Multiplier systems for Siegel modular groups

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Abstract

Deligne proved in [De] (s. also [Hi], 7.1) that the weights of Siegel modular forms on any congruence subgroup of the Siegel modular group of genus $g > 1$ must be integral or half integral. Actually he proved that for a system $v(M)$ of complex numbers of absolute value 1

$$v(M)\det(CZ+D)^r \qquad (r \in \mathbb{R})$$

can be an automorphy factor only if $2r$ is integral. We give a different proof for this. It uses Mennicke's result that subgroups of finite index of the Siegel modular group are congruence subgroups and some techniques from the paper [BMS] of Bass-Milnor-Serre.

Introduction

We fix a natural number g (which later will be 2). We denote by $E = E^{(g)}$ the $g \times g$-unit matrix and by

$$I = I^{(g)} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

the standard alternating matrix. The symplectic group $\mathrm{Sp}(g, \mathbb{R})$ consists of all $M \in \mathrm{GL}(2g, \mathbb{R})$ with the property $M'IM = I$. Here M' denotes the transposed matrix of M. We consider the usual action $MZ = (AZ+B)(CZ+D)^{-1}$ of the real symplectic group $\mathrm{Sp}(g, \mathbb{R})$ on the Siegel upper half plane. The function

$$J(M, Z) = \det(CZ + D)$$

has no zeros on the half plane. Since the half plane is convex, there exists a continuous choice $L(M, Z) = \arg J(M, Z)$ of the argument. We normalize it such that it is the principal value for $Z = \mathrm{i}E$ where E denotes the unit matrix. Recall that the principal value $\mathrm{Arg}(a)$ is defined such that it is in the interval $(-\pi, \pi]$. So we have

$$L(M, \mathrm{i}E) = \mathrm{Arg}(J(M, \mathrm{i}E)) \in (-\pi, \pi].$$

We consider

$$w(M,N) := \frac{1}{2\pi}\big((L(MN,Z) - L(M,NZ) - L(N,Z)\big).$$

Obviously,

$$e^{2\pi \mathrm{i} w(M,N)} = 1.$$

Hence $w(M,N)$ is a constant (independent of Z),

$$w(M,N) \in \mathbb{Z}.$$

Remark. *The function $w : \mathrm{Sp}(n,\mathbb{R}) \times \mathrm{Sp}(n,\mathbb{R}) \to \mathbb{Z}$ is a cocycle in the following sense:*

$$\begin{aligned} w(M_1M_2,M_3) + w(M_1,M_2) &= w(M_1,M_2M_3) + w(M_2,M_3), \\ w(E,M) = w(M,E) &= 0. \end{aligned}$$

The computation of $w(M,N)$ in genus 1 is easy for the following reason. From the definition we have

$$2\pi w(M,N) = \mathrm{Arg}((c\alpha + d\gamma)\mathrm{i} + c\beta + d\gamma) - \arg(cN(\mathrm{i}) + d) - \mathrm{Arg}(\gamma\mathrm{i} + \delta)$$

for

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\arg(cN(\mathrm{i}) + d)$ is obtained from the principal value of $\arg(c\mathrm{i} + d)$ through continuous continuation. But $cz + d$ for z in the upper half plane never crosses the real axis. Hence the result of the continuation is the principal value too. So all three arguments in the definition of $w(M,N)$ are the principal values (in genus 1). This makes it easy to compute w. We rely on tables for the values of w which have been derived by Petersson and reproduced by Maass [Ma1], Theorem 16.

0.1 Lemma. *Let $M = \binom{*\ \ *}{m_1\, m_2}$, $S = \binom{a\, b}{c\, d}$ be two real matrices with determinant 1 and (m_1', m_2') the second row of the matrix MS. Then*

$$4w(M,S) = \begin{cases} \mathrm{sgn}\, c + \mathrm{sgn}\, m_1 - \mathrm{sgn}\, m_1' - \mathrm{sgn}(m_1 c m_1') & \text{if } m_1 c m_1' \neq 0, \\ -(1 - \mathrm{sgn}\, c)(1 - \mathrm{sgn}\, m_1) & \text{if } c m_1 \neq 0, m_1' = 0, \\ (1 + \mathrm{sgn}\, c)(1 - \mathrm{sgn}\, m_2) & \text{if } c m_1' \neq 0, m_1 = 0, \\ (1 - \mathrm{sgn}\, a)(1 + \mathrm{sgn}\, m_1) & \text{if } m_1 m_1' \neq 0, c = 0, \\ (1 - \mathrm{sgn}\, a)(1 - \mathrm{sgn}\, m_2) & \text{if } c = m_1 = m_1' = 0. \end{cases}$$

Corollary. *Assume that $m_1 c m_1' \neq 0$ and that $m_1 m_1' > 0$ or $m_1 c < 0$. Then* $w(M,S) = 0$.

We give an example.

0.2 Lemma. *We have*

$$w\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = w\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 0.$$

We denote by $\Gamma_g[q]$ the principal congruence subgroup level q. This is the kernel of the natural homomorphism $\mathrm{Sp}(g,\mathbb{Z}) \to \mathrm{Sp}(g,\mathbb{Z}/q\mathbb{Z})$.

1. Some special values of the cocycle

We give some examples for values of w in genus $g > 1$.

1.1 Lemma. *One has*

$$w\left(\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, M\right) = 0.$$

The proof is trivial and can be omitted. □

1.2 Lemma. *Let $g = 2$ and*

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have

$$w(P, M) = w(M, P) = \begin{cases} 0 & \text{if } \operatorname{Im}\det(\mathrm{i}C + D) < 0, \\ -1 & \text{if } \operatorname{Im}\det(\mathrm{i}C + D) > 0. \end{cases}$$

Proof. Let $z := \det(C\mathrm{i} + D)$. One computes

$$2\pi w(P, M) = 2\pi w(M, P) = \operatorname{Arg}(-z) - \operatorname{Arg}(z) - \operatorname{Arg}(-1). \qquad \square$$

1.3 Definition. *The **Siegel parabolic group** consists of all symplectic matrices of the form*

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

*The two **Klingen parabolic groups** in the case $g = 2$ consist of all symplectic matrices of the form*

$$\begin{pmatrix} a_1 & 0 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \quad \textit{resp.} \quad \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ 0 & a_4 & b_3 & b_4 \\ 0 & 0 & d_1 & 0 \\ 0 & c_4 & d_3 & d_4 \end{pmatrix}.$$

There is a character on the Siegel parabolic group

$$\varepsilon\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(D).$$

For an element M of the Siegel parabolic group, the expression $\det(CZ+D) = \det(D)$ is independent of Z. Hence

$$L(M,Z) = 0 \quad \text{if } \varepsilon(M) > 0.$$

An immediate consequence is the following lemma.

1.4 Lemma. *For two elements M, N of the Siegel parabolic group we have $w(M,N) = 0$ if $\varepsilon(M) > 0$.*

1.5 Lemma. *Let $g = 2$ and let M be a Klingen parabolic matrix and N a Siegel parabolic matrix with $\varepsilon(N) > 0$. Then $w(M,N) = 0$.*

Proof. Since $L(N, \mathrm{i}E) = 1$, we have to show that the arguments of of $J(MN, \mathrm{i}E)$ and of $L(M, N(\mathrm{i}E))$ are the same. Both determinants are equal. But the argument of the first is the principal part and that of the second is defined by continuation from the argument of $J(M, \mathrm{i}E)$. Hence it is sufficient to show that the principal part of the argument of $L(M,Z)$ is continuous. This is the case if $\operatorname{Im} J(M,Z)$ is always ≥ 0 or always < 0. Actually, for the first Klingen parabolic group

$$\operatorname{Im} J(M,Z) = c_1 d_4 \operatorname{Im} z_0 \qquad \text{where} \quad Z = \begin{pmatrix} z_0 & * \\ * & * \end{pmatrix}.$$

The argument for the second Klingen parabolic group is the same. This proves the lemma. □

1.6 Lemma. *Let $g = 2$ and let*

$$M = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}.$$

Then

$$w(I,M) = \begin{cases} 0 & \textit{if } \operatorname{tr}(S) \geq 0, \\ -1 & \textit{else.} \end{cases}$$

Proof. From the definition we have

$$2\pi w(I,M) = \operatorname{Arg}\det(\mathrm{i}E + S) - \operatorname{Arg}\det(E) - \arg\det(\mathrm{i}E + S).$$

The third argument is defined through continuation along $\det(\mathrm{i}E + tS)$, beginning from $t = 0$ to $t = 1$. For $t = 0$ we have to take the principal value which is π. The imaginary part of $\det(\mathrm{i}E + tS)$ equals $t\operatorname{tr}(S)$. In the case $\operatorname{tr}(S) \geq 0$ we keep the principal value. But if it is negative we make a jump by -2π. □

1.7 Lemma. *Let $g = 2$ and let*

$$M = \begin{pmatrix} E & 0 \\ S & E \end{pmatrix}.$$

Then

$$w(M, I) = \begin{cases} -1 & \text{if } \operatorname{tr}(S) \geq 0, \\ 0 & \text{else.} \end{cases}$$

Proof. Let $z = \det(\mathrm{i}S + E)$. One computes $2\pi w(M, I) = \operatorname{Arg}(-z) - \pi - \operatorname{Arg}(z)$. This depends on the imaginary part of z which is $\operatorname{tr}(S)$. □

2. Multipliers

2.1 Definition. *Let $\Gamma \subset \operatorname{Sp}(g, \mathbb{R})$ be an arbitrary subgroup and let r be a real number. A system $v(M)$, $M \in \Gamma$, of complex numbers of absolute value 1 is called a multiplier system of weight r if*

$$v(MN) \equiv v(M)v(N)\sigma(M, N)$$

where

$$\sigma(M, N) = \sigma_r(M, N) := e^{2\pi \mathrm{i} r w(M,N)}.$$

The elliptic modular group $\operatorname{Sp}(1, \mathbb{Z}) = \operatorname{SL}(2, \mathbb{Z})$ admits multipliers for every real r. One can construct them by means of the discriminant function Δ. This is a modular form without zeros. Hence we can choose a holomorphic power $f(z) = \Delta(z)^{r/12}$. This can be used to construct a multiplier.

Maass [Ma2] proved that the full Siegel modular group of genus $g > 1$ admits only multipliers for integral r and their values can be only ± 1. As a consequence (s. [Ch]), for every multiplier system on a subgroup Γ of finite index of the modular group the weight r is rational and the values of v are contained in a finite subgroup of S^1.

Let $\Gamma_{g,\vartheta}$ be the theta group of degree g. It consists of all integral symplectic matrices such that AB' and CD' have even diagonal. The function

$$\vartheta(Z) = \sum_{n \in \mathbb{Z}^g} e^{2\pi \mathrm{i} n' Z n}$$

is a modular form of weight 1/2 on the theta group. It can be used to construct a multiplier system of weight 1/2.

The result of Deligne states:

2.2 Theorem. *Let $g > 1$ and let $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$ be any subgroup of finite index of the Siegel modular group. Multiplier systems of weight r can only exist if $2r$ is integral.*

It is sufficient to prove this in the case $g = 2$. So we assume from now on $g = 2$.

We assume that a natural number q' is given and that v is a multiplier system of weight r on $\Gamma_2[q']$.

For any $L \in \mathrm{Sp}(2, \mathbb{Z})$ we can consider a conjugate multiplier system [FB] that is defined by

$$\tilde{v}(M) = v(LML^{-1}) \frac{\sigma(LML^{-1}, L)}{\sigma(L, M)}.$$

It is easy to check that this is a multiplier system. The quotient of two multiplier systems of the same weight is a homomorphism, as we know into a finite group. Since every subgroup of finite index of the Siegel modular group is a congruence subgroup, we obtain $\tilde{v}(M) = v(M)$ on some subgroup $\Gamma_2[q] \subset \Gamma_2[q']$ (where q may depend on L).

2.3 Lemma. *For given L in the full modular group there exists a multiple q of q' such that such that*

$$v(M) = v(LML^{-1}) \frac{\sigma(LML^{-1}, L)}{\sigma(L, M)}$$

for each $M \in \Gamma_2[q]$.

This will be used for several matrices, in particular for $M = I$.

2.4 Proposition. *There exists a multiple q of q' such that the following holds. Let U be an element from the subgroup that is generated by the matrices $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$. Let M be a matrix from $\Gamma_2[q]$ of the form*

$$M = \begin{pmatrix} U' & * \\ 0 & U^{-1} \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} E & 0 \\ * & E \end{pmatrix}.$$

Then $v(M) = 1$.

Proof. The matrices of the first type build a finitely generated group. The number of generators is independent on q. It is enough to prove $v(M) = 1$ for the generators, since $w(M, N) = 0$ for all M, N in this group. We also have $v(M)^n = v(M^n)$. Since the values of v are contained in a finite group, we find an n such that $v(M^n) = 1$ for all of the generators.

The second case is more difficult. Due to Lemma 2.3 it is sufficient to prove $\sigma(IMI^{-1}, I) = \sigma(I, M)$ for translation matrices M. This follows from the Lemmas 1.6 and 1.7. □

3. Embedded subgroups

We have to consider three embeddings of $\mathrm{SL}(2,\mathbb{Z})$ into $\mathrm{Sp}(2,\mathbb{Z})$, namely

$$\iota_1, \iota_2, \iota_3 : \mathrm{SL}(2) \longrightarrow \mathrm{Sp}(2),$$

$$\iota_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \iota_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix},$$

$$\iota_3(M) = \begin{pmatrix} M & 0 \\ 0 & M'^{-1} \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{pmatrix}.$$

We have $w(\iota_3(M), \iota_3(N)) = 0$. Hence $M \mapsto v(\iota_3(M))$ is a homomorphism into a finite group. Its kernel is a subgroup of finite index in $\mathrm{SL}(2,\mathbb{Z})$. We will show that it is in fact a congruence subgroup.

Let P as in Lemma 1.2. We have

$$P\iota_1(M)P^{-1} = \iota_2(M).$$

From Lemma 1.2 follows $w(\iota_2(M), P) = w(P, \iota_1(M))$. Hence we obtain from Lemma 2.3 the following result.

3.1 Lemma. *We have*

$$v(\iota_1(M)) = v(\iota_2(M))$$

for $M \in \Gamma_1[q]$.

For sake of simplicity we write

$$v(M) = v(\iota_1(M)) = v(\iota_2(M)).$$

This is a multiplier system in genus 1. We have

$$w(M, N) = w(\iota_\nu(M), \iota_\nu(N)), \quad \text{for} \quad \nu = 1, 2.$$

3.2 Lemma. *The value $v(M)$, $M \in \Gamma_1[q]$, depends only on the second row of M.*

Proof. When M, N have the same second row, then $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)M = N$. We know $w\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), M\big) = 0$ and $v\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = 1$ (Proposition 2.4). □

3.3 Lemma. *Assume that v is a multiplier system of weight r on $\Gamma_2[q']$. There exists a multiple q of q' such that for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1[q]$ we have*

$$v\begin{pmatrix} d_1 & -c_1 & 0 & 0 \\ -b_1 & a & 0 & 0 \\ 0 & 0 & a & b_1 \\ 0 & 0 & c_1 & d_1 \end{pmatrix} \cdot v\begin{pmatrix} a & 0 & b_2 & 0 \\ 0 & 1 & 0 & 0 \\ c_2 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = v\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b_1^2 b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_1^2 c_2 & 0 & y \end{pmatrix}$$

where

$$y = d_1 - b_1 c_1 d_2 + c_1 c_2 b_1 b_2 d_1.$$

Proof. The proof depends on a certain relation which occurs in [BMS] during the proof of Lemma 13.3. We reproduce it here. We set

$$H_1 = \begin{pmatrix} d_1 & -c_1 & 0 & 0 \\ -b_1 & a & 0 & 0 \\ 0 & 0 & a & b_1 \\ 0 & 0 & c_1 & d_1 \end{pmatrix},\ H_2 = \begin{pmatrix} a & 0 & b_2 & 0 \\ 0 & 1 & 0 & 0 \\ c_2 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b_1^2 b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_1^2 c_2 & 0 & y \end{pmatrix}.$$

We consider the matrices

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -b_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ ac_2 & c_1 c_2 & 1 & 0 \\ c_1 c_2 & 0 & 0 & 1 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 1 & c_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c_1 & 1 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 1 & 0 & -ad_1^2 b_2 & b_1 b_2 d_1 \\ 0 & 1 & b_1 b_2 d_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now a direct computation shows

$$R_2 H_3 = H_1 H_2 R_1 R_3 R_4.$$

We have to compute w-values. We assume that $c_1 c_2 \neq 0$. First we treat $w(R_2, H_3)$. We have

$$R_2 H_3 = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ ac_2 & ac_1 c_2 & 1 & b_1^2 b_2 c_1 c_2 \\ c_1 c_2 & c_1^2 c_2 & 0 & y \end{pmatrix}.$$

We are going to compute $w(R_2, H_3)$. A direct computation gives

$$\operatorname{Im} J(R_2H_3, \mathrm{i}E) = c_2(1 + c_1^2).$$

Next we treat $J(R_2, H_3(\mathrm{i}E))$. Here the argument has to be defined by continuation from the principal value of the argument of $J(R_2, \mathrm{i}E)$. We can do this along the straight line from $\mathrm{i}E$ to $H_3(\mathrm{i}E))$. The points on this line are of the form $\left(\begin{smallmatrix}\mathrm{i} & 0\\ 0 & \tau\end{smallmatrix}\right)$ where τ is in the upper half plane. One computes

$$J\Big(R_2, \begin{pmatrix}\mathrm{i} & 0\\ 0 & \tau\end{pmatrix}\Big) = \det\begin{pmatrix}1 + ac_2\mathrm{i} & c_1c_2\tau\\ c_1c_2\mathrm{i} & 1\end{pmatrix}.$$

The real part is $1 + c_1^2c_2^2 \operatorname{Im}\tau$ which is positive. Hence the principal value of the argument is continuous along the line. So we see

$$L(R_2, H_3(\mathrm{i}E)) \in (-\pi, \pi].$$

Finally we compute

$$\operatorname{Im} J(H_3, \mathrm{i}E) = c_1^2c_2.$$

Now we see that the imaginary part of $J(R_2H_3, \mathrm{i}E)$ and $J(H_3, \mathrm{i}E)$ have the same sign (namely the sign of c_2). Hence their arguments are both contained in $(0, \pi)$ or in $(-\pi, 0)$. This means that $2\pi w(R_2, H_3)$ is contained in $(0, \pi) - (-\pi, \pi] - (0, \pi)$ or in $(-\pi, 0) - (-\pi, \pi] - (-\pi, 0)$. This is $(-2\pi, 2\pi)$ in both cases. We obtain

$$w(R_2, H_3) = 0.$$

The case $c_1c_2 = 0$ is easy and can be omitted.

From Lemma 1.5 we can take $w(H_2, R_1R_3R_4) = 0$. For trivial reason one has $w(H_1, H_2R_1R_3R_4) = 0$. Now we evaluate

$$v(R_2H_3) = v(H_1H_2R_1R_3R_4).$$

The left hand side is

$$v(R_2)v(H_3)\sigma(R_2, H_3) = v(R_2)v(H_3).$$

But $v(R_2) = 1$ (Proposition 2.4). Hence the left hand side is just $v(H_3)$. The right hand side is

$$v(H_1)v(H_2R_1R_3R_4)\sigma(H_1, H_2R_1R_3R_4) = v(H_1)v(H_2R_1R_3R_4).$$

Similarly we see

$$v(H_2R_1R_2R_3) = v(H_2)v(R_1R_2R_3)\sigma(H_2, R_1R_3R_4) = v(H_2)v(R_1R_3R_4).$$

From Proposition 2.4 we know $v(R_1R_3R_4) = 1$. Hence we get $v(H_3) = v(H_1)v(H_2)$. □

4. Mennicke symbol

We have seen that $v(\iota_1(M)) = v(\iota_2(M))$ depends only on the second row of $M \in \Gamma_1[q]$. Hence we can define

$$\left\{ \begin{matrix} c \\ d \end{matrix} \right\} = v \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = v \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}^{-1}.$$

We also can define

$$\begin{bmatrix} b \\ a \end{bmatrix} = v \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{pmatrix}.$$

It is clear that this does not depend on the choice of c, d.

4.1 Proposition. *For a suitable multiple $q > 2$ of q' the bracket $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$ is a Mennicke symbol. This means that it is a function on the set of all coprime (a, b) with the property $a \equiv 1 \bmod q$ and $b \equiv 0 \bmod q$ such that the following properties hold.*

MS1 *It is invariant under the transformations $(a, b) \mapsto (a + xb, b)$ and $(a, b) \mapsto (a, b + qay)$ for integral x, y.*

MS2 *It satisfies the rule*

$$\begin{bmatrix} b_1 b_2 \\ a \end{bmatrix} = \begin{bmatrix} b_1 \\ a \end{bmatrix} \begin{bmatrix} b_2 \\ a \end{bmatrix}.$$

Proof of MS1. We notice that w is trivial on the image of ι_3. Hence v is a character on this group. The invariance under $(a, b) \mapsto (a, b+qay)$ follows from the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & qy \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b + qay \\ * & * \end{pmatrix}.$$

To prove the invariance under $(a, b) \mapsto (a + xb, b)$, we consider

$$\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} a + xb & b \\ * & * \end{pmatrix}.$$

Due to Lemma 2.3 we can assume that $v(\iota_3(M))$ is invariant under conjugation with $\iota_3\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. This proves MS1.

Proof of MS2. The proof of MS2 needs two Lemmas which we now have to formulate and prove now. We make use of

$$v(\iota_\nu(M^{-1})) = v(\iota_\nu(M))^{-1}.$$

This is true since in genus 1 one has $w(M, M^{-1}) = 0$. (This is a general rule for $c \neq 0$ but also for $c = 0$ and $a > 0$. But in our case $c = 0$ implies $a = 1$ since we assume $q > 2$.) This relation implies

$$\begin{Bmatrix} c \\ d \end{Bmatrix} = \begin{Bmatrix} -c \\ a \end{Bmatrix}^{-1}.$$

From Lemma 3.3 we get after the replacement, $c_2 \mapsto -c_2$ the following general rule (compare Lemma 13.3 in [BMS]).

4.2 Lemma. *Let $a - 1 \equiv c_1 \equiv c_2 \equiv 0$ mod q and let a, c_1 and a, c_2 be coprime. Then*

$$\begin{bmatrix} c_1 \\ a \end{bmatrix} \begin{Bmatrix} c_2 \\ a \end{Bmatrix} = \begin{Bmatrix} c_1^2 c_2 \\ a \end{Bmatrix}.$$

We need also the following simple lemma.

4.3 Lemma. *We have*

$$\begin{Bmatrix} 1 - a \\ a \end{Bmatrix} = 1$$

for $a \equiv 1 \bmod q$.

Proof. We use

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 - a & a - 1 \\ 1 - a & a \end{pmatrix}. \qquad \square$$

We insert un Lemma 4.2 now $c_2 = 1 - a$ to obtain the following formula.

$$\begin{bmatrix} c \\ a \end{bmatrix} = \begin{Bmatrix} c^2(1 - a) \\ a \end{Bmatrix}.$$

Before we continue, we mention that $\{\}$ is not a Mennicke symbol. It does not satisfy MS1.

4.4 Lemma. *We have*

$$\begin{Bmatrix} c \\ d \end{Bmatrix} = \begin{Bmatrix} c \\ d + yc \end{Bmatrix}$$

and

$$\begin{Bmatrix} c + xqd \\ d \end{Bmatrix} = \begin{Bmatrix} c \\ d \end{Bmatrix} e^{2\pi i r s}.$$

where

$$s = w\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ qx & 1 \end{pmatrix} \right).$$

Proof. The first relation can be derived from

$$\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c & d+cy \end{pmatrix}.$$

To derive the second one we consider the relation

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ qx & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c+dxq & d \end{pmatrix}.$$

It shows

$$\left\{ \begin{matrix} c+dxq \\ d \end{matrix} \right\} = \left\{ \begin{matrix} c \\ d \end{matrix} \right\} e^{2\pi i rs}.$$

The w-value s is usually not zero. □

Proof of Proposition 4.1 (MS2) continued. Now we use

$$\begin{pmatrix} * & * \\ c^2 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c^2 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c^2-ac^2 & a \end{pmatrix}.$$

From the corollary of the table of Maass in the introduction we get

$$w\left(\begin{pmatrix} * & * \\ c^2 & a \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -c^2 & 1 \end{pmatrix} \right) = 0$$

and hence

$$\left\{ \begin{matrix} c^2(1-a) \\ a \end{matrix} \right\} = \left\{ \begin{matrix} c^2 \\ a \end{matrix} \right\}.$$

So we obtain

$$\begin{bmatrix} c \\ a \end{bmatrix} = \left\{ \begin{matrix} c^2 \\ a \end{matrix} \right\}$$

and moreover

$$\begin{bmatrix} c_1 c_2 \\ a \end{bmatrix} = \left\{ \begin{matrix} c_1^2 c_2^2 \\ a \end{matrix} \right\} = \begin{bmatrix} c_1 \\ a \end{bmatrix} \left\{ \begin{matrix} c_2^2 \\ a \end{matrix} \right\} = \begin{bmatrix} c_1 \\ a \end{bmatrix} \begin{bmatrix} c_2 \\ a \end{bmatrix}.$$

This finishes the proof of Proposition 4.1. □

The main result about Mennicke symbols is that they are trivial [BMS], Theorem 3.6. Hence we obtain now the important result.

4.5 Proposition. *The multiplier system v is identically one on all*

$$\begin{pmatrix} M & 0 \\ 0 & M'^{-1} \end{pmatrix} \quad \textit{for } M \in \Gamma_1[q].$$

From Lemma 4.2 follows now

$$\left\{ {c^2 \atop d} \right\} = 1 \quad \text{and} \quad \left\{ {c_1 \atop d} \right\} = \left\{ {c_1 c_2^2 \atop d} \right\}$$

for $c \equiv c_1 \equiv c_2 \equiv 0 \bmod q$ and $d \equiv 1 \bmod q$. This can be generalized. We have to consider the Kronecker symbol $\left(\frac{c}{d}\right)$. For its definition and properties we refer to [Di]. We will need it only for $c \neq 0$ and for odd d. We collect some properties (always assuming this condition)

$$\left(\frac{c_1 c_2}{d}\right) = \left(\frac{c_1}{d}\right)\left(\frac{c_2}{d}\right), \quad \left(\frac{c}{d_1 d_2}\right) = \left(\frac{c}{d_1}\right)\left(\frac{c}{d_2}\right).$$

Assume $d > 0$ or $c_1 c_2 > 0$. Then

$$\left(\frac{c_1}{d}\right) = \left(\frac{c_2}{d}\right) \quad \text{if} \quad c_1 \equiv c_2 \bmod d.$$

Also the relation

$$\left(\frac{c}{d_1}\right) = \left(\frac{c}{d_2}\right) \quad \text{if} \quad \begin{cases} d_1 \equiv d_2 \bmod c \text{ and } c \equiv 0 \bmod 4, \\ d_1 \equiv d_2 \bmod 4c \text{ and } c \equiv 2 \bmod 4 \end{cases}$$

is valid. Finally we mention

$$\left(\frac{c}{-1}\right) = \begin{cases} 1 & \text{for } c > 0, \\ -1 & \text{for } c < 0. \end{cases}$$

Since one of the rules demands $c \equiv 0 \bmod 4$, we will from now on assume that $q \equiv 0 \bmod 4$.

4.6 Proposition. *Let q be a suitable multiple of q' and let*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1[q], \quad \left(\frac{c}{d}\right) = 1.$$

Then $v(M) = 1$.

Proof. We use the invariance under $(c, d) \mapsto (c, d+xc)$. We can apply Dirichlet's prime number theorem and therefore assume that $d = p$ is a (positive) prime. But then the Kronecker symbol is the usual Legendre symbol. Since $d \equiv 1 \bmod q$ we have $\left(\frac{q}{d}\right) = 1$. This implies $\left(\frac{c/q}{d}\right) = 1$. Since d is a prime, we get a solution of $c/q = x^2 + dy$ or $c = qx^2 + dqy$. Now use

$$\begin{pmatrix} * & * \\ qx^2 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ qy & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

In the case $c > 0$ the w-value is zero. This follows from the corollary in the table of Maass in the introduction. In the case $c < 0$ we must have $y < 0$ and again from this corollary follows that the w-value is zero. (In the notation of the table the sign distribution of (m_1, c, m_1') is $(+,*,+)$ or $(+,-,*)$.) Now we get

$$v(M) = v\begin{pmatrix} * & * \\ c & d \end{pmatrix} = v\begin{pmatrix} * & * \\ qx^2 & d \end{pmatrix} = \begin{Bmatrix} qx^2 \\ d \end{Bmatrix}.$$

Lemma 4.2 now shows

$$\begin{Bmatrix} x^2q \\ d \end{Bmatrix} = \begin{Bmatrix} x^2q^3 \\ d \end{Bmatrix} = \begin{Bmatrix} q(qx)^2 \\ d \end{Bmatrix} = \begin{Bmatrix} q \\ d \end{Bmatrix} = \begin{Bmatrix} q \\ 1 \end{Bmatrix} = 1. \qquad \square$$

4.7 Lemma. *Assume that the matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is contained in $\Gamma_1[q]$ and has the following properties. All entries are positive and $dq < c(q-1)$. Then*

$$v(M) = e^{-2\pi \mathrm{i} r} \quad \textit{if} \quad \left(\frac{c}{d}\right) = -1.$$

Proof. We consider

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1-q & -q \\ q & 1+q \end{pmatrix} = \begin{pmatrix} * & * \\ c-qc+dq & -cq+d+dq \end{pmatrix}.$$

Clearly $\left(\frac{q}{1+q}\right) = 1$. We also claim

$$\left(\frac{c-qc+dq}{-cq+d+dq}\right) = 1.$$

To prove this, we observe

$$\left(\frac{c-qc+dq}{-cq+d+dq}\right) = \left(\frac{c-qc+dq}{d-c}\right) = \left(\frac{c-qc+dq}{-1}\right)\left(\frac{c-qc+dq}{c-d}\right).$$

Now we use $c - qc + dq < 0$. It follows $c - d > 0$. Hence we get

$$= -\left(\frac{c}{c-d}\right) = -\left(\frac{c}{-d}\right) = -\left(\frac{c}{d}\right) = -(-1) = 1.$$

Now we have proved

$$v\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1-q & -q \\ q & 1+q \end{pmatrix}\right) = 1.$$

The left hand side equals

$$v\begin{pmatrix} a & b \\ c & d \end{pmatrix} \exp\left\{2\pi \mathrm{i} r w\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 1-q & -q \\ q & 1+q \end{pmatrix}\right)\right\} = 1.$$

From Maass' table in the introduction follows that the w-value is 1. (The sign distribution of (m_1, c, m_1') is $(+,+,-)$.) This proves Lemma 4.7. $\square$

There exist two coprime natural numbers c, d such that $c \equiv 0 \bmod q$ and $d \equiv 1 \bmod q$ and such that $\left(\frac{c}{d}\right) = -1$. We also can assume $dq < c(q-1)$. The pair (c, d) is the second row of a matrix $M = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) \in \Gamma_1[q]$. We want to compute $v(M)$. Since we can add a multiple of the second row to the first one, we can assume that a and b are also positive. From Lemma 4.7 we know $v(M) = e^{-2\pi i r}$. Now we consider

$$v(M^2) = v(M)^2 e^{2\pi i r w(M,M)}.$$

Since all entries from M are positive, we have $w(M, M) = 0$. So we get

$$v(M^2) = e^{-4\pi i r}.$$

We compute $\left(\frac{\gamma}{\delta}\right)$ for the matrix

$$N = M^2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

We get

$$\left(\frac{\gamma}{\delta}\right) = \left(\frac{c(a+d)}{cb+d^2}\right) = \left(\frac{c}{cb+d^2}\right)\left(\frac{a+d}{cb+d^2}\right).$$

We have

$$\left(\frac{c}{cb+d^2}\right) = \left(\frac{c}{d^2}\right) = 1$$

and

$$\left(\frac{a+d}{cb+d^2}\right) = \left(\frac{a+d}{d(a+d)-1}\right).$$

Since $a + d \equiv 2 \bmod 4$ we only can change the denominator mod $4(a+d)$. Since $d \equiv 1$ mod 4 we see

$$\left(\frac{a+d}{d(a+d)-1}\right) = \left(\frac{a+d}{a+d-1}\right) = \left(\frac{1}{a+d-1}\right) = 1.$$

This shows $v(N) = 1$ and we get the relation

$$e^{-4\pi i r} = 1$$

which implies that $2r$ is integral. This finishes the proof of the main result.

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