Dimension formulae for vector valued automorphic forms

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Introduction

In his paper [Bo1], at the end of Sect. 4, Borcherds mentions without proof a beautiful formula for dimensions of vector valued modular forms of weight ≥ 2 with respect to the full modular group (here reproduced as Theorem 6.1).

Skoruppa informed me that he derived already 1985 in his Ph.D. thesis [Sk] these dimension formulas for all weights by means of the Shimura trace formula.

More general results, including arbitrary Fuchsian groups, can be found in the paper [Bo2] of Borcherds, Sect. 7. Most of them have been proved by the Selberg trace formula, see [Iv] and also [Fi]. The Selberg trace formula in its standard form causes the restriction that the weight is > 2. Borcherds mentions that "with a bit more care this also works for weight 2". As we mentioned, this bit more care was taken already 1985 in the thesis of Skoruppa.

The purpose of this paper is to produce the dimension formula in all weights, for general Fuchsian groups and for arbitrary rational weights.

We consider arbitrary discrete subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{R})/\pm$ with finite volume of the fundamental domain. For them we consider vector valued modular forms of a rational weight r. They have the transformation property

$$f(\gamma\tau) = \gamma'(\tau)^{-r/2}\varrho(\gamma)f(\tau),$$

where $\varrho(\gamma)$ is matrix valued. We assume that all $\varrho(\gamma)$ are of finite order and that the can be diagonalized simultaneously for all γ in a subgroup of finite index Γ_0. We make some further weak assumption for ϱ (see Assumption 3.3) which is fulfilled at least for arithmetic groups. These assumptions are not really necessary. In principle, one could consider arbitrary real weights r and we could take arbitrary unitary multiplier systems in the sense of [Fi]. But we found that these restrictions are convenient and they cover all cases which occur usually in the theory of modular forms.

Since we allow arbitrary rational weights, there is an ambiguity in the definition of $\gamma'(\tau)^{-r/2}$. One standard way to overcome this, is, to use covering groups of $\mathrm{SL}(2,\mathbb{R})$. Instead of this we found it convenient to use the old-fashioned method of multiplier systems, here in a matrix-valued sense. They could be also called "projective representations". In Sect. 6 we reformulate the results for representations of the two-fold metaplectic covering of $\mathrm{SL}(2,\mathbb{Z})$ and reproduce Borcherds' formula in [Bo1].

Our proof rests on the Riemann–Roch formula for vector bundles (and not on the Selberg trace formula). The idea is to use a sufficiently small normal subgroup Γ_0 for which the vector bundle splits into a sum of line bundles. This gives a reduction to the well-known case of scalar valued modular forms.

1. Riemann–Roch for vector bundles

Riemann surfaces are always assumed to be connected. We need the notion of the degree of a coherent sheaf. It can be characterized as follows:

1.1 Theorem. *Let X be a compact Riemann surface. There exists a unique function which associates to an arbitrary coherent sheaf $\mathcal{M}$ on X a non-negative integer* $\deg \mathcal{M}$ *such that the following properties are satisfied:*

1) $\deg \mathcal{M}$ *depends only on the isomorphy class of $\mathcal{M}$.*
2) *For a skyscraper sheaf $\mathcal{W}$*

$$\deg(\mathcal{W}) = \sum_{a\in X} \dim \mathcal{W}_a.$$

3) *If D is a divisor and $\mathcal{O}_D$ the associated line-bundle then*

$$\deg(\mathcal{O}_D) = \deg D.$$

4) *For a short exact sequence of coherent sheaves*

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

one has

$$\deg(\mathcal{M}_2) = \deg(\mathcal{M}_1) + \deg(\mathcal{M}_3).$$

5) *If : $X \to Y$ is non-constant holomorphic map between compact Riemann surfaces, then*

$$\deg(f^*\mathcal{M}) = \deg(X/Y)\deg(\mathcal{M}).$$

Here $\deg(X/Y)$ *denotes the covering degree of $X \to Y$.*

We also need the rank of a coherent sheaf. If $\mathcal{M}$ is a coherent sheaf on a compact Riemann surface then there exists a finite set such the restriction of $\mathcal{M}$ to $X - S$ is a vector bundle. By definition, $\mathrm{Rank}(\mathcal{M})$ is the rank of this vector bundle.

1.2 Theorem (Riemann-Roch Theorem). *Let $\mathcal{M}$ be a coherent sheaf on a compact Riemann surface X. Then*

$$\dim H^0(X,\mathcal{M}) - \dim H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\Omega)) = \deg(\mathcal{M}) + \mathrm{Rank}(\mathcal{M})(1-g).$$

Here Ω denotes the canonical sheaf (sheaf of holomorphic differentials).

We mention that this theorem can be reduced to the classical one for divisors. First one *defines* the degree by means of the formula

$$\chi(\mathcal{M}) = \deg(\mathcal{M}) + \mathrm{Rank}(\mathcal{M})(1-g)$$

where

$$\chi(\mathcal{M}) = \dim H^0(X,\mathcal{M}) - \dim H^1(X,\mathcal{M})$$

is the Euler–Poincarè-characteristic of $\mathcal{M}$. The conditions 1) and 2) then are trivial, 3) is the classical Riemann–Roch and 4) is true since $\chi(\mathcal{M})$ and $\mathrm{Rank}(\mathcal{M})$ are additive in this sense. 5) also can easily be reduced to divisors. What remains to be done is the construction of the duality pairing

$$H^1(X,\mathcal{M}) \times H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\Omega)) \longrightarrow \mathcal{O}_X$$

and the proof that it is non-degenerate.

2. Vector valued modular forms

Let $\alpha : D \to D'$ be a biholomorphic mapping between two domains in the complex plane. Once for ever we choose a holomorphic logarithm $\log \alpha'(z)$ and define then

$$j_r(\alpha, z) := \alpha'(z)^{-r/2} := e^{-r\log\alpha'(z)/2}$$

for an arbitrary rational number r. (This definition is possible for arbitrary real and even more complex r. For sake of simplicity we restrict here to automorphic forms of rational weight.)

There holds a kind of chain rule for two biholomorphic mappings $\alpha : D \to D'$, $\beta : D' \to D''$:

$$j_r(\beta\alpha, z) = w_r(\alpha,\beta)^{-1} j_r(\beta, \alpha z) j_r(\alpha z).$$

Here $w_r(\alpha,\beta)$ is a certain root of unity. For even r it is one.

Let f be a function on D'. We define the function $f|\alpha$ on D as

$$(f|\alpha)(z) = (f\underset{r}{|}\alpha)(z) = f(\alpha(z)) j_r(\alpha, z).$$

Then the chain rule reads as

$$f|(\beta\alpha) = w_r(\alpha, \beta)(f|\beta)|\alpha.$$

2.1 Definition. *Let $D \subset \mathbb{C}$ be a domain and Γ a group of biholomorphic transformations of D. By a (vector valued) multiplier system of weight $r \in \mathbb{Q}$ with values in a finite dimensional complex vector space V we understand a map*

$$\varrho : \Gamma \longrightarrow \mathrm{GL}(V)$$

with the following properties:

1) $\varrho(\gamma_1\gamma_2) = w_r(\gamma_1, \gamma_2)\varrho(\gamma_1)\varrho(\gamma_2)$.
2) *The matrices $\varrho(\gamma)$ are of finite order.*
3) *There exists a subgroup $\Gamma_0 \subset \Gamma$ of finite index such that $\varrho(\gamma)$ can be simultaneously diagonalized for $\gamma \in \Gamma_0$,*

Property 1) means that

$$J(\gamma, z) = j_r(\gamma, z)\varrho(\gamma)$$

is a (vector valued) factor of automorphy, i.e.

$$J(\beta\alpha, \tau) = J(\beta, \alpha\tau)J(\alpha, \tau).$$

So it makes sense to consider functions $f : D \to V$ with the transformation property

$$f(\gamma z) = J(\gamma, z)f(z).$$

2.2 Lemma. *Let $\alpha : D \to \tilde{D}$ be a biholomorphic map of domains and Γ a group of biholomorphic transformations of D. Then $\tilde{\Gamma} = \alpha\Gamma\alpha^{-1}$ is a group of biholomorphic transformations of $\tilde{D}$. Let ϱ be a multiplier system of weight r for (D, Γ) then*

$$\tilde{\varrho}(\gamma) = \varrho(\alpha^{-1}\gamma\alpha)w_r(\alpha^{-1}, \gamma)w_r(\alpha^{-1}\gamma\alpha, \alpha^{-1})$$

is a multiplier system for $(\tilde{D}, \tilde{\Gamma})$ with corresponding automorphy factor

$$\tilde{J}(\gamma, w) = J(\alpha^{-1}\gamma\alpha, \alpha^{-1}w) = \tilde{\varrho}(\gamma)j_r(\gamma, w)^{-1}.$$

Let $f : D \to V$ be a function with the property $f(\gamma z) = J(\gamma, z)f(z)$ for $\gamma \in \Gamma$ then the transformed function $\tilde{f} = f|\alpha^{-1}$ has the transformation property $\tilde{f}(\gamma w) = \tilde{J}(\gamma, w)f(w)$ for $\gamma \in \tilde{\Gamma}$.

From now on Γ denotes a group of biholomorphic transformations of the upper half plane $\mathbb{H}$. We assume that Γ acts properly discontinuously. We denote by $S \subset \mathbb{R} \cup \{\infty\}$ the set of cusps and by $\mathbb{H}^* = \mathbb{H} \cup S$ the extended upper half plane. The quotient

$$X = X_\Gamma := \mathbb{H}^*/\Gamma$$

carries a structure as Riemann surface. We assume that this surface is compact.

In the following we fix a group Γ, a rational number r and a multiplier system $\varrho : \Gamma \to \mathrm{GL}(V)$ of weight r for Γ. Let n be the dimension of V. We want to define for each point $a \in \mathbb{H}^*$ an unordered n-tuple of numbers

$$x_1, \dots, x_n, \quad 0 \le x_i < 1 \qquad (n = \dim V).$$

We will call them the characteristic numbers of a (with respect to Γ, ϱ, r).

We start with the case where $a \in \mathbb{H}$ is an inner point. We transform it to the origin of the unit disk $\mathbb{E}$ by means of the transformation

$$\alpha(\tau) = \frac{\tau - a}{\tau - \bar{a}}.$$

We consider the conjugate group $\tilde{\Gamma} = \alpha\Gamma\alpha^{-1}$. We also consider the conjugate multiplier system $\tilde{\varrho}$ and corresponding automorphy factor $\tilde{J}$ in the sense of Lemma 2.2.

2.3 Remark and Definition. *For a point $a \in \mathbb{H}$ we consider the conjugate group*

$$\tilde{\Gamma} = \alpha\Gamma\alpha^{-1}, \quad \alpha(\tau) = \frac{\tau - a}{\tau - \bar{a}},$$

and the transformed automorphy factor $\tilde{J}(\gamma, w)$. The stabilizer of the origin in $\tilde{\Gamma}$ is generated by the transformation $r_e(w) = e^{2\pi \mathrm{i}/e} w$ where e is the order of the stabilizer Γ_a. The transformation $R = \tilde{J}(r_e, w)$ is independent of w and has the property $R^e = \mathrm{id}$. We define the characteristic numbers

$$x_1 \dots, x_n, \quad 0 \le x_i < 1,$$

such that $\mathrm{e}^{2\pi \mathrm{i} x_\nu}$ are the eigenvalues of R. The numbers $e x_\nu$ are integral.

The proof is rather trivial. Every element $\gamma \in \tilde{\Gamma}$ which stabilizes the origin must be of the form $w \mapsto \zeta w$ where ζ is a complex number of absolute number one. The only subgroup of order e of the multiplicative group of complex numbers is the group generated by $e^{2\pi \mathrm{i}/e}$. Since the derivative of γ is constant, $\tilde{J}(\gamma, w)$ is independent of w. The automorphy property implies that it is a homomorphism. The image is a group of some order that divides e. $\square$

Another way to describe the characteristic numbers is as follows (use Lemma 2.2).

2.4 Remark. *(Notations as in Remark and Definition 2.3.) Consider in the stabilizer Γ_a the generator γ that corresponds to the element r_e in the unit-disk. Then the characteristic numbers x_ν are defined such that $0 \le x_\nu < 1$ and that $\mathrm{e}^{2\pi \mathrm{i} x_\nu}$ are the eigen values of $J(\gamma, a)$.*

Next we define the characteristic numbers in the case where a is the cusp ∞. The stabilizer Γ_∞ is generated by a translation

$$t^N(\tau) := \tau + N, \quad N > 0.$$

The matrix

$$R = J(t^N, \tau)$$

is independent of τ and has finite order. The characteristic numbers are defined such that $e^{2\pi \mathrm{i} x_\nu}$ are the eigenvalues of R.

Next we treat the case where a is an arbitrary cusp. We choose a transformation $\alpha \in \mathrm{Aut}(\mathbb{H})$ with the property $\alpha(a) = \infty$. We can consider the conjugate group $\tilde{\Gamma} = \alpha\Gamma\alpha^{-1}$ and the conjugate multiplier system $\tilde{\varrho}$ (and of course the same r). The group $\tilde{\Gamma}$ has the cusp ∞. We want to define the characteristic numbers of (Γ, r, ϱ) at the cusp a to be the characteristic numbers of $(\tilde{\Gamma}, r, \tilde{\varrho})$ at ∞. It is easy to prove that this definition does not depend on the choice of α.

2.5 Lemma. *Let a be a cusp and $\alpha \in \mathrm{Aut}(\mathbb{H})$ be a transformation with the property $\alpha(a) = \infty$. We consider the conjugate group $\tilde{\Gamma} = \alpha\Gamma\alpha^{-1}$ and the conjugate multiplier system $\tilde{\varrho}$. The characteristic numbers of $(\tilde{\Gamma}, r, \tilde{\varrho})$ at ∞ are independent of the choice of α.*

Proof. We can assume that $a = \infty$. Then α is of the form $\alpha(\tau) = u\tau + v$. Let $t^N(\tau) = \tau + N$ be the generator of Γ_∞. We set $\tilde{N} = uN$. Then $t^{\tilde{N}} = \alpha t^N \alpha^{-1}$ and this is the generator of $\tilde{\Gamma}_\infty$. Taking a suitable basis of V we can assume that the matrix of $R = J(t^N, \cdot)$ is diagonal. Then we can assume that V has dimension one and that R acts by multiplication by $e^{2\pi \mathrm{i} x}$ where x is the characteristic number. The function $f(\tau) = e^{2\pi \mathrm{i} x\tau}$ then has the property $f(\tau + N) = J(t^N, \tau) f(\tau)$. From the second part of Lemma 2.2 follows that the function $\tilde{f}(\tau) = f(u\tau + v)$ has the property

$$\tilde{f}(\tau + \tilde{N}) = \tilde{J}(t^{\tilde{N}}, \tau)\tilde{f}(\tau).$$

This formula implies that $J(t^{\tilde{N}}, \tau)$ is also the multiplication by $e^{2\pi \mathrm{i} x}$. $\square$

2.6 Lemma. *The characteristic numbers depend only on the Γ-orbit of a point $a \in \mathbb{H}^*$. Hence the can be considered for points $x \in X_\Gamma$. They can be different from $(0, \dots, 0)$ only for cusps or elliptic fixed pints.*

We want to introduce local automorphic forms. Let $U \subset X_\Gamma$ be an open subset. We denote its inverse image in $\mathbb{H}^*$ by $\tilde{U}$. This is a Γ-invariant subset. Hence we can consider all holomorphic functions $f : \tilde{U} - S \to \mathbb{C}$ with the transformation property

$$f(\gamma\tau) = J(\gamma, \tau) f(\tau), \qquad \gamma \in \Gamma.$$

Assume that Γ has cusp ∞ and that it is contained in $\tilde{U}$. Then $\tilde{U}$ contains some upper half plane $\operatorname{Im}\tau > C > 0$ and f has the transformation property $f(\tau+N) = Rf(\tau)$. Since R has finite order, f has some multiple of N as period. We call f regular at ∞ if f is bounded for $\operatorname{Im}\tau \to \infty$ and cuspidal if it tends to 0. We can diagonalize R and describe f by components

$$f_\nu(\tau+N) = e^{2\pi \mathrm{i} x_\nu} f_\nu(\tau).$$

The function $g_\nu(\tau) = f_\nu(\tau)e^{-2\pi \mathrm{i} x_\nu \tau/N}$ has period N. Hence we have a Fourier expansion

$$f_\nu(\tau) = e^{\frac{2\pi \mathrm{i}}{N} x_\nu \tau} \sum_{m=-\infty}^{\infty} a_\nu(m) e^{\frac{2\pi \mathrm{i}}{N} m\tau}.$$

The function f is regular at ∞ if $a_\nu(m) \neq 0$ implies $x_\nu + m \geq 0$ and cuspidal if it implies $x_\nu + m > 0$. Since $0 \leq x_\nu < 1$ the condition $x_\nu + m \geq 0$ is equivalent to $m \geq 0$.

Using "transformation to ∞" one can define the notions "regular" and "cuspidal" also for other cusps. It is clear that this notion does not depend on the choice of the transformation. It is also clear that this notion depends only on the Γ-orbit of a cusp.

We define a certain sheaf $\mathcal{M} = \mathcal{M}_\Gamma(r, \varrho)$ on X_Γ. For open U in $X = X_\Gamma$ the space $\mathcal{M}(U)$ consists of all local automorphic forms $f : \tilde{U} - S \to V$ which are regular at the cusps. This defines a sheaf and even more an $\mathcal{O}_X$-module. For even r and the trivial one-dimensional representation ϱ we write $\mathcal{M}(r)$ instead of $\mathcal{M}(r, \varrho)$. We also can consider the subsheaf $\mathcal{M}^{\mathrm{cusp}} = \mathcal{M}_\Gamma^{\mathrm{cusp}}(r, \varrho)$ of all cuspidal local automorphic forms. This is also an $\mathcal{O}_X$-module.

2.7 Lemma. *The sheafs $\mathcal{M} = \mathcal{M}_\Gamma(r, \varrho)$, $\mathcal{M}^{\mathrm{cusp}} = \mathcal{M}_\Gamma^{\mathrm{cusp}}(r, \varrho)$ are vector bundles of rank $n = \dim V$, hence coherent.*

Proof. We have to show that $\mathcal{M}_x$ is a free $\mathcal{O}_{X,x}$-module for each $x \in X_\Gamma$. Let $a \in \mathbb{H}^*$ be a representant of x. We have to treat two cases. The first case is that a is a cusp. We can assume that $a = \infty$. As explained above, the elements of $\mathcal{M}_x$ can be considered as Fourier series of the kind

$$f_\nu(\tau) = \sum_{m+x_\nu \geq 0} a_\nu(m) e^{\frac{2\pi \mathrm{i}}{N}(m+x_\nu)\tau}.$$

Recall that $m + x_\nu \geq 0$ means the same as $m \geq 0$. If we map $f_\nu(\tau)$ to

$$\sum_{m\geq 0} a_\nu(m) e^{\frac{2\pi \mathrm{i}}{N} m\tau}$$

we get an isomorphism from $\mathcal{M}_x$ to $\mathcal{O}_{X,x}^n$. This shows that $\mathcal{M}_x$ is free. The case $\mathcal{M}^{\mathrm{cusp}}$ is similar.

Next we consider the case where a is an interior point. In this case we can identify $\mathcal{M}_x$ with holomorphic functions f in a small disc around $w = 0$ which transform as

$$f(e^{2\pi \mathrm{i}/e} w) = Rf(w), \quad R^e = \mathrm{id}\,.$$

The components of f with respect to a basis of eigenvectors satisfy

$$f_\nu(e^{2\pi \mathrm{i}/e} w) = e^{2\pi \mathrm{i} x_\nu} f_\nu(w).$$

From the Taylor expansion one can derive that $f_\nu(w) = w^{ex_\nu} g_\nu(w^e)$. The local ring $\mathcal{O}_{X,x}$ can be identified with the ring of power series $\mathbb{C}\{w^e\}$. The map $f_\nu \mapsto g_\nu$ gives an $\mathcal{O}_{X,x}$-linear isomorphism from $\mathcal{M}_x$ to $\mathcal{O}^n_{X,x}$. $\square$

For a multiplier system ϱ of weight r with values in the vector space V one can define the dual multiplier system ϱ'. It is realized on $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. By definition $\varrho'(\gamma)$ is the transposed of $\varrho(\gamma^{-1})$. It is easy to check that this is a multiplier system of weight $-r$. We mention that a multiplier system of weight r can be considered as multiplier system of weight r' for each $r' \equiv r \bmod 2$.

As in the case of the sheaf $\mathcal{M}$, we write $\mathcal{M}^{\mathrm{cusp}}(r)$ instead of $\mathcal{M}^{\mathrm{cusp}}(r, \varrho)$ for even r and the trivial one-dimensional representation ϱ.

2.8 Lemma. *The sheaf $\mathcal{M}^{\mathrm{cusp}}(2)$ is a canonical sheaf. The dual sheaf of $\mathcal{M}(r, \varrho)$ is isomorphic to $\mathcal{M}^{\mathrm{cusp}}(2 - r, \varrho')$, where ϱ' denotes the dual multiplier system.*

Proof. The canonical sheaf on a compact Riemann surface is the sheaf of holomorphic differentials. Let ω be a holomorphic differential on an open subset $U \subset X_\Gamma$. Its inverse image on $\tilde{U} - S$ is of the form $f(\tau)d\tau$. The function f transforms like an automorphic form of weight two (and trivial multiplier system). Using the formula

$$2\pi \mathrm{i} dz = dq/q \quad \text{for} \quad q = e^{2\pi \mathrm{i} \tau}$$

it is easy to show that the regularity of ω at the cusp classes means that f is cuspidal. For the elliptic fixed points a similar argument works. We omit it. $\square$

Next we define a pairing

$$\mathcal{M}(r, \varrho) \times \mathcal{M}^{\mathrm{cusp}}(2 - r, \varrho') \longrightarrow \mathcal{M}^{\mathrm{cusp}}(2).$$

For this we use the natural pairing

$$V \times \mathrm{Hom}(V, \mathbb{C}) \longrightarrow \mathbb{C}, \quad \langle v, L \rangle = L(v).$$

Let $f \in \mathcal{M}(r, \varrho)$ and $g \in \mathcal{M}^{\mathrm{cusp}}(2 - r, \varrho')$ be local automorphic forms on some $U \subset X_\Gamma$. Then $\langle f, g \rangle$ transforms like an automorphic form of weight 2 with respect to the trivial multiplier system. It is clear that it is cuspidal. So the pairing has been defined. It has to be checked that it is non-degenerated. This can be done by a local computation at points $x \in X_\Gamma$. We restrict to the case where x is the image of the cusp ∞. Recall that – using a suitable basis of V – the elements of $\mathcal{M}(r,\varrho)_x$ can be identified with Fourier series

$$f_\nu(\tau) = \sum_{m+x_\nu\geq 0}^{\infty} a_\nu(m) e^{\frac{2\pi \mathrm{i}}{N}(m+x_\nu)\tau}.$$

The characteristic numbers y_ν of the dual multiplier system have the property $x_\nu + y_\nu \equiv 0 \bmod 1$. Hence – using the dual basis – the elements of $\mathcal{M}^{\mathrm{cusp}}(2-r,\varrho')_x$ can be identified with Fourier series

$$g_\nu(\tau) = \sum_{m-x_\nu>0}^{\infty} b_\nu(m) e^{\frac{2\pi \mathrm{i}}{N}(m-x_\nu)\tau}$$

and the pairing is just $\sum f_\nu g_\nu$. The condition $m + x_\nu \geq 0$ is equivalent to $m \geq 0$ and the condition $m - x_\nu > 0$ is equivalent to $m \geq 1$. Finally $\mathcal{M}(2)_x$ can be identified with all Fourier series

$$h(\tau) = \sum_{m\geq 1} c(m) e^{\frac{2\pi \mathrm{i}}{N} m\tau}.$$

Let $q = e^{\frac{2\pi \mathrm{i}}{N}\tau}$. Using the isomorphisms

$$\begin{aligned}
&\mathcal{M}(r,\varrho)_x \xrightarrow{\sim} \mathbb{C}\{q\}^n, && f \longmapsto \Big(\sum a_\nu(m) q^m\Big),\\
&\mathcal{M}^{\mathrm{cusp}}(2-r,\varrho')_x \xrightarrow{\sim} \mathbb{C}\{q\}^n, && f \longmapsto \Big(\sum b_\nu(m) q^{m-1}\Big),\\
&\mathcal{M}(2)_x \xrightarrow{\sim} \mathbb{C}\{q\}^n, && h \longmapsto \Big(\sum c_\nu(m) q^{m-1}\Big),
\end{aligned}$$

the pairing gets equivalent to the standard pairing

$$\mathbb{C}\{q\}^n \times \mathbb{C}\{q\}^n \longrightarrow \mathbb{C}\{q\}, \quad \langle P, Q\rangle = \sum_\nu P_\nu Q_\mu,$$

which is obviously non-degenerated. □

3. The computation of the degree

We consider a subgroup $\Gamma_0 \subset \Gamma$ of finite index. We restrict ϱ to Γ_0 and consider the sheaf

$$\mathcal{M}_0 = \mathcal{M}_{\Gamma_0}(r,\varrho)$$

on the Riemann surface X_{Γ_0}. We want to compare the degrees of $\mathcal{M}$ and $\mathcal{M}_0$. Let

$$\pi : X_{\Gamma_0} \longrightarrow X_\Gamma$$

be the natural covering. We know $\deg \mathcal{M}^* = \deg(X_0/X) \deg \mathcal{M}$. There is an obvious inclusion of sheaves $\mathcal{M} \hookrightarrow \pi_* \mathcal{M}_0$. By functoriality this induces a map

$$\pi^* \mathcal{M} \longrightarrow \mathcal{M}_0.$$

Let $x \in X_{\Gamma_0}$. The stalk of $\pi^*\mathcal{M}$ is

$$(\pi^*\mathcal{M})_a \cong \mathcal{M}_{\pi(x)} \otimes_{\mathcal{O}_{X_{\Gamma,\pi(x)}}} \mathcal{O}_{X_{\Gamma_0,x}}.$$

Since $\mathcal{O}_{X_{\Gamma_0,x}}$ is a free $\mathcal{O}_{X_{\Gamma,\pi(x)}}$-module we see that

$$(\pi^*\mathcal{M})_{\pi(x)} \longrightarrow (\mathcal{M}_0)_x$$

is injective. Outside a finite set (images of cusps and of elliptic fixed points of Γ) it is an isomorphism. So we get an exact sequence

$$0 \longrightarrow \pi^*\mathcal{M} \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{K} \longrightarrow 0$$

with a skyscraper sheaf $\mathcal{K}$. We have to compute its degree

$$\deg \mathcal{K} = \sum_{x \in X_{\Gamma_0}} \dim \mathcal{K}_x.$$

We compute

$$\mathcal{K}_x = (\mathcal{M}_0)_x \;/\; (\mathcal{M}_{\pi(x)} \otimes_{\mathcal{O}_{X_{\Gamma,\pi(x)}}} \mathcal{O}_{X_{\Gamma_0,x}})$$

first in the case where x comes from an inner point $a \in \mathbb{H}$. Let $w = (\tau - a)/(\tau - a)$. For sake of simplicity we assume that a is not an elliptic fixed point of Γ_0. Then the local ring $\mathcal{O}_{X_{\Gamma_0,x}}$ can be identified with the ring of power series $\mathbb{C}\{w\}$. The ring $\mathcal{O}_{X_{\Gamma,\pi(x)}}$ can be identified with $\mathbb{C}\{w^e\}$. As in section two we take a basis of V such that all $\varrho(\gamma)$ are diagonal. Then we have natural isomorphisms

$$(\mathcal{M}_0)_x \cong \mathbb{C}\{w\}^n.$$

and

$$\mathcal{M}_{\pi(x)} = \prod_{\nu=1}^{n} w^{ex_\nu} \mathbb{C}\{w^e\}.$$

If we tensor this with $\mathbb{C}\{w\}$ we get

$$\mathcal{M}_{\pi(x)} \otimes_{\mathcal{O}_{X_{\Gamma,\pi(x)}}} \mathcal{O}_{X_{\Gamma_0,x}} = \prod_{\nu=1}^{n} w^{ex_\nu} \mathbb{C}\{w\}.$$

This shows the following result.

3.1 Lemma. *Let $\Gamma_0 \subset \Gamma$ be a subgroup of finite index and let $\pi : X_{\Gamma_0} \to X_\Gamma$ be the natural projection. We consider a point $x \in X_{\Gamma_0}$ which is the image of an inner point $a \in \mathbb{H}$. We assume that a is no fixed point of Γ_0. Let ϱ be a multiplier system of weight r for Γ. We denote by*

$$\sigma(a) = x_1 + \cdots + x_n$$

the sum of the characteristic numbers at a. Then the formula

$$\dim \mathcal{K}_x = e\sigma(a), \qquad \mathcal{K} = \mathcal{M}_{\Gamma_0}(\varrho, r)/\pi^* \mathcal{M}_\Gamma(\varrho, r),$$

holds.

Now we consider the case that $x \in X_{\Gamma_0}$ is the image of the cusp ∞. Analogously to N for Γ, we denote by N_0 the smallest positive number such that $\tau \mapsto \tau + N_0$ is in Γ_0. The number N_0/N is integral. We set

$$q = e^{\frac{2\pi \mathrm{i}}{N_0}\tau}.$$

Then the local ring of X_{Γ_0} at x is $\mathbb{C}\{q\}$ and the local ring of X_Γ at $\pi(x)$ is $\mathbb{C}\{q^{N_0/N}\}$. The stalk of $\mathcal{M} = \mathcal{M}_\Gamma(\varrho, r)$ at $\pi(x)$ is (after diagonalization)

$$\mathcal{M}_{\pi(x)} = \prod_{\nu=1}^{n} e^{\frac{2\pi \mathrm{i}}{N} x_\nu \tau} \mathbb{C}\{q^{N_0/N}\}.$$

We get

$$\mathcal{M}_{\pi(x)} \otimes_{\mathcal{O}_{\Gamma,\pi(x)}} \mathcal{O}_{X_{\Gamma_0},x} = \prod_{\nu=1}^{n} e^{\frac{2\pi \mathrm{i}}{N} x_\nu \tau} \mathbb{C}\{q\}.$$

The characteristic numbers $y_1, \dots, y_n$ of with respect to Γ_0 are defined by

$$y_\nu \equiv (N_0/N) x_\nu \mod 1, \qquad 0 \le y_\nu < 1,$$

or, using the Gauss bracket,

$$y_\nu = (N_0/N)x_\nu - [(N_0/N)x_\nu].$$

Hence the stalk of $\mathcal{M}_0 = \mathcal{M}_{\Gamma_0}(\varrho, r)$ at x is

$$\mathcal{M}_{0,x} = \prod_{\nu=1}^{n} e^{\frac{2\pi \mathrm{i}}{N_0} y_\nu \tau} \mathbb{C}\{q\}.$$

This shows

$$\dim \mathcal{K}_x = \sum_{\nu=1}^{n} [(N_0/N)x_\nu].$$

For sake of simplicity we assume that the characteristic numbers of the cusp ∞ with respect to Γ_0 are zero. Then $(N_0/N)x_\nu$ is integral. We also mention that N_0/N is the index of $\Gamma_{0,\infty}$ in Γ_∞. Hence we get

$$\dim \mathcal{K}_x = \sum_{\nu=1}^{n} [\Gamma_\infty : \Gamma_{0,\infty}] x_\nu.$$

We recall that the characteristic numbers for a point $x \in X_\Gamma$ depend on (ϱ, r). To point out the r-dependency we will write frequently $x_\nu = x_\nu(r)$ and $\sigma(x, r) = \sigma(x)$ for their sum.

3.2 Lemma. *Let $\Gamma_0 \subset \Gamma$ be a subgroup of finite index and let $\pi : X_{\Gamma_0} \to X_\Gamma$ be the natural projection. We consider a point $x \in X_{\Gamma_0}$ which is the image of a cusp a. Let ϱ be a multiplier system of weight r for Γ. We denote by $\sigma(a, r)$ the sum of the characteristic numbers. We assume that the characteristic numbers of the cusps with respect to Γ_0 are zero. Then the formula*

$$\dim \mathcal{K}_x = [\Gamma_a : \Gamma_{0,a}]\sigma(a, r), \qquad \mathcal{K} = \mathcal{M}_{\Gamma_0}(\varrho, r)/\pi^*\mathcal{M}_\Gamma(\varrho, r),$$

holds.

For the rest of this paper we make the following assumption.

3.3 Assumption. *The triple Γ, ϱ, r has the following property. There exists a subgroup of finite index $\Gamma_0 \subset \Gamma$ such that Γ_0 acts fixed point free on $\mathbb{H}$ and that the characteristic numbers of all cusps with respect to Γ_0 are zero.*

This assumption is harmless. It is fulfilled for arithmetic groups since there are many subgroups of finite index in form of congruence subgroups.

We use the notation

$$\pi : X_{\Gamma_0} \longrightarrow X_\Gamma$$

for the canonical map. We get a formula for the degree of $\mathcal{K}$.

3.4 Proposition. *The formula*

$$\deg \mathcal{K} = [\Gamma : \Gamma_0] \sum_{x \in X_\Gamma} \sigma(x, r)$$

holds. Here $\sigma(x, r)$ is the sum of the characterstic numbers at (a representative of) x.

We now get the link between the degrees of $\mathcal{M}_\Gamma(\varrho, r)$ and $\mathcal{M}_{\Gamma_0}(\varrho, r)$. The covering degree of $\pi : X_{\Gamma_0} \to X_\Gamma$ equals the index $[\Gamma : \Gamma_0]$. Using 5) from Theorem 1.1 we get the following formula.

$$[\Gamma : \Gamma_0] \deg \mathcal{M}_\Gamma(\varrho, r) = \deg \mathcal{M}_{\Gamma_0}(\varrho, r) - [\Gamma : \Gamma_0] \sum_{x \in X_\Gamma} \sigma(x, r).$$

The group Γ_0 can be chosen small enough such that the multiplier system is diagonal, that it acts fixed point free on $\mathbb{H}$ and that the characteristic numbers of the cusps are zero. Then $\mathcal{M}_{\Gamma_0}(\varrho, r)$ is a direct sum of line bundles and we are reduced to the well-known case $V = \mathbb{C}$ which has been treated at various places in the literature. For sake of completeness we repeat shortly the argument. Since every line-bundle has a meromorphic section (i.e. a mermorphic automorphic form) f. We associate to f a divisor $D = (f)$. such that $\mathcal{O}_D$ is isomorphic to $\mathcal{M}_{\Gamma_0}(\varrho, r)$. If $x \in X_{\Gamma_0}$ is the image of an inner point $a \in \mathbb{H}$, then $D(x)$ is the usual order of f at a. Let a be the cusp ∞. Since the characteristic number are zero, we can consider f as a holomorphic function in $q = e^{\frac{2\pi i}{N_0}\tau}$ and we define $D(x)$ to be the order of this function at $q = 0$. For an arbitrary cusp we use "transformation to ∞". It is easy to check that the order is independent of the choice of the transformation and even more that it depends only one the Γ_0-orbit of a. Let m be a natural number. Then on has $(f^m) = m(f)$ (since the characteristic numbers of the cusps vanish). We can take m such that mr is even and such the multiplier system of f is trivial. Now we can compare with modular forms of weight two.

$$\deg \mathcal{M}_{\Gamma_0}(\varrho, r) = \frac{r}{2} \deg \mathcal{M}(2).$$

We use that $\mathcal{M}^{\mathrm{cusp}}(2)$ is a canonical bundle and that the degree of the canonical bundle is $2g_0 - 2$. We obtain that the degree of $\mathcal{M}(2)$ is $2g_0 - 2 + h_0$ where h_0 denotes the number of cusp classes of Γ_0. Collecting together we obtain the following formula.

3.5 Proposition. *The formula*

$$\deg \mathcal{M}_\Gamma(\varrho, r) = \frac{rn}{2[\Gamma : \Gamma_0]}(2g_0 - 2 + h_0) - \sum_{x \in X_\Gamma} \sigma(x, r).$$

holds. Here g_0 denotes the genus of X_{Γ_0}. The rank of ϱ is denoted by n and h_0 denotes the number of cusp classes of Γ_0.

We want to express the formula above in data of the group Γ alone. For this we have to use the Riemann–Hurwitz ramification formula. Let $\pi : X_0 \to X$ a holomorphic non-constant map between compact Riemann surfaces. Let g be the genus of X and g_0 the genus of X_0. For $a \in X_0$ we denote by $\mathrm{Ord}(\pi, a)$ the order of π at a. That the order is m means that f looks locally around a like $z \mapsto z^m$. The ramification formula states

$$g_0 - 1 = \deg(f)(g - 1) + \frac{1}{2} \sum_{a \in X} (\mathrm{Ord}(f, a) - 1).$$

We assume that $f : X_0 \to X$ is Galois. This means that there is a finite group G of biholomorphic transformations of X_0 such that two points in X_0 are G-equivalent if and only if they have the same image in X. In the Galois case the order at a point $a \in X_0$ depends only on its image $b \in X$. Hence we can define

$$e(f, b) := \mathrm{Ord}(f, a) \qquad (b \in X).$$

The number of points $a \in X_0$ over $b \in X$ is $\deg f / e(f, b)$. So the ramification formula can be written as

$$\frac{g_0 - 1}{\deg f} = (g - 1) + \frac{1}{2} \sum_{b \in Y} \Big(1 - \frac{1}{e(f, b)}\Big)$$

in this case. We want to use to reformulate the degree formula in Proposition 3.5. We notice that the degree of $\pi : X_{\Gamma_0} \to X_\Gamma$ equals the index $[\Gamma : \Gamma_0]$. The number of inverse points of a given cusp class $b \in X_\Gamma$ is $[\Gamma_0 : \Gamma]/e(\pi, b)$. Hence we have

$$h_0 = \sum_{b \in X_\Gamma \text{ cusp}} \frac{[\Gamma_0 : \Gamma]}{e(\pi, b)}.$$

Now we obtain the following result.

3.6 Proposition. *Assume in addition to the conditions of Proposition 3.5 that Γ_0 is normal in Γ. Then the degree formula can be rewritten as*

$$\deg \mathcal{M}_\Gamma(\varrho, r) = rn\Big(g - 1 + \frac{h}{2} + \frac{1}{2} \sum_{b \in X_\Gamma \text{ not cusp}} \Big(1 - \frac{1}{e(\pi, b)}\Big)\Big) - \sum_{x \in X_\Gamma} \sigma(x, r).$$

4. The dimension formula

The Riemann-Roch formula states

$$\chi(\mathcal{M}_\Gamma(\varrho, r)) = \deg(\mathcal{M}_\Gamma(\varrho, r)) + \operatorname{Rank}(\mathcal{M}_\Gamma(\varrho, r))(1 - g).$$

We are more interested in the spaces of automorphic forms

$$[\Gamma, \varrho, r] := H^0(X_\Gamma, \mathcal{M}_\Gamma(\varrho, r)).$$

The Serre dual space is the subspace of cusp forms of $[\Gamma, \varrho', 2 - r]$. In the case $r > 2$ this space vanishes. In the case $r = 2$ there is a difference depending on the fact whether Γ has a cusp or not. Modular forms of weight 0 are constants. Hence $[\Gamma, \varrho', 0]$ is just the space of ϱ'-invariants of V. This is ismorphic to the space of ϱ-invariants of V. Since constant cusp forms are zero if there is a cusp, we obtain the following dimension formula.

4.1 Theorem. *In the case $r > 2$ we have*

$$\begin{aligned}\dim[\Gamma, \varrho, r] =& rn\Big(g - 1 + \frac{h}{2} + \frac{1}{2} \sum_{b \in X_\Gamma \text{ not cusp}} \Big(1 - \frac{1}{e(b)}\Big)\Big) \\ &+ n(1 - g) - \sum_{x \in X_\Gamma} \sigma(x, r).\end{aligned}$$

Here g is the genus of X_Γ, the number of cusps is denoted by h. The order of the stabilizer of a representant of b is denoted by $e(b)$. The dimension of V is n and $x_1(r), \dots, x_n(r)$ are the characteristic numbers. (They depend on r.)

Supplement. *When Γ has a cusp then this formula remains true in the case $r = 2$. Otherwise one has to add $\dim V^\varrho$ to the right hand side.*

We denote by $[\Gamma,\varrho,r]_0$ the subspace of cusp forms of $[\Gamma,\varrho,r]$. This is the space of global sections of the sheaf $\mathcal{M}^{\rm cusp}=\mathcal{M}^{\rm cusp}(\Gamma,\varrho,r)$. Since the quotient $\mathcal{M}/\mathcal{M}^{\rm cusp}$ is a skyscraper sheaf we have

$$\chi(\mathcal{M})-\chi(\mathcal{M}^{\rm cusp})=\sum_{x\in X_\Gamma,\ {\rm cusp}}\dim(\mathcal{M}_x/\mathcal{M}_x^{\rm cusp}).$$

Recall that $\mathcal{M}_x$ is given by Fourier series with summation over integers m such that $m+x_\nu(r)\ge 0$ and in the subspace $\mathcal{M}_x^{\rm cusp}$ the summation is restricted to $m+x_\nu(r)>0$. There is only a difference if the characteristic number x_ν is zero. We see

$$\chi(\mathcal{M}^{\rm cusp})=\chi(\mathcal{M})-\sum_{x\in X_\Gamma,\ {\rm cusp}}\#\{\nu;\ x_\nu(r)=0\}.$$

4.2 Remark. *Assume that Γ has cusp ∞. The number*

$$\#\{\nu;\ x_\nu=0\}$$

equals the dimension of the subspace of invariants of V under the transformations $J(\gamma,\tau)$, $\gamma\in\Gamma_\infty$. (These transformations do not depend on τ.)

Finally we formulate the dimension formula for the space of cusp forms. In the case of weight 2 we have to be careful, since

$$\chi(\mathcal{M}^{\rm cusp}(\Gamma,\varrho,2))=\dim[\Gamma,\varrho,2]_0-\dim[\Gamma,\varrho',0].$$

In the case of an even weight, ϱ is a representation and $[\Gamma,\varrho',0]$ can be identified with the space of invariants of ϱ'. We obtain the following result.

4.3 Theorem. *Assume that Γ has at least one cusp. In the case $r>0$ the dimension of the space of cusp forms is*

$$\dim[\Gamma,\varrho,r]_0=\dim[\Gamma,\varrho,r]-\sum_{x\in X_\Gamma,\ {\rm cusp}}\#\{\nu;\ x_\nu(r)=0\}.$$

In the case $r=2$ we have to add $\dim V^\varrho$ to the right hand side.

5. The full modular group

We specialize the dimension formula to the group $\Gamma = \mathrm{SL}(2,\mathbb{Z})/\pm$. As usual it acts on the upper half plane by $(a\tau+b)(c\tau+d)^{-1}$. In this case $g=0$ and $h=1$. We have two classes of elliptic fixed points of order $e=2$ resp. $e=3$. In the dimension formula we get

$$\sum_{b\in X_\Gamma \text{ not cusp}} \Bigl(1-\frac{1}{e(b)}\Bigr) = \Bigl(1-\frac{1}{2}\Bigr)+\Bigl(1-\frac{1}{3}\Bigr)=\frac{7}{6}.$$

So the dimension formula gives

$$\dim[\Gamma,\varrho,r] = \frac{rn}{12}+n-\sum_{x\in X_\Gamma}\sigma(x,r).$$

We use the usual generators

$$T=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad \begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

Representatives of the elliptic fixed points are i and $\zeta_3=-1/2+\mathrm{i}\sqrt{3}/2$. The elements in their stabilizers which correspond to the rotation with factor $e^{2\pi\mathrm{i}/e}$ can be computed easily as S resp. $(ST)^{-1}$.

Let A be a complex matrix of finite order. The eigenvalues are roots of unity which we can write in the form

$$\lambda=\exp(2\pi\mathrm{i}\alpha) \text{ with } 0\le\alpha<1.$$

We use the notation

$$\alpha(A)=\sum_\lambda \alpha,$$

where λ runs through all eigenvalues (counted with multiplicity).

The contributions of the characteristic numbers in the dimension formula can be written as

$$\sum_{x\in X_\Gamma}(x_1(r)+\cdots+x_n(r)) = \alpha(J(S,\mathrm{i}))+\alpha(J((ST)^{-1},\zeta_3))+\alpha(J(T,\cdot)).$$

(The function $J(T,\tau)$ is independent of τ.)

5.1 Theorem. *In case of the full modular group the dimension formula is valid for $r \geq 2$ (including $r = 2$) and reads as*

$$\dim[\Gamma, \varrho, r] = \frac{rn}{12} + n - \alpha(J(S, \mathrm{i})) - \alpha(J((ST)^{-1}, \zeta_3)) - \alpha(J(T, \cdot)).$$

For the subspace of cusp forms one has

$$\dim[\Gamma, \varrho, r]_0 = \dim[\Gamma, \varrho, r] - \dim V^{J(T,\cdot)} + \begin{cases} 0 & \text{if } r > 2, \\ \dim V^{\varrho} & \text{if } r = 2. \end{cases}$$

We treat a simple example just to get a feeling how the formula works. The weight r is assumed to be even and we consider the case of a trivial multiplier system. This means $J(\gamma, \tau) = (c\tau + d)^{r/2}$. So we get

$$J(S, \mathrm{i}) = e^{2\pi \mathrm{i} r/4}$$

and

$$J((ST)^{-1}, \zeta_3) = e^{2\pi \mathrm{i} r/6}.$$

The sum of both is

$$\begin{cases} 0 & \text{for } r \equiv 0 \bmod 12, \\ 7/6 & \text{for } r \equiv 2 \bmod 12, \\ 1/3 & \text{for } r \equiv 4 \bmod 12, \\ 1/2 & \text{for } r \equiv 6 \bmod 12, \\ 2/3 & \text{for } r \equiv 8 \bmod 12, \\ 5/6 & \text{for } r \equiv 10 \bmod 12. \end{cases}$$

Using the table above, one gets

$$\begin{cases} \left[\frac{r}{12}\right] & \text{if } r \equiv 2 \bmod 12, \\ \left[\frac{r}{12}\right] + 1 & \text{else.} \end{cases}$$

This formula is true for all even $r > 0$ (also for $r = 2$).

6. The metaplectic group

There is a different way to express multiplier systems using the metaplectic group. We recall this concept briefly in the case of half-integral weight The metaplectic group

$$\mathrm{Mp}(2, \mathbb{R}) \longrightarrow \mathrm{SL}(2, \mathbb{R})$$

can be described as the set of all pairs (M, J), where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$ and where $J = \sqrt{c\tau + d}$ is one of the two holomorphic square roots of the function $c\tau + d$ on the upper half plane $\mathbb{H}$. The group law is

$$(M, \sqrt{c\tau + d})(M', \sqrt{c'\tau + d'}) = (MM', \sqrt{c'\tau + d'}\sqrt{cM'\tau + d}).$$

One knows that $\mathrm{Mp}(2, \mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \ \mathrm{Re}\,\tau > 0,$$

and that the relations

$$S^2 = (ST)^3 = Z, \quad Z = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathrm{i} \right), \quad Z^4 = 1$$

are defining ones.

Let

$$\varrho : \mathrm{Mp}(2, \mathbb{Z}) \longrightarrow \mathrm{GL}(V)$$

a representation of $\mathrm{Mp}(2, \mathbb{Z})$ on some finite dimensional complex vector space. Let r be an integer or a half integer $(2r \in \mathbb{Z})$. An (entire) modular form of weight r with respect to ϱ is holomorphic function $f : \mathbf{H} \to V$ with the transformation law

$$f(M\tau) = \sqrt{c\tau + d}^{2r} \varrho(M) f(\tau) \quad \text{for all} \quad (M, \sqrt{c\tau + d}) \in \mathrm{Mp}(2, \mathbb{Z})$$

and such that f is bounded for $\mathrm{Im}\,\tau \geq 1$.

We denote by $[\mathrm{Mp}(2, \mathbb{Z}), r, \varrho]$ the space of all entire modular forms. Let

$$\varrho : \mathrm{Mp}(2, \mathbb{Z}) \longrightarrow \mathrm{GL}(V)$$

a representation of $\mathrm{Mp}(2, \mathbb{Z})$ on some finite dimensional complex vector space. We assume that it is trivial on a subgroup of finite index. Let r be an integer or a half integer $(2r \in \mathbb{Z})$.

Let $V_0 \subset V$ be the subspace on which $\varrho(-E, \mathrm{i})$ (E denotes the unit matrix) acts by multiplication with $e^{-\pi \mathrm{i} r} = \mathrm{i}^{-2r}$. It is quite clear that the values of f are contained in V_0 and that V_0 is invariant under $\mathrm{Mp}(2, \mathbb{Z})$. Let $\gamma \in \mathrm{SL}(2, \mathbb{Z})/\pm$ a modular transformation. We choose a pre-image $(M, \sqrt{c\tau + d}) \in \mathrm{Mp}(2, \mathbb{C})$ and define the operator

$$J(\gamma, \tau)a = \sqrt{c\tau + d}^{2r} \varrho(M)a \quad \text{for} \quad a \in V_0.$$

This is independent of the choice of the pre-image (since we restrict to V_0). By trivial reason

$$J(\gamma, \tau)\alpha'(\tau)^{r/2}$$

is a V_0-valued multiplier system and Assumption 3.3 is satisfied. The space of modular forms of weight r with respect to this multiplier coincides with $[\mathrm{Mp}(2, \mathbb{Z}), \varrho, r]$. Hence the dimension formula gives the following result.

6.1 Theorem. *Let $\varrho : \mathrm{Mp}(2,\mathbb{Z}) \to \mathrm{GL}(V)$ be a representation on a finite dimensional vector whose image is finite. Let V_0 be the biggest subspace of V where $\varrho(-E,\mathrm{i})$ acts by multiplication with $\mathrm{e}^{-\pi\mathrm{i}r}$. We denote by d the dimension of V_0. Then one has for $r \geq 2$ (including $r = 2$)*

$$\dim[\mathrm{Mp}(2,\mathbb{Z}),\varrho,r] = \frac{rd}{12} + d - \alpha(e^{\pi\mathrm{i}r/2}\varrho(S)) - \alpha\Big(\big(e^{\pi\mathrm{i}r/3}\varrho(ST)\big)^{-1}\Big) - \alpha(\varrho(T)).$$

The invariants α have to be taken with respect to the action on V_0.

For the subspace of cusp forms one has

$$\dim[\Gamma,\varrho,r]_0 = \dim[\Gamma,\varrho,r] - \dim V_0^{\varrho(T)} + \begin{cases} 0 & \text{if } r > 2, \\ \dim V_0^{\varrho} & \text{if } r = 2. \end{cases}$$

The operator $e^{\pi\mathrm{i}r/2}\varrho(S)$ (considered on V_0) has order 2 and $\big(e^{\pi\mathrm{i}r/3}\varrho(ST)\big)^{-1}$ has order 3. Their α-invariants can be computed very easily be means of the following lemma.

6.2 Lemma. *Let A be a $d \times d$-matrix. We have*

$$\alpha(A) = \begin{cases} \frac{d}{4} - \dfrac{\mathrm{tr}(A)}{4} & \text{if } A^2 = E \\[2ex] \frac{d}{3} - \frac{1}{3}\,\mathrm{Re}(\mathrm{tr}(A^{-1})) + \dfrac{1}{3\sqrt{3}}\,\mathrm{Im}(\mathrm{tr}(A^{-1})) & \text{if } A^3 = E. \end{cases}$$

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