Multiplier systems for Hermitian modular groups

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Abstract

The Hermitian modular groups of degree $n > 1$ and their subgroups of finite index admit multiplier systems of weight $r \in \mathbb{R}$ only if $2r \in \mathbb{Z}$. This follows from the corresponding result of Deligne [De, Hi] for the Siegel modular group, since the Siegel modular group can be embedded into the Hermitian modular groups. In this paper we will prove that in the Hermitian case $n > 2$ only multiplier systems of integral weight exist. The case $n = 2$ is exceptional. Haowu Wang [Wa] gave a remarkable example of a modular form of half integral weight on a certain congruence group Γ in a Hermitian modular group of degree 2. Actually he constructs a Borcherds product of weight $23/2$ for a group of type $\mathrm{O}(2,4)$. This group is isogenous to the group $\mathrm{U}(2,2)$ that contains the Hermitian modular groups of degree two. In this paper we want to study such multiplier systems. If one restricts them to the unimodular group

$$\mathcal{U} = \left\{ U;\quad \begin{pmatrix} \bar{U}'^{-1} & 0 \\ 0 & U \end{pmatrix} \in \Gamma \right\}$$

one obtains a usual character. Our main result states that the kernel of this character is a non-congruence subgroup. Our method works also in the case of the Siegel modular group. Here we obtain a new proof of the mentioned result of Deligne.

Introduction

We fix a natural number g (which later will be 2). We denote by $E = E^{(g)}$ the $g \times g$-unit matrix and by

$$I = I^{(g)} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

the standard alternating matrix. The unitary group $\mathrm{U}(g,g)$ consists of all $M \in \mathrm{GL}(2g,\mathbb{C})$ with the property $\bar{M}'IM = I$. The special unitary group $\mathrm{SU}(g,g)$ is the subgroup of elements with determinant one. One has $\mathrm{SU}(1,1) = \mathrm{SL}(2,\mathbb{R})$.

From now on we fix an imaginary quadratic field $F = \mathbb{Q}(\sqrt{d})$ of discriminant

$d < 0$ and denote by

$$\mathfrak{o} = \mathbb{Z} + \mathbb{Z}\omega, \quad \omega = \frac{\delta + \sqrt{\delta}}{2},$$

its ring of integers. The Hermitian modular group $\Gamma_{F,g}$ is the subgroup of $\mathrm{U}(g,g)$ of matrices with entries in $\mathfrak{o}$. Let $\mathfrak{q} \subset \mathfrak{o}$ be a non zero ideal. Then

$$\Gamma_{F,g}[\mathfrak{q}] = \mathrm{kernel}(\Gamma_{F,g} \longrightarrow \mathrm{GL}(2g, \mathfrak{o}/\mathfrak{q}))$$

is the (principal) congruence subgroup of level $\mathfrak{q}$. Since the field F is fixed, we can omit the label F and write

$$\Gamma_g := \Gamma_{F,g} \quad \text{and} \quad \Gamma_g[\mathfrak{q}] = \Gamma_{F,g}[\mathfrak{q}].$$

For sufficiently small $\mathfrak{q}$ the group $\Gamma_g[\mathfrak{q}]$ is contained in $\mathrm{SL}(2g, \mathfrak{o})$. Then the group $\Gamma_1[\mathfrak{q}]$ is the usual principal congruence subgroup of the elliptic modular group $\mathrm{SL}(2, \mathbb{Z})$ of level $\mathfrak{q} \cap \mathbb{Z}$.

1. Multiplier systems

We consider the usual action $MZ = (AZ+B)(CZ+D)^{-1}$ of the unitary group $\mathrm{U}(g,g)$ on the Hermitian upper half plane

$$\mathcal{H}_g = \{Z \in \mathbb{C}^{n\times n}; \quad Z = X + \mathrm{i}Y, \quad X = \bar{X}', \ Y = \bar{Y}' > 0 \text{ (positive definite)}\}.$$

This an open convex domain in $\mathbb{C}^{n\times n}$. The function

$$J(M, Z) = \det(CZ + D)$$

has no zeros on the half plane. Since the half plane is convex, there exists a continuous choice $L(M, Z) = \arg J(M, Z)$ of the argument. We normalize it such that it is the principal value for $Z = \mathrm{i}E$ where E denotes the unit matrix. Recall that the principal value $\mathrm{Arg}(a)$ is defined such that it is in the interval $(-\pi, \pi]$. So we have

$$L(M, \mathrm{i}E) = \mathrm{Arg}(J(M, \mathrm{i})) \in (-\pi, \pi].$$

We consider

$$w(M, N) := \frac{1}{2\pi}\big((L(MN, Z) - L(M, NZ) - L(N, Z)\big).$$

Obviously,

$$e^{2\pi \mathrm{i} w(M,N)} = 1.$$

Hence $w(M, N)$ is independent of Z and $w(M, N) \in \mathbb{Z}$. Usually we will compute $w(M, N)$ by evaluation at $Z = \mathrm{i}E$. Then $L(MN, \mathrm{i}E)$, $J(N, \mathrm{i}E)$ are given by the principal values, but $L(M, N(\mathrm{i}E))$ is obtained through continuous continuation of the principal value $L(M, \mathrm{i}E)$ along a path from $\mathrm{i}E$ to $N(\mathrm{i}E)$.

1.1 Remark. *The function $w : \mathrm{U}(g,g)\times\mathrm{U}(g,g)\to\mathbb{Z}$ is a cocycle in the following sense:* Coc

$$\begin{aligned}w(M_1M_2,M_3)+w(M_1,M_2)&=w(M_1,M_2M_3)+w(M_2,M_3),\\ w(E,M)=w(M,E)&=0.\end{aligned}$$

The computation of $w(M,N)$ in genus 1 is easy for the following reason. From the definition we have

$$2\pi w(M,N)=\mathrm{Arg}((c\alpha+d\gamma)\mathrm{i}+c\beta+d\gamma)-\arg(cN(\mathrm{i})+d)-\mathrm{Arg}(\gamma\mathrm{i}+\delta)$$

for

$$M=\begin{pmatrix}a&b\\c&d\end{pmatrix},\quad N=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$$

where $\arg(cN(\mathrm{i})+d)$ is obtained from the principal value of $\arg(c\mathrm{i}+d)$ through continuous continuation. But $cz+d$ for z in the upper half plane never crosses the real axis. Hence the result of the continuation is the principal value too. So all three arguments in the definition of $w(M,N)$ are the principal values (in genus 1). This makes it easy to compute w. We rely on tables for the values of w which have been derived by Petersson and reproduced by Maass [Ma], Theorem 16.

1.2 Lemma. *Let $M=\left(\begin{smallmatrix}*&*\\m_1&m_2\end{smallmatrix}\right)$, $S=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ be two real matrices with determinant 1 and (m_1',m_2') the second row of the matrix MS. Then* MP

$$4w(M,S)=\begin{cases}\operatorname{sgn}c+\operatorname{sgn}m_1-\operatorname{sgn}m_1'-\operatorname{sgn}(m_1cm_1') & \text{if } m_1cm_1'\neq0,\\ -(1-\operatorname{sgn}c)(1-\operatorname{sgn}m_1) & \text{if } cm_1\neq0, m_1'=0,\\ (1+\operatorname{sgn}c)(1-\operatorname{sgn}m_2) & \text{if } cm_1'\neq0, m_1=0,\\ (1-\operatorname{sgn}a)(1+\operatorname{sgn}m_1) & \text{if } m_1m_1'\neq0, c=0,\\ (1-\operatorname{sgn}a)(1-\operatorname{sgn}m_2) & \text{if } c=m_1=m_1'=0.\end{cases}$$

Corollary. *Assume that $m_1cm_1'\neq0$ and that $m_1m_1'>0$ or $m_1c<0$. Then $w(M,S)=0$.*

We give an example.

1.3 Lemma. *We have* raTr

$$w\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\begin{pmatrix}1&x\\0&1\end{pmatrix}\right)=w\left(\begin{pmatrix}1&x\\0&1\end{pmatrix},\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)=0.$$

2. Some special values of the cocycle

We give some examples for values of w in genus $g > 1$.

2.1 Lemma. *One has* LTra

$$w\left(\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, M\right) = 0.$$

The proof is trivial and can be omitted. □

2.2 Lemma. *Let* Pval

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have

$$w(P, M) = w(M, P) = \begin{cases} 0 & \text{if } \operatorname{Im}\det(\mathrm{i}C + D) < 0, \\ -1 & \text{if } \operatorname{Im}\det(\mathrm{i}C + D) > 0. \end{cases}$$

Proof. Let $z := \det(\mathrm{i}C + D)$. One computes

$$2\pi w(P, M) = 2\pi w(M, P) = \operatorname{Arg}(-z) - \operatorname{Arg}(z) - \operatorname{Arg}(-1). \qquad \square$$

2.3 Definition. *The **Siegel parabolic group** consists of all elements from* DefSK
$\mathrm{SU}(g, g)$ *of the form*

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

There is a character on the Siegel parabolic group

$$\varepsilon\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(D).$$

For an element M of the Siegel parabolic group, the expression $\det(CZ + D) = \det(D)$ is independent of Z. Hence

$$L(M, Z) = 0 \quad \text{if } \varepsilon(M) = 1.$$

An immediate consequence is the following lemma.

2.4 Lemma. *For two elements P,Q of the Siegel parabolic group we have $w(P,Q)=0$ if $\varepsilon(P)=1$.* ParM

We have to consider two embeddings $\iota_1,\iota_2:\mathrm{SL}(2,\mathbb{R})\to\mathrm{SU}(2,2)$, namely

$$\iota_1\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&0&b&0\\0&1&0&0\\c&0&d&0\\0&0&0&1\end{pmatrix},\quad \iota_2\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&a&0&b\\0&0&1&0\\0&c&0&d\end{pmatrix}.$$

2.5 Lemma. *Let M be in the image of one of the embeddings ι_ν and N a Siegel parabolic matrix with $\varepsilon(N)=1$. Then $w(M,N)=0$.* KSz

Proof. Since $L(N,\mathrm{i}E)=0$, we have to show that the arguments of $J(MN,\mathrm{i}E)$ and of $J(M,N(\mathrm{i}E))$ are the same. Both determinants are equal. But the argument of the first is the principal part and that of the second is defined by continuation from the argument of $L(M,\mathrm{i}E)$. Hence it is sufficient to show that the principal part of the argument of $L(M,Z)$ is continuous. This is the case if $\operatorname{Im}L(M,Z)$ is always ≥ 0 or always ≤ 0. Actually,

$$\operatorname{Im}L(M,Z)=c\operatorname{Im}z_0 \qquad \text{where}\quad Z=\begin{pmatrix}z_0&*\\ *&*\end{pmatrix}.$$

This proves the lemma. □

2.6 Lemma. *Assume $g=2$. Let* ITra

$$M=\begin{pmatrix}E&S\\0&E\end{pmatrix},\quad S=\bar S'.$$

Then

$$w(I,M)=\begin{cases}0 & \text{if } \operatorname{tr}(S)\ge 0,\\ -1 & \text{else.}\end{cases}$$

Proof. From the definition we have

$$2\pi w(I,M)=\operatorname{Arg}\det(\mathrm{i}E+S)-\operatorname{Arg}\det(E)-\arg\det(\mathrm{i}E+S).$$

The third argument is defined through continuation along $\det(\mathrm{i}E+tS)$, beginning from $t=0$ to $t=1$. For $t=0$ we have to take the principal value which is π. The imaginary part of $\det(\mathrm{i}E+tS)$ equals $t\operatorname{tr}(S)$. In the case $\operatorname{tr}(S)\ge 0$ we keep the principal value. But if it is negative we make a jump by -2π. □

2.7 Lemma. *Assume $g=2$. Let* TraI

$$M=\begin{pmatrix}E&0\\S&E\end{pmatrix},\quad S=\bar S'.$$

Then

$$w(M,I)=\begin{cases}-1 & \text{if } \operatorname{tr}(S)\ge 0,\\ 0 & \text{else.}\end{cases}$$

Proof. Let $z=\det(\mathrm{i}S+E)$. One computes $w(M,I)=\operatorname{Arg}(-z)-\pi-\operatorname{Arg}(z)$. This depends on the imaginary part of z which is $\operatorname{tr}(S)$. □

3. Multipliers

3.1 Definition. *Let $\Gamma \subset \mathrm{U}(g,g)$ be an arbitrary subgroup and let r be a real number. A system $v(M)$, $M \in \Gamma$, of complex numbers of absolute value 1 is called a* ***multiplier system*** *of weight r if* TriVw

$$v(MN) \equiv v(M)v(N)\sigma(M,N)$$

where

$$\sigma(M,N) = \sigma_r(M,N) := e^{2\pi \mathrm{i} r w(M,N)}.$$

Let now Γ is a normal subgroup of finite index of Γ_g, $g \geq 2$. Since the congruence subgroup has been solved we know that Γ contains a congruence subgroup $\Gamma_n[\mathfrak{q}]$. It is easy to show that weights r of multiplier systems are rational [Ch]. Hence a sutable power of v is trivial on some congruence subgroup. This shows that there exists a natural number l such the all values of v are lth roots of unity.

For any $L \in \Gamma_g$ we can consider a conjugate multiplier system on Γ that is defined by

$$\tilde{v}(M) = v(LML^{-1})\frac{\sigma(LML^{-1},L))}{\sigma(L,M)}.$$

It is easy to check that this is a multiplier system and that this defines an action of Γ_n on the set of all multiplier systems on Γ. The quotient of two multiplier systems of the same weight is a homomorphism, as we know into a finite group. Since the congruence subgroup problem has been solved for the Hermitian modular group, we obtain $\tilde{v}(M) = v(M)$ on some congruence subgroup. Since the Hermitian modular group is finitely generated, we can find a joint congruence subgroup

3.2 Lemma. *Let v be a multiplier system on a subgroup $\Gamma \subset \Gamma_n$ of finite index. In the case $n \geq 2$ there exists an ideal $\mathfrak{q} \neq 0$ such that $\Gamma[\mathfrak{q}] \subset \Gamma$ and such that* EleO

$$v(M) = v(LML^{-1})\frac{\sigma(LML^{-1},L)}{\sigma(L,M)} \qquad (M \in \Gamma[\mathfrak{q}])$$

for all $L \in \Gamma_g$.

Several times we will replace by $\mathfrak{q}$ by a smaller ideal. We then just say "for suitable $\mathfrak{q}$".

3.3 Proposition. *Let v be a multiplier system on a subgroup $\Gamma \subset \Gamma_2$ of finite index. For suitable $\mathfrak{q}$ the group $\Gamma_2[\mathfrak{q}]$ is contained in Γ and for each matrix M from $\Gamma_2[\mathfrak{q}]$ of the form* ProOe

$$M = \begin{pmatrix} E & 0 \\ * & E \end{pmatrix}.$$

we have $v(M) = 1$.

Proof. Due to Lemma 3.2 it is sufficient to prove $\sigma(IMI^{-1}, I) = \sigma(I, M)$ for translation matrices M. This follows from the Lemmas 2.6 and 2.7. $\square$

3.4 Proposition. *Let v be a multiplier system on a subgroup $\Gamma \subset \Gamma_2$ of finite index. For suitable $\mathfrak{q}$ we have $\Gamma_2[\mathfrak{q}] \subset \Gamma$ and such the following holds. Let U be an element from the subgroup that is generated by the matrices $\left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ q & 1 \end{smallmatrix}\right)$ for $q \in \mathfrak{q}$ and let* ProOz

$$M = \begin{pmatrix} \bar{U}'^{-1} & * \\ 0 & U \end{pmatrix}.$$

Then $v(M) = 1$.

Proof. The matrices of this type generate a finitely generated group. The number of generators is independent of q. It is enough to prove $v(M) = 1$ for the generators, since $w(M, N) = 0$ for all M, N in this group. We also have $v(M)^n = v(M^n)$. Since the values of v are contained in a finite group, we find an n such that $v(M^n) = 1$ for all of the generators.

4. Embedded subgroups

Besides the embeddings ι_1, ι_2 we have to consider the embedding

$$\iota : \mathrm{GL}(2, \mathbb{C}) \longrightarrow \mathrm{U}(2,2), \quad \iota(U) = \begin{pmatrix} \bar{U}'^{-1} & 0 \\ 0 & U \end{pmatrix}.$$

This gives us an embedding $\mathrm{GL}(2, \mathfrak{o}) \hookrightarrow \Gamma_2$. We use the notation

$$\mathrm{SL}(2, \mathfrak{o})[\mathfrak{q}] = \mathrm{kernel}(\mathrm{SL}(2, \mathfrak{o}) \longrightarrow \mathrm{SL}(2, \mathfrak{o}/\mathfrak{q})).$$

We have $w(\iota(U), \iota(V)) = 1$. Hence, for suitable $\mathfrak{q}$

$$\mathrm{SL}(2, \mathfrak{o})[\mathfrak{q}] \longrightarrow S^1, \quad U \longmapsto v(\iota(U)),$$

is a homomorphism. We mentioned that the values of v are lth roots of unity. Hence the kernel is a subgroup of finite index in $\mathrm{SL}(2, \mathfrak{o})$.

Our method depends on some game between the embeddings ι_1, ι_2 and ι. We have

$$P\iota_1(M)P^{-1} = \iota_2(M), \quad P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

From Lemma 2.2 follows $w(\iota_2(M), P) = w(P, \iota_1(M))$. Hence we obtain from Lemma 3.2 the following result.

4.1 Lemma. *Let v be a multiplier system on a subgroup $\Gamma \subset \Gamma_2$ of finite index. For suitable $\mathfrak{q}$ we have $\Gamma_2[\mathfrak{q}] \subset \Gamma$ and* iEiZ

$$v(\iota_1(M)) = v(\iota_2(M))$$

for $M \in \Gamma_1[\mathfrak{q}]$.

For sake of simplicity we write

$$v(M) = v(\iota_1(M)) = v(\iota_2(M)).$$

This is a multiplier system in genus 1. We have

$$w(M, N) = w(\iota_\nu(M), \iota_\nu(N)), \quad \text{for} \quad \nu = 1, 2.$$

4.2 Lemma. *Let v be a multiplier system on a subgroup $\Gamma \subset \Gamma_2$ of finite index. For suitable $\mathfrak{q}$ the value $v(M)$, $M \in \Gamma_1[\mathfrak{q}]$, depends only on the second row of M.* SRd

Proof. When M, N have the same second row, then $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)M = N$. We know $w\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), M\big) = 0$ and $v\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = 1$ (Proposition 3.3). $\square$

4.3 Lemma. *Let v be a multiplier system on a subgroup $\Gamma \subset \Gamma_2$ of finite index. For suitable $\mathfrak{q}$ we have $\Gamma_2[\mathfrak{q}] \subset \Gamma$ and for any* ZweiV

$$M_1 \in \begin{pmatrix} a & b_1 \\ c_1 & d_1 \end{pmatrix} \in \mathrm{SL}(2, \mathfrak{o})[\mathfrak{q}] \quad \textit{and} \quad M_2 = \begin{pmatrix} a & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma_1[\mathfrak{q}]$$

(in particular $a \in \mathbb{Z}$) the relation

$$v\begin{pmatrix} \bar{d}_1 & -\bar{c}_1 & 0 & 0 \\ -\bar{b}_1 & a & 0 & 0 \\ 0 & 0 & a & b_1 \\ 0 & 0 & c_1 & d_1 \end{pmatrix} \cdot v\begin{pmatrix} a & 0 & b_2 & 0 \\ 0 & 1 & 0 & 0 \\ c_2 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = v\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b_1\bar{b}_1 b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_1\bar{c}_1 c_2 & 0 & y \end{pmatrix}$$

holds (where $y = d_2 - b_2 c_2 \bar{d}_1 + \bar{b}_1 b_2 \bar{c}_1 c_2 d_1$).

Proof. The proof depends on a certain relation which occurs in [BMS] during the proof of Lemma 13.3. We reproduce it here. We set

$$H_1 = \begin{pmatrix} \bar d_1 & -\bar c_1 & 0 & 0 \\ -\bar b_1 & a & 0 & 0 \\ 0 & 0 & a & b_1 \\ 0 & 0 & c_1 & d_1 \end{pmatrix},\ H_2 = \begin{pmatrix} a & 0 & b_2 & 0 \\ 0 & 1 & 0 & 0 \\ c_2 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b_1\bar b_1 b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_1\bar c_1 c_2 & 0 & y \end{pmatrix}.$$

We consider the matrices

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \bar b_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -b_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ ac_2 & \bar c_1 c_2 & 1 & 0 \\ c_1c_2 & 0 & 0 & 1 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 1 & \bar c_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c_1 & 1 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 1 & 0 & -ad_1\bar d_1 b_2 & b_1b_2\bar d_1 \\ 0 & 1 & \bar b_1 b_2 d_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now direct computations give

$$R_2H_3 = H_1H_2R_1R_3R_4.$$

We have to compute w-values. We assume that $c_1c_2 \neq 0$. First we treat $w(R_2, H_3)$. We have

$$R_2H_3 = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ ac_2 & a\bar c_1 c_2 & 1 & b_1\bar b_1 b_2 \bar c_1 c_2 \\ c_1c_2 & c_1\bar c_1 c_2 & 0 & y \end{pmatrix}.$$

We are going to compute $w(R_2, H_3)$. A direct computation gives

$$\operatorname{Im} J(R_2H_3, \mathrm{i}E) = c_2(1 + c_1\bar c_1).$$

Next we treat $J(R_2, H_3(\mathrm{i}E))$. Here the argument has to be defined by continuation from the principal value of the argument of $J(R_2, \mathrm{i}E)$. We can do this along the straight line from $\mathrm{i}E$ to $H_3(\mathrm{i}E))$. The points on this line are of the form $\left(\begin{smallmatrix} \mathrm{i} & 0 \\ 0 & \tau \end{smallmatrix}\right)$ where τ is in the upper half plane. One computes

$$J\Bigl(R_2, \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \tau \end{pmatrix}\Bigr) = \det \begin{pmatrix} 1 + ac_2\mathrm{i} & \bar c_1 c_2 \tau \\ c_1c_2\mathrm{i} & 1 \end{pmatrix}.$$

The real part is $1 + c_1\bar{c}_1 c_2^2 \operatorname{Im}\tau$ which is positive. Hence the principal value of the argument is continuous along the line. So we see

$$L(R_2, H_3(\mathrm{i}E)) \in (-\pi, \pi].$$

Finally we compute

$$\operatorname{Im} J(H_3, \mathrm{i}E) = c_1\bar{c}_1 c_2.$$

Now we see that the imaginary part of $J(R_2H_3, \mathrm{i}E)$ and $J(H_3, \mathrm{i}E)$ have the same sign (namely the sign of c_2). Hence their arguments are both contained in $(0, \pi)$ or in $(-\pi, 0)$. This means that $2\pi w(R_2, H_3)$ is contained in $(0, \pi) - (-\pi, \pi] - (0, \pi)$ or in $(-\pi, 0) - (-\pi, \pi] - (-\pi, 0)$. This is $(-2\pi, 2\pi)$ in both cases. We obtain

$$w(R_2, H_3) = 0.$$

The case $c_1c_2 = 0$ is easy and can be omitted.

From Lemma 2.5 we can take $w(H_2, R_1R_3R_4) = 0$. Since H_1 is a unimodular transformation with $\varepsilon(H_1) > 0$ we have $w(H_1, H_2R_1R_3R_4) = 0$. Now we evaluate

$$v(R_2H_3) = v(H_1H_2R_1R_3R_4).$$

The left hand side is

$$v(R_2) + v(H_3) + w(R_2, H_3) = v(R_2) + v(H_3).$$

But $v(R_2) = 0$ (Proposition 3.3). Hence the left hand side is just $v(H_3)$. The right hand side is

$$v(H_1) + v(H_2R_1R_3R_4) + w(H_1, H_2R_1R_3R_4) = v(H_1) + v(H_2R_1R_3R_4).$$

Similarly we see

$$v(H_2R_1R_2R_3) = v(H_2) + v(R_1R_2R_3) + w(H_2, R_1R_3R_4) = v(H_2) + v(R_1R_3R_4).$$

From Proposition 3.4 we know $v(R_1R_3R_4) = 0$. Hence we get $v(H_3) = v(H_1)v(H_2)$. □

Now we assume that the multiplier system is of half integral weight. We can restrict it to a subgroup of finite index of the Siegel modular group. Since this group has the congruence subgroup property v must agree with the theta multiplier system on a suitable congruence subgroup. We obtain the existence of a natural number $q \equiv 0 \bmod 4$ such that $q \in \mathfrak{q}$ and such that

$$v(\iota_1 M) = v(\iota_2 M) = \left(\frac{c}{d}\right) \quad \text{for} \quad M \in \Gamma_1[q].$$

Here the symbol $(\cdot)$ means the Kronecker symbol. We refer to [Di] for it. Next we assume that v is trivial on a congruence subgroup inside $\mathcal{U}$. So we can $v(H_2) = 1$. From Lemma 4.3 we get the following result.

There exists a natural number $q \equiv 0 \bmod 4$ such that for

$$\begin{aligned} &a \in \mathbb{Z}, \quad c_1 \in \mathfrak{o}, \quad c_2 \in \mathbb{Z}, \\ &a \equiv 1 \bmod q, \quad c_1 \equiv 0 \bmod q, \quad c_2 \equiv 0 \bmod q, \\ &a\mathfrak{o} + c_1\mathfrak{o} = \mathfrak{o}, \quad a\mathbb{Z} + c_2\mathbb{Z} = \mathbb{Z} \end{aligned}$$

the relation

$$\Bigl(\frac{c_2}{a}\Bigr) = \Bigl(\frac{c_1\bar{c}_1 c_2}{a}\Bigr)$$

holds. This implies

$$\Bigl(\frac{c\bar{c}}{a}\Bigr) = 1 \quad \text{for} \quad a \in 1 + q\mathbb{Z},\ c \in q\mathfrak{o},\ (a,c) = 1.$$

One can apply this relation to qc for an arbitrary $c \in \mathfrak{o}$ to obtain

$$\Bigl(\frac{c\bar{c}}{a}\Bigr) = 1 \quad \text{for} \quad a \in 1 + q\mathbb{Z},\ c \in \mathfrak{o},\ (a,c) = 1.$$

It is known that there are infinitely many primes of the form $p = c\bar{c}$ [Co]. We choose one such that p and q are coprime. Then we have

$$\Bigl(\frac{p}{a}\Bigr) = (-1)^{\frac{p-1}{2}} \Bigl(\frac{a}{p}\Bigr).$$

Since $a = 1 + xq$ meets every coset mod p we can find a such that

$$\Bigl(\frac{a}{p}\Bigr) = -(-1)^{\frac{p-1}{2}}$$

or

$$\Bigl(\frac{c\bar{c}}{p}\Bigr) = -1.$$

This is contradiction. This gives one of our main results.

4.4 Theorem. *Let $\Gamma \subset \Gamma_2$ be any subgroup of finite index of a Hermitian modular group of degree two. Let v be a multiplier system of half integral weight. The restriction of v to the subgroup* ThDz

$$\mathcal{U} = \Bigl\{ U; \quad \begin{pmatrix} \bar{U}'^{-1} & 0 \\ 0 & U \end{pmatrix} \in \Gamma \Bigr\}$$

is a usual character. Its kernel is a non-congruence subgroup of finite index.

Now we consider arbitrary $g \geq 2$. There is a standard embedding $\Gamma_2[\mathfrak{o}]$ into $\Gamma_g[\mathfrak{q}]$. We can restrict a multiplier system to the case $g = 2$. Since the congruence group property holds for $\mathrm{SL}(3,\mathfrak{o})$ we can apply Theorem 4.4 to obtain the following result.

4.5 Theorem. *Let $n > 2$ and let $\Gamma \subset \Gamma_g$ be any subgroup of finite index of the Hermitian modular group. Multiplier systems of weight r can only exist if r is integral.* ThD

In the next section we investigate the non-congruence subgroup of $\mathcal{U}$ in the case $n = 2$ in more detail.

5. Mennicke symbol

We recall the notion of a Mennicke symbol. Let R be a commutative ring with unity and $\mathfrak{q} \subset R$ a non-zero ideal. We introduce the set

$$\mathcal{C}(R,\mathfrak{q}) := \big\{ (a,b) \in R \times R; \quad Ra + Rb = R,\ a \equiv 1 \bmod \mathfrak{q},\ b \equiv 0 \bmod \mathfrak{q} \big\}.$$

5.1 Definition. *A Mennicke symbol mod $\mathfrak{q}$ is a map* MenS

$$\mathcal{C}(R,\mathfrak{q}) \longrightarrow G, \quad (a,b) \longmapsto \begin{bmatrix} b \\ a \end{bmatrix},$$

into some group G such that the following properties hold.

MS1 *It is invariant under the transformations $(a,b) \mapsto (a + xb, b)$ and $(a,b) \mapsto (a, b + qay)$ for integral x, y.*

MS2 *It satisfies the rule*

$$\begin{bmatrix} b_1 b_2 \\ a \end{bmatrix} = \begin{bmatrix} b_1 \\ a \end{bmatrix} \begin{bmatrix} b_2 \\ a \end{bmatrix}.$$

In our context, the group G will be the group of complex numbers of absolute value one. Mennicke symbols have been classified in [BSM] for Dedekind domains of arithmetic type. If R is the ring of algebraic integers in a number field that is not totally imaginary, then the Mennicke symbols are trivial. In the case of a totally imaginary field they can be described explicitly by means of power residue symbols.

The main result of this section is

5.2 Theorem. *Let v be a multiplier system of half integral weight on a subgroup of finite index of a Hermitian modular group of degree two. Then there exists a non-zero ideal $\mathfrak{q} \subset \mathfrak{o}$ with the following properties.* PisM

1) $\Gamma_2[\mathfrak{q}] \subset \Gamma$.

2) *There exists a Mennicke symbol $[\cdot]$ for $(\mathfrak{o}, \mathfrak{q})$ such that for all $M \in \mathrm{SL}(2, \mathfrak{o})[q]$ one has*

$$\begin{bmatrix} c \\ a \end{bmatrix} = v \begin{pmatrix} \bar{U}'^{-1} & 0 \\ 0 & U \end{pmatrix}, \quad U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proof. The proof is given during the rest of this section.

We have to consider also the embeddings $\iota_1, \iota_2 : \Gamma_1[q] \longrightarrow \Gamma_2[q]$. As in the Hermitian case we have $v(i_1(M)) = v(i_2(M))$ an this depends only on the second row of $M \in \Gamma_1[q]$. Hence we can define

$$\begin{Bmatrix} c \\ d \end{Bmatrix} = v \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = v \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}^{-1}.$$

The elements of $\mathcal{C}(\mathfrak{o},\mathfrak{q})$ are the second columns of the matrices in $\mathrm{GL}(2,\mathfrak{o})[\mathfrak{q}]$. Hence

$$\begin{bmatrix} b \\ d \end{bmatrix} = \left(\iota\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

is well-defined on $\mathcal{C}(\mathfrak{o},\mathfrak{q})$. We claim that this symbol satisfies MS1. We notice that w is trivial on the image of ι. Hence v is a character on this group. The invariance under $(a,b)\mapsto(a,b+qay)$ follows from the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & qy \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b+qay \\ * & * \end{pmatrix}.$$

To prove the invariance under $(a,b)\mapsto(a+xb,b)$, we consider

$$\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} a+xb & b \\ * & * \end{pmatrix}.$$

Due to Lemma 3.2 we can assume that $v(\iota(M)))$ is invariant under conjugation with $\iota\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$. This proves MS1.

We would like to have also MS2. To get a result in this direction, we make use of

$$v(\iota_\nu(M^{-1})) = v(\iota_\nu(M))^{-1}, \quad \nu = 1,2.$$

This is true since in genus 1 one has $w(M,M^{-1}) = 0$. (This is a general rule for $c \neq 0$ and also for $c = 0$ and $a > 0$. But in our case $c = 0$ implies $a = 1$ since we assume $q > 2$.) From Lemma 4.3 we get the general rule (compare Lemma 13.3 in [BMS].)

$$\begin{bmatrix} c_1 \\ a \end{bmatrix}\begin{Bmatrix} c_2 \\ a \end{Bmatrix} = \begin{Bmatrix} c_1\bar{c}_1c_2 \\ a \end{Bmatrix}.$$

We insert $c_2 = 1 - a$.

5.3 Lemma. *We have* emaA

$$\begin{Bmatrix} 1-a \\ a \end{Bmatrix} = 1$$

for $a \equiv 1 \bmod q$.

Proof. We use

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-a & a-1 \\ 1-a & a \end{pmatrix}. \qquad \square$$

Now we obtain

$$\begin{bmatrix} c \\ a \end{bmatrix} = \begin{Bmatrix} c^2(1-a) \\ a \end{Bmatrix}.$$

Before we continue, we mention that $\{\}$ is not a Mennicke symbol. It does not satisfy MS1. Nevertheless it is closely related to $[\cdot]$.

5.4 Lemma. *We have* GnM

$$\begin{Bmatrix} c \\ d \end{Bmatrix} = \begin{Bmatrix} c \\ d + yc \end{Bmatrix}$$

and

$$\begin{Bmatrix} c + xqd \\ d \end{Bmatrix} = \begin{Bmatrix} c \\ d \end{Bmatrix} e^{2\pi i r s} \quad \textit{where} \quad s = w\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ qx & 1 \end{pmatrix}\right).$$

Proof. The first relation can be derived from

$$\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c & d + cy \end{pmatrix}.$$

To derive the second one we consider the relation

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ qx & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c + dxq & d \end{pmatrix}.$$

It shows

$$\begin{Bmatrix} c + dxq \\ b \end{Bmatrix} = \begin{Bmatrix} c \\ d \end{Bmatrix} e^{2\pi i r s}.$$

The w-value s is usually not zero. □

But from the corollary of the table of Maass in the introduction we get

$$w\left(\begin{pmatrix} * & * \\ c\bar{c} & a \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -qc\bar{c} & 1 \end{pmatrix}\right) = 0.$$

Using this, we get

$$\begin{Bmatrix} c\bar{c}(1-a) \\ a \end{Bmatrix} = \begin{Bmatrix} c\bar{c} \\ a \end{Bmatrix}.$$

So we obtain

$$\begin{bmatrix} c \\ a \end{bmatrix} = \begin{Bmatrix} c\bar{c} \\ a \end{Bmatrix}$$

and moreover

$$\begin{bmatrix} c_1 c_2 \\ a \end{bmatrix} = \begin{Bmatrix} c_1\bar{c}_1 c_2\bar{c}_2 \\ a \end{Bmatrix} = \begin{bmatrix} c_1 \\ a \end{bmatrix} \begin{Bmatrix} c_2\bar{c}_2 \\ a \end{Bmatrix} = \begin{bmatrix} c_1 \\ a \end{bmatrix} \begin{bmatrix} c_2 \\ a \end{bmatrix}.$$

This is part of the condition MS2. (We assume up to now $a \in \mathbb{Z}$).

5.5 Lemma. *Let (a, b) be two elements of $\mathfrak{o}$ such that $(a, b) = \mathfrak{o}$. Then there exists $x \in \mathfrak{o}$ such that $a + xb$ is not divisible by any natural number > 1.* abX

Proof. We write an element $a \in \mathfrak{o}$ in the form

$$a = \dot{a} + \ddot{a}\omega, \quad \omega = \frac{\delta + \sqrt{d}}{2}.$$

We will use

$$\omega^2 = -N(\omega) + d\omega.$$

From a solution $ax + by = 1$ we derive that the 4 integers

$$\dot{a},\ \dot{b},\ \ddot{a}N(\omega),\ \ddot{b}N(\omega)$$

are coprime. We have to find $x \in \mathfrak{o}$ such that

$$\dot{a} + \dot{x}\dot{b} - \ddot{x}\ddot{b}N(\omega), \quad \ddot{a} + \dot{x}\ddot{b} + \ddot{x}(\dot{b} + \ddot{b}d)$$

are coprime. We consider the greatest common divisors

$$d = \mathrm{ggT}(\ddot{a}, \dot{b}, \ddot{b})$$

By Dirichlet's prime number theorem we can find $y \in \mathfrak{o}$ such that

$$\ddot{a} + \dot{y}\ddot{b} + \ddot{y}(\dot{b} + \ddot{b}d) = dp$$

where p is a prime number. There are infinitely many choices for p. Hence we can get that p is coprime to $\ddot{b}^2(d^2 - d)/4 + \dot{b}^2 + \dot{b}\ddot{b}d$. (This expression equals $(\ddot{b}d/2 + \dot{b})^2 - \ddot{b}^2 d/4$ which is positive.) Now we set

$$\dot{x} = \dot{y} + t(\dot{b} + \ddot{b}d), \quad \ddot{x} = \ddot{y} - t\ddot{b} \qquad (t \in \mathbb{Z}).$$

Then we have still

$$\ddot{a} + \dot{x}\ddot{b} + \ddot{x}(\dot{b} + \ddot{b}d) = dp$$

but

$$\dot{a} + \dot{x}\dot{b} - \ddot{x}\ddot{b}N(\omega) = \dot{a} + \dot{y}\dot{b} - \ddot{y}\ddot{b}N(\omega) + t(\dot{b}^2 + \dot{b}\ddot{b}d + \ddot{b}^2 N(\omega)).$$

Now we consider the greatest common divisor

$$d' = (\dot{a} + \dot{y}\dot{b} - \ddot{y}\ddot{b}N(\omega), \dot{b}^2 + \dot{b}\ddot{b}d + \ddot{b}^2 N(\omega)).$$

We can choose t such that

$$\dot{a} + \dot{x}\dot{b} - \ddot{x}\ddot{b}N(\omega) = d'p'$$

where p' is a prime. We can choose p' coprime to dp. Our goal was to get dp and $d'p'$ coprime. This means that d, d' are coprime. But

$$\mathrm{ggT}(d, d') = \mathrm{ggT}(\ddot{a}, \dot{b}, \ddot{b}, \dot{a} + \dot{y}\dot{b} - \ddot{y}\ddot{b}N(\omega), \dot{b}^2 + \dot{b}\ddot{b}d + \ddot{b}^2 N(\omega)) = \mathrm{ggT}(\dot{a}, \dot{b}, \ddot{a}, \ddot{b}) = 1.$$

This proves Lemma 5.5. $\square$

Now we investigate the equation

$$\begin{bmatrix} q^2b_1b_2 \\ a \end{bmatrix} = \begin{bmatrix} qb_1 \\ a \end{bmatrix}\begin{bmatrix} qb_2 \\ a \end{bmatrix}.$$

It is invariant under the replacement $b_1 \mapsto b_1 + xa$. By Lemma 5.5 we can assume that b_1 is not divisible by any natural number. We also want to make an replacement for b_2. For this we consider the ray class of the principal ideal (b_2) mod the ideal (a). (Recall that two ideals $\mathfrak{b}_1$, $\mathfrak{b}_2$ are in the same ray class mod an ideal $\mathfrak{a}$ if there exist $\beta_1 \equiv \beta_2 \equiv 1 \bmod \mathfrak{a}$ such that $\beta_1\mathfrak{b}_1 = \beta_2\mathfrak{b}_2$.) Our product formula does not change if one replaces b_2 by βb_2 for $\beta \equiv 1 \bmod (a)$. Hence we may replace (b_2) by any other (b_2') in the same ray class. In each ray class there are infinitely many primes. Hence we can assume that b_2 is coprime to $N(b_1)$. Now we make the replacement $b_2 + xaN(b_1)$. Again we make use of Lemma 5.5 to reduce to the case where b_2 is not divisible by any natural number, and, in addition, is coprime to b_2 and to $\bar{b}_2$. Then b_1b_2 is also not divisible by any natural number.

Now we make the stronger assumption $a \equiv 1 \bmod q^2$. Then $\ddot{a} \equiv 0 \bmod q^2$. We can make the replacement $a \mapsto a + a + xq^2b_1b_2$ without changing the product formula. This means that we replace $\ddot{a} \longmapsto q^2(\ddot{a}/q^2 + \ddot{y})$ where $y = xb_1b_2$ Since b_1b_2 is not divisible by any natural number, we can choose x such that $\ddot{y}$ runs through all integers. This shows that we can assume $a \in \mathbb{Z}$. But then the product formula has been proved. Now we replace q by q^2 to obtain Theorem 5.2. $\square$

6. Multiplier systems for the Siegel modular group

In our proof we used the result of Deligne about the multiplier systems of subgroups of finite index of the Siegel modular group. Our methods are sufficient to give a new short proof for this result. In this section we denote by $\Gamma_g = \mathrm{Sp}(g, \mathbb{Z})$ the Siegel modular group and by $\Gamma_g[q]$ the principal congruence subgroup of level q.It is clear how to define the notion of a multiplier system in this case. So let v be a multiplier system of weight $r \in \mathbb{R}$ on $\Gamma_g[q]$. We consider the embedding

$$\iota : \Gamma_1[q] \longrightarrow \Gamma_2[q], \quad M \longmapsto \begin{pmatrix} M'^{-1} & 0 \\ 0 & M \end{pmatrix}$$

and we define

$$\begin{bmatrix} c \\ a \end{bmatrix} = v(\iota(M)), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is rather clear that this is independent on the choice of b, d. Each coprime pair (a, b), $a \equiv 1 \bmod q$, $b \equiv 0 \bmod q$ is part of a matrix $M \in \Gamma_1[q]$. Hence we have constructed a function on $\mathcal{C}(\mathbb{Z}, q\mathbb{Z})$.

The same proof as in the case of the Hermitian modular group shows that $[\cdot]$ is Mennicke symbol. Now, in contrast to the Hermitian case we can use the basic result ([BMS], Theorem 3.6) that Mennicke symbols on $\mathcal{C}(\mathbb{Z}, q\mathbb{Z})$ are trivial! This gives the following result.

6.1 Proposition. *The multiplier system v is identically one on all* Triio

$$\begin{pmatrix} M & 0 \\ 0 & M'^{-1} \end{pmatrix} \quad \textit{for } M \in \Gamma_1[q].$$

As in the Hermitian case we can define in the Siegel case

$$\left\{ \begin{matrix} c \\ d \end{matrix} \right\} = v \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = v \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}^{-1}.$$

From the basic relations follows now

$$\left\{ \begin{matrix} c^2 \\ d \end{matrix} \right\} = 1 \quad \text{and} \quad \left\{ \begin{matrix} c_1 \\ d \end{matrix} \right\} = \left\{ \begin{matrix} c_1 c_2^2 \\ d \end{matrix} \right\}$$

for $c \equiv c_1 \equiv c_2 \equiv 0 \bmod q$ and $d \equiv 1 \bmod q$. This can be generalized. Again we have to consider the Kronecker symbol $\left(\frac{c}{d}\right)$ (see [Di]). We will need it only for $c \neq 0$ and for odd d. We collect some properties (always assuming this condition)

$$\left(\frac{c_1 c_2}{d}\right) = \left(\frac{c_1}{d}\right)\left(\frac{c_2}{d}\right), \quad \left(\frac{c}{d_1 d_2}\right) = \left(\frac{c}{d_1}\right)\left(\frac{c}{d_2}\right).$$

Assume $d > 0$ or $c_1 c_2 > 0$. Then

$$\left(\frac{c_1}{d}\right) = \left(\frac{c_2}{d}\right) \quad \text{if} \quad c_1 \equiv c_2 \bmod d.$$

Also the relation

$$\left(\frac{c}{d_1}\right) = \left(\frac{c}{d_2}\right) \quad \text{if} \quad \begin{cases} d_1 \equiv d_2 \bmod c \text{ and } c \equiv 0 \bmod 4, \\ d_1 \equiv d_2 \bmod c \text{ and } c \equiv 2 \bmod 4 \end{cases}$$

is valid. Finally we mention

$$\left(\frac{c}{-1}\right) = \begin{cases} 1 & \text{for } c > 0, \\ -1 & \text{for } c < 0. \end{cases}$$

Since one of the rules demands $c \equiv 0 \bmod 4$, we will from now on assume that $q \equiv 0 \bmod 4$.

6.2 Proposition. *Let q be a suitable multiple of q' and let* KronS

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1[q], \qquad \left(\frac{c}{d}\right) = 1.$$

Then $v(M) = 1$.

Proof. We use the invariance under $(c, d) \mapsto (c, d+xc)$. We can apply Dirichlet's prime number theorem and therefore assume that $d = p$ is a (positive) prime. But then the Kronecker symbol is the usual Legendre symbol. Since $d \equiv 1 \bmod q$ we have $\left(\frac{q}{d}\right) = 1$. This implies $\left(\frac{c/q}{d}\right) = 1$. Since d is a prime, we get a solution of $c/q = x^2 + dy$ or $c = qx^2 + dqy$. Now use

$$\begin{pmatrix} * & * \\ qx^2 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ qy & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

In the case $c > 0$ the w-value is zero. This follows from the corollary in the table of Maass in the introduction. In the case $c < 0$ we must have $y < 0$ and again from this corollary follows that the w-value is zero. (In the notation of the table the sign distribution of (m_1, c, m_1') is $(+, *, +)$ or $(+, -, *)$.) Now we get

$$v(M) = v\begin{pmatrix} * & * \\ c & d \end{pmatrix} = v\begin{pmatrix} * & * \\ qx^2 & d \end{pmatrix} = \left\{ \begin{matrix} qx^2 \\ d \end{matrix} \right\}.$$

The basic relation now gives

$$\left\{ \begin{matrix} x^2q \\ d \end{matrix} \right\} = \left\{ \begin{matrix} x^2q^3 \\ d \end{matrix} \right\} \left\{ \begin{matrix} q(qx)^2 \\ d \end{matrix} \right\} = \left\{ \begin{matrix} q \\ d \end{matrix} \right\} = 1. \qquad \square$$

6.3 Lemma. *Assume that the matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is contained in $\Gamma_1[q]$ and has the following properties. All entries are positive and $dq < c(q-1)$. Then* zPir

$$v(M) = e^{-2\pi i r} \quad \textit{if} \quad \left(\frac{c}{d}\right) = -1.$$

Proof. We consider

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1-q & -q \\ q & 1+q \end{pmatrix} = \begin{pmatrix} * & * \\ c - qc + dq & -cq + d + dq \end{pmatrix}.$$

Clearly $\left(\frac{q}{1+q}\right) = 1$. We also claim

$$\left(\frac{c - qc + dq}{-cq + d + dq}\right) = 1.$$

To prove this, we observe

$$\Big(\frac{c-qc+dq}{-cq+d+dq}\Big)=\Big(\frac{c-qc+dq}{d-c}\Big)=\Big(\frac{c-qc+dq}{-1}\Big)\Big(\frac{c-qc+dq}{c-d}\Big)$$

Now we use $c-qc+dq<0$. It follows $c-d>0$. Hence we get

$$=-\Big(\frac{c}{c-d}\Big)=-\Big(\frac{c}{-d}\Big)=-\Big(\frac{c}{d}\Big)=-(-1)=1.$$

Now we have proved

$$v\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1-q & -q\\ q & 1+q\end{pmatrix}\right)=1.$$

The left hand side equals

$$v(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\exp\left\{2\pi\mathrm{i}rw\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},\begin{pmatrix} 1-q & -q\\ q & 1+q\end{pmatrix}\right)\right\}=1.$$

From Maass' table in the introduction follows that the w-value is 1. (The sign distribution of (m_1,c,m_1') is $(+,+,-)$.) This proves Lemma 6.3. □

There exist two coprime natural numbers c,d such that $c\equiv 0 \bmod q$ and $d\equiv 1 \bmod q$ and such that $\left\{\frac{c}{d}\right\}=-1$. We also can assume $dq<c(q-1)$. The pair (c,d) is the second row of a matrix $M=\left(\begin{smallmatrix} a & b\\ b & d\end{smallmatrix}\right)\in\Gamma_1[q]$. We want to compute $v(M)$. Since we can add a multiple of the second row to the first one, we can assume that a and b are also positive. From Lemma 6.3 we know $v(M)=e^{-2\pi\mathrm{i}r}$. Now we consider

$$v(M^2)=v(M)^2e^{2\pi\mathrm{i}rw(M,M)}.$$

Since all entries from M are positive, we have $w(M,M)=0$. So we get

$$v(M^2)=e^{-4\pi\mathrm{i}r}.$$

We compute $\left(\frac{\gamma}{\delta}\right)$ for the matrix

$$N=M^2=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

We get

$$\Big(\frac{\gamma}{\delta}\Big)=\Big(\frac{c(a+d)}{cb+d^2}\Big)=\Big(\frac{c}{cb+d^2}\Big)\Big(\frac{a+d}{cb+d^2}\Big).$$

We have

$$\Big(\frac{c}{cb+d^2}\Big)=\Big(\frac{c}{d^2}\Big)=1$$

and

$$\Bigl(\frac{a+d}{cb+d^2}\Bigr) = \Bigl(\frac{a+d}{d(a+d)-1}\Bigr).$$

Since $a+d \equiv 2 \bmod 4$ we only can change the denominator mod $4(a+d)$. Since $d \equiv 1$ mod 4 we see

$$\Bigl(\frac{a+d}{d(a+d)-1}\Bigr) = \Bigl(\frac{a+d}{a+d-1}\Bigr) = \Bigl(\frac{1}{a+d-1}\Bigr) = 1.$$

This shows $v(N) = 1$ and we get the relation

$$e^{-4\pi\mathrm{i}r} = 1$$

which implies that $2r$ is integral. In this way we obtain Delignes result.

6.4 Theorem. *The weight r of a multiplier system on a subgroup of finite index of the Siegel modular subgroup of degree > 1 is integral or half integral.* TD

References

[BMS] Bass, H. Milnor, J. Serre, J.P.: *Solution of the congruence subgroup problem for* SL_n $(n \geq 3)$ *and* Sp_{2n} $(n \geq 2)$*,* Publications mathèmatiques l'I.H.É.S., tome **33**, p. 59–137 (1967)

[Ch] Christian, U.: *Hilbert-Siegelsche Modulformen und Poincarésche Reihen,* Mathematische Annalen **148**, 257307 (1962)

[Co] Cox, D.A.: *Primes of the Form* $x^2 + ny^2$*,* Wiley Series in Pure and Applied Mathematics (2013)

[De] Deligne, P.: *Extensions centrales non résiduellement finies de groupes arithmetiques,* C. R. Acad. Sci. Paris **287**, p. 203-208 (1978)

[Di] Dickson, L.E.: *Introduction to the Theory of Numbers,* Dover Publications, New York, Dover (1957).

[Hi] Hill, R.: *Fractional weights and non-congruence subgroups,* Automorphic Forms and Representations of algebraic groups over local fields, Saito, H., Takahashi, T. (ed.) Surikenkoukyuroku series **1338**, 71-80 (2003)

[Ma] Maass, H.: *Lectures on Modular Functions of One Complex Variable,* Notes by Sunder Lal, Tata Institute Of Fundamental Research, Bombay, Revised 1983 (1964)

[Me] Mennicke, J.: *Zur Theorie der Siegelschen Modulgruppe,* Math. Annalen **159**, 115–129 (1965)