A simple proof of some Macdonald identities

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2007

Introduction

Let $\mathfrak{g}$ be a simple complex Lie-algebra of dimension d, Δ its set of roots (with respect to some Cartan algebra) and Δ^+ the subset of positive roots (with respect to an ordering),

$$\varrho = \frac{1}{2}\sum_{\alpha>0}\alpha$$

half of the sum of the positive roots and

$$M := \sum_{\alpha\in\Delta}\frac{\alpha}{\langle\alpha,\alpha\rangle}\mathbb{Z}$$

the modified root lattice. We set

$$D(\mu) := \prod_{\alpha>0}\frac{\langle\mu,\alpha\rangle}{\langle\varrho,\alpha\rangle}.$$

The Macdonald identity which we mean*) can be written in the form

$$\sum_{\mu\in M} D(\mu+\varrho)X^{\langle\mu+\varrho,\mu+\varrho\rangle} = X^{\frac{d}{24}}\prod_{n=1}^{\infty}(1-X^n)^d.$$

If we set $X = e^{2\pi\mathrm{i}\tau}$ it is an identity of modular forms

$$\eta(\tau)^d = \sum_{\mu\in M} D(\mu+\varrho)e^{2\pi\mathrm{i}\langle\mu+\varrho,\mu+\varrho\rangle\tau}.$$

The left hand side is the d-th power of the well-known Dedekind η-function and the right hand side is a theta series with a polynomial coefficient. In the theta transformation formalism on uses usually $e^{\pi\mathrm{i}}$ instead of $e^{2\pi\mathrm{i}}$. Hence we introduce

$$L := \sqrt{2}M,\quad a = \sqrt{2}\varrho,\quad P(z) = D(z/\sqrt{2})$$

*) *Affine Roots Systems and Dedekind's η-function*, Inv. Math. **15**, 91–143 (1972)

and rewrite the right hand side as

$$\vartheta_{L,P,a}(\tau) := \sum_{g\in L} P(g+a)e^{\pi \mathrm{i}\langle g+a,g+a\rangle\tau}.$$

We have to prove that this is a modular form of weight $d/2$ with respect to the full modular group and that the vanishing order at ∞ measured in $e^{\pi\mathrm{i}\tau/12}$ is (at least) d. Then it can be divided by η^d. The quotient will be a modular form of weight zero, hence constant. Comparing the first non-zero Fourier coefficient one can see that the constant is 1.

In the next section we will give a general criterion under which $\vartheta_{L,P,a}$ is a modular form with respect to the full modular group. In section 3 we will see that this criterion applies to our situation.

1. Theta series with harmonic coefficients

Let $L \cong \mathbb{Z}^r$ be a lattice with a positive definit bilinear form

$$L\times L \longrightarrow \mathbb{R}, \quad (a,b)\longmapsto \langle a,b\rangle.$$

We set $V = L\otimes_{\mathbb{Z}}\mathbb{R}$ and extend $\langle\cdot,\cdot\rangle$ to an $\mathbb{R}$-bilinear map. The dual lattice is

$$L' = \{\, a\in V; \quad \langle a,x\rangle\in\mathbb{Z} \text{ for all } x\in L \,\}.$$

We denote by $v(L)$ the volume of V/L. A polynomial $P: V\to\mathbb{C}$ is called harmonic, if $\Delta P = 0$. Here Δ is the Laplace operator with respect to $\langle\cdot,\cdot\rangle$. The well-known theta inversion formula states:

1.1 Lemma. *Let $P: V\to\mathbb{C}$ a homogenous harmonic polynomial of degree k. Then*

$$\sum_{g\in L} P(g+a)e^{\pi\mathrm{i}\langle g+a,g+a\rangle(-1/\tau)} = \frac{\mathrm{i}^{-k}}{v(L)}\sqrt{\frac{\tau}{\mathrm{i}}}^{\,r+2k}\sum_{g\in L'}P(g)e^{\pi\mathrm{i}\{\langle g,g\rangle\tau+2\langle g,a\rangle\}}.$$

We are interested in triples (L,P,a), such that the series $\vartheta_{L,P,a}$ equals $\eta(\tau)^{r+2k}$.

1.2 Lemma. *The theta series*

$$\vartheta_{L,P,a} = \sum_{g\in L} P(g+a)e^{\pi\mathrm{i}\langle g+a,g+a\rangle\tau}$$

equals $\eta(\tau)^{r+2k}$ up to a constant factor if the following conditions are satisfied.

1. *P is a homogenous harmonic polynomial of degree k.*
2. *L is an even lattice of rank r (even means $\langle x, x\rangle \in 2\mathbb{Z}$ for $x \in L$).*
3. *$a \in L'$, $\langle a, a\rangle = (r+2k)/12$ and $\langle g+a, g+a\rangle \geq \langle a, a\rangle$ for $g \in L$.*
4. *There exist a group of isometries G of L and a character $\det : G \to \mathbb{C}^*$ with the property $P(\sigma z) = \det(\sigma)P(z)$ for $\sigma \in G$ and such that for every element $b \in L'/L$, which is not in the G-orbit of the image of a in L'/L, there exists $\sigma \in G$ such that $\sigma(b) = b$ and $\det(\sigma) \neq 1$.*

Remark. *The constant factor is one if $P(a) = 1$ and if $\langle g+a, g+a\rangle = \langle a, a\rangle$ only for $g = 0$.*

Proof. From 2) and 3) we see

$$\vartheta_{L,P,a}(\tau+1) = e^{\pi \mathrm{i}\langle a,a\rangle}\vartheta_{L,P,a}(\tau).$$

The essential point is the transformation formula under $\tau \mapsto -1/\tau$. In the theta transformation formula occurs the series

$$\sum_{g\in L'} P(g)e^{\pi\mathrm{i}\{\langle g,g\rangle\tau+2\langle g,a\rangle\}} = \sum_{b\in L'/L} e^{2\pi\mathrm{i}\langle a,b\rangle}\sum_{g\in L} P(g+b)e^{\pi\mathrm{i}\langle g+b,g+b\rangle\tau}.$$

The conditions 1) and 4) show that only the b which are in the G-orbit of (the image of) a give a non-zero contribution. Hence the right hand side is a constant multiple of $\vartheta_{L,P,a}(\tau)$. We obtain

$$\vartheta_{L,P,a}(-1/\tau) = \varepsilon\sqrt{\frac{\tau}{\mathrm{i}}}^{\,r+2k}\vartheta_{L,P,a}(\tau)$$

with some constant ε. We can assume that $\vartheta_{L,P,a}$ is different from zero. Since $\tau \mapsto -1/\tau$ is an involution we must have $\varepsilon = \pm 1$. Now we see that $\vartheta_{L,P,a}/\eta(\tau)^{r+2k}$ is an entire modular form of weight zero. Because of condition 3) it is regular at the cusp and hence a constant. (As a consequence $\varepsilon = 1$.) This completes the proof of 1.2. □

2. The proof

We have to show that

$$L := \sqrt{2}M, \quad a = \sqrt{2}\varrho, \quad P(z) = D(z/\sqrt{2})$$

(notations as in section 1) satisfy the assumptions of 1.2.

Proof of 1. We use Weyl's formula

$$e^{\varrho}\prod_{\alpha>0}\left(1-e^{-\alpha}\right) = \sum_{\sigma\in W}\det(\sigma)\, e^{\sigma\varrho},$$

where W denotes the Weyl group. An immediate consequence is

$$\prod_{\alpha>0}\left(e^{t\langle\alpha/2,\mu\rangle}-e^{-t\langle\alpha/2,\mu\rangle}\right)=\sum_{\sigma\in W}\det(\sigma)\,e^{t\langle\sigma\varrho,\mu\rangle}.$$

Expanding the exponential function and comparing coefficients one obtains

$$\frac{1}{l!}\sum_{\sigma\in W}\det(\sigma)\langle\sigma\varrho,\mu\rangle^l=\begin{cases}0 & \text{for } l<k,\\ \prod_{\alpha>0}\langle\alpha,\mu\rangle & \text{for } l=m,\end{cases}$$

where k denotes the numbers of positive roots. (This equals the degree of P.) It follows that the function $f(\mu)=\prod_{\alpha>0}\langle\alpha,\mu\rangle$ is harmonic, since

$$\Delta f(\mu)=\frac{\langle\varrho,\varrho\rangle}{(k-2)!}\sum_{\sigma\in W}\det(\sigma)\langle\sigma\varrho,\mu\rangle^{k-2}=0. \qquad \square$$

Proof of 2. One has to use that $\langle\alpha,\alpha\rangle^{-1}$ is integral for every root. $\square$

Proof of 3. Since reflexion with respect to a simple root α maps ϱ to $\varrho-\alpha$, we have $2\langle\alpha,\varrho\rangle/\langle\alpha,\alpha\rangle=1$ for simple roots and therefore $\alpha\in L'$.

The assumption $\langle a,a\rangle=(r+2k)/12$ is a consequence of Freudenthals formula $\dim\mathfrak{g}=24\langle\varrho,\varrho\rangle$.

The assumption $\langle g+a,g+a\rangle\geq\langle a,a\rangle$ for $g\in L$ follows from the identity

$$\langle\mu,\mu\rangle=\sum_{\alpha\in\Delta}\langle\mu,\alpha\rangle^2,$$

which es evident from the definition of the Killing form and which implies

$$\langle\mu+\varrho,\mu+\varrho\rangle-\langle\varrho,\varrho\rangle=2\sum_{\beta>0}\left\{\left(\langle\mu,\beta\rangle+\frac{1}{4}\right)^2-\frac{1}{16}\right\}.$$

Since $\langle\mu,\beta\rangle$ is an integral multiple of $1/2$ for $\mu\in M$ and roots β, we obtain that the sum is non-negative.

Proof of 4. For the group G we take the Weyl group W and for the character det the determinant. The formula $P(\sigma z)=\det\sigma P(z)$ is obvious. We use now the lattice

$$\Lambda=2M=\sqrt{2}L=\sum_{\alpha\in\Delta}\mathbb{Z}\,\check\alpha\quad\text{where}\quad\check\alpha=\frac{2}{\langle\alpha,\alpha\rangle}\alpha.$$

It is known that this lattice is generated by the $\check\alpha$ for simple α. We recall $\langle\varrho,\check\alpha\rangle=1$ for simple α.

We will work with the affine Weyl group $W^{\mathrm{aff}}=W\cdot\Lambda$ which acts on $V=\Lambda\otimes_{\mathbb{Z}}\mathbb{R}$ by $wx=\sigma x+\lambda$. We set $\varepsilon(w)=\det(\sigma)$. The following Lemma is equivalent to condition 4 in 1.2.

2.1 Lemma. *Let $b \in \Lambda'$ an element such that $2b$ is not in the $W^{\rm aff}$-orbit of 2ϱ. Then there exists $w \in W^{\rm aff}$ such that $w(2b) = 2b$ and $\varepsilon(w) = -1$.*

Proof. We denote by $\alpha_1, \dots, \alpha_r$ the simple roots and by δ the highest root (the dominant weight of the adjoint representation). Since the eigenvalue of the Casimir operator for the adjoint representation is one, we have

$$\langle \varrho + \delta, \varrho + \delta\rangle - \langle \varrho, \varrho\rangle = 1.$$

Recall that the affine Weyl-group $W^{\rm aff}$ acts simply transitive on the affine Weyl chambers, which are the connected components of the complement of the union of the affine spaces $\langle \alpha, x\rangle = n$, where α is a root and n an integer. The standard Weyl-chamber is

$$C = \bigl\{\, x \in V; \quad \langle x, \alpha_i\rangle > 0 \text{ for } 1 \le i \le r \quad \text{and} \quad \langle x, \delta\rangle < 1 \,\bigr\}.$$

Now we take $b \in \Lambda'$ such such that $2b$ is not in the $W^{\rm aff}$-orbit of 2ϱ. We will prove that $2b$ is on the boundary of a Weyl chamber. Since the reflection along a wall is contained in $W^{\rm aff}$, this will prove 2.1.

We argue by contradiction and assume that $2b$ is in the interior of the standard Weyl chamber C, especially $\langle 2b, \alpha_i\rangle > 0$. Since $\langle b, \check\alpha\rangle \in \mathbb{Z}$, we get

$$\langle 2b, \check\alpha_i\rangle = 2k_i, \quad k_i \ge 1 \text{ (integer)}.$$

Since $\langle \varrho, \check\alpha_i\rangle = 1$ and since b is different from ϱ, there exists an index $l \in \{1, \dots, r\}$ with $k_l \ge 2$.

Next we use that the highest root δ is of the form

$$\delta = \sum_{i=1}^{r} m_i \alpha_i, \quad m_i \ge 1 \text{ (integer)}.$$

We will need a little more. If there are roots of different length (types B_n, C_n, F_4, G_2), then $m_i \ge 2$ for all short roots α_i and even more $m_i \ge 3$ for the type G_2. We obtain

$$\begin{aligned}\langle 2b, \delta\rangle = \sum_{i=1}^{r} m_i \frac{\langle \alpha_i, \alpha_i\rangle}{2} 2k_i &\ge m_l \langle \alpha_l, \alpha_l\rangle + \sum_{i=1}^{r} m_i \langle \alpha_i, \alpha_i\rangle \\ &= m_l \langle \alpha_l, \alpha_l\rangle + 2\langle \varrho, \delta\rangle \\ &= m_l \langle \alpha_l, \alpha_l\rangle + 1 - \langle \delta, \delta\rangle.\end{aligned}$$

By the above remark about the m_i we get $\langle 2b, \delta\rangle \ge 1$ in all cases. This implies that $2b$ is not in the interior of the standard affine Weyl chamber, which contradicts to our assumption. This completes the proof of 1.2. □

Finally we have to show the assumption of the remark in 1.2. We have to show that

$$\sum_{\beta > 0} \left\{ \Bigl(\langle \alpha, \beta\rangle + \frac{1}{4} \Bigr)^2 - \frac{1}{16} \right\} = 0 \qquad (\alpha \in M)$$

implies $\alpha = 0$. Since $2\langle \alpha, \beta\rangle \in \mathbb{Z}$, the sum is 0 if and only if $\langle \alpha, \beta\rangle$ is contained in $\{0, -1/2\}$ for all $\beta > 0$. We have to show $\alpha = 0$. This follows from the following lemma applied to $\lambda = -2\alpha \in \Lambda$.

2.2 Lemma. *Let $\lambda \in \Lambda$ and $\langle \lambda, \alpha \rangle \in \{0, 1\}$ for all positive roots α. Then $\lambda = 0$.*

Proof. It is sufficient to show that $\langle \lambda, \alpha \rangle = 0$ for all simple α. We argue by contradiction and assume $\langle \lambda, \alpha \rangle = 1$ for some simple root α. Since the relation $\langle \lambda, \delta \rangle \in \{0, 1\}$ holds also for the highest root

$$\delta = \sum_{i=1}^{r} m_i \alpha_i, \quad m_i \geq 1 \text{ (integer)},$$

we obtain that $\langle \lambda, \delta \rangle = 1$ and $\langle \lambda, \alpha_i \rangle = 1$ for precisely one simple root α_i (with corresponding $m_i = 1$) and $\langle \lambda, \alpha_j \rangle = 0$ for the others. Hence λ is a vertex of the standard affine Weyl chamber C. The translation $Tx = x - \lambda$ belongs to the affine Weyl group W^{aff}. Hence $T\lambda = 0$ is a vertex of the chamber TC. We consider the (unbounded) standard Weyl chamber of the finite Weyl group W

$$\tilde{C} = \left\{ x \in V; \quad \langle x, \alpha_i \rangle > 0 \text{ for } 1 \leq i \leq r \right\}.$$

We choose an element σ from the (finite) Weyl group W, which transforms TC into $\tilde{C}$. So σTC is one of the affine Weyl chambers chambers contained in $\tilde{C}$. The only affine Weyl chamber in $\tilde{C}$, which contains 0 in its closure, is the affine standard chamber C. Since $\sigma T\lambda = \sigma 0 = 0$, we must have $\sigma TC = C$. This implies $\sigma T = \text{id}$, which is not possible. □

Final remark. The formula $\langle \varrho + \delta, \varrho + \delta \rangle - \langle \varrho, \varrho \rangle = 1$ shows that 2ϱ lies in the interior of the chamber C. Hence the conjugates of 2ϱ under W^{aff} are all different. Therefore the sum $\sum_{b \in L'/L}$ in the proof of lemma 1.2, where only the b in the orbit of 2ϱ survive, can be written as a sum over the Weyl group W. The equality $\varepsilon = 1$ at the end of section one now implies the formula

$$\sum_{\sigma \in W} \det \sigma \cdot e^{4\pi \mathrm{i} \langle \sigma\varrho, \varrho \rangle} = \sqrt{2}^{r} \mathrm{i}^{k} v(M).$$