The Göpel variety

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Abstract

In this paper we will prove that the six-dimensional *Göpel variety* in P^{134} is generated by 120 linear, 35 cubic and 35 quartic relations. This result was already obtained in [RS], but the authors used a statement in [Co] saying that the Göpel variety set theoretically is generated by the linear and cubic relations alone. Unfortunately this statement is false. There are 120 extra points. Nevertheless the results stated in [RS] are correct. There are required several changes that we will illustrate in some detail.

Introduction

In this paper we will consider the so-called, in **[RS], Sect. 4**, *Göpel variety*. It is an interesting six dimensional variety arising in connection with the moduli space of principally polarized abelian arieties of level 2 and genus 3. In fact, these two varieties are birational equivalent, since the Göpel variety parameterizes the coefficients of the Coble quartic. Moreover, this can be seen as the the GIT quotient describing the moduli space of seven points in P^2, i.e. the moduli space of Del Pezzo surfaces of degree 2 , cf. [DO] . It has also a realization as ball quotient, cf. [Ko]. We start with a description.

We consider the vector space of "theta characteristics" $\mathbb{F}_2^{2g}$ equipped with the standard symplectic pairing. A *Göpel group* is a maximal isotropic subspace G. Its dimension is g. Every Göpel group has exactly only translate $M = a+G$ that consists of even characteristics. We ssociate two modular forms (Sect. 1)

$$\vartheta_G = \prod_{m\in M} \vartheta[m], \quad s_G = \prod_{m\notin M} \vartheta[m].$$

These are modular forms of of level two. In the cases $g \geq 3$ the multipliers are trivial. The graded algebras and their associated projective varieties

$$A = \mathbb{C}[\ldots\vartheta_G\ldots], \quad B = \mathbb{C}[\ldots s_G\ldots]$$

are of great interest. We are mainly interested in the case $g = 3$. The 6-dimensional variety $\mathrm{proj}(B)$ is the *Göpel variety*. It has been studied several times. There are 135 Göpel groups. Hence this variety sits in P^{134}. As a consequence of the quartic Riemann theta relations one obtains 120 linear relations between the s_G. Hence the Göpel variety can be considered as a subvariety of P^{14}. In the literature there have been described also cubic [Co], [DO] and quartic relations [RS] and it is stated in this paper that these linear cubic and quartic relations generate the full ideal of relations, cf. [RS], Theorem 5.1. As far as we understand, the proof of this theorem depends on a statement of Coble that the Göpel variety set theoretically is generated by the linear and cubic relations alone. **(The reference to Coble is in [RS] on p. 22.)** Unfortunately this statement of Coble is false. There are 120 extra points that satisfy the linear and cubic relations but not all quartic relations. Nevertheless it remains true that the linear, cubic and quartic relations generate the full vanishing ideal (hence a prime ideal). The proof is quite involved, so we decided to reopen the story again and to give a complete proof. Needless to say that nowadays computers have been a basic tool. For sake of completeness we restate the part of Theorem 5.1 from [RS] that we will "reprove".

Theorem. *The six-dimensional Göpel variety $\mathcal{G}$ has degree 175 in P^{14}. The homogeneous coordinate ring of $\mathcal{G}$ is Gorenstein and its defining prime ideal is minimally generated by 35 cubics and 35 quartics.*

As in [RS] computer computations are necessary. We used the computer algebra system MAGMA [BC].

1. Thetanullwerte

In our context, theta characteristics in genus g are elements of $\mathbb{F}_2^{2g}$. Usually they are written as columns. Sometimes we have to associate to an element of $\mathbb{F}_2$ an integer. This is done by means of a section

$$\mathbb{F}_2 \xrightarrow{\iota} \mathbb{Z} \longrightarrow \mathbb{F}_2$$

We take

$$\iota(x) = \begin{cases} 0 \in \mathbb{Z} & \text{if } x = 0 \text{ in } \mathbb{F}_2, \\ 1 \in \mathbb{Z} & \text{if } x = 1 \text{ in } \mathbb{F}_2. \end{cases}$$

A characteristic m is called even if

$$a'b = 0 \quad \text{where} \quad m = \begin{pmatrix} a \\ b \end{pmatrix}.$$

There are $(2^g+1)2^{g-1}$ even characteristics. The full modular group $\Gamma_g := \mathrm{Sp}(g,\mathbb{Z})$ acts on the set of (even) characteristics by

$$M\{m\} = M'^{-1}m + \begin{pmatrix}(CD')_0\\(AB')_0\end{pmatrix} \qquad (\mathrm{mod}\ 2).$$

Here we denote by S_0 the diagonal of the matrix S written as column. This action is double transitive on the set of all even characteristics.

Thetanullwerte are defined by

$$\vartheta[m](Z) = \sum_{n\in\mathbb{Z}^n} e^{\pi \mathrm{i}\{Z[n+a/2]+b'(2n+a)\}}, \qquad m = \begin{pmatrix}a\\b\end{pmatrix}.$$

Here $m \in \mathbb{F}_2^{2g}$ is a characteristic **(to be precise, one has to replace m be a vector in $\mathbb{Z}^{2g}$ using the section ι)**. A thetanullwert is different from zero if and only if m is even. Hence we have $(2^g+1)2^{g-1}$ non-zero thetanullwerte. The thetanullwerte satisfy the *quartic Riemann relations.* They are of the following type. For each pair of characteristics $p, q \in \mathbb{F}^{2g}$ one has

$$\sum_{m\ \text{even}} \pm\vartheta[m]\vartheta[m+p]\vartheta[m+q]\vartheta[m+p+q] = 0.$$

We will not describe the quite delicate signs here.

We number the 36 even theta characteristics in genus 3 as follows:

1	$(0,0,0,0,0,0)$	13	$(0,0,0,0,1,0)$	25	$(0,0,0,1,0,1)$
2	$(1,0,0,0,0,0)$	14	$(1,0,0,0,1,0)$	26	$(0,1,0,1,0,1)$
3	$(0,1,0,0,0,0)$	15	$(0,0,1,0,1,0)$	27	$(1,0,1,1,0,1)$
4	$(1,1,0,0,0,0)$	16	$(1,0,1,0,1,0)$	28	$(1,1,1,1,0,1)$
5	$(0,0,1,0,0,0)$	17	$(0,0,0,1,1,0)$	29	$(0,0,0,0,1,1)$
6	$(1,0,1,0,0,0)$	18	$(1,1,0,1,1,0)$	30	$(1,0,0,0,1,1)$
7	$(0,1,1,0,0,0)$	19	$(0,0,1,1,1,0)$	31	$(0,1,1,0,1,1)$
8	$(1,1,1,0,0,0)$	20	$(1,1,1,1,1,0)$	32	$(1,1,1,0,1,1)$
9	$(0,0,0,1,0,0)$	21	$(0,0,0,0,0,1)$	33	$(0,0,0,1,1,1)$
10	$(0,1,0,1,0,0)$	22	$(1,0,0,0,0,1)$	34	$(1,1,0,1,1,1)$
11	$(0,0,1,1,0,0)$	23	$(0,1,0,0,0,1)$	35	$(1,0,1,1,1,1)$
12	$(0,1,1,1,0,0)$	24	$(1,1,0,0,0,1)$	36	$(0,1,1,1,1,1)$

2. Thetanullwerte and modular forms

Recall that the real symplectic group

$$\mathrm{Sp}(g,\mathbb{R}) = \left\{ M \in \mathrm{GL}(2g,\mathbb{R});\quad M'IM = I \right\} \qquad \left(I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \right)$$

acts on the generalized half plane

$$\mathcal{H}_g := \left\{ Z = X + \mathrm{i}Y;\quad Z = Z',\ Y > 0 \text{ (positive definite)} \right\}$$

by

$$MZ = (AZ+B)(CZ+D)^{-1}, \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let $\Gamma_g := \mathrm{Sp}(g,\mathbb{Z})$ be the Siegel modular group. The principal congruence subgroup of level l is

$$\Gamma_g[l] := \mathrm{kernel}(\mathrm{Sp}(g,\mathbb{Z}) \longrightarrow \mathrm{Sp}(g,\mathbb{Z}/l\mathbb{Z}))$$

and Igusa's subgroup is

$$\Gamma_g[l,2l] := \left\{ M \in \Gamma_g[l];\quad AB'/l \text{ and } CD'/l \text{ have even diagonal} \right\}.$$

For even l, Igusa's subgroup is a normal subgroup of Γ_g.

For each $M \in \mathrm{Sp}(g,\mathbb{R})$ and each $Z \in \mathcal{H}_g$ we take a holomorphic root $\sqrt{\det(CZ+D)}$. To be concrete, we make the choice such that for $Z = \mathrm{i}E$ it is the principal value of the square root. Let $\Gamma \subset \mathrm{Sp}(g,\mathbb{Z})$ be a subgroup of finite index and r an integer. A multiplier system of weight $r/2$ is map $v : \Gamma \to S^1$ such that $v(M)\sqrt{\det(CZ+D)}^{r/2}$ is an automorphy factor. This means that

$$(f|M)(Z) := v(M)^{-1}\sqrt{\det(CZ+D)}^{-r/2} f(MZ), \quad M \in \Gamma,$$

is an action of Γ (from the right). A modular form of weight $r/2$ and multiplier system v on Γ is a holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ with the property

$$f(MZ) = v(M)\sqrt{\det(CZ+D)}^{r} f(Z), \qquad M \in \Gamma.$$

In the case $g = 1$ the usual regularity condition at the cusps has to be added. We denote the space of all these forms by $[\Gamma, r/2, v]$. In the case that $r/2$ is integral and that v is trivial we omit v in this notation. For given (r_0, v_0) we can define the algebra

$$A(\Gamma,(r_0,v_0)) := \bigoplus_{r\in\mathbb{Z}} [\Gamma, rr_0, v^r].$$

If it is clear which starting weight and multiplier system are used, we write

$$A(\Gamma) = A(\Gamma, (r_0, v_0)).$$

By the theory of Satake compactification, $A(\Gamma)$ is a finitely generated algebra whose associated projective variety, considered as complex space, is biholomorphic equivalent to the Satake compactification of $\mathcal{H}_g/\Gamma$,

$$\operatorname{proj}(A(\Gamma)) = X(\Gamma) = \overline{\mathcal{H}_g/\Gamma}.$$

There is a fundamental multiplier system on the theta group $\Gamma_\vartheta := \Gamma_g[1,2]$, of 8-th roots of unity $v(M)$ for $M \in \Gamma_\vartheta$, such that

$$\vartheta[0] \in [\Gamma_\vartheta, 1/2, v_\vartheta].$$

More generally, a formula

$$\vartheta[M\{m\}](MZ) = v(M,m)\sqrt{\det(CZ+D)}\vartheta[m](Z) \quad \text{for all } M \in \Gamma_g,$$

with a certain system of numbers $v(M,m)$ of 8th roots of unity holds.

It is well-known that

$$\vartheta[m] \in [\Gamma_g[4,8], 1/2, v_\vartheta] \quad \text{for all } m.$$

We use $(1/2, v_\vartheta)$ as starting weight and multiplier system to define the ring

$$A(\Gamma_g[4,8]) := \bigoplus_{r\in\mathbb{Z}} [\Gamma_g[4,8], r/2, v_\vartheta^r].$$

The fundamental lemma of Igusa [Ig1] says that this ring is the normalization of the ring generated by the thetanullwerte. In the cases $g \le 2$ both rings agree [Ig2].

3. Thetanullwerte of second kind

The 2^g thetanullwerte of second kind are defined by

$$f_a(Z) := \vartheta\begin{bmatrix} a \\ 0 \end{bmatrix}(2Z) = \sum_{n\in\mathbb{Z}^g} e^{2\pi \mathrm{i} Z[n+a/2]}.$$

They are modular forms on the group $\Gamma_g[2,4]$ with a joint multiplier system v_Θ on $\Gamma_g[2,4]$. One has $v_\Theta^2 = v_\vartheta^2$ on $\Gamma_g[2,4]$ but v_Θ and v_ϑ are different there. We take $(1/2, v_\Theta)$ as starting weight and multiplier system to define

$$A(\Gamma_g[2,4]) := \sum_{r\in\mathbb{Z}} [\Gamma_g[2,4], r/2, v_\Theta^r].$$

In the cases $g = 1, 2$ one has

$$A(\Gamma_g[2,4]) = \mathbb{C}[\dots f_a \dots]$$

and this ring is a polynomial ring in 2 or 4 variables. Hence the Satake compactification is P^1 or P^3. In the case $g = 3$ Runge [Ru] determined the structure of this ring.

To describe Runge's result, we associate to each $a \in \mathbb{F}_2^3$ a variable $F = F(a)$. We number them as follows:

1) (0,0,0) 2) (1,0,0) 3) (0,1,0) 4) (1,1,0)
5) (0,0,1) 6) (1,0,1) 7) (0,1,1) 8) (1,1,1)

3.1 Theorem (Runge). *There exists a homogenous polynomial $R \in \mathbb{C}[F_1 \dots F_8]$ of degree 16 such that R generates the kernel of the natural homomorphism $F_a \mapsto f_a$. This homomorphism is surjective. Hence we have*

$$A(\Gamma_3[2,4]) := \sum_{r \in \mathbb{Z}} [\Gamma_3[2,4], r/2, v_\Theta^r] = \mathbb{C}[\dots f_a \dots] = \mathbb{C}[\dots F(a) \dots]/(R).$$

The subring of forms of integral weight is

$$A^{(2)}(\Gamma_3[2,4])) = \mathbb{C}[\dots f_a f_b \dots] = \mathbb{C}[\dots \vartheta[m]^2 \dots].$$

All relations between the $\vartheta[m]^2$ are contained in the ideal generated by the Riemann relations.

Runge also described the algebra of modular forms of even weight for $\Gamma_3[2]$. We use the notation (for integral k only)

$$A^{(k)}(\Gamma_g[2]) = \sum_{r \equiv 0 \bmod k} [\Gamma_g[2], r]$$

and omit (k) if $k = 1$. From Runge' s result a modified structure theorem can be derived which we are going to discuss now.

A linear subspace $G \subset \mathbb{F}_2^{2g}$ is called a *Göpel group* if it is a maximal isotropic subspace with respect to the symplectic pairing

$$\langle m, n \rangle = a'\beta + b'\alpha \qquad m = \begin{pmatrix} a \\ b \end{pmatrix}, \ n = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

One can show that each Göpel group has a unique translate $M = a + G$ that consists of even characteristics. The standard example is the subspace consisting of all $\binom{a}{b}$, $b = 0$. Each Göpel group contains 2^g elements. We define

$$\vartheta_G = \prod_{m \in M} \vartheta[m].$$

These are modular forms on $\Gamma_g[2]$ but possibly with non-trivial multipliers. But the following holds. Assume that $k2^{g-1}$ is divisible by 4. Then ϑ_G^k has trivial multipliers on $\Gamma_g[2]$. We denote the smallest k with this property by k_g. Then we have

$$k_g = \begin{cases} 4 & \text{if } g = 1, \\ 2 & \text{if } g = 2, \\ 1 & \text{if } g > 2. \end{cases}$$

More precisely we have

3.2 Lemma. *We have*

a) *The forms $\vartheta_G(\tau)^{k_g}$ are in $[\Gamma_g(2), k_g 2^{g-1}]$.*

b) *The forms $\vartheta_G(\tau)^{k_g}$ are linearly independent.*

Proof. The first statement is a consequence of the results of [Ig2] and it is reduced to the fact that the matrix of characteristics M satisfy the congruences

$$k_g MM' \equiv 0 \bmod 2, \qquad \operatorname{diag}(k_g MM') \equiv 0 \bmod 4.$$

About the linear independence, we know that the group Γ_g acts transitively on the even cosets of Göpel groups. Moreover, among the $\vartheta_G(\tau)^{k_g}$ only one (namely the standard example of all m with $b = 0$) has the Fourier coefficient $a(0) \neq 0$. □

In low genera, these monomials are of particular interest. In the case $g = 1$ Göpel groups have order 2. There are three Göpel groups which are generated by the three elements of $\mathbb{F}^2$ that are different from 0. The corresponding ϑ_G are

$$\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4 \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4, \quad \vartheta\begin{bmatrix}0\\0\end{bmatrix}^4 \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4, \quad \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4 \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4.$$

They generate the algebra $A^{(4)}(\Gamma_1[2])$.

Now we consider $g = 2$. There are 15 Göpel groups. Hence we obtain 15 modular forms $\vartheta(G)^2$ of weight 4 one $\Gamma_2[2]$. They have trivial multipliers. It is known that these 15 modular forms generate the ring $A^{(4)}(\Gamma_2[2])$. The associated projective variety is the Satake compactification $\overline{\mathcal{H}_2/\Gamma_2[2]}$. It equals $\operatorname{proj} A^{(2)}(\Gamma_2[2])$. Due to a result of Igusa [Ig3] this algebra is generated by 5 forms of weight 2 that satisfy a quartic relation. Hence $\operatorname{proj} A^{(4)}(\Gamma_2[4])$ can be thought as image of the Igusa quartic in P^4 followed by the Veronese embedding $P^4 \to P^{14}$. Hence the variety in P^{14} can described as the intersection of 51 quadrics, 50 define the Veronese embedding and one is induced by Igusa's quartic.

Let us assume $g = 3$. There are 135 Göpel groups and hence 135 modular forms ϑ_G of weight 4. They are linearly independent. It is know through results of Runge [Ru2] that they generate the algebra of modular forms whose weight is divisible by 4

$$A^{(4)}(\Gamma_3[2]) = \mathbb{C}[\ldots \vartheta_G \ldots].$$

Hence $\operatorname{proj}(\mathbb{C}[\dots\vartheta_G\dots])$ is the Satake compactification $\overline{\mathcal{H}_3/\Gamma_3[2]}$. Actally Runge proved more. The algebra $A^{(4)}(\Gamma_3[2])$ is generated by 15 forms $\vartheta[m]^4$ and 15 of the ϑ_G.

4. Reciprocal maps

Let $\Gamma \subset \operatorname{Sp}(g,\mathbb{Z})$ be a subgroup of finite index. Then $\mathcal{H}_g/\Gamma$ carries a structure as quasi-projective algebraic variety. Let $f_0,\dots,f_N$ be holomorphic modular forms on Γ, different from 0, with the same weight and the same multipliers. Then we can consider the projective variety

$$\operatorname{proj}(\mathbb{C}[f_0,\dots,f_N]) \subset P^N.$$

We use the notation $A = \mathbb{C}[f_0,\dots,f_N]$. There is a natural rational map

$$\mathcal{H}_g/\Gamma \longrightarrow \operatorname{proj} A, \quad Z \longmapsto [f_0(Z),\dots,f_N(Z)].$$

It is regular outside the set of joint zeros of the forms f_i. But the full (=biggest) domain of regularity can be larger of course. Now we make the assumption that there is given a modular form φ on Γ (with some multiplier system) such that

$$g_i = \frac{\varphi}{f_i}$$

is a holomorphic modular form. One can take for example the product of all f_i. Then we define $B = \mathbb{C}[g_1,\dots,g_N]$. This ring depends (up to canonical isomorphism) not on the choice of φ. We can consider

$$\mathcal{H}_g/\Gamma \longrightarrow \operatorname{proj} B, \quad Z \longmapsto [g_0(Z),\dots,g_M(Z)].$$

The diagram

$$\begin{tikzcd} & \operatorname{proj} A \arrow[dd] \\ \mathcal{H}_g/\Gamma \arrow[ur] \arrow[dr] & \\ & \operatorname{proj} B \end{tikzcd}$$

commutes where the vertical map is induced by the rational map

$$P^N \longrightarrow P^N, \quad [x_0,\dots,x_N] \longmapsto [x_0^{-1},\dots,x_N^{-1}].$$

It is a birational map. On the level of the graded algebras A, B it is associated to the homomorphisms of graded algebras

$$\begin{aligned} B \longrightarrow A, &\quad g_i \longmapsto \frac{f_0 \cdots f_N}{\varphi} g_i, \\ A \longrightarrow B, &\quad f_i \longmapsto g_0 \cdots g_N f_i. \end{aligned}$$

We want to apply this for the system of modular forms $\vartheta_G^{k_g}$. We will describe the corresponding rings and varieties in the cases $g \leq 3$. We define

$$s_G := \frac{\Theta_g}{\vartheta_G} \quad \text{where} \quad \Theta_g = \prod_m \theta[m].$$

So our pair of reciprocal rings is

$$A = \mathbb{C}[\dots \vartheta_G^{k_g} \dots], \quad B = \mathbb{C}[\dots s_G^{k_g} \dots].$$

Genus 1

We introduced already in the case $n = 1$ for f_i the three modular forms

$$f_0 = \vartheta\begin{bmatrix}0\\0\end{bmatrix}^4 \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4, \quad f_1 = \vartheta\begin{bmatrix}0\\0\end{bmatrix}^4 \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4, \quad f_2 = \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4 \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4.$$

They generate the algebra

$$A = A^{(4)}(\Gamma_1[2]).$$

We can take

$$\varphi = \left(\vartheta\begin{bmatrix}0\\0\end{bmatrix} \vartheta\begin{bmatrix}0\\1\end{bmatrix} \vartheta\begin{bmatrix}1\\0\end{bmatrix}\right)^4.$$

Hence the functions g_i are

$$g_0 = \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4, \quad g_1 = \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4, \quad g_2 = \vartheta\begin{bmatrix}0\\0\end{bmatrix}^4.$$

They generate the algebra

$$B = A^{(2)}(\Gamma_1[2]).$$

The relation in A is

$$f_0 f_1 = f_1 f_2 + f_1 f_2.$$

Hence $\operatorname{proj} A$ is a quadric in P^2 (isomorphic to P^1). The relation in B is $g_2 = g_0 + g_1$. This is a linear subspace P^2 (isomorphic to P^1). Hence the map $\operatorname{proj} B \to \operatorname{proj} A$ is biholomorphic in this case. It is induced by the natural embedding $A \subset B$ and it describes an isomorphism of a quadric in P^2 onto a linear P^1. Moreover the relations in A and B are reciprocally induced.

Genus 2

In the case $g = 2$ we have $k_g = 2$. We mentioned already that the 15 forms ϑ_G^2 generate $A^{(4)}(\Gamma_2[2])$ and that the associated variety is a copy of the Igusa quartic. Now we study the reciprocal variety. The s_G are products of 6 theta constants. Hence s_G^2 are of weight 6. So we have 15 forms s_G^2 of weight 6. They are not linearly independent. They span a space of dimension 5. There is one additional cubic relation. This defines a copy of the Segre cubic [GS]. We obtain a birational map from the Igusa quartic and the Segre cubic. Actually one knows that they are dual hypersurfaces. We give some details.

4.1 Proposition. *In genus two, the reciprocal map induces a rational map*

$$\operatorname{proj} A(\Gamma_2[2]) \to P^{14}$$

which is birational unto its image. The (closure of the) image is defined by 10 trinomials such as

$$s_{G_1}^2 - s_{G_2}^2 + s_{G_3}^2 = 0$$

and one cubic binomial such as

$$s_{G_1}^2 s_{G_2}^2 s_{G_3}^3 = s_{G_4}^2 s_{G_5}^2 s_{G_6}^2.$$

The closure of the image is isomorphic to the Segre cubic.

Proof. This is consequence of the results in [GSM] and [RSS]. □

We will explain, as one can obtain the relations defining the Segre cubic, using Riemann relations.

We know that, in genus two, there are two types of Riemann's quartic relations :

$$1) \quad \sum_{m \text{ even}} \pm\vartheta_m^4 = 0,$$

$$2) \quad \sum_{m,n \text{ even}} \pm\vartheta_m^2\vartheta_n^2 = 0.$$

The relations of the form 2) can be obtained in this way: for each one dimensional totally isotropic subspace N there are 3 even cosets $N + a = \{a, n_1 + a, n_2 + a, n_3 + a\}$. To each coset we associate the monomial $\vartheta_a^2\vartheta_{a+n_1}^2$. The three monomials span a two dimensional space. Thus we have a relation with 3 terms. There are 15 independent such relations.

We shall write $r_1 + r_2 = r_3$ for such a relation. Along $r_1r_2r_3 \neq 0$, we have

$$1/r_2r_3 + 1/r_1r_3 = 1/r_1r_2.$$

Multiplying by $\prod_{m,\text{even}} \theta_m^2$, we get a trinomial relation as in the proposition. At the end we get 10 independent relations. To obtain the cubic relations we use relations of the form 1). A relation looks like

$$\vartheta_{m_1}^4 \pm \vartheta_{m_2}^4 \pm \vartheta_{m_3}^4 \pm \vartheta_{m_4}^4 = 0$$

with m_1, m_2, m_3, m_4 an azygetic quadruplet and we apply an argument similar to the previous. We will illustrate a specific case.

We will list the ten even characteristics.

1	$(0,0,0,0)$	6	$(0,1,1,0)$
2	$(0,0,0,1)$	7	$(1,0,0,0)$
3	$(0,0,1,0)$	8	$(1,0,0,1)$
4	$(0,0,1,1)$	9	$(1,1,0,0)$
5	$(0,1,0,0)$	10	$(1,1,1,1)$

We have the relation

$$\vartheta_1^4 - \vartheta_3^4 - \vartheta_7^4 - \vartheta_{10}^4 = 0.$$

Hence

$$1/(\vartheta_3\vartheta_7\vartheta_{10})^4 - 1/(\vartheta_1\vartheta_7\vartheta_{10})^4 - 1/(\vartheta_1\vartheta_3\vartheta_{10})^4 - 1/(\vartheta_1\vartheta_3\vartheta_7)^4 = 0.$$

Now we multiply this relation by

$$\prod_{i=1}^{10} \vartheta_i^6/((\vartheta_2\vartheta_4\vartheta_5\vartheta_6)^2\vartheta_9^4).$$

We observe that in the first denominators we have

$$(\vartheta_3\vartheta_7\vartheta_9\vartheta_{10})^4(\vartheta_2\vartheta_4\vartheta_5\vartheta_6)^2 = (\vartheta_2\vartheta_3\vartheta_9\vartheta_{10})^2(\vartheta_4\vartheta_5\vartheta_7\vartheta_{10})^2(\vartheta_3\vartheta_6\vartheta_7\vartheta_9)^2$$

In the second

$$(\vartheta_1\vartheta_7\vartheta_9\vartheta_{10})^4(\vartheta_2\vartheta_4\vartheta_5\vartheta_6)^2 = (\vartheta_1\vartheta_5\vartheta_7\vartheta_9)^2(\vartheta_1\vartheta_4\vartheta_9\vartheta_{10})^2(\vartheta_2\vartheta_6\vartheta_7\vartheta_{10})^2$$

and so on. Hence the quotients will be of the required form.

Applying the reciprocal map to the above relations, one can get the 51 quadrics defining $\operatorname{proj} A(\Gamma_2(2))^{(4)}$ in P^{14}. This is the result of a private communication with Bernd Sturmfels.

5. The Göpel variety

Now we study the case $g = 3$. It will be necessary to perform some computer calculation. We used the computer algebra system MAGMA [BC]. We will not reproduce any program here, but we only give the hint to some commands which all refer to MAGMA.

Since $g = 3$ we have $k_g = 1$. We mentioned already that the 135 forms ϑ_G generate $A^{(4)}(\Gamma_3[2])$. Now we want to study the reciprocal variety. For this we have to describe the ring that is generated by the 135 forms s_M. They are products of 28 thetas and hence have weight 14. We recall that they are related to Coble's quartics, in fact, they span a 15 dimensional space that is an irreducible representation of the group Γ_3. This space is spanned also by the 15 coefficients of Coble's quartics, cf. [GS].

We consider the polynomial ring in 36 variables

$$\mathbb{C}[\dots T_m \dots] = \mathbb{C}[T_1, \dots, T_{36}]$$

where we use the ordering of the 36 even characteristics m of Sect. 1. We also associate to each Göpel group variables X_G and Y_G to build two polynomials rings in 135 variables

$$\mathbb{C}[\dots Y_G \dots], \quad \mathbb{C}[\dots X_G \dots].$$

The variables X_G represent the modular forms ϑ_G and the variables Y_G represent the forms s_G. This can be expressed through the homomorphisms

$$\mathbb{C}[\dots X_G \dots] \longrightarrow \mathbb{C}[T_1, \dots, T_{36}], \quad X_G \longmapsto \prod_{m \in M} T_m,$$
$$\mathbb{C}[\dots Y_G \dots] \longrightarrow \mathbb{C}[T_1, \dots, T_{36}], \quad Y_G \longmapsto \prod_{m \notin M} T_m.$$

Here M is the even coset of G. We want to determine the inverse image of the ideal generated by the quartic Riemann relations in $\mathbb{C}[\dots Y_G \dots]$. First we describe some examples in this inverse image. We begin with linear relations. A Riemann relation of the third kind is a relation of the form

$$r_3 = r_1 + r_2$$

Here the r_i are products of 4 theta constants or the negative of it, and the 12 involved characteristics are pairwise different. Then the products $r_i r_j$ belong to the system of X_G and, as a consequence, the complementary expressions

$$Y_{ij} = \frac{\prod T_m}{r_i r_j}$$

belong to the system Y_G. So

$$Y_{23} = Y_{12} + Y_{13}$$

is a linear relation between the Y_G which is in the inverse image of the ideal generated by the quartic Riemann relations.

We construct 630 cubic relations as follows. One can find sixtuplets of Göpel groups $G_1, \dots, G_6$ such that $\vartheta_{G_1}\vartheta_{G_2}\vartheta_{G_3}$ and $\vartheta_{G_4}\vartheta_{G_5}\vartheta_{G_6}$ give identical monomials in the variables $T_1, \dots, T_{36}$. We give an example. (The digits represent theta characteristics in the numbering defined above)

$$\begin{aligned} M_1 &= (3, 12, 13, 19, 24, 28, 30, 35) \\ M_2 &= (2, 8, 9, 12, 30, 32, 33, 36) \\ M_3 &= (5, 12, 16, 20, 21, 26, 30, 34) \end{aligned}$$

$$\begin{aligned} M_4 &= (8, 12, 16, 19, 24, 26, 30, 33) \\ M_5 &= (3, 5, 9, 12, 30, 32, 34, 35) \\ M_6 &= (2, 12, 13, 20, 21, 28, 30, 36) \end{aligned}$$

The reciprocal forms satisfy the same relation

$$s_{G_1}s_{G_2}s_{G_3} = s_{G_4}s_{G_5}s_{G_6}.$$

This gives elements of the kernel of $\mathbb{C}[\dots Y_G \dots] \longrightarrow \mathbb{C}[T_1, \dots, T_{36}]$.

In a similar way one constructs 12 285 quartic relations in the kernel of $\mathbb{C}[\dots Y_G \dots] \longrightarrow \mathbb{C}[T_1, \dots, T_{36}]$. There are two orbits with respect to the full modular group. We give two representatives.

1) Consider the eight Göpel groups $G_1, \dots, G_8$ with even orbits

$$\begin{aligned} M_1 &= (8, 12, 16, 19, 24, 26, 30, 33) \\ M_2 &= (6, 12, 13, 18, 24, 25, 31, 35) \\ M_3 &= (1, 11, 13, 19, 22, 27, 30, 35) \\ M_4 &= (1, 3, 6, 8, 25, 26, 27, 28) \end{aligned}$$

$$\begin{aligned} M_5 &= (3, 12, 13, 19, 24, 28, 30, 35) \\ M_6 &= (6, 8, 11, 12, 22, 24, 25, 26) \\ M_7 &= (1, 8, 18, 19, 26, 27, 30, 31) \\ M_8 &= (1, 6, 13, 16, 25, 27, 33, 35) \end{aligned}$$

Then we have the tautological relation

$$\vartheta_{G_1} \cdots \vartheta_{G_4} = \vartheta_{G_5} \cdots \vartheta_{G_8}.$$

The Göpel forms s_G satisfy the same relation.

2) Consider the eight Göpel groups $G_1, \dots, G_8$. They agree with their even orbits, $G_i = M_i$.

$$
\begin{aligned}
M_1 &= (1, 8, 18, 19, 26, 27, 30, 31)\\
M_2 &= (1, 6, 13, 16, 25, 27, 33, 35)\\
M_3 &= (1, 3, 5, 7, 9, 10, 11, 12)\\
M_4 &= 1, 11, 15, 17, 24, 28, 32, 348)\\
\\
M_5 &= (1, 3, 6, 8, 25, 26, 27, 28)\\
M_6 &= (1, 12, 16, 18, 24, 27, 31, 33)\\
M_7 &= (1, 7, 10, 11, 30, 32, 34, 35)\\
M_8 &= (1, 5, 9, 11, 13, 15, 17, 19)
\end{aligned}
$$

Then we have again a tautological relation

$$\vartheta_{G_1} \cdots \vartheta_{G_4} = \vartheta_{G_5} \cdots \vartheta_{G_8}$$

and the Göpel forms s_G satisfy the same relation.

The space of linear relations has dimension 120. We selected 15 independent $Y_{G_1}, \dots Y_{G_{15}}$. The following list contains their reciprocal forms.

1	$T_3T_{12}T_{13}T_{19}T_{24}T_{28}T_{30}T_{35}$
2	$T_2T_{11}T_{16}T_{17}T_{23}T_{28}T_{31}T_{34}$
3	$T_2T_8T_9T_{12}T_{30}T_{32}T_{33}T_{36}$
4	$T_6T_{10}T_{13}T_{20}T_{23}T_{27}T_{32}T_{33}$
5	$T_7T_{10}T_{13}T_{19}T_{22}T_{27}T_{32}T_{34}$
6	$T_5T_8T_{19}T_{20}T_{21}T_{24}T_{33}T_{34}$
7	$T_4T_6T_9T_{12}T_{29}T_{31}T_{34}T_{35}$
8	$T_2T_7T_{17}T_{20}T_{25}T_{28}T_{30}T_{31}$
9	$T_3T_4T_{15}T_{16}T_{21}T_{22}T_{31}T_{32}$
10	$T_3T_6T_{18}T_{19}T_{25}T_{28}T_{30}T_{31}$
11	$T_4T_{12}T_{16}T_{17}T_{21}T_{27}T_{31}T_{34}$
12	$T_7T_8T_{15}T_{16}T_{23}T_{24}T_{29}T_{30}$
13	$T_7T_9T_{14}T_{20}T_{22}T_{28}T_{31}T_{33}$
14	$T_4T_7T_{14}T_{15}T_{26}T_{28}T_{33}T_{35}$
15	$T_1T_2T_{15}T_{16}T_{23}T_{24}T_{31}T_{32}$

We set $Y_i = Y_{G_i}$. Then we consider the polynomial ring $\mathbb{C}[Y_1, \dots, Y_{15}]$ and the homomorphism

$$\mathbb{C}[Y_1, \dots, Y_{15}] \longrightarrow \mathbb{C}[\dots Y_G \dots] \longrightarrow \mathbb{C}[T_1, \dots, T_{36}], \quad Y_i \longmapsto T_{G_i}.$$

One can check that the cubic relations, considered in $\mathbb{C}[Y_1,\dots,Y_{15}]$, define a 35-dimensional space of relations. We denote the ideal generated by them in $\mathbb{C}[Y_1\cdots Y_{15}]$ by $\mathfrak{a}$. The ideal generated by the cubic and quartic relations is denoted by $\mathfrak{b}$. One can check that the ideal $\mathfrak{b}$ can be generated by the 35 cubic and by 35 quartic relations, **cf. [RS], Sect. 5**. We mention that the quartic relations can be determined also in the following way. They have the property that their product with $Y_1\cdots Y_{15}$ is contained in $\mathfrak{a}$.

By means of the command `quotient(a,b)` we determined the ideal

$$\mathfrak{x}=\mathfrak{a}:\mathfrak{b}=\{P\in\mathbb{C}[Y_1\cdots Y_{15}];\quad P\mathfrak{b}\subset\mathfrak{a}\}.$$

We computed that the zero locus of $\mathfrak{x}$ in P^{14} is zero dimensional. Hence it consists of finitely many points in P^{14}. In the complement of these points the zero loci of $\mathfrak{a}$ and $\mathfrak{b}$ agree.

5.1 Proposition. *The variety of $\mathfrak{a}$ is the union of the variety of $\mathfrak{b}$ and* 120 *isolated points, In particular, the variety of $\mathfrak{a}$ is not irreducible.*

The proof is given by means of a computer. $\square$

One of the 120 points in the coordinates $Y_1,\dots,Y_{15}$ is

$$[-1,0,1,0,1,0,1,-1,1,0,0,0,0,0,0].$$

Fortunately the variety of $\mathfrak{b}$ is irreducible. Even more, $\mathfrak{b}$ is a prime ideal. The proof in [RS] seems to rely on the false assumption that the variety of $\mathfrak{a}$ is irreducible. It seems us to be worthwhile to present a correct proof. First we formulate this result.

5.2 Theorem. *The ideal $\mathfrak{b}\subset\mathbb{C}[Y_1,\dots,Y_{15}]$ is a prime ideal. It is the inverse image of the ideal generated by the quartic Riemann relations with respect to the homomorphism $\mathbb{C}[Y_1,\dots,Y_{15}]\to\mathbb{C}[T_1,\dots,T_{36}]$. The associated projective variety is the six-dimensional Göpel variety.*

Proof. **Similarly to [RS] during the proof of Theorem 5.1** we start with a Noether normalization which can be computed over $\mathbb{Q}$ by means of the command `NoetherNormalization`. The ideal $\mathfrak{b}$ is defined over $\mathbb{Q}$. The ring $\mathbb{Q}[Y_1,\dots,Y_{15}]/\mathfrak{b}_{\mathbb{Q}}$ is integral over the polynomial ring in the following 7 variables

$3Y_1+2Y_2+Y_3-Y_6+Y_9,\quad 9Y_1+4Y_2+3Y_3+3Y_4+Y_5-Y_8+Y_{10},\quad -9Y_1-2Y_2-3Y_3-3Y_4-2Y_5+Y_7-Y_{10}+Y_{11},\quad 11Y_1+5Y_2+4Y_3+4Y_4+Y_5-Y_6+Y_9+Y_{10}-Y_{11}+Y_{12},\quad 24Y_1+9Y_2+7Y_3+9Y_4+2Y_5-3Y_6-Y_7-Y_8+2Y_9+2Y_{10}-2Y_{11}+Y_{12}+Y_{13},\quad 9Y_1+Y_2+4Y_3+4Y_4+4Y_5+2Y_6-Y_7+Y_8+Y_{10}-Y_{11}+Y_{12}+Y_{14},\quad -25Y_1-8Y_2-6Y_3-9Y_4-Y_5+4Y_6+Y_7+2Y_8-Y_9-Y_{10}+2Y_{11}-Y_{13}+Y_{14}+Y_{15}.$

Computing Hilbert polynomials (`HilbertSeries`) shows that the ring $\mathbb{Q}[Y_1,\dots,Y_{15}]/\mathfrak{b}_{\mathbb{Q}}$ is free over this polynomial ring and hence is a Cohen-Macaulay ring (the argument is similar as in the proof of Theorem 5.1 in [RS]). Since there can be found regular points (see below), the ideal $\mathfrak{b}$ is a radical ideal and it is equi-dimensional. It remains to show that $\mathrm{proj}(\mathbb{C}[Y_1,\dots,Y_{15}]/\mathfrak{b})$ has only one irreducible component. To prove this, we intersect it with an 8-dimensional linear subspace of P^{14}. Actually we take the zero locus of the last 6 linear forms in the above Noether basis. These 6 forms and $\mathfrak{b}$ generate an ideal $\mathfrak{c}$. It is defined over $\mathbb{Q}$. One can check that it is reduced and that the dimension of $\mathrm{proj}(\mathbb{Q}[Y_1,\dots,Y_{15}]/\mathfrak{c}_{\mathbb{Q}})$ is zero. A zero dimensional scheme is called a cluster. The degree of this cluster is 175 **(compare [RS], p. 21)**. Since all irreducible components of the variety of $\mathfrak{b}$ are 6-dimensional, this cluster must hit every irreducible component. We claim now that the 175 points of this cluster are all regular points of the variety of $\mathfrak{b}$. Their explicit determination seems not to be possible, since they have coordinates in $\bar{\mathbb{Q}}$. The situation gets better if we perform the calculation not in $\bar{\mathbb{Q}}$ but in characteristic p. We took $p = 557$. This reduction is allowed by a standard flatness argument, since the Hilbert polynomials in characteristic zero and 557 agree. The points in the algebraic closure of $\mathbb{F}_p$ can be computed by means of the command `RationalPoints`. They consist of 4 Galois orbits. Actually the orbits are 21 points in $\mathbb{F}_{p^{21}}$, 22 points in $\mathbb{F}_{p^{22}}$, 32 points in $\mathbb{F}_{p^{32}}$, 100 points in $\mathbb{F}_{p^{100}}$.

Since we have all points explicitly, it is no problem to show that all 175 points are regular points of the variety of $\mathfrak{b}$. Hence each of them is contained in only one of the components of the variety of $\mathfrak{b}$. In a final step we will show that all 175 points are contained in the same component.

We need some more information. The forms ϑ_G and s_G have level two and trivial multipliers. Hence they must be expressible by the thetas f_a of second kind. This must be reflected by homomorphims

$$\mathbb{C}[\dots X_G \dots] \longrightarrow \mathbb{C}[\dots F_a \dots], \quad \mathbb{C}[\dots Y_G \dots] \longrightarrow \mathbb{C}[\dots F_a \dots].$$

Both homomorphisms can be constructed explicitly but the expressions of s_G in terms of the f_a are extremely big (polynomials of degree 28 in 8 variables) and seem to be useless for further computations. So we concentrate on the first of these homomorphisms and explain, how the ϑ_G can be expressed as polynomials (of degree 8) in the f_a. This can be done as follows.

Consider a quartic Riemann relation of the kind $r_3 = r_1 + r_2$ where the r_i are products of 4 theta constants $\vartheta[m]$ such that the 12 involved theta constants are pairwise different. Then consider

$$2r_1r_2 = r_3^2 - r_1^2 - r_2^2,$$

It turns out that r_1r_2 is one of the ϑ_G. So this can be expressed by squares of theta constants, which can be expressed by the f_a.

One can check that each of the 175 points is in general position in the sense that none of the 135 linear forms vanish on it. Hence we can invert the values of these 135 linear forms and consider the reciprocal point $x = [x_1, \dots, x_{135}]$. We can consider the natural map

$$\operatorname{proj}(\mathbb{C}[\dots f_a \dots]) \longrightarrow P^{134}.$$

The image of this map is irreducible, it can be identified with

$$\operatorname{proj}(\mathbb{C}[\dots \vartheta_G \dots]) = X(\Gamma_3[2]).$$

Hence it is sufficient to prove that x has an inverse image in

$$\operatorname{proj}(\mathbb{C}[\dots f_a \dots]) = X(\Gamma_3[2,4]).$$

Since the covering degree is 64, there will be 64 inverse images. So all what one has to prove is the following. Take the vanishing ideal of the point x and take its image under the homomorphism $\mathbb{C}[\dots X_G \dots] \longrightarrow \mathbb{C}[\dots F_a \dots]$. Consider the ideal that is generated by this image and the Runge polynomial. This defines a cluster. A computer computations shows that this cluster is not empty (and that its degree is 64). This finishes the proof of Theorem 5.2. $\square$

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