A simple proof of some Macdonald identities

Sigrid Böge and Eberhard Freitag

Mathematisches Institut
Im Neuenheimer Feld 288
D69120 Heidelberg

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Introduction

Let \( \mathfrak{g} \) be a simple complex Lie-algebra of dimension \( d \), \( \Delta \) its set of roots (with respect to some Cartan algebra) and \( \Delta^+ \) the subset of positive roots (with respect to an ordering),

\[
\varrho = \frac{1}{2} \sum_{\alpha > 0} \alpha
\]

half of the sum of the positive roots and

\[
M := \sum_{\alpha \in \Delta} \frac{\alpha}{\langle \alpha, \alpha \rangle} \mathbb{Z}
\]

the modified root lattice. We set

\[
D(\mu) := \prod_{\alpha > 0} \frac{\langle \mu, \alpha \rangle}{\langle \varrho, \alpha \rangle}.
\]

The Macdonald identity which we mean\(^*\) can be written in the form

\[
\sum_{\mu \in M} D(\mu + \varrho) X^{\langle \mu + \varrho, \mu + \varrho \rangle} = X^{d/2} \prod_{n=1}^{\infty} (1 - X^n)^d.
\]

If we set \( X = e^{2\pi i \tau} \) it is an identity of modular forms

\[
\eta(\tau)^d = \sum_{\mu \in M} D(\mu + \varrho) e^{2\pi i \langle \mu + \varrho, \mu + \varrho \rangle \tau}.
\]

The left hand side is the \( d \)-th power of the well-known Dedekind \( \eta \)-function and the right hand side is a theta series with a polynomial coefficient. In the theta transformation formalism on uses usually \( e^{\pi i} \) instead of \( e^{2\pi i} \). Hence we introduce

\[
L := \sqrt{2} M, \quad a = \sqrt{2} \varrho, \quad P(z) = D(z/\sqrt{2})
\]

\(^*\) Affine Roots Systems and Dedekind’s \( \eta \)-function, Inv. Math. 15, 91–143 (1972)
and rewrite the right hand side as

\[ \vartheta_{L,P,a}(\tau) := \sum_{g \in L} P(g + a) e^{\pi i (g+a.g+a) \tau}. \]

We have to prove that this is a modular form of weight \( d/2 \) with respect to the full modular group and that the vanishing order at \( \infty \) measured in \( e^{\pi i \tau/12} \) is (at least) \( d \). Then it can be divided by \( \eta^d \). The quotient will be a modular form of weight zero, hence constant. Comparing the first non-zero Fourier coefficient one can see that the constant is 1.

In the next section we will give a general criterion under which \( \vartheta_{L,P,a} \) is a modular form with respect to the full modular group. In section 3 we will see that this criterion applies to our situation.

1. Theta series with harmonic coefficients

Let \( L \cong \mathbb{Z}^r \) be a lattice with a positive definite bilinear form

\[ L \times L \rightarrow \mathbb{R}, \quad (a,b) \mapsto \langle a,b \rangle. \]

We set \( V = L \otimes_{\mathbb{Z}} \mathbb{R} \) and extend \( \langle \cdot, \cdot \rangle \) to an \( \mathbb{R} \)-bilinear map. The dual lattice is

\[ L' = \{ a \in V; \quad \langle a, x \rangle \in \mathbb{Z} \text{ for all } x \in L \}. \]

We denote by \( v(L) \) the volume of \( V/L \). A polynomial \( P : V \rightarrow \mathbb{C} \) is called harmonic, if \( \Delta P = 0 \). Here \( \Delta \) is the Laplace operator with respect to \( \langle \cdot, \cdot \rangle \). The well-known theta inversion formula states:

1.1 Lemma. Let \( P : V \rightarrow \mathbb{C} \) a homogenous harmonic polynomial of degree \( k \). Then

\[ \sum_{g \in L} P(g + a) e^{\pi i (g+a.g+a)(-1/\tau)} = \frac{i^{-k}}{v(L)} \sqrt{\frac{\tau}{i}} \sum_{g \in L'} P(g) e^{\pi i \{ \langle g,g \rangle \tau + 2 \langle g,a \rangle \}}. \]

We are interested in triples \( (L, P, a) \), such that the series \( \vartheta_{L,P,a} \) equals \( \eta(\tau)^{r+2k} \).

1.2 Lemma. The theta series

\[ \vartheta_{L,P,a} = \sum_{g \in L} P(g + a) e^{\pi i (g+a.g+a) \tau} \]

equals \( \eta(\tau)^{r+2k} \) up to a constant factor if the following conditions are satisfied.
1. $P$ is a homogenous harmonic polynomial of degree $k$.

2. $L$ is an even lattice of rank $r$ (even means $\langle x, x \rangle \in 2\mathbb{Z}$ for $x \in L$).

3. $a \in L'$, $\langle a, a \rangle = (r + 2k)/12$ and $(g + a, g + a) \geq \langle a, a \rangle$ for $g \in L$.

4. There exist a group of isometries $G$ of $L$ and a character $\det : G \to \mathbb{C}^*$ with the property $P(\sigma z) = \det(\sigma) P(z)$ for $\sigma \in G$ and such that for every element $b \in L'/L$, which is not in the $G$-orbit of the image of $a$ in $L'/L$, there exists $\sigma \in G$ such that $\sigma(b) = b$ and $\det(\sigma) \neq 1$.

**Remark.** The constant factor is one if $P(a) = 1$ and if $(g + a, g + a) = \langle a, a \rangle$ only for $g = 0$.

**Proof.** From 2) and 3) we see

$$\vartheta_{L,P,a}(\tau + 1) = e^{\pi i \langle a, a \rangle} \vartheta_{L,P,a}(\tau).$$

The essential point is the transformation formula under $\tau \mapsto -1/\tau$. In the theta transformation formula occurs the series

$$\sum_{g \in L'} P(g) e^{\pi i \{\langle g, g \rangle \tau + 2\langle g, a \rangle\}} = \sum_{b \in L'/L} e^{2\pi i \langle a, b \rangle} \sum_{g \in L} P(g + b) e^{\pi i \langle g + b, g + b \rangle \tau}.$$

The conditions 1) and 4) show that only the $b$ which are in the $G$-orbit of (the image of) $a$ give a non-zero contribution. Hence the right hand side is a constant multiple of $\vartheta_{L,P,a}(\tau)$. We obtain

$$\vartheta_{L,P,a}(-1/\tau) = \varepsilon \sqrt{\frac{\tau + 2k}{1}} \vartheta_{L,P,a}(\tau)$$

with some constant $\varepsilon$. We can assume that $\vartheta_{L,P,a}$ is different from zero. Since $\tau \mapsto -1/\tau$ is an involution we must have $\varepsilon = \pm 1$. Now we see that $\vartheta_{L,P,a}/\eta(\tau)^{r+2k}$ is an entire modular form of weight zero. Because of condition 3) it is regular at the cusp and hence a constant. (As a consequence $\varepsilon = 1$.) This completes the proof of 1.2. \qed

**2. The proof**

We have to show that

$$L := \sqrt{2}M, \quad a = \sqrt{2} \varrho, \quad P(z) = D(z/\sqrt{2})$$

(notations as in section 1) satisfy the assumptions of 1.2.

**Proof of 1.** We use Weyl’s formula

$$e^\varrho \prod_{\alpha > 0} (1 - e^{-\alpha}) = \sum_{\sigma \in W} \det(\sigma) e^{\sigma \varrho},$$
where $W$ denotes the Weyl group. An immediate consequence is

$$\prod_{\alpha>0} (e^{t\langle \alpha/2, \mu \rangle} - e^{-t\langle \alpha/2, \mu \rangle}) = \sum_{\sigma \in W} \det(\sigma) e^{t\langle \sigma \rho, \mu \rangle}.$$ 

Expanding the exponential function and comparing coefficients one obtains

$$\frac{1}{l!} \sum_{\sigma \in W} \det(\sigma) \langle \sigma \rho, \mu \rangle^l = \begin{cases} 0 & \text{for } l < k, \\ \prod_{\alpha>0} \langle \alpha, \mu \rangle & \text{for } l = m, \end{cases}$$

where $k$ denotes the numbers of positive roots. (This equals the degree of $P$.)

It follows that the function $f(\mu) = \prod_{\alpha>0} \langle \alpha, \mu \rangle$ is harmonic, since

$$\Delta f(\mu) = \frac{\langle \rho, \rho \rangle}{(k-2)!} \sum_{\sigma \in W} \det(\sigma) \langle \sigma \rho, \mu \rangle^{k-2} = 0. \quad \Box$$

Proof of 2. One has to use that $\langle \alpha, \alpha \rangle^{-1}$ is integral for every root. \quad \Box

Proof of 3. Since reflexion with respect to a simple root $\alpha$ maps $\rho$ to $\rho - \alpha$, we have $2\langle \alpha, \rho \rangle / \langle \alpha, \alpha \rangle = 1$ for simple roots and therefore $\alpha \in L'$.

The assumption $\langle a, a \rangle = (r+2k)/12$ is a consequence of Freudenthal's formula $\dim g = 24\langle \rho, \rho \rangle$.

The assumption $\langle g+a, g+a \rangle \geq \langle a, a \rangle$ for $g \in L$ follows from the identity

$$\langle \mu, \mu \rangle = \sum_{\alpha \in \Delta} \langle \mu, \alpha \rangle^2,$$

which is evident from the definition of the Killing form and which implies

$$\langle \mu + \rho, \mu + \rho \rangle - \langle \rho, \rho \rangle = 2 \sum_{\beta > 0} \left\{ \left( \frac{\langle \mu, \beta \rangle}{4} \right)^2 - \frac{1}{16} \right\}.$$ 

Since $\langle \mu, \beta \rangle$ is an integral multiple of $1/2$ for $\mu \in M$ and roots $\beta$, we obtain that the sum is non-negative.

Proof of 4. For the group $G$ we take the Weyl group $W$ and for the character $\det$ the determinant. The formula $P(\sigma z) = \det \sigma P(z)$ is obvious. We use now the lattice

$$\Lambda = 2M = \sqrt{2}L = \sum_{\alpha \in \Delta} \mathbb{Z} \tilde{\alpha} \quad \text{where} \quad \tilde{\alpha} = \frac{2}{\langle \alpha, \alpha \rangle} \alpha.$$ 

It is known that this lattice is generated by the $\tilde{\alpha}$ for simple $\alpha$. We recall $\langle \rho, \tilde{\alpha} \rangle = 1$ for simple $\alpha$.

We will work with the affine Weyl group $W^{aff} = W \cdot \Lambda$ which acts on $V = \Lambda \otimes \mathbb{R}$ by $wx = \sigma x + \lambda$. We set $\varepsilon(w) = \det(\sigma)$. The following Lemma is equivalent to condition 4 in 1.2.
2.1 Lemma. Let $b \in \Lambda'$ an element such that $2b$ is not in the $W^\text{aff}$-orbit of $2 \varrho$. Then there exists $w \in W^\text{aff}$ such that $w(2b) = 2b$ and $\varepsilon(w) = -1$.

Proof. We denote by $\alpha_1, \ldots, \alpha_r$ the simple roots and by $\delta$ the highest root (the dominant weight of the adjoint representation). Since the eigenvalue of the Casimir operator for the adjoint representation is one, we have
\[
\langle \varrho + \delta, \varrho + \delta \rangle - \langle \varrho, \varrho \rangle = 1.
\]
Recall that the affine Weyl-group $W^\text{aff}$ acts simply transitive on the affine Weyl chambers, which are the connected components of the complement of the union of the affine spaces $\langle \alpha, x \rangle = n$, where $\alpha$ is a root and $n$ an integer. The standard Weyl-chamber is
\[
C = \{ x \in V; \langle x, \alpha \rangle > 0 \text{ for } 1 \leq i \leq r \text{ and } \langle x, \delta \rangle < 1 \}.
\]
Now we take $b \in \Lambda'$ such that $2b$ is not in the $W^\text{aff}$-orbit of $2 \varrho$. We will prove that $2b$ is on the boundary of a Weyl chamber. Since the reflection along a wall is contained in $W^\text{aff}$, this will prove 2.1.

We argue by contradiction and assume that $2b$ is in the interior of the standard Weyl chamber $C$, especially $\langle 2b, \alpha_i \rangle > 0$. Since $\langle b, \tilde{\alpha} \rangle \in \mathbb{Z}$, we get $\langle 2b, \tilde{\alpha}_i \rangle = 2k_i$, $k_i \geq 1$ (integer).

Next we use that the highest root $\delta$ is of the form
\[
\delta = \sum_{i=1}^{r} m_i \alpha_i, \quad m_i \geq 1 \text{ (integer)}.
\]
We will need a little more. If there are roots of different length (types $B_n, C_n, F_4, G_2$), then $m_i \geq 2$ for all short roots $\alpha_i$ and even more $m_i \geq 3$ for the type $G_2$. We obtain
\[
\langle 2b, \delta \rangle = \sum_{i=1}^{r} m_i \frac{\langle \alpha_i, \alpha_i \rangle}{2} 2k_i \geq m_l \langle \alpha_l, \alpha_l \rangle + \sum_{i=1}^{r} m_i \langle \alpha_i, \alpha_i \rangle = m_l \langle \alpha_l, \alpha_l \rangle + 2 \langle \varrho, \delta \rangle = m_l \langle \alpha_l, \alpha_l \rangle + 1 - \langle \delta, \delta \rangle.
\]
By the above remark about the $m_i$ we get $\langle 2b, \delta \rangle \geq 1$ in all cases. This implies that $2b$ is not in the interior of the standard affine Weyl chamber, which contradicts to our assumption. This completes the proof of 1.2.

Finally we have to show the assumption of the remark in 1.2. We have to show that
\[
\sum_{\beta > 0} \left\{ \left( \frac{\langle \alpha, \beta \rangle}{4} + \frac{1}{16} \right)^2 - \frac{1}{16} \right\} = 0 \quad (\alpha \in M)
\]
implies $\alpha = 0$. Since $2\langle \alpha, \beta \rangle \in \mathbb{Z}$, the sum is 0 if and only if $\langle \alpha, \beta \rangle$ is contained in $\{0, -1/2\}$ for all $\beta > 0$. We have to show $\alpha = 0$. This follows from the following lemma applied to $\lambda = -2\alpha \in \Lambda$. 

\[\square\]
2.2 Lemma. Let $\lambda \in \Lambda$ and $\langle \lambda, \alpha \rangle \in \{0, 1\}$ for all positive roots $\alpha$. Then $\lambda = 0$.

Proof. It is sufficient to show that $\langle \lambda, \alpha \rangle = 0$ for all simple $\alpha$. We argue by contradiction and assume $\langle \lambda, \alpha \rangle = 1$ for some simple root $\alpha$. Since the relation $\langle \lambda, \delta \rangle \in \{0, 1\}$ holds also for the highest root

$$\delta = \sum_{i=1}^{r} m_{i} \alpha_{i}, \quad m_{i} \geq 1 \text{ (integer)},$$

we obtain that $\langle \lambda, \delta \rangle = 1$ and $\langle \lambda, \alpha_{i} \rangle = 1$ for precisely one simple root $\alpha_{i}$ (with corresponding $m_{i} = 1$) and $\langle \lambda, \alpha_{j} \rangle = 0$ for the others. Hence $\lambda$ is a vertex of the standard affine Weyl chamber $C$. The translation $Tx = x - \lambda$ belongs to the affine Weyl group $W^{\text{aff}}$. Hence $T\lambda = 0$ is a vertex of the chamber $TC$. We consider the (unbounded) standard Weyl chamber of the finite Weyl group $W$

$$\hat{C} = \{ x \in V; \quad \langle x, \alpha_{i} \rangle > 0 \text{ for } 1 \leq i \leq r \}.$$ 

We choose an element $\sigma$ from the (finite) Weyl group $W$, which transforms $TC$ into $\hat{C}$. So $\sigma TC$ is one of the affine Weyl chambers chambers contained in $\hat{C}$. The only affine Weyl chamber in $\hat{C}$, which contains 0 in its closure, is the affine standard chamber $C$. Since $\sigma T\lambda = \sigma 0 = 0$, we must have $\sigma TC = C$. This implies $\sigma T = \text{id}$, which is not possible. \qed

Final remark. The formula $\langle \varrho + \delta, \varrho + \delta \rangle - \langle \varrho, \varrho \rangle = 1$ shows that $2\varrho$ lies in the interior of the chamber $C$. Hence the conjugates of $2\varrho$ under $W^{\text{aff}}$ are all different. Therefore the sum $\sum_{b \in L'/L}$ in the proof of lemma 1.2, where only the $b$ in the orbit of $2\varrho$ survive, can be written as a sum over the Weyl group $W$. The equality $\varepsilon = 1$ at the end of section one now implies the formula

$$\sum_{\sigma \in W} \det \sigma \cdot e^{4\pi i \langle \sigma \varrho, \varrho \rangle} = \sqrt{2} r^{k} v(M).$$