0$. Therefore the usual proof (compare [5], IV.4.7) applies. □

Now we have proved

4.3 Proposition. *Let*

$$f \in [\Gamma_n[r], q], \quad r > n+1, \; r \;\; \textit{even if} \;\; q = 1 \;\; \textit{or} \;\; 2,$$

be an eigenform of a suitable power of a $T(p)$, $p \equiv \pm 1 \pmod q$, which does not vanish at all 0-dimensional cusps. Then f is a linear combination of Eisenstein series:

$$f(Z) = \sum_s f(s) E_r^s(Z).$$

Here s runs through a system of elements of $\mathfrak{M}$, which represents all geometrical cusp classes, i.e. the elements from $\Gamma_n[q]\backslash\Gamma_n/\Gamma_{n,0}$.

Corollary. *f is eigenform of a suitable power $T(p)^m$ for **every** prime $p \equiv \pm 1 \pmod q$.*

5 Siegel's Hauptsatz

Let $S = S^{(m)}$ be a positive definite symmetric integral matrix with even diagonal elements. We assume that the order m is even and define

$$r = m/2 \quad \text{and} \quad q = 4 \det S.$$

We denote by

$$S_1, \ldots, S_h$$

a system of representatives of the unimodular classes of the genus of S and define for arbitrary $n \in \mathbb{N}$ the genus invariant

$$F(S,Z) = F(S,Z^{(n)}) = \sum_{\nu=1}^{n} m_\nu \vartheta(S_\nu, Z).$$

Here $\vartheta(S,Z)$ denotes the theta series

$$\vartheta(S,Z) = \sum_{G=G^{(m,n)} \text{ integral}} e^{\pi i \sigma(S[G]Z)}$$

and m_ν the weight

$$m_\nu = \frac{E(S_\nu)^{-1}}{E(S_1)^{-1} + \cdots + E(S_h)^{-1}},$$

where $E(S)$ denotes the order of the unit group

$$\mathscr{E}(S) = \{U \in \mathrm{GL}(m,\mathbb{Z}); \quad U'SU = S\}.$$

By a result of Andrianov [1] $F(S,Z)$ is an eigenform of $T(p)^2$ for $p \equiv 1 \pmod q$. By Proposition 3.3 F is a linear combination of Eisenstein series in the case $r > n+1$,

$$F(S,Z) = \sum F(S,s) E_r^s(Z) \quad (r > n+1).$$

The value $F(S,s)$ can be computed without difficulty by means of the usual theta transformation formula. For the sake of completeness we give some details:

Let R be a symmetric rational $n \times n$-Matrix. We consider the symplectic matrix

$$L(R) := \begin{pmatrix} E & R \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

and define the value of a modular form $f \in [\Gamma[q], r]$ at R by

$$f(R) := \lim_{t \to \infty} (f | L(R))(\mathrm{i}\, tE).$$

This is up to a trivial constant factor the value in a certain cusp class. This depends on the fact that every such matrix R can be written in the form $R = AC^{-1}$ where A and C are the blocs in the first column of a modular matrix $L \in \Gamma_n$. A simple calculation shows

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E & R \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \begin{pmatrix} C & D \\ 0 & C'^{-1} \end{pmatrix}.$$

This implies that $f(R)$ is essentially the value of f at the element $s \in \mathfrak{M}$ represented by L^{-1},

$$f(s) = \det C^r f(R).$$

Every double coset $\Gamma_n[q] \backslash \Gamma_n / \Gamma_{n,0}$ contains representatives with $\det C \neq 0$. This means that a modular form f vanishes at all 0-dimensional cusps if and only if $f(R) = 0$ for all rational symmetric matrices.

We want to compute $f(R)$ for certain theta series. The theta series which we have in mind are of the type

$$\vartheta_\varphi(S;Z) = \sum_{G=G^{(m,n)} \text{ integral}} \varphi(G) e^{\pi i \sigma S[G] Z},$$

where $S = S^{(m)}$ is a rational symmetric positive definite $m \times m$-matrix and φ a periodic function on the set of all integral $m \times n$-matrices. "Periodic" means that there exists a natural number q with the property $\varphi(G) = \varphi(G + qX)$. We always assume that the order m of the quadratic form is even, $m = 2r$. Then $\vartheta_\varphi(S;Z)$ is a modular form (of integral weight r) with respect to a suitable level. We want to compute the value of this series at $R = 0$. Of course this value does not depend on the choice of the level.

5.1 Lemma. *The value of $\vartheta_\varphi(S;Z)$ at $R = 0$ is*

$$\vartheta_\varphi(S;0) = \det S^{-n/2}\, \mathrm{i}^{-nr}\, q^{-2rn} \sum_{X \bmod q} \varphi(X).$$

It is sufficient to prove this formula for coefficients of the type $\varphi(G) = \exp(\pi i \sigma(V'G))$. In this case the theta series is a standard theta series with characteristics and the well-known inversion formula ([5], I.0.13) can be applied.

Applying 5.1 to $\vartheta_\varphi(S;Z+R)$ instead of $\vartheta_\varphi(S;Z)$ we obtain

5.2 Lemma. *Let R be a rational symmetric $m \times m$-matrix and Q a natural number such that $\varphi(X)\exp(\pi i \sigma S[X]R)$ depends only on X* mod *Q. Then the value of $\vartheta_\varphi(S;Z)$ at R is*

$$\vartheta_\varphi(S;R) = \det S^{-n/2}\, \mathrm{i}^{-nr}\, Q^{-2rn} \sum_{X \bmod Q} \varphi(X) e^{\pi i \sigma S[X]R}.$$

This value is an invariant of the genus of S.

We apply this calculation to L^{-1} instead of L and for $\varphi \equiv 1$.

5.3 Lemma. *There exists (for given even S) a unique function*

$$H : \Gamma_n \longrightarrow \mathbb{C}$$

with the following properties:

1) *$H(M) = H(S,M)$ depends only on the double coset $\Gamma_n[q] M \Gamma_{n,0}^+$;*
2) *In the case*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n, \quad \det C \neq 0,$$

one has

$$\begin{aligned} H(S,M) &= H(S,C,D) \\ &= \det S^{-n/2}\, \mathrm{i}^{nr} \det C^{-r} \sum_{G=G^{(m,n)} \bmod C} e^{-\pi i \sigma(S[G]C^{-1}D)}. \end{aligned}$$

We obtain the analytic form of Siegel's Hauptsatz (under the restrictions mentioned):

5.4 Siegel's Hauptsatz.

$$F(S,Z) = \sum_{M:\Gamma_{n,0}\backslash\Gamma_n} H(S,C,D)\det(CZ+D)^{-r}$$

6 Representation of Eisenstein series by theta series

We want to investigate the values of theta series at 0-dimensional cusps in more detail. Two new congruence subgroups will play a role.

6.1 Definition. The group $\Gamma_{n,1}[q]$ consists of all matrices $M \in \Gamma_n$ with the property

$$C \equiv 0 \pmod q, \quad \det A \equiv 1 \pmod q.$$

The group $\Gamma_{n,\square}[q]$ is defined by

$$C \equiv 0 \pmod q, \quad \det A \text{ is a square mod } q.$$

We need two very easy statements about these groups. The proofs can be omitted.

6.2 Remark. The group $\Gamma_{n,1}[q]$ is normal in $\Gamma_{n,\square}[q]$. The quotient is isomorphic to the group of squares in $(\mathbb{Z}/q\mathbb{Z})^*$. The group $\Gamma_{n,1}[q]$ is the semidirect product of the principal congruence subgroup and the group $\Gamma_{n,0}^+$,

$$\Gamma_{n,1}[q] = \Gamma_n[q]\Gamma_{n,0}^+.$$

We recall that the values of a modular form $f \in [\Gamma_n[q], r]$ are defined by a certain function on the set $\Gamma_n[q]\backslash\Gamma_n/\Gamma_{n,0}^+$. As the group $\Gamma_n[q]$ is normal in the full modular group, we deduce from 6.2

6.3 Remark. There is a natural bijection

$$\Gamma_n[q]\backslash\Gamma_n/\Gamma_{n,0}^+ \longleftrightarrow \Gamma_n/\Gamma_{n,1}[q].$$

From this remark we obtain that considering the values at 0-dimensional cusps, we can attach to a modular form $f \in [\Gamma_n[q], r]$ a function

$$\Phi = \Phi_f : \Gamma_n/\Gamma_{n,1}[q] \longrightarrow \mathbb{C}.$$

The function Φ satisfies a further relation, namely

$$\Phi\left(M\begin{pmatrix} U' & 0 \\ 0 & U^{-1}\end{pmatrix}\right) = \det U^r \Phi(M) \quad \text{for} \quad U \in \mathrm{GL}(n,\mathbb{Z}).$$

If the weight is even, this condition can be replaced by saying that Φ is defined on $\Gamma_n/\Gamma_{n,\pm 1}[q]$, where $\Gamma_{n,\pm 1}[q]$ is defined by the conditions $\det C \equiv 0 \pmod q$ and $\det A \equiv \pm 1 \pmod q$. One has $\Gamma_{n,\pm 1}[q] = \Gamma_n[q]\Gamma_{n,0}$ and hence $\Gamma_n/\Gamma_{n,\pm 1}[q] = \Gamma_n[q]\backslash\Gamma_n/\Gamma_{n,0}$, which is the set of cusp classes and not only a covering of this set.

The theory of Eisenstein series as discussed above (4.1) shows:

6.4 Remark. Assume $r > n+1$. If

$$\Phi : \Gamma_n/\Gamma_{n,1}[q] \longrightarrow \mathbb{C}$$

is an arbitrary function with the additional property

$$\Phi\left(M\begin{pmatrix} U' & 0 \\ 0 & U^{-1}\end{pmatrix}\right) = (\det U)^r\Phi(M), \quad \text{for} \quad U \in \mathrm{GL}(n,\mathbb{Z}),$$

then there exists a modular form (actually a linear combination of Eisenstein series) $f \in [\Gamma_n[q], r]$, such that $\Phi = \Phi_f$.

The functions Φ which belong to theta series have special properties.

6.5 Proposition. *Let $f \in [\Gamma_n[q], r]$ be a modular form of level q which can be expressed as linear combination of theta series of the type*

$$\vartheta_\varphi(S;Z) = \sum_{G=G^{(m,n)}\text{ integral}} \varphi(G)e^{\pi i\sigma S[G]Z}$$

with respect to arbitrary rational positive matrices $S = S^{(2r)}$ and arbitrary periodic coefficients φ. Then the corresponding function

$$\Phi = \Phi_f : \Gamma_n/\Gamma_{n,1}[q] \longrightarrow \mathbb{C}$$

comes from a function on the quotient

$$\Gamma_n/\Gamma_{n,\square}[q].$$

Proof. In the following we consider the functions Φ as functions on Γ_n. Let Q be a natural number, which is a multiple of the given q. By the remainder theorem the natural projection $(\mathbb{Z}/Q\mathbb{Z})^* \to (\mathbb{Z}/q\mathbb{Z})^*$ is surjective. It maps the subgroup of squares onto the subgroup of squares. From this one easily deduces that the group $\Gamma_{n,\square}[q]$ is generated by the subgroups $\Gamma_{n,\square}[Q]$ and $\Gamma_{n,1}[q]$,

$$\Gamma_{n,\square}[q] = \langle \Gamma_{n,\square}[Q], \Gamma_{n,1}[q]\rangle.$$

For this reason it is sufficient to prove the invariance of Φ in 6.5 under $\Gamma_{n,\square}[Q]$ for some suitably chosen Q. This means that Φ is constant on the orbits $M\Gamma_{n,\square}[Q]$ for $M \in \Gamma_n$. The modular group Γ_n acts on the space generated by all theta series and every theta series of the considered type is contained in a finite dimensional Γ_n-invariant space. Therefore it is sufficient to prove that the function Φ is constant on $\Gamma_{n,\square}[Q]$ for some suitable Q.

Each rational symmetric matrix S can be transformed by means of an invertible rational matrix into a diagonal matrix $D = S[U]$. From this it follows easily that the vector space generated by all theta functions is generated by all theta series with respect to diagonal matrices S. Hence we may assume that S is diagonal. For any positive integer there is an obvious formula $\vartheta_\varphi(S;Z) = \vartheta_\psi(t^2S;Z)$ with some new periodic coefficient ψ. Therefore we may assume that S is an even diagonal matrix. We now choose a natural number q such that the entries of S divide q and such that Φ has period q. We choose Q such that $2q^2$ divides Q. Then $\vartheta_\varphi(S;Z)$ is a modular form on $\Gamma_n[Q]$. The function Φ is constant on $\Gamma_{n,1}[Q]$. Therefore it is sufficient to find a system of representatives of $\Gamma_{n,\square}[Q]/\Gamma_{n,1}[Q]$, such that Φ is constant on this system. We choose for each $a \in \mathbb{Z}$, $(a,Q) = 1$ a matrix

$$\begin{pmatrix} a^2 & b \\ Q & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \qquad (c = Q).$$

The set of matrices

$$\begin{pmatrix} a^2 & & & b & & \\ & 1 & & & 0 & \\ & & \ddots & & & \ddots \\ & & 1 & & & 0 \\ Q & & & d & & \\ & 0 & & & 1 & \\ & & \ddots & & & \ddots \\ & & 0 & & & 1 \end{pmatrix}$$

containes a system of representatives. We consider Φ as function of a. Since the matrices S, A, B, C, D which we consider now are diagonal matrices, the Gauss sum defining Φ splits into a product of nm sums. Up to a factor independent of a the function Φ is a product of Gauss sums of the form

$$\sum_{n \bmod Q} \varphi(n) e^{\pi i \frac{t a^2 n^2}{Q}}$$

with some function $\varphi : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$. The entry t is one of the diagonal elements of S. It divides q. We still have the possibility to enlarge Q and hence can assume $t = 1$.

Proposition 6.5 now has been reduced to the following

Claim. *Let q be natural number and $\varphi : \mathbb{Z} \to \mathbb{C}$ a function with period q. Define $Q = 2q^2$. Then the Gauss sum*

$$\sum_{n \bmod Q} \varphi(n) e^{\pi i \frac{a^2 n^2}{Q}}, \quad (a,Q) = 1$$

is independent of a.

We define $Q_1 := Q/q$, which is an even multiple of q. Therefore φ has period Q_1 and we obtain

$$\begin{aligned} \sum_{n \bmod Q} \varphi(n) e^{\pi i \frac{a^2 n^2}{Q}} &= q^{-1} \sum_{n \bmod Q} \varphi(n) \sum_{x \bmod q} e^{\pi i \frac{a^2 (n + x Q_1)^2}{Q}} \\ &= q^{-1} \sum_{n \bmod Q} \varphi(n) e^{\pi i \frac{a^2 n^2}{Q}} \sum_{x \bmod q} e^{2\pi i \frac{a^2 n x Q_1}{Q}}. \end{aligned}$$

The inner sum is q if $n \equiv 0 \pmod q$ and 0 elsewhere. Hence we obtain

$$\varphi(0) \sum_{\substack{n \bmod Q, \\ n \equiv 0 (q)}} e^{\pi i \frac{a^2 n^2}{Q}}.$$

The independence of a ($(a,Q)=1$) now is obvious because we may sum over an instead of n. □

We now want to prove a converse theorem of 6.5. For this purpose we have to construct certain special theta series.

Let m be the smallest even number greater than n,

$$m = 2\left(\left[\frac{n}{2}\right]+1\right).$$

We consider the theta series

$$f(Z) := \sum_{G\in\mathbb{Z}^{(m,n)}} e^{2\pi i\sigma(G)/q} e^{\pi i\sigma(Z[G])},$$

where $\sigma(G) = g_{11}+\cdots+g_{nn}$ denotes the trace of the rectangular matrix $G = G^{(m,n)}$. It is well known that $f(Z)$ is a modular form of weight $m/2$ on Igusa's group

$$\Gamma_n[q,2q] := \{M \in \Gamma_n[q];\ \text{ the diagonal of } AB'/q \text{ and } CD'/q \text{ is even.}\}$$

The multiplier system is trivial, if q is divisible by 4. Hence we assume in the following that $q \equiv 0 \pmod 4$:

$$f \in [\Gamma_n[q,2q], m/2] \quad (q \equiv 0 \pmod 4).$$

The corresponding cusp function Φ can be computed.

6.6 Lemma. *The cusp function $\Phi_f : \Gamma_n \to \mathbb{C}$, which corresponds to f has the properties $\Phi_f(E) \neq 0$ and*

$$\Phi_f(M) \neq 0 \Longrightarrow C \equiv 0 \pmod q.$$

Proof. For the proof it is convenient to express f by means of the classical theta nullwerte

$$\vartheta[\mathfrak{m}](Z) = \sum_{g\in\mathbb{Z}^n} \exp \pi i(Z[g+a]+2(g+a)'b), \quad \mathfrak{m} = \begin{pmatrix} a \\ rb \end{pmatrix}, \ a,b \in \mathbb{C}^n.$$

Obviously

$$f(Z) = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(Z)^{m-n} \prod_{\nu=1}^{n} \vartheta\begin{bmatrix}0\\e_\nu/q\end{bmatrix}(Z),$$

where $e_1,\ldots,e_n$ denote the standard unit vectors. The action of the modular group on the theta nullwerte is well known:

$$\vartheta[M\{\mathfrak{m}\}](MZ) = v(M,\mathfrak{m})\sqrt{\det(CZ+D)}\,\vartheta[\mathfrak{m}](Z) \quad (M \in \Gamma_n),$$

where

$$M\{\mathfrak{m}\} = (M')^{-1}\cdot\mathfrak{m} + \frac{1}{2}\begin{pmatrix}(CD')_0\\(AB')_0\end{pmatrix}.$$

As usual we denote be S_0 the column formed by the diagonal elements of a matrix S.

The limit $\lim_{t\to\infty}\theta[\mathfrak{m}](\mathrm{i}\,tE)$ is 0 if a is not integral. Making use of the fact that $\vartheta[\mathfrak{m}]$ for $\mathfrak{m}=0$ occurs as a factor of f, we see that $\Phi(M)$ vanishes if the diagonal of AC' is not even. Looking at the remaining factors we conclude that $\Phi(M)$ vanishes if Ce_ν is not divisible by q. □

We need modular forms on the principal congruence subgroup rather than on Igusa's group. Therefore we consider the symmetrization

$$F = \sum_{M \in \Gamma_n[q,2q]\backslash\Gamma_n[q]} f|M.$$

This is a modular form on $\Gamma_n[q]$ which inherits the cusp condition 6.6 of f.

6.7 Remark. The symmetrization F of f is a modular form on $\Gamma_n[q]$ of weight $\left[\frac{n}{2}\right]+1$. The associated cusp function has the properties $\Phi_F(E) \neq 0$ and

$$\Phi_F(M) \neq 0 \Longrightarrow C \equiv 0 \pmod{q}.$$

We consider now special theta series

$$g_t(Z) := \vartheta(2Z)\vartheta(2tZ), \quad t \in \mathbb{N},$$

where $\vartheta(Z) = \vartheta(Z)[0]$ denotes the theta nullwert with respect to the characteristics $\mathfrak{m} = 0$. It is well known that this is a modular form of weight 1 on $\Gamma_{n,0}[q]$, if $4t$ divides q but with respect to a multiplier system ε_t. More precisely,

$$g_t(MZ) = \varepsilon_t(M)\det(CZ+D)g_t(Z) \quad \text{for } M \in \Gamma_{n,0}[q] \ \ (4t|q).$$

The multiplier can be expressed by the generalized Jacobi symbol [2]:

$$\varepsilon_t(M) = \operatorname{sgn}(\det D)\left(\frac{-t}{|\det D|}\right).$$

The multiplier is trivial on the principal congruence subgroup,

$$g_t \in [\Gamma_n[q], 1] \quad (4t|q).$$

We denote by Φ_t the corresponding cusp function. We are interested in the values $\Phi_t(M)$ only for $M \in \Gamma_{n,0}[q]$. The above transformation formula yields

$$\Phi_t(M) = \varepsilon_t(M) \quad \text{for } \ M \in \Gamma_{n,0}[q].$$

6.8 Lemma. *Let q be a natural number which is divisible by 4. There exists a linear combination*

$$g(Z) = \sum_{4t|q} g_t(Z),$$

such that the corresponding cusp function

$$\Phi(M) := \sum_{4t|q} \Phi_t(M)$$

has the properties

1) $\Phi(E) \neq 0$;
2) *Let $M \in \Gamma_{n,0}[q]$ be an element such that $\Phi(M) \neq 0$. Then* $\det A$ *or* $-\det A$ *is a square* mod q.

Proof. The values $\Phi(M)$ are equal to 1 on $\Gamma_{n,\square}[q]$. They are characters on the group $\Gamma_{n,0}[q]/\Gamma_{n,\square}[q]$. The application $M \mapsto \det D$ defines an isomorphism of this group onto

$$\mathcal{G} := (\mathbb{Z}/q\mathbb{Z})^* / (\mathbb{Z}/q\mathbb{Z})^{\square},$$

where $(\mathbb{Z}/q\mathbb{Z})^{\square}$ denotes the subgroup of all squares in $(\mathbb{Z}/q\mathbb{Z})^*$. The order $\#\mathcal{G}$ of $\mathcal{G}$ is $4 \cdot 2^u$, where u is the number of odd prime divisors of q.

We are interested in characters ε on $\mathcal{G}$ with the additional property $\varepsilon(-1) = -1$. Their number is

$$\frac{1}{2}\#\mathcal{G} = 2^a, \quad \text{where } a \text{ is the number of } \textit{all} \text{ prime divisors of } q.$$

The characters

$$\varepsilon_t(d) := \operatorname{sgn}(d)\left(\frac{-t}{d}\right)$$

have this property. They are pairwise distinct if t runs through all square free divisors of q. Hence they exhaust all characters ε with $\varepsilon(-1) = -1$. It is well known that an arbitrary function $\alpha : \mathcal{G} \to \mathbb{C}$ with this property can be written as a linear combination of characters with this property, hence as a linear combination of the ε_t. We obtain Lemma 6.8, if we apply this to the function

$$\alpha(d) = \begin{cases} 1 & \text{if } d \text{ is a square,} \\ -1 & \text{if } -d \text{ is a square,} \\ 0 & \text{elsewhere.} \end{cases}$$

□

Now we take the product of the functions F from 6.7 and g from 6.8 and obtain a modular form

$$G := Fg \in \left[\Gamma[q], \left[\frac{n}{2}\right] + 2\right].$$

The cusp function Φ_G does not vanish identically but is supported on the group generated by $\Gamma_{n,\square}[q]$ and by all $M \in \Gamma_n$ with $B = C = 0$. This is an extension of index 2 of $\Gamma_{n,\square}[q]$. We multiply G by an even power of $\vartheta(Z)$, and obtain a modular form $H = G\vartheta^k$ with the same cusp behaviour, where the weight r can be an arbitrary natural number which is greater or equal $\left[\frac{n}{2}\right] + 2$. We now consider the space generated by all $H|M$, $M \in \Gamma_n$. It is obvious that any function

$$\Phi : \Gamma_n/\Gamma_{n,\square}[q] \longrightarrow \mathbb{C}$$

with the additional property

$$\Phi\left(M\begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix}\right) = (\det U)^r \Phi(M), \quad \text{for} \quad U \in \mathrm{GL}(n, \mathbb{Z})$$

is the cusp function of a function in this space.

We still need more. We want to have an *eigenform* with respect to a power of some $T(p)$. Therefore we choose some prime $p \equiv 1 \pmod q$. In Section 1 we proved for a suitable natural number m the commutation rule

$$h|T(p)^m|L = h|L|T(p)^m, \quad L \in \Gamma_n.$$

This implies that the cusp function of $h|T(p)^m$ is a multiple of that of h. The space generated by $h|T(p)^{mk}$, $k \geq 0$, contains an $T(p)^m$-eigenform with the same cusp function as h.

6.9 Theorem. *Let n, $q \equiv 0 \pmod 4$ and $r > n+1$ be natural numbers. Assume that $E(Z)$ is a linear combination of Eisenstein series of level q whose corresponding cusp function $\Phi(M)$, $M \in \Gamma_n$ is right invariant under $\Gamma_{n,\square}$. Then E can be expressed as linear combination of theta series.*

The theta series which have to be used are contained in the closure of $Fg\vartheta^k$ (s. 6.7, 6.8) under the action of the full modular group and of some power of some $T(p)$, $p \equiv 1 \pmod q$.

ANDRIANOV's explicit formula for the action of $T(p)$ on theta series [1] and the standard theta transformation formula show that all the theta series which have to be used in 6.9 are contained in the span of all

$$\vartheta_\varphi(S;Z) = \sum_{G=G^{(m,n)} \text{ integral}} \varphi(G) e^{\pi i \sigma S[G]Z},$$

where φ has period $2q$ and S and $2qS^{-1}$ have to be integral.

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