A remark on a theorem of Runge

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1 Introduction

In this note we present a few calculations concerning the Siegel modular variety of genus three and level $(2, 4)$. We obtain a new proof of the result of Runge about the structure of the ring of modular forms. In very complicated papers Tsuymine found generators for the ring of Siegel modular forms of genus 3 following a method proposed by Igusa. In a remarkable paper Runge found a different much easier approach. He determined the structure of the ring of modular forms even for the subgroup of level $(2, 4)$ in the sense of Igusa. There are 8 theta constants

$$\theta_a(Z) := \sum_{g\in\mathbf{Z}^3} exp(2\pi iZ[g + a/2]),$$

where a runs through the 8 elements of $(\mathbf{Z}/2\mathbf{Z})^3$. These theta constants define a map

$$Th : \Gamma_3(2, 4)\backslash\mathbf{S}_3 \longrightarrow \mathbf{P}^7(\mathbf{C}).$$

The image is a hypersurface defined by a classical theta relation of degree 16. By a variant of the so-called fundamental lemma of Igusa this map is birational. Runge proved that this hypersurface is normal and deduced immediately from this result that the ring of modular form is generated by those 8 thetas and that the theta relation is the defining relation of this ring. Using a method different from that used by Runge we prove directly that the hypersurface defined by the theta relation is normal. As we will see below it is sufficient to prove that the codimension of the singular locus is greater or equal 2. It is sufficient to find a prime p such that this is true after reduction mod p. The computer algebra program Macaulay delivers the tools to prove this by a computer calculation.

2 The hypersurface X

2.1 Equation

As in [2] we encode the 8 thetas and hence our coordinates in $\mathbf{P}^7$ by triples (a_1, a_2, a_3), where the components are 0 or 1. We write the coordinates $(X_0 : \ldots : X_7)$ on $\mathbf{P}^7$ in the following way. We identify the 8 triples (a_1, a_2, a_3) with the numbers $\{0, \ldots, 7\}$by the correspondence given in the table (this is the standard binary notation of the $j \in \{0, \ldots, 7\}$)

$0 \leftrightarrow (0,0,0)$	$1 \leftrightarrow (0,0,1)$	$2 \leftrightarrow (0,1,0)$	$3 \leftrightarrow (0,1,1)$
$4 \leftrightarrow (1,0,0)$	$5 \leftrightarrow (1,0,1)$	$6 \leftrightarrow (1,1,0)$	$7 \leftrightarrow (1,1,1)$

Then notations such as $X_{\sigma+\xi}$ ($\sigma,\ \xi \in \mathbf{F}_2^3$) make sense. For $x = (\xi, \xi') \in \mathbf{F}_2^3 \times \mathbf{F}_2^3$ there is a quadric Q_x in the X_σ defined by

$$Q_{(\xi,\xi')} = \sum_\sigma < \sigma, \xi' > X_\sigma X_{\sigma+\xi}, \tag{1}$$

where $< \sigma, \xi > := \exp(\pi i^t \sigma \xi)$ which is $+1$ (resp. -1) when ${}^t\sigma\xi = 0$ (resp. 1). This quadric vanishes identically for $\xi \cdot \xi' = 1$ (the odd thetas), and there are 36 non-vanishing Q_x (even thetas). We consider the following products

$$\begin{aligned}
r_1 &= Q_{\left[\begin{smallmatrix} 0&0&0\\0&0&0 \end{smallmatrix}\right]} Q_{\left[\begin{smallmatrix} 0&0&0\\1&0&0 \end{smallmatrix}\right]} Q_{\left[\begin{smallmatrix} 0&0&0\\0&1&0 \end{smallmatrix}\right]} Q_{\left[\begin{smallmatrix} 0&0&0\\1&1&0 \end{smallmatrix}\right]} \\
r_2 &= Q_{\left[\begin{smallmatrix} 0&0&1\\0&0&0 \end{smallmatrix}\right]} Q_{\left[\begin{smallmatrix} 0&0&1\\1&0&0 \end{smallmatrix}\right]} Q_{\left[\begin{smallmatrix} 0&0&1\\0&1&0 \end{smallmatrix}\right]} Q_{\left[\begin{smallmatrix} 0&0&1\\1&1&0 \end{smallmatrix}\right]} \\
r_3 &= Q_{\left[\begin{smallmatrix} 0&0&0\\0&0&1 \end{smallmatrix}\right]} Q_{\left[\begin{smallmatrix} 0&0&0\\1&0&1 \end{smallmatrix}\right]} Q_{\left[\begin{smallmatrix} 0&0&0\\0&1&1 \end{smallmatrix}\right]} Q_{\left[\begin{smallmatrix} 0&0&0\\1&1&1 \end{smallmatrix}\right]}
\end{aligned}$$

The hypersurface X is defined by

$$X = \{r_1^2 + r_2^2 + r_3^2 - 2(r_1 r_2 + r_1 r_3 + r_2 r_3) = 0\}. \tag{2}$$

This is a degree 16 hypersurface in $\mathbf{P}^7$. This hypersurface is the image of $\mathcal{A}_3(2,4)$ under the map Th. More details can be found for example in *loc. cit.*.

2.2 Automorphism group

The Heisenberg group in the present case is a group of order 2^8, which is given in [3] the following projective representation. The group is

$$G = \{(t, x) | t \in \mathbf{C}, t^4 = 1,\ x = (\xi, \xi') \in \mathbf{F}_2^3 \times \mathbf{F}_2^3\},$$

with identity element $(1, 0)$ and mulitiplication $(t, (\xi, \xi')) \cdot (s, (\eta, \eta')) = (ts(-1)^{\xi\eta'}, (\xi + \eta, \xi' + \eta'))$. Letting V denote the vector space spanned by the X_σ above (so $\mathbf{P}(V) = \mathbf{P}^7$ into which Th maps), the Heisenberg representation U of G (*loc. cit.*, 4.2.2), is determined by its action on the X_σ, which is given by

$$U((t, (\xi, \xi'))X_\sigma = t(-1)^{(\sigma+\xi)\xi'} X_{\sigma+\xi}.$$

Generators are given by taking $(\xi, \xi') = (\xi, 0)$ and $(0, \xi')$, respectively; the former elements give pure permutations (for $t = 1$), while the latter are pure sign-changes. The subgroup $A(G) \subset G$ is defined as the subgroup consisting of elements acting trivially on the center of G. By *loc. cit.* 4.2.3 there is an extension

$$1 \longrightarrow \mathbf{F}_2^6 \longrightarrow A(G) \longrightarrow Sp(6, \mathbf{F}_2) \longrightarrow 1, \tag{3}$$

and a projective representation

$$\tilde{T} : A(G) \longrightarrow PGL(V),$$

which induces an action of $A(G)$ on $\mathbf{P}^7$. The elements of $PGL(V)$ corresponding to $\mathbf{F}_2^6$ are given by: for $x \in \mathbf{F}_2^6 - \{0\}$, choose $(t_x, x) \in G$ of order 2, and set

$$U_x := U(t_x, x).$$

Theorem 4.2.6 of *loc. cit.* describes elements *not* in the kernel of $A(G) \longrightarrow Sp(6, \mathbf{F}_2)$: for $x \in \mathbf{F}_2^6 - \{0\}$, set

$$\tilde{T}_x := U_x + iI.$$

This element is a "square root" of U_x in the sense that $\tilde{T}_x^2 = Int_x$. It is easy now to identify the sequence (3) with the sequence

$$1 \longrightarrow H_{2,4} \longrightarrow \Gamma_{2,4} \longrightarrow Sp(6, \mathbf{F}_2) \longrightarrow 1,$$

where $H_{2,4} = \Gamma(2)/\Gamma(2,4)$, $\Gamma_{2,4} = \Gamma/\Gamma(2,4)$, thus describing the automorphism group of X.

2.3 Subloci

A glance at the equation (2) shows that the ideal which is generated by any three of the quadrics, one of each in the definition of the r_i, is completely contained in X. A computer calculation shows that this ideal defines a locus which is of dimension four and degree eight. If we adjoin to this ideal one further of the quadrics (1), then either the dimension drops by one, or the dimension remains the same and the degree drops to four. In this case the ideal is now the ideal defining a Segre embedded copy of $\mathbf{P}^1 \times \mathbf{P}^3$. The original ideal was then the ideal of the union of two such varieties. This fits in with the modular interpretation of X by the general fact that the Siegel modular variety of level $(2,4)$ always contains a Segre embedded image of Siegel varieties of level $(2,4)$ of *lower* dimension (see [4], II, §6); in the case at hand, the varieties of degrees one and two are $\mathbf{P}^1$ and $\mathbf{P}^3$, respectively. It is known that these images define the *reducibility locus*, i.e., the set of moduli points corresponding to abelian varieties which are reducible (cf. *.loc. cit.* I, p. 81), where the locus of irreducible varieties is denoted $\mathcal{A}^\circ$. As an example, we consider the ideal generated by the Q_x, where $x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Then it is easily verified that this ideal also contains the two quadrics Q_x for $x = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. The ideal generated by the first three of these four quadrics is the ideal of the union of two such Segre embedded products, as just mentioned. The other copy of Segre embedded product has the ideal generated by the quadrics for $x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ in addition to the first three mentioned above.

In addition to these loci, there are 126 special $\mathbf{P}^3$'s, which are eigenspaces of the Heisenberg group. These turn out to be contained in the Segre-embedded products just discussed, in the following way. There are, on each copy of $\mathbf{P}^1$, six cusps (recall that $\Gamma_1(2,4) \cong \Gamma_1(4)$), and for each such the image of $\{cusp\} \times \mathbf{P}^3$ is one of these linear $\mathbf{P}^3$'s. These loci correspond to the real "cusps" of the Siegel modular variety of degree three, i.e., they are the (compactifications of) boundary components of maximal dimension.

3 Determination of the singular locus

3.1 Method

We will compute the singular locus of X with the computer algebra system Macaulay, and in this section we provide the mathematical basis for its use, stating that for the information we desire (dimension and degree of the singular locus) one can apply characteristic p results to a certain extent.

Lemma 3.1 *Let $f \in \mathbf{Z}[x_1, \ldots, x_n]$ be a homogenous polynomial with coprime coefficients, and let $X \subset \mathbf{P_Z}^{n-1}$ be the projective hypersurface defined by f, i.e. $X = \mathrm{Proj}(\mathbf{Z}[x_1, \ldots, x_n]/(f))$. For an algebraically closed field F let $X_F = X \otimes F$ denote the fibre over F, and let $\Sigma_F \subset X_F$ be the singular locus. Then for any algebraically closed field L of characteristic 0 we have*

$$dim \Sigma_L \leq dim \Sigma_F.$$

Proof: There exists a closed subscheme $\Sigma \subset X$, whose fibers over an algebraically closed field F coincide with the singular locus Σ_F. Let Σ_i be an irreducible component of Σ which meets X_L in a non-empty set. The subscheme Σ_i is flat over $\mathbf{Z}$. The inequality

$$dim(\Sigma_i)_L \leq dim(\Sigma_i)_F$$

is then a consequence of the semicontinuity of fiber dimension. This proves the lemma. □

It follows from this result that a computation of the dimension (and degree) of the singular locus in finite characteristic gives an *upper bound* for the (actual) result in characteristic 0.

3.2 Description

The computer algebra program Macaulay computes a Gröbner basis of a homogenous ideal $\mathcal{I}$. It is possible from the properties of the Gröbner basis to compute the Hilbert function of the ideal, and in particular the dimension and degree can be calculated in this manner. This computation, which was carried out in the Macaulay default characteristic 31991, together with the Lemma above, yield

Proposition 3.2 *The singular locus of X has dimension at most four and degree at most 1344. In particular, X is normal.*

In fact, the singular locus has *exactly* dimension four: the Segre embedded products described in section 1.3 are of dimension four, and are contained in that singular locus (see below).

3.3 New proof of Runge's theorem

Theorem 3.3 ([4], 2.8) *Let $\oplus_k[\Gamma_3(2,4), k]$ denote the ring of modular forms for $\Gamma_3(2,4)$. Then*

$$\oplus_k[\Gamma_3(2,4), k] = \mathbf{C}[X_\sigma X_\tau].$$

Proof: As we have mentioned, it is sufficient to show that X is normal. But a hypersurface is normal if and only if the dimemsion of the singular locus has codimension at least two. □

Remark: Runge's original proof is based on Sasaki's result [5], which implies that X is smooth *outside* the reducible locus, which as we have seen consists of components of dimension 4. We will show below that in fact each such component is simply contained (i.e., with multiplicity one) in that singular locus.

4 Other calculations

4.1 Cusp forms of weight 6

By the fact that X is a projective *hypersurface*, it follows that the sections of the canonical bundle are given by the set of octics with certain vanishing properties. These sections correspond to cusp forms of weight 6. The condition on the octics is that they vanish at the 126 $\mathbf{P}^3$'s described in section 1.3, i.e., that these octics are in the ideal of the 126 $\mathbf{P}^3$'s. To check whether there are any such octics, we calculated the ideal of the 126 $\mathbf{P}^3$'s up to degree 9. While there are elements of degree 9, there are no octics. This gives

Proposition 4.1 *There are no cusp forms of weight* ≤ 6 *for* $\Gamma_3(2,4)$.

Remark: R. Salvati-Manni has proved that X is *not* rational. For example a modification of the proof of III.5.29 in [1] shows that X is of general type. Salvati-Manni also constructed a holomorphic differential form of degree three.

4.2 Multiplicity of the singular locus

First we checked, using the Macaulay *reduce* command, whether the Segre embedded products are contained in the singular locus; they are (note that it suffices to check this for *one* such, since the automorphism group acts transitively on these components). However, there is a priori the possibility that these components may have higher multiplicity in the singular locus. To exclude this possibility, we proceeded as follows. It turned out not to be feasible to use the Macaulay *quotient* command to divide out a component of the singular locus. Instead, we considered the following sublocus. Let H_σ denote the hyperplane $\{X_\sigma = 0\}$; we checked that the singular locus of $H_\sigma \cap X$ still has degree 1344, but now only dimension 3. Then we intersected with another H_τ such that the dimension was still 3 but the degree dropped to 72. It is easy to see that this is the union of Segre embedded $\mathbf{P}^1 \times \mathbf{P}^2$'s (setting a coordinate in $\mathbf{P}^3$ to zero), each of which has degree 3. Here we were able to use the *quotient* command; a given component was contained *simply* in $H_\sigma \cap H_\tau \cap \Sigma$. The same then clearly holds for the components of Σ. Since the total degree is 1344 and each component has degree 4, it follows from this:

Proposition 4.2 *The singular locus consists of 336 Segre embedded copies of* $\mathbf{P}^1 \times \mathbf{P}^3$*'s of degree four (and possibly components of lower dimension).*

If we use Sasaki's result, then there are no lower-dimensional components.

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