A graded algebra related to cubic surfaces

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1. Introduction

In the papers [ACT1,2] the moduli space of marked cubic surfaces is described as a certain modular variety which belongs to the Picard modular group of level $\sqrt{-3}$ of the ring of Eisenstein integers acting on the 4-ball . In [AF] the ring of modular forms on this variety has been investigated. There are ten basic modular forms of weight three and 270 cubic relations between them. Hence we are lead to consider a certain ideal*)

$$\mathcal{J} \subset \mathbb{C}[Y_0, \ldots, Y_9]$$

generated by 270 cubic polynomials. We will redefine this ideal here in §2. Our main result states that the algebra $\mathbb{C}[Y_0, \ldots, Y_9]/\mathcal{J}$ can be understood as the *full ring of modular forms* (theorem 4.6). The corresponding projective variety

$$\mathcal{V} := \operatorname{proj}(\mathbb{C}[Y_0, \ldots, Y_9]/\mathcal{J})$$

is isomorphic to the moduli space of marked cubic surfaces in the sense of geometric invariant theory. The Hilbert function of this ring is determined (4.4).

The proof depends on the fact that the Weyl group of the lattice E_6 acts on this ring. The 10-dimensional representation of $W(E_6)$ on the space generated

*) A similar ideal has been defined also by Yoshida [Yo]. Actually Yoshida recovered a model of the moduli space of marked cubic surfaces which is due to Coble [Co]. Meanwhile we can identify our Y_i with Coble's polynomials. This and also the relation to recent work of van Geemen [vG] and also of Matsumoto and Terasoma [MT] will be described in another paper.

by $Y_0, \dots, Y_9$ is irreducible. Under the simple subgroup $G_{25\,920}$ of $W(E_6)$ this representation splits into two non-isomorphic 5-dimensional representations. We denote one of them by

$$\mathbb{C}A_1 + \cdots + \mathbb{C}A_5 \subset \mathbb{C}Y_0 + \cdots + Y_9.$$

The natural homomorphism

$$\mathbb{C}[A_1, \dots, A_5] \longrightarrow \mathbb{C}[Y_0, \dots, Y_9]/\mathcal{J}$$

turns out to be injective. We will show that $\mathbb{C}[Y_0, \dots, Y_9]/\mathcal{J}$ is a *free module* over $\mathbb{C}[A_1, \dots, A_5]$. As a consequence we obtain a $G_{25\,920}$-invariant finite mapping

$$\mathcal{V} \longrightarrow P^4(\mathbb{C}).$$

The covering degree of this map is 27.

Up to now some of the proofs of this paper use computer algebra, especially calculations with Gröbner bases. We used the computer algebra system Singular. We will not print any program here. Instead of this we explain the computations in a way such an interested reader can reproduce them quickly by means of own programs.

A reader who feels that computer based proofs are not mathematical proofs in the strong sense is invited to eliminate the computer calculations and to find intrinsic proofs.

The explicit determination of the 5-dimensional subspace is basic for this paper. A connection between the moduli space $\mathcal{M}$ and the $G_{25\,920}$-invariant $P^4(\mathbb{C})$ was conjectured by D. Allcock. The idea how to exhibit the 5-dimensional subspace inside $\mathbb{C}Y_1 + \cdots + \mathbb{C}Y_9$ is due to B. van Geemen. This paper is the result of many discussions with both and I want to express my gratitude for their support.

2. Definition of the ideal

We recall the definition of the ideal $\mathcal{J}$ from [AF]: Up to now I don't know a description of this ideal in simple invariant terms (compare the remark at the end of §2). So we have to repeat the concrete description as it came up in [AF]. First of all one needs a realization of the 10-dimensional representation of $W(E_6)$. There are many ways to realize this representation. We use the realization of [AF]:

Let $\mathbb{F}_3$ the field of three elements. We consider the bilinear form

$$(x, y) := x_0y_0 - x_1y_1 - \cdots - x_4y_4$$

on $\mathbb{F}_3^5$. Its orthogonal group is denoted by $\mathrm{O}(5,3)$. This group contains the Weyl group $W(E_6)$ as normal subgroup of index 2. The Weyl group is generated by the negatives of reflections along vectors of norm -1 (a typical being $(0,1,0,0,0)$). There are 81 isotropic vectors in $\mathbb{F}_3^5$. The group $\mathrm{O}(5,3)$ permutes the 40 non-zero pairs $\pm a$ of isotropic vectors. This defines an action of $\mathrm{O}(5,3)$ on $\mathbb{Q}^{40}$. We will describe the 10-dimensional representation as a subspace of this $\mathbb{Q}^{40}$:

The following list is a system of representatives of the 40 non-zero pairs $\pm a$ of isotropic vectors.

(1,1,1,1,1)	(1,1,1,1,-1)	(1,1,1,-1,1)	(1,1,1,-1,-1)	(1,1,0,0,0)	(1,1,-1,1,1)
(1,1,-1,1,-1)	(1,1,-1,-1,1)	(1,1,-1,-1,-1)	(1,0,1,0,0)	(1,0,0,1,0)	(1,0,0,0,1)
(1,0,0,0,-1)	(1,0,0,-1,0)	(1,0,-1,0,0)	(1,-1,1,1,1)	(1,-1,1,1,-1)	(1,-1,1,-1,1)
(1,-1,1,-1,-1)	(1,-1,0,0,0)	(1,-1,-1,1,1)	(1,-1,-1,1,-1)	(1,-1,-1,-1,1)	(1,-1,-1,-1,-1)
(0,1,1,1,0)	(0,1,1,0,1)	(0,1,1,0,-1)	(0,1,1,-1,0)	(0,1,0,1,1)	(0,1,0,1,-1)
(0,1,0,-1,1)	(0,1,0,-1,-1)	(0,1,-1,1,0)	(0,1,-1,0,1)	(0,1,-1,0,-1)	(0,1,-1,-1,0)
(0,0,1,1,1)	(0,0,1,1,-1)	(0,0,1,-1,1)	(0,0,1,-1,-1)		

The group $\mathrm{O}(5,3)$ permutes this list. We consider the vector space of dimension 40 which has the elements of this list as basis. We write an element of this space as a row-vector with 40 components. The 10-dimensional subspace $W \subset \mathbb{Q}^{40}$ is generated by the following ten vectors:

```
3 0 0 0 0 0 0 0 0 1 1 1-2-2-2 0 1 1 0-1 1 0 0-2-1-1-1-1-1-1-1 2-1-1 2 2-2-1-1 1
0 3 0 0 0 0 0 0 0 1 1-2 1-2-2 0 1 1 0-1 1 0-3 1-1-1-1-1-1-1 2-1-1 2-1 2-2-1 2-2
0 0 3 0 0 0 0 0 0 1-2 1-2 1-2 0 1 1 0-1 1-3 0 1-1-1-1-1-1 2-1-1 2-1 2-1-2 2-1-2
0 0 0 3 0 0 0 0 0 1-2-2 1 1-2 0 1 1 0-1-2 0 0 1-1-1-1-1 2-1-1-1 2 2-1-1 1-1-1-2
0 0 0 0 3 0 0 0 0 1 1 1 1 1 1 0-2-2 0-1-2 0 0-2-1-1-1-1-1-1-1-1-1-1-1 1-1-1 1
0 0 0 0 0 3 0 0 0-2 1 1-2-2 1 0 1 1-3-1 1 0 0 1-1-1 2 2-1-1-1 2-1-1-1-1-2 2 2 1
0 0 0 0 0 0 3 0 0-2 1-2 1-2 1 0 1-2 0-1 1 0 0 1-1 2-1 2-1-1 2-1-1-1-1-1 1-1 2 1
0 0 0 0 0 0 0 3 0-2-2 1-2 1 1 0-2 1 0-1 1 0 0 1 2-1 2-1-1 2-1-1-1-1-1-1 1 2-1 1
0 0 0 0 0 0 0 0 3-2-2-2 1 1 1 0-2-2 3-1-2 3 3-2 2 2-1-1 2-1-1-1-1-1-1-1 4-1-1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3-3-3 3 0-3 3 3-3 0 0 0 0 0 0 0 0 0 0 0 0 3-3-3 3
```

The given basis of W (the rows of the above matrix) is denoted by $Y_0, \dots, Y_9$. It is easy to verify (at least if one uses a computer) that the group $\mathrm{O}(5,3)$ acts on the space W and that this is a realization of the unique 10-dimensional representation of $\mathrm{O}(5,3)$ such that $-\mathrm{id}$ acts by multiplication with -1. Irreducibility is understood in the absolute sense (over $\mathbb{C}$) but it is important to notice that the representation is defined over $\mathbb{Q}$. For the proof one can use the character tables of the ATLAS of finite groups or a computer algebra system like GAP3. This free software is developed at: *University of Aachen, RWTH, Templergraben 65, 5100 Aachen, Germany.*

It is available from *http://www-gap.dcs.st-and.ac.uk/ gap.*

Our realization of the 10-dimensional representation looks artificial. Later (4.7) we will see that it is very natural in our context.

We use the projective space of the dual space of $W \otimes_{\mathbb{Q}} \mathbb{C}$ and identify it with $P^9(\mathbb{C})$ using the above basis $Y_0, \dots Y_9$ of W. We also consider the ring of polynomials $\mathbb{C}[Y_0, \dots, Y_9]$.

2.1 Remark. *The* $\mathrm{O}(5,3)$*-orbit of the one dimensional space generated by the cubic polynomial*

$$\begin{aligned}(Y_1 - Y_2)(Y_0 - Y_3 - Y_5 + Y_8 - Y_9)(Y_4 + Y_6 + Y_7)\\ - (Y_1 - Y_2 - Y_6 + Y_7)(Y_0 - Y_3)(Y_4 + Y_5 + Y_8 - Y_9)\end{aligned}$$

consists of 270 *lines. The space generated by these cubic polynomials has dimension* 30*. The group* $\mathrm{O}(5,3)$ *acts (absolutely) irreducibly on this space.*

This is also a very easy calculation which can be done by means of a calculator or with some patience also by hand. The space of all cubic polynomials in $Y_0, \dots, Y_9$ contains a 10-dimensional representation of $\mathrm{O}(5,3)$ with multiplicity two and two 30-dimensional representations each with multiplicity two and a 80-dimensional representation with multiplicity one. Therefore it is clear that an invariant subspace of dimension 30 is irreducible. Unfortunately this representation occurs with multiplicity two and for this reason it is not clear how to distinguish the subspace of the cubic polynomials in 2.1.

We denote by $\mathcal{J}_{\mathbb{Q}} \subset \mathbb{Q}[Y_0, \dots, Y_9]$ the ideal which is generated by the 270 cubic polynomials. We are more interested in $\mathcal{J}$, which is its extension to $\mathbb{C}$, but we have also to work with $\mathcal{J}_{\mathbb{Q}}$ (mainly because we want to apply computer algebra).

3. Irreducibility

The following result is taken from [AF] (see also [Y]):

3.1 Proposition. *The radical of the ideal* $\mathcal{J} \subset \mathbb{C}[Y_0, \dots, Y_9]$ *is a prime ideal. The dimension of the associated projective variety* $\mathcal{V}$ *is* 4*.*

The paper [AF] contains already a sketch of the proof. It is necessary to know more details for the following. Hence we give more details of the proof here:

Proof of 3.1. In a first step one has to prove that all irreducible components of $\mathcal{V}$ have the dimension 4.

3.2 Lemma. *The dimension of* $\mathcal{V}$ *is four.*

The dimension of $\mathcal{V}$ equals the dimension of $\mathcal{V}_{\mathbb{Q}} = \mathrm{proj}(\mathbb{Q}[Y_0, \dots, Y_9]/J_{\mathbb{Q}})$. This dimension can be computed by means of the command dim of the computer algebra system Singular. This free software is developed at:

University of Kaiserslautern, Departement of Mathematics and Centre of Computer Algebra, 67663 Kaiserslautern, Germany.

It is available from *http://www.mathematik.uni-kl.de/~zca/Singular.*

The variety $\mathcal{V}$ could still have components of dimension < 4. But then the projective dimension of $\mathcal{J}_{\mathbb{Q}}$ would be > 5.

3.3 Lemma. *The projective dimension of*

$$\mathbb{Q}[Y_0, \dots, Y_9]/\mathcal{J}_{\mathbb{Q}}$$

is five.

This means that there exists a free resolution

$$0 \longrightarrow F_5 \longrightarrow \cdots \longrightarrow F_0 \longrightarrow \mathbb{Q}[Y_0, \dots, Y_9]/\mathcal{J}_{\mathbb{Q}} \longrightarrow 0.$$

Such a resolution can be computed my means of the command mres of Singular. □

From 3.2 and 3.3 we know that all irreducible components of $\mathcal{V}$ have dimension 4.

We have to show that only one irreducible component occurs. The idea to prove this is to consider hyperplane sections of $\mathcal{V}$ and to decompose them into irreducible components. We will use the following trivial criterion:

3.4 Remark. *Let X be a projective variety such that every irreducible component has the same positive dimension and let H be a hyperplane. Assume that there exists a chain $Y_1, \dots Y_m$ of irreducible components of $X \cap H$ in which all irreducible components of $X \cap H$ occur and such that every intersection $Y_i \cap Y_{i+1}$ $(1 \leq i < m)$ contains a smooth point of X. Then X is (set-theoretically) irreducible.*

This is clear because a hyperplane must meet every irreducible component of X and because the intersection of two irreducible components of X is contained in the singular locus of X.

Proof of 3.1 continued. The hyperplanes which we have in mind correspond to so-called crosses, a notion which is fundamental in [AF]. Recall that a cross is a set of five pairs $\pm a$ of vectors in $\mathbb{F}_3^5$, which are pairwise orthogonal and such that $(a, a) = 1$ for one of the five and -1 for the four others. Let C be a cross. As in [AF] has been pointed out, there exists up to a constant factor a unique element $F \in \mathrm{Hom}(W, \mathbb{Q})$ such that F changes its sign under the five reflections along the five vectors of the cross. We can consider hyperplane H which is defined by $F(z) = 0$ in $P^9(\mathbb{C})$. All crosses are equivalent under $\mathrm{O}(5, 3)$, an example of a cross is

$$(1, 0, 0, 0, 0) \quad (0, 1, 0, 0, 0) \quad \dots \quad (0, 0, 0, 0, 1).$$

We want to intersect the hyperplane H with $\mathcal{V}$. It seems to be impossible to use the standard commands of a computer algebra system which provide commands for primary decomposition or something similar. The ideal $\mathcal{J}$ is too complicated. The problem is comparable to the the factorization of a big natural number. It can be very hard to find a divisor of a natural number. But if one has one conjecturally from elsewhere, it is trivial to verify or disprove it. We are in a similar situation with our hyperplane section, because we can guess its components. For this one has to use the connection of $\mathcal{V}$ with the moduli space of marked cubic surfaces, which has been worked out in [AF]. This connection leads to the following

3.5 Definition.

1. *Let C be a cross. We denote by $\mathcal{J}(C) \subset \mathbb{Q}[Y_0, \dots, Y_9]$ the ideal generated by $\mathcal{J}_{\mathbb{Q}}$ and the linear form F which belongs to the cross.*
2. *Let $a \in \mathbb{F}_3^5$ be a non-isotropic vector ($(a,a) \neq 0$). The ideal $\mathcal{J}(a) \subset \mathbb{Q}[Y_0, \dots, Y_9]$ is generated by the ideals $\mathcal{J}(C)$, where C runs through all crosses which contain $\pm a$.*

We mention that $a \in \mathbb{F}_3^5$ is contained in three crosses if $(a,a) = 1$ and in 15 crosses if $(a,a) = -1$.

3.6 Lemma. *Let $C = \{\pm a_1, \dots, \pm a_5\}$ be a cross. Then*

$$\mathcal{J}(C) = \mathcal{J}(a_1) + \cdots + \mathcal{J}(a_5).$$

The dimension of the projective variety attached to an ideal $\mathcal{J}(a)$ is three.

It is no problem to decide by calculation whether two ideals given by generators are the same and it is also no problem to compute the dimension. We don't give any details. □

From the connection with the moduli space of marked cubic surfaces [AF] one has to expect that 3.6 gives the decomposition of the hyperplane section associated to C into 5 absolutely irreducible components. Recall that this connection rests on 3.1 so we can use it only as a guide.

We have to investigate the ideals $\mathcal{J}(a)$. We have to differ between the cases $(a,a) = \pm 1$. The case $(a,a) = -1$ is easier. As we mentioned the ideal $\mathcal{J}(a)$ is generated in this case by $\mathcal{J}$ and 15 linear forms. But the 15 linear forms are not independent. A simple computation shows that thy span a five dimensional space. This means that we can consider the projective variety $\mathcal{V}(a)$ as a variety in a $P^4(\mathbb{C})$ (because $4 = 9 - 5$). Hence $\mathcal{V}(a)$ is hypersurface in $P^4(\mathbb{C})$. It must be the set of zeros of one polynomial and this polynomial must be cubic. After some calculation one finds:

3.7 Lemma. *Let $a \in \mathbb{F}_3^5$ with $(a,a) = -1$. The ideal $\mathcal{J}(a)$ is generated by five linear forms and a cubic polynomial. It is an absolute prime ideal. The projective variety $\mathcal{V}(a)$ is a copy of the* ***Segre cubic****.*

We recall that the Segre-cubic is the unique (up to a projective transformation) cubic threefold in $P^4(\mathbb{C})$ with ten nodes as singular locus.

Next we investigate the intersection $\mathcal{V}(a) \cap \mathcal{V}(b)$ for two orthogonal a, b, where at least on of them has norm -1.

3.8 Lemma. *Let $a, b \in \mathbb{F}_3^5$ be two orthogonal vectors of norm -1 ($(a,a) = (b,b) = -1$). Then $\mathcal{J}(a) + \mathcal{J}(b)$ is generated by seven linear forms, $\mathcal{V}(a) \cap \mathcal{V}(b)$ is a copy of the projective plane $P^2(\mathbb{C})$.*

As soon as one knows the seven linear forms, 3.8 can be verified. One obtains the generators by means of the Singular command std($\mathcal{J}(a) + \mathcal{J}(b)$). □

3.9 Lemma. *Let $a, b \in \mathbb{F}_3^5$ be two orthogonal vectors of different norm. Then $\mathcal{J}(a) + \mathcal{J}(b)$ is generated by six linear forms and a cubic form, $\mathcal{V}(a) \cap \mathcal{V}(b)$ is a copy of the* ***Cayley cubic*** *in $P^3(\mathbb{C})$.*

We recall that the Cayley-cubic is the unique (up to a projective transformation) cubic surface in $P^3(\mathbb{C})$ with four nodes as singular locus.

Again the result can be found by means of Singular and then verified directly. □

Using the results 3.8 and 3.9 it is easy to find in the intersection $\mathcal{V}(a) \cap \mathcal{V}(b)$ of two orthogonal vectors of different norm a point which is smooth in $\mathcal{V}$. So the proof of 3.1 is reduced to

3.10 Proposition. *The radical of the ideal $\mathcal{J}(a)$, $(a,a) = 1$, is an absolute prime ideal. The dimension of the associated projective variety $\mathcal{V}(a)$ is 3.*

The ideal $\mathcal{J}(a)$ is rather complicated. It contains only three linear forms. Inspection shows that only two of them are linearly independent. Hence $\mathcal{V}(a)$ should be considered in this case as a subvariety of $P^7(\mathbb{C})$.

Proof of 3.10 (and hence of 3.1). The strategy will be the same. One intersects $\mathcal{V}(a)$ with a hyperplane and applies 3.4. Again one takes a hyperplane H which comes from a cross.

3.11 Lemma. *let a be a vector of norm one and C a cross whose member $\pm b$ of norm one is orthogonal to a. Then the intersection $H \cap \mathcal{V}(a)$ consists of five absolutely irreducible components of dimension two. There exists a chain of components as in 3.4 such that the intersections $Y_i \cap Y_{i+1}$ are (linearly embedded) rational projective lines.*

Proof. We take the ideal $J(a)$ and the linear form F attached to the cross C. We have to consider the ideal

$$J(a, C) := (J(a), F) \subset \mathbb{Q}[Y_0, \ldots, Y_9].$$

This ideal is simple enough to compute the minimal prime ideals containing it by means of the Singular command minAssGTZ. One obtains 5 prime ideals in $\mathbb{Q}[Y_0,\dots,Y_9]$. The pairwise sums can be computed and on finds a chain as claimed in 3.11. Still their remains a problem. We need that the five prime ideals are *absolutely* prime. The following simple criterion can be used.

3.12 Remark. *Let X be an algebraic variety defined over $\mathbb{Q}$, which is irreducible over $\mathbb{Q}$. Assume that there exists a rational smooth point of X. Then X is absolutely irreducible.*

This is clear because the absolute Galois group permutes the absolute irreducible components of X. Hence rational points are contained in all components, hence they are singular points if there is more than one component. □

4. The Cohen-Macaulay property

From 3.3 follows that the graded ring $\mathbb{Q}[Y_0,\dots,Y_9]/\mathcal{J}_{\mathbb{Q}}$ is a Cohen-Macaulay ring. In this section we construct explicitly a subring, which is isomorphic to a polynomial ring in 5 variables, such that the full ring is a free module of this subring.

The 10-dimensional representation decomposes under $G_{25\,920}$ into two non-isomorphic five dimensional pieces. They are defined over the field of Eisenstein integers $\mathbb{Q}(\zeta)$, where $\zeta = -1/2 + \mathrm{i}\sqrt{3}/2$.

Following a suggestion of B. van Geemen we computed bases $A_0,\dots,A_4$ and $B_0,\dots B_4$ of the two subspaces. They are related with the original basis by

$$18Y0 =$$
$$(-3\zeta-1)A0 + (3\zeta+3)A1 + (-7\zeta-6)A2 + (-2\zeta-1)A3 + 2A4+$$
$$(-\zeta-1)B0 + (1-\zeta)B1 + (-3\zeta-2)B2 + (-2\zeta-1)B3 - 2\zeta B4$$

$$18Y1 =$$
$$2A0 - 4\zeta A2 + (2-2\zeta)A3 + 2A4+$$
$$(-4\zeta-4)B0 + (2\zeta+4)B1 + (-6\zeta-8)B2 + (-2\zeta-4)B3 - 2\zeta B4$$

$$18Y2 =$$
$$(-6\zeta+2)A0 - 4\zeta A2 + (2-2\zeta)A3 + 2A4+$$
$$(-4\zeta-4)B0 + (-4\zeta-2)B1 - 2B2 + (-2\zeta-4)B3 - 2\zeta B4$$

$$18Y3 = (-3\zeta-1)A0+(-3\zeta-3)A1-\zeta A2+(-2\zeta-1)A3+2A4+ (-7\zeta-1)B0+(1-\zeta)B1+(-3\zeta-2)B2+(-2\zeta-1)B3-2\zeta B4$$

$$18Y4 = -4A0+2\zeta A2+(-2\zeta-4)A3-4A4+ (2\zeta+2)B0+(-2+2\zeta)B1+4B2+(2-2\zeta)B3+4\zeta B4$$

$$18Y5 = 2A0+6\zeta A1-4\zeta A2+(2-2\zeta)A3+2A4+ (2\zeta+2)B0+(2\zeta+4)B1-2B2+(-2\zeta-4)B3-2\zeta B4$$

$$18Y6 = (3\zeta+5)A0+(3\zeta+3)A1-\zeta A2+(-2\zeta-1)A3+2A4+ (-\zeta-1)B0+(5\zeta+1)B1+(-3\zeta-2)B2+(-2\zeta-1)B3-2\zeta B4$$

$$18Y7 = (-3\zeta-1)A0+(3\zeta+3)A1-\zeta A2+(-2\zeta-1)A3+2A4+ (-\zeta-1)B0+(1-\zeta)B1+(3\zeta-2)B2+(-2\zeta-1)B3-2\zeta B4$$

$$18Y8 = (-3\zeta-7)A0+(3-3\zeta)A1+(5\zeta-6)A2+(-2\zeta-7)A3+(-6\zeta-4)A4+ (-\zeta+5)B0+(5\zeta+1)B1+(-3\zeta+4)B2+(-2\zeta+5)B3+(4\zeta+6)B4$$

$$18Y9 = (-3\zeta-9)A0+(3-3\zeta)A1+(3\zeta-6)A2-9A3+(-6\zeta-6)A4+ (3\zeta+9)B0+(-3+3\zeta)B1+(-3\zeta+6)B2+9B3+(6\zeta+6)B4$$

It is easy to invert this relation and to verify that the spaces generated by the A-s and B-s are invariant under the reflections along elements of norm -1. We will of course not reproduce this calculation here, instead of this we explain how it was found: Recall that there exist 135 crosses, three for each pair $\pm a \in \mathbb{F}_3^5$ of norm 1. Let C be a cross. We mentioned already that the elements of the 10-dimensional representation space, which change their sign under the five reflections along the elements of the cross, constitute a one-dimensional subspace. Each element of norm 1 is contained in three crosses, we obtain three one dimensional spaces. Let F, G, H be generators. They turn out to be linearly dependent. One can normalize them [AF] such that $F+G+H=0$. Put $A=F+\zeta G+\zeta^2 H$ and apply reflections along vectors of norm -1 to A. It turns out that the images of A generate a five dimensional space, invariant under $G_{25\,920}$. The B-space is constructed in a similar way.

From now on the application of computer algebra is slightly more complicated because we have to replace the base field $\mathbb{Q}$ by the field $K=\mathbb{Q}(\sqrt{-3})$

of Eisenstein integers. The following calculations have been performed over K and then be extended to $\mathbb{C}$ by means of simple arguments.

4.1 Lemma. *The natural homomorphisms*

$$\begin{aligned}\mathcal{A} &:= \mathbb{C}[A_0,\dots,A_4] \longrightarrow \mathbb{C}[Y_0,\dots,Y_9]/\mathcal{J},\\ \mathcal{B} &:= \mathbb{C}[B_0,\dots,B_4] \longrightarrow \mathbb{C}[Y_0,\dots,Y_9]/\mathcal{J}\end{aligned}$$

are injective.

We also consider the homomorphism

$$\mathbb{C}[Y_0,\dots,Y_9] = \mathbb{C}[A_0,\dots,A_4,\,B_0,\dots,B_4] \longrightarrow \mathbb{C}[B_0,\dots,B_4],$$

which maps A_i to 0 and B_i to B_i. The image of $\mathcal{J}$ is denoted by

$$\mathfrak{b} \subset \mathbb{C}[B_0,\dots,B_4].$$

A short computation with the computer algebra system Singular shows

4.2 Lemma. *The radical of $\mathfrak{b}$ is the prime ideal $(B_0,\dots B_4)$. More precisely: Let $\mathfrak{b}_n$ resp. $\mathcal{B}_n$ denote the homogenous part of degree n of the ideal $\mathfrak{b}$ resp. of the polynomial ring $\mathcal{B} = \mathbb{C}[B_0,\dots,B_4]$. Then*

$$\dim(\mathcal{B}_n/\mathfrak{b}_n) = \begin{cases} 1 & \text{if } n = 0\\ 5 & \text{if } n = 1\\ 15 & \text{if } n = 2\\ 5 & \text{if } n = 3\\ 1 & \text{if } n = 4\\ 0 & \text{if } n > 4\end{cases}$$

The dimension of the vector space $\mathcal{B}/\mathfrak{b}$ is 27. We denote a basis by $b_1,\dots,b_{27}$, where b_1 is of degree zero, the next five of degree 1, and so on. We consider them as elements of $\mathbb{C}[Y_0,\dots,Y_9]/\mathcal{J}$ by means of the natural homomorphism

$$\mathcal{B} := \mathbb{C}[B_0,\dots,B_4] \longrightarrow \mathbb{C}[Y_0,\dots,Y_9]/\mathcal{J},$$

which is injective by 4.1. We also can consider the natural homomorphism

$$\mathcal{A} := \mathbb{C}[A_0,\dots,A_4] \longrightarrow \mathbb{C}[Y_0,\dots,Y_9]/\mathcal{J},$$

which is also injective.

4.3 Lemma. *The ring $\mathbb{C}[Y_0,\dots,Y_9]/\mathcal{J}$ is a finitely generated module over the subring $\mathbb{C}[A_0,\dots,A_4]$, which is a polynomial ring. A system of generators is given by $b_1,\dots,b_{27}$.*

Proof. It is sufficient to show that monomials M in the B-s are contained in the module $\mathcal{A}b_1+\cdots\mathcal{A}b_{27}$. This will be proved by induction on the degree d of the monomial M. We can assume (because of 4.2 and the definition of the b-s) that there exists a polynomial $P(A_0,\dots,A_4,\,B_0,\dots,B_4)$ of the degree d, which is contained in $\mathcal{J}$ and such that $b=P(0,0,0,0,0,B_0,\dots,B_4)$. The element $M-P$ is congruent to M mod $\mathcal{J}$. Each monomial which occurs in $M-P$ contains at least one of the A-s. hence all occurring B-degrees are $<d$ and we can use the induction hypothesis. □

We want to compare Hilbert functions: A Singular computation shows:

4.4 Proposition. *The Hilbert function of the ring $\mathbb{C}[Y_0,\dots,Y_9]/\mathcal{J}$ is given by*

$$P(X)=\frac{9}{8}X^4+\frac{9}{4}X^3+\frac{27}{8}X^2+\frac{9}{4}X+1,$$

i.e. $P(n)$ is the dimension of the degree n-part for $n\ge 0$.

The Hilbert function of the polynomial ring in 5 variables is

$$Q(X)=\frac{1}{24}X^4+\frac{5}{12}X^3+\frac{35}{24}X^2+\frac{25}{12}X+1.$$

One verifies

$$P(X)=Q(X)+5Q(X-1)+15Q(X-2)+5Q(X-3)+Q(X-4).$$

The occurring coefficients are the weights of the generators b_i. This means that they must be linearly independent:

4.5 Theorem. *The algebra $\mathbb{C}[Y_0,\dots,Y_9]/J$ is a **free** module of rank 27 over the polynomial ring $\mathbb{C}[A_0,\dots,A_4]$. It is especially a **Cohen-Macaulay ring**. (All its localizations by prime ideals are Cohen-Macaulay rings in the sense of [Se]).*

Now we are able to prove that J is a prime ideal and even more, that the ring $A=\mathbb{C}[Y_0,\dots,Y_9]/J$ is normal. By Serre's criterion for normality it is sufficient to show that the singular locus of $\mathrm{Spec}(A)$ has codimension >1. It is sufficient to prove this for $\mathrm{proj}(A)$ instead of $\mathrm{Spec}(A)$. Hence we have to prove that the singular locus of $\mathcal{V}$ has codimension >1.

It is very difficult to get hold of the singular locus in its whole by means of computer algebra, because this is defined by a rather involved ideal including determinants of derivatives. Therefore we proceed as follows:

If the dimension of the singular locus were three, the intersection with an arbitrary subvariety of dimension 6 of $P^9(\mathbb{C})$ would be not empty. But we will construct such a subvariety which contains no singular point of $\mathcal{V}$. We take actually take the linear space V defined by the three linear equations

$$\begin{aligned} Y_1 - Y_2 - Y_6 + Y_7 = 0, \\ Y_1 + Y_2 + Y_4 = 0, \\ Y_0 - Y_1 - Y_2 + Y_3 - Y_5 + Y_6 + Y_7 - Y_8 - Y_9 = 0. \end{aligned}$$

It has to be mentioned that these are cross-hyperplanes, i.e. their intersection with $\mathcal{V}$ are crosses. The reason of this choice is that the intersection with $\mathcal{V}$ is very simple. It is (set theoretically) the union of 11 linear subspaces (defined over $\mathbb{Q}$) of dimension 1. One of them is

$$Y_0 = Y_1 = Y_2 = Y_3 = Y_4 = Y_5 = Y_5 + Y_8 + Y_9 = 0.$$

The calculation can be done in Singular. Now we have to prove that no singular point of $\mathcal{V}$ lies on this line.

It is trivial to decide numerically from a given point whether it is singular or not. For this one has to compute partial derivatives in this point of the cubic relations and then one has to compute the rank of a certain numerical matrix.

It is usually also easy to decide whether a singular point of $\mathcal{V}$ is on a (rational) line. One computes the Jacobian of the generators of $\mathcal{J}$ and reduces this Jacobian modulo the ideal, which defines the line. This simplifies the Jacobian considerably. Then one looks for a 5×5-subdeterminant, which does not vanish on the line. Its zero locus inside the line consists of a finite set of points. There might be a problem, that these points are not rational. Therefore it is better to take more than one minor.

Our results have applications to modular forms. We use now results and notations of [AF]. We consider the Eisenstein lattice

$$L = \mathcal{E}^5, \quad \mathcal{E} = \mathbb{Z}[-1/2 + \sqrt{-3}/2]$$

equipped with the Hermitean form $\bar{a}_0 b_0 - \bar{a}_1 b_1 - \cdots - \bar{a}_4 b_4$. The group $\Gamma \subset \mathrm{U}(1,4)$ is the congruence subgroup of level 2, i.e. the subgroup of the integral unitary group $\mathrm{U}(L)$, which acts trivial on $L/(\sqrt{-3}L)$. The space of modular forms on Γ of weight r is denoted by $[\Gamma, r]$. We refer to [AF] for its definition. The thirds root of unity are contained in $\sqrt{-3}\mathcal{E}$. As mentioned already in [AF] this implies that modular forms are zero if their weight is not divisible by three.

4.6 Theorem. *The ring $\mathbb{C}[Y_0, \dots, Y_9]/\mathcal{J}$ is a normal integral domain. It equals the ring of all modular forms*

$$A(\Gamma) = \bigoplus_{r \in \mathbb{Z}} [\Gamma, r] = \mathbb{C}[Y_0, \dots, Y_9]/\mathcal{J}$$

on Γ, if we count the weights of the Y_j as three. We especially have:

1. *If $[\Gamma, r]$ is different from 0, then r is divisible by three.*
2. *The dimension of $[\Gamma, 3r]$ is given by 4.4.*
3. *The natural inclusion $\mathbb{C}[A_0, \ldots, A_4] \to A(\Gamma)$ defines an everywhere regular and finite map*
$$\mathcal{V} \longrightarrow P^4(\mathbb{C}).$$
This map is equivariant with respect to $G_{25\,920}$ and has covering degree 27.
4. *The ring $A(\Gamma)$ is a free module over the polynomial ring $\mathbb{C}[A_0, \ldots, A_4]$ generated by one modular form of weight 0, five of weight 3, fifteen of weight 6, five of weight 9 and one of weight 12.*

The variety $\mathrm{proj}(A(\Gamma))$ is isomorphic to the Baily-Borel compactification of the ball quotient $\mathcal{B}_4/\Gamma$ (s. [AF]). The singular locus of this compactification consists of the 40 cusps. From [AF] one can see the following connection with the 10×40-matrix which we introduced at the beginning to get a model for the 10-dimensional representation: The columns of this matrix define 40 singular points in $P^9(\mathbb{C})$. This means that this matrix defines 40 points in the projective space:

4.7 Proposition. *The variety $\mathcal{V}$ has exactly 40 points. They are given by the columns of the 10×40-matrix given at the beginning. For example the first column gives the singular point*

$$[1,0,0,0,0,0,0,0,0,0] \in \mathcal{V}.$$

This result explains a little the strategy which we used to prove that the singular locus of $\mathcal{V}$ has codimension > 1. We looked for a curve, which doesn't run into a cusp. We constructed this curve as intersection of three crosses. This means that the irreducible components of this curve are quotients of a one-dimensional ball, wich is nothing else but a model of the usual upper half-plane. Hence the 11 lines which we got are *compact* quotients of the upper half plane (so-called Shimura curves).

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