A Siegel cusp form of degree 12 and weight 12.

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It has been conjectured by Witt [Wi] (1941) and proved later (1967) independently by Igusa [I] and M. Kneser [K] that the theta series with respect to the two unimodular even positive definite lattices of rank 16 are linearly dependent in degree ≤ 3 and linearly independent in degree 4. In this paper we consider the next case of the 24 Niemeier lattices of rank 24. The associated theta series are linearly dependent in degree ≤ 11 and linearly independent in degree 12. The resulting Siegel cusp form of degree 12 and weight 12 is a Hecke eigenform which seems to have interesting properties. We would like to thank G. Höhn for helpful comments and hints.

Construction of Siegel cusp forms by theta series.

Let Λ be an even unimodular positive definite lattice, i.e. a free abelian group equipped with a positive definite symmetric bilinear form (x, y), such that Λ coincides with its dual and such that

$$Q(x) := \frac{1}{2}(x, x)$$

is integral. By reduction mod 2 we obtain a quadratic form

$$q : E := \Lambda/2\Lambda \longrightarrow \mathbf{Z}/2\mathbf{Z}, \qquad q(a + 2\Lambda) = Q(a) \bmod 2.$$

on the $\mathbf{Z}/2\mathbf{Z}$-vector space E. The standard theta series of degree n with respect to Λ is

$$\vartheta_\Lambda(Z) = \sum_{g \in \Lambda^n} \exp \pi \mathrm{i} \sigma(T(g)Z) \qquad (\sigma = \text{trace}),$$

where

$$T(g) := \big((g_i, g_j)\big)_{1 \leq i,j \leq n} \qquad (g = (g_1, \ldots, g_n)).$$

The variable Z varies on the Siegel upper half plane of degree n. This is a modular form with respect to the full Siegel modular group $\mathrm{Sp}(2n, \mathbf{Z})$, but is not a cusp form. The weight is $m/2$ if m denotes the rank of Λ, and m is divisible by 8. To obtain a cusp from we modify this definition.

Assume that a function $\epsilon(F)$ is given which depends on subspaces $F \subset E$. For $g \in \Lambda^n$ we denote by $F(g)$ the image of $\mathbf{Z}g_1 + \cdots + \mathbf{Z}g_n$ in E. For an arbitrary degree n we define

$$f^{(n)}(Z) := \sum_{g \in \Lambda^n} \epsilon(F(g)) \exp \frac{\pi \mathrm{i}}{2} \sigma(T(g)Z) \qquad (\sigma = \text{trace}).$$

In general this will not be a modular form with respect to the full modular group.

To construct a suitable function $\epsilon(F)$ we use the orthogonal group $\mathrm{O}(E)$ of the vector space E. It consists of all elements from the general linear group $\mathrm{GL}(E)$ which preserve the quadratic form q. It is a basic fact for our construction that $\mathrm{O}(E)$ admits a subgroup of index 2. It is the kernel of the so-called *Dickson invariant*. We refer to [B] for some details. To define the Dickson invariant we chose a basis $e_1, \ldots e_m$ of E such that q is of the form

$$q\Big(\sum_{i=1}^{m} x_i e_i\Big) = \sum_{j=1}^{m/2} x_j x_{m/2+j},$$

which is possible because all even unimodular lattices are equivalent over $\mathbf{Z}/p\mathbf{Z}$ for any natural number p. The orthogonal group $\mathrm{O}(E)$ now appears as a subgroup of the symplectic group $\mathrm{Sp}(m, \mathbf{Z}/2\mathbf{Z})$. It consists of all symplectic matrices $M = \left(\begin{smallmatrix} A B \\ C D \end{smallmatrix} \right)$ such that the diagonals of $A'C$ and $B'D$ are zero. This is the image of the so-called theta group. It is easy to check that

$$D : \mathrm{O}(E) \longrightarrow \mathbf{Z}/2\mathbf{Z}, \quad D(M) = \sigma(C'B),$$

is a homomorphism. This is the Dickson invariant. It is non-trivial because if $a \in E$ is an element with $q(a) \neq 0$ then the "transvection" $x \mapsto x - (a, x)a$ has non-zero Dickson invariant.

A subspace $F \subset E$ is called *isotropic* if the restriction of q to F vanishes. We now consider maximal isotropic subspaces of E. Their dimension is $m/2$. The orthogonal group $\mathrm{O}(E)$ acts transitively on the set of these spaces. But under the kernel of the Dickson invariant D there are two orbits. Two spaces F_1 and F_2 are in the same orbit if and only if their intersection has even dimension. We select one of the two orbits and call it the first orbit and call the other the second orbit.

We now define a special $\epsilon(F)$ as follows. It is different from 0 if and only if F is maximal isotropic. It is 1 on the first orbit and -1 on the second one.

In the following we consider the system of modular forms $f^{(n)}$ constructed by means of this special $\epsilon(F)$.

Our first observation is that the functions $f^{(n)}(Z)$ have period 1 in all variables and hence admit a Fourier expansion

$$f^{(n)}(Z) = \sum_{T} a_n(T) \exp(\pi \mathrm{i} \sigma(TZ)),$$

where T runs over all integral symmetric matrices with even diagonal. Our next observation is that the coefficients $a_n(T)$ are invariant under unimodular substitutions $T \mapsto U'TU$, where $U \in \mathrm{GL}(n, \mathbf{Z})$. Let L be an arbitrary even lattice of rank n. The Gram matrix $T = \big((e_i, e_j)\big)$ with respect to a basis of the lattice is determined up to unimodular equivalence. We can define

$$a(L) := a_n(T).$$

An easy computation gives

$$a(L) = \#\mathrm{Aut}(L) \sum_M \epsilon(M/2M),$$

where the sum is over all M such that
  1. M is a n-dimensional sublattice of Λ.
  2. M is isomorphic to $L(2)$. ($L(2)$ denotes the doubled lattice L. It has the same underlying group as L but the norms (x, x) are doubled.)
  3. $M/2M$ is maximal isotropic in $\Lambda/2\Lambda$.

The group $\mathrm{Aut}(\Lambda)$ acts on the set of all M. It acts also on the subspaces $F \subset E$. We later need to know that this group preserves the Dickson invariant. This is the case if $\mathrm{Aut}(\Lambda)$ is contained in the special orthogonal group. For this one has to use that the composition of the natural homomorphism $\mathrm{Aut}(\Lambda) \to \mathrm{O}(E)$ with $(-1)^D$ is the determinant [B].

In the following we assume that all automorphisms of Λ have determinant $+1$. Otherwise all $f^{(n)}$ vanish. So we have to exclude all lattices Λ which contain a vector of norm 2. We can reformulate the formula for the Fourier coefficients as

Lemma 1. *The Fourier coefficients $a(L)$ of the functions $f^{(n)}$ are given by*

$$a(L) = \#\mathrm{Aut}(\Lambda)\#\mathrm{Aut}(L) \sum_M \frac{\epsilon(F)}{\#\mathrm{Aut}(\Lambda, M)} \qquad (F = M/2M).$$

Here M runs over a set of representatives of $\mathrm{Aut}(\Lambda)$-orbits of sublattices of Λ which are isomorphic to $L(2)$. The group $\mathrm{Aut}(\Lambda, M)$ consists of all elements of $\mathrm{Aut}(\Lambda)$ which preserve M as a set.

We want to prove now that $f^{(n)}$ is a modular form with respect to the full modular group. More precisely $f := (f^{(n)})$ is a stable system of Siegel modular forms, i.e. $f^{(n)}$ can be obtained from $f^{(n+1)}$ by applying the Siegel Φ-operator. It is known that every stable system can be written in a canonical way as linear combination of the standard theta functions ϑ_L. This leads us to the following construction of a linear combination of standard theta series.

Let $F \subset E$ be a maximal isotropic space. We consider the inverse image $\pi^{-1}(F)$ of F under the natural projection $\pi : \Lambda \to E$. The bilinear form $(x, y)/2$ is even and unimodular on $\pi^{-1}(F)$. In this way we obtain a new m-dimensional even unimodular lattice Λ_F. This is the so-called perestroika of Λ with respect to F in the notation of Koch and Venkov [KV].

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We need some more notation. Let Λ' be an even unimodular positive definite lattice of dimension m. We introduce the mass and the modified mass by

$$\mathrm{mass}(\Lambda') = \sum_{\Lambda_F \cong \Lambda'} \frac{1}{\#\mathrm{Aut}(\Lambda, F)},$$

$$\mathrm{mass}^\epsilon(\Lambda') = \sum_{\Lambda_F \cong \Lambda'} \frac{\epsilon(F)}{\#\mathrm{Aut}(\Lambda, F)},$$

where F runs over a system of representatives of $\mathrm{Aut}(\Lambda)$-orbits of maximal isotropic subspaces of E with perestroika of type Λ'.

We fix a system $\Lambda_1, \ldots, \Lambda_h$ of representatives of isomorphism classes of such lattices Λ' and write

$$\mathrm{mass}(i) := \mathrm{mass}(\Lambda_i), \quad \mathrm{mass}^\epsilon(i) := \mathrm{mass}^\epsilon(\Lambda_i).$$

Theorem 2. *We have*

$$f = \#\mathrm{Aut}(\Lambda) \sum_{i=1}^{h} \mathrm{mass}^\epsilon(i) \vartheta_{\Lambda_i}.$$

In particular the $f^{(n)}$ are modular forms with respect to the full modular group. The forms $f^{(n)}$ vanish for $n < m/2$, and are cusp forms for $n = m/2$.

Proof. The right hand side of the equation in theorem 2 can be written as $\sum_F \epsilon(F) \vartheta_{\Lambda_F}$. So the difference between both sides is

$$\sum_F \epsilon(F) \sum_{g:\ F(g) \subset F,\ F(g) \neq F} \exp \frac{\pi \mathrm{i}}{2} \sigma(T(g)Z).$$

We have to show that this series vanishes. We even show that the partial sum for each fixed g vanishes. This means:

Let $F' \subset E$ be an isotropic subspace which is not maximal. Then

$$\sum_{F' \subset F} \epsilon(F) = 0.$$

This follows from the existence of an element $g \in \mathrm{O}(E)$ which stabilizes F' and which has non-trivial Dickson invariant. The existence of such a g can be proved easily by using the above normal form of E. This proves Theorem 2.

The main problem is whether the cusp form $f^{(m/2)}$ vanishes identically or not. This depends on the lattice Λ. In the next section we show that it does not vanish if Λ is the Leech lattice.

The Siegel cusp form is nonzero in case of the Leech lattice.

From now on we assume that Λ is the Leech lattice. In this section we show that the Siegel form $f^{(12)}$ from the previous section does not vanish in this case. It is a Siegel cusp form of degree 12 and weight 12. Actually we will give two proofs. The first one uses computer calculations and uses the representation as linear combination of standard theta series. The second proof uses the original definition and is independent of computer calculations.

A first proof for the non vanishing of the cusp form $f^{(12)}$.

By theorem 2 the Siegel modular form $f^{(12)}$ is a linear combination of degree 12 theta functions of the 24 Niemeier lattices. We refer to [CS] for a detailed description of the Niemeier lattices. If L is a Niemeier lattice different from the Leech lattice the vectors of norm 2 generate a sublattice L_0 which determines L up to isomorphism. We use the notation $L = L_0^+$. Hence D_{24}^+ is (up to isomorphism) the unique Niemeier lattice which contains the root lattice D_{24}. We use the usual notations [CS] for the root lattices. We want to compute the modified mass of D_{24}^+.

Lemma 3. *The group* $\mathrm{Aut}(\Lambda)$ *acts transitively on the set of all sublattices of* Λ *which are isomorphic to* $D_{24}(2)$. *The same is true for* $D_{24}^+(2)$.

Proof. Recall that a frame of Λ is a set of 24 distinct pairs $\pm v_i$ of norm 8 vectors of Λ all congruent mod 2Λ. (See lecture 3 of chapter 10 of [CS].) To every frame we may associate a copy of $D_{24}(2)$ in Λ. It is generated by the vectors $(\pm v_i \pm v_j)/2$. We also get an embedding of $D_{24}^+(2)$ into Λ because the glue vector is contained in Λ. It is easy to see that this defines bijections between frames and sublattices of type $D_{24}(2)$ and between frames and sublattices of type $D_{24}^+(2)$. The group $\mathrm{Aut}(\Lambda)$ permutes the frames transitively. This proves Lemma 3.

The image F_0 of an embedded $D_{24}^+(2)$-lattice is a maximal isotropic subspace of $\Lambda/2\Lambda$. All these F_0 have the same Dickson sign because of lemma 3. We normalize the Dickson sign that it is 1 on these F_0. If F is any other maximal isotropic subspace then

$$\epsilon(F) = (-1)^{\dim_{\mathbf{F}_2}(F \cap F_0)}.$$

From lemma 3 now follows that the mass and the modified mass with respect to D_{24}^+ agree. The 24 masses $\mathrm{mass}(i)$ have been computed in the paper [DLMN]. The Niemeier lattice D_{24}^+ has index $i = 24$. We will denote the number of frames by $n_F = 8292375 = 3^6.5^3.7.13$. From Lemma 3 we obtain

$$\mathrm{mass}(24) = \mathrm{mass}^\epsilon(24) = \frac{n_F}{\#\mathrm{Aut}(\Lambda)} = \frac{1}{2} \cdot \frac{1}{501397585920}$$

in accordance with [DLMN].

The following table shows a 24×24-matrix. The columns correspond to the 24 Niemeier lattices in order of their Coxeter numbers. This order is not unique. We will use the order given in theorem 4. The rows correspond to the following lattices of degree ≤ 12: the 0 dimensional lattice, A_j for $1 \leq j \leq 11$, D_j for $4 \leq j \leq 12$, and E_6, E_7, E_8. The matrix entry is the number of sublattices of the Niemeier lattice isomorphic to the lattice of each row. This means that each column contains suitable normalized Fourier coefficients of the theta function of the corresponding Niemeier lattice.

0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
A_1	0	24	36	48	60	72	72	84	96	108	120	120	144	144	156	168	192	216	216	264	300	360	360	552
A_2	0	0	12	32	60	96	96	140	192	252	320	320	480	480	572	672	896	1152	1152	1760	2300	3360	3360	8096
A_3	0	0	0	8	30	72	72	140	240	378	560	560	1080	1080	1430	1848	2912	4320	4320	8360	12650	22680	22680	87032
A_4	0	0	0	0	6	24	0	84	144	378	600	384	1344	864	2574	2688	6384	8064	10584	25344	53130	94080	72576	680064
A_5	0	0	0	0	0	4	0	28	56	252	452	128	1184	144	3432	2688	10696	9408	19908	59136	177100	296576	120960	4307072
A_6	0	0	0	0	0	0	4	16	108	240	0	856	0	3432	1536	13744	8256	32112	101376	480700	766720	103680	22150656	
A_7	0	0	0	0	0	0	0	2	27	90	0	495	0	2574	384	14022	5832	43794	126720	1081575	1660320	38880	94140288	
A_8	0	0	0	0	0	0	0	0	3	20	0	220	0	1430	0	11696	2560	48620	112640	2042975	2929600	2880	334721024	
A_9	0	0	0	0	0	0	0	0	0	2	0	66	0	572	0	8008	512	43758	67584	3268760	4100096	0	1004163072	
A_{10}	0	0	0	0	0	0	0	0	0	0	0	12	0	156	0	4368	0	31824	24576	4457400	4472832	0	2556051456	
A_{11}	0	0	0	0	0	0	0	0	0	0	0	1	0	26	0	1820	0	18564	4096	5200300	3727360	0	5538111488	
D_4	0	0	0	0	0	1	6	0	10	0	15	60	80	180	0	210	126	840	315	990	0	4970	9450	10626
D_5	0	0	0	0	0	0	2	0	6	24	48	108	0	168	126	1008	378	1584	0	11928	22680	42504		
D_6	0	0	0	0	0	0	0	0	1	4	7	0	0	84	84	336	63	1848	0	11788	11340	134596		
D_7	0	0	0	0	0	0	0	0	0	0	1	0	0	24	36	120	0	1584	0	12520	3240	346104		
D_8	0	0	0	0	0	0	0	0	0	0	0	0	3	9	45	0	990	0	13005	405	735471			
D_9	0	0	0	0	0	0	0	0	0	0	0	0	1	10	0	440	0	11440	0	1307504				
D_{10}	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	132	0	8008	0	1961256				
D_{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0	4368	0	2496144				
D_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	1820	0	2704156				
E_6	0	0	0	0	0	0	0	0	0	0	1	4	0	0	0	56	28	0	0	1120	3360	0		
E_7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	0	0	120	360	0		
E_8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	3	0		

The rank of this matrix is 24. Therefore the 24 theta functions are linearly independent in degree 12. All lattices of rows in the above table other than D_{12} are of rank < 12. From this we see that the space of degree 12 cusp forms spanned by the 24 theta functions is at most one dimensional.

In the following we use the notation $\vartheta(L_0) = \vartheta_L$ for a root lattice L_0 contained in a Niemeier lattice $L = L_0^+$. We deduce from the above matrix:

Theorem 4. *The 24 theta functions are linearly independent in degree 12. Every degree 12 cusp form spanned by the 24 theta functions is a constant multiple of*

$$\frac{1}{152769576960}\vartheta(\text{Leech}) \qquad -\frac{1}{3183476736}\vartheta(A_1^{24}) \qquad +\frac{1}{591224832}\vartheta(A_2^{12})$$

$$-\frac{5}{1146617856}\vartheta(A_3^8) \qquad +\frac{13}{1990656000}\vartheta(A_4^6) \qquad -\frac{83}{11943936000}\vartheta(A_5^4 D_4)$$

$$+\frac{19}{16307453952}\vartheta(D_4^6) \qquad +\frac{41}{15676416000}\vartheta(A_6^4) \qquad -\frac{1}{37158912000}\vartheta(A_7^2 D_5^2)$$

$$-\frac{197}{351151718400}\vartheta(A_8^3) \qquad +\frac{59}{214990848000}\vartheta(A_9^2 D_6) \qquad -\frac{13}{229323571200}\vartheta(D_6^4)$$

$$-\frac{1}{35831808000}\vartheta(A_{11}D_7 E_6) \qquad +\frac{1}{143327232000}\vartheta(E_6^4) \qquad +\frac{31}{7685922816000}\vartheta(A_{12}^2)$$

$$+\frac{37}{11415217766400}\vartheta(D_8^3) \qquad -\frac{29}{21069103104000}\vartheta(A_{15}D_9) \qquad -\frac{1}{7023034368000}\vartheta(D_{10}E_7^2)$$

$$+\frac{1}{3511517184000}\vartheta(A_{17}E_7) \qquad +\frac{53}{4237899595776000}\vartheta(D_{12}^2) \qquad -\frac{1}{1332620771328000}\vartheta(A_{24})$$

$$-\frac{1}{3595793596416000}\vartheta(D_{16}E_8) \qquad +\frac{1}{10787380789248000}\vartheta(E_8^3) \qquad +\frac{1}{2729207339679744000}\vartheta(D_{24})$$

Remark. *This linear combination has been normalized so that the Fourier coefficient of the D_{12} lattice is 1.*

We know that at least one coefficient in theorem 2 is different from 0. From the linear independence of theorem 4 we obtain:

Theorem 5. *The form $f^{(12)}$ of theorem 2 does not vanish when Λ is the Leech lattice.*

We know already that $f^{(12)}$ is a cusp form. This cusp form must be a constant multiple of the linear combination of theta functions in theorem 4. The constant factor can be determined by looking at the coefficient of D_{24}^+:

Theorem 6. *Let g be the linear combination from theorem 4. This linear combination is a cusp form. We have*

$$f = \#\mathrm{Aut}(\Lambda) \cdot 2^6 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot g.$$

The following corollary has been obtained independently by Igusa [I] and Kneser [K]. They gave an affirmative answer to a question raised by Witt [W]. Igusa used deep results about modular forms. The proof of M. Kneser [K] is elementary.

Corollary. *If E_8^2 and D_{16}^+ are the two 16 dimensional even unimodular lattices and L is a lattice of dimension at most 3 then the numbers of embeddings of L into E_8^2 and D_{16}^+ are the same.*

Proof. Look at the Fourier coefficient in f of the lattice $M = L \oplus E_8$. This coefficient is given in terms of the numbers of embeddings of M into various Niemeier lattices, and must vanish as f is a cusp form of degree 12 and M has dimension at most 11. On the other hand any Niemeier lattice containing M must contain an E_8 sublattice so must be E_8^3 or $D_{16}^+E_8$, and the number of embeddings of M into E_8^3 or $D_{16}E_8$ is given up to some fixed factors by the number of embeddings of L into E_8^2 or D_{16}^+ respectively. This easily implies that the numbers of embeddings of L into E_8^2 or D_{16}^+ are equal. This proves the corollary.

A second proof for the non vanishing of the cusp form.

We will give another proof that $f^{(12)}$ is not identically 0, without using computer calculations, by showing that $a(M)$ is nonzero if M is the D_{12} lattice. We use the formula for the Fourier coefficients from lemma 1 coming from the first representation of f.

Lemma 7. *There is a maximal isotropic subspace F_0 of $\Lambda/2\Lambda$ which has an odd number of conjugates under $Aut(\Lambda)$.*

Proof. The (unique) orbit of maximal isotropic subspaces with perestroika D_{24}^+ has as representative F_0 the span of all vectors of the form $\sum_i 4n_i e_i$ where the sum is over the usual orthogonal basis e_i of norm 1/8 vectors of $\Lambda \otimes \mathbf{R}$ ([CS], p.287) and the n_i's are either all in $\mathbf{Z}$ or all in $\mathbf{Z} + 1/2$ and have even sum. Then F_0 is fixed by the standard subgroup $2^{12}.M_{24}$ of $Aut(\Lambda)$ which has odd index, so that F_0 has an odd number of conjugates under $Aut(\Lambda)$. This proves lemma 7.

The lattice D_{12} is the 12 dimensional lattice of determinant 4 generated by the roots of as D_{12} root system, and $D_{12}(2)$ is this lattice with all inner products multiplied by 2.

As we have mentioned, a frame of Λ is a set of 24 distinct pairs $\pm v_i$ of norm 8 vectors of Λ all congruent mod 2Λ. For each frame there is an action of the Mathieu group M_{24} on this 24 element set.

Lemma 8. *Sublattices L of Λ isomorphic to $D_{12}(2)$ correspond to frames in Λ together with a choice of 12 element subset of the 24 element subset of the frame.*

Proof. Twice the images of the norm 1 vectors of the dual of the D_{12} in Λ give a set of 12 pairs $\pm v_i$ of norm 8 vectors of Λ, all congruent mod 2, and these determine a unique frame and a 12 element subset of the 24 element set of the frame. The image L of the $D_{12}(2)$ is then spanned by the set of vectors of the form $(\pm v_i \pm v_j)/2$.

Conversely any choice of frame and 12 element subset gives a sublattice isomorphic to $D_{12}(2)$ by the construction above. This gives a one to one correspondence between such sublattices and pairs consisting of a frame and a 12 element subset, which proves lemma 8.

Lemma 9. *There are exactly 5 orbits of sublattices of Λ isomorphic to $D_{12}(2)$, of sizes $2^2.5.7.11.23n_F$, $2^4.3^2.5.7.11.23n_F$, $2^6.3^2.7.11.23n_F$, $2^8.3^2.7.23n_F$, and $2^4.7.23n_F$.*

Proof. This follows immediately from lemma 3 and the classification of the five M_{24}-orbits of 12 element subsets in [CS, chapter 10, theorem 22] and the fact that $Aut(\Lambda)$ acts transitively on the n_F frames.

Lemma 10. *A sublattice L of Λ isomorphic to $D_{12}(2)$ represents a maximal isotropic subspace of $\Lambda/2\Lambda$ if and only if there are no vectors $v \in \Lambda$ with $2v \in L$, $v \notin L$.*

Proof. The image of L in $\Lambda/2\Lambda$ is isotropic as all vectors of L have norm divisible by 4, so we have to check whether this image has dimension 12, in other words we have to check whether the map from $L/2L$ to $\Lambda/2\Lambda$ is injective. But this is the same as asking whether there exist no vectors v as above, and this proves lemma 10.

Lemma 11. *There is a vector $v \in \Lambda$ with $2v \in L$, $v \notin L$, if and only if the 12 element set S corresponding to L contains no nonzero elements of the Golay code.*

Proof. If there is such a set S, then we can construct v as a sum $\sum_{i \in S} \pm v_i/4$. Conversely given v, we can construct S as the set of coordinates where the coefficient of v is 2 mod 4 (which is nonempty as $v \notin L$). This proves lemma 11.

Lemma 12. *There are exactly two $Aut(\Lambda)$-orbits of sublattices L of Λ isomorphic to $D_{12}(2)$ such that $L/2L$ is a maximal isotropic subspace of $\Lambda/2\Lambda$, and they have sizes $2^6.3^2.7.11.23n_F$ and $2^8.3^2.7.23n_F$.*

Proof. By lemmas 4, 5 and 6 we have to find the orbits of 12 element subsets of a 24 element set acted on by M_{24} which contain no nonzero elements of the Golay code. These can be read off from the discussion in section 2.6 of chapter 10 of [CS]. In Conway's terminology, the "special" and "extraspecial" 12 element sets (and no others) contain octads of the Golay code, and the "umbral" 12 element sets are already in the Golay code. This leaves the "transverse" and "penumbral" as the 12 element sets containing no element of the Golay code, and they have the orbit sizes as stated in the lemma. This proves lemma 12. We now obtain a new proof for

Theorem. *In case of the Leech lattice the Siegel cusp form $f^{(12)}$ of theorem 2 is nonzero.*

Proof. We use the formula of lemma 1 to compute the Fourier coefficient $a(D_{12})$. The Dickson signs are constant on each of the two orbits. Using the formulas

$$\#\mathrm{Aut}(D_{12}) = 2^{12} \cdot 12!, \qquad \#\mathrm{Aut}(\Lambda) = 2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$$

we obtain

$$a(D_{12}) = 2^{28} \cdot 3^{13} \cdot 5^5 \cdot 7^3 \cdot 11 \cdot 13 \cdot 23 \cdot (\pm 11 \pm 4)$$

Whatever the signs of the Dickson invariants might be, this number is different from 0. This proves the theorem. (From Lemma 3 and 8 it follows that in fact both signs are $+1$. This is in accordance with the formula $a(D_{12}) = 2^{28} \cdot 3^{14} \cdot 5^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 23$, which comes from lemma 1.)

The coefficient we have shown is nonzero is in some sense the first nonzero coefficient, or more precisely a nonzero coefficient corresponding to a 12 dimensional lattice of smallest possible determinant. This follows easily from the fact that there are no even 12 dimensional lattices with determinant less than 4.

We observed that the Fourier coefficient of a lattice M of determinant n often seems to be closely related to the coefficient of q^n of the weight $13/2$ ordinary modular form

$$
\begin{aligned}
\eta(8\tau)^{12}\theta(\tau) = {} & q^4 + 2q^5 + 2q^8 - 12q^{12} - 22q^{13} - 24q^{16} + 56q^{20} + 84q^{21} + 108q^{24} \\
& - 112q^{28} - 66q^{29} - 176q^{32} + 9q^{36} - 398q^{37} - 196q^{40} \\
& + 364q^{44} + 990q^{45} + 1056q^{48} - 616q^{52} + 70q^{53} - 728q^{56} \\
& + 432q^{60} - 2354q^{61} - 1472q^{64} - 240q^{68} + 1080q^{69} + 990q^{72} \\
& - 484q^{76} + 1848q^{77} + 2752q^{80} + 2352q^{84} + 2292q^{85} + 1276q^{88} \\
& - 2608q^{92} - 3852q^{93} - 9504q^{96} + O(q^{100})
\end{aligned}
$$

at least when n is 0, 4, or 5 mod 8. It is often the same when n is divisible by 4 and often differs by a factor of -2 when $n \equiv 5$ mod 8. We have not been able to find such similar properties when $n \equiv 1$ mod 8.

Here is a table of the Fourier coefficients corresponding to lattices M that have determinant at most 96 and are generated by their norm 2 vectors. This table was calculated using the expression of f as a linear combination of theta functions, and using the fact that if M is generated by norm 2 vectors then the number of embeddings of M into any other lattice L can easily be worked out knowing the root system of L. A larger version of this table and the programs used to calculate it can be found on R. E. Borcherds' home page http://www.dpmms.cam.ac.uk/~reb.

det	coef	Lattice	det	coef	Lattice	det	coef	Lattice
4	1	D_{12}	4	1	D_4E_8	5	-1	A_4E_8
8	2	$A_1A_3E_8$	8	2	A_1D_{11}	8	2	D_5E_7
9	6	$A_2^2E_8$	9	6	E_6^2	12	-12	$A_1^2A_2E_8$
12	-12	A_2D_{10}	12	-12	A_5E_7	12	-12	D_6E_6
13	11	A_{12}	16	40	$A_1D_4E_7$	16	40	$A_1^2D_{10}$
16	40	$A_1^4E_8$	16	40	D_6^2	16	-24	A_3D_9
16	-24	D_5D_7	16	-88	D_4D_8	20	-8	$A_1A_4E_7$
20	56	A_4D_8	21	-42	A_6E_6	24	108	A_1A_{11}
24	108	$A_1A_2D_9$	24	108	$A_1D_5E_6$	24	108	$A_2A_3E_7$
24	108	A_5D_7	28	-112	A_6D_6	32	-48	$A_1A_3D_8$
32	-48	$A_1D_4D_7$	32	-48	$A_1D_5D_6$	32	-48	$A_1^2A_3E_7$
32	-48	$A_1^3D_9$	32	-176	A_7D_5	33	54	A_2A_{10}
36	9	A_8D_4	36	48	$A_2^2D_8$	36	738	$A_2D_4E_6$

36	−336	$A_1A_2^2E_7$	36	−336	$A_1A_5E_6$	40	−196	$A_1A_4D_7$
40	−196	A_3A_9	44	364	$A_1^2A_{10}$	45	234	$A_2A_4E_6$
45	−495	A_4A_8	48	288	$A_1A_5D_6$	48	288	$A_1^2A_2D_8$
48	288	$A_1^2D_4E_6$	48	288	$A_1^3A_2E_7$	48	288	$A_2D_4D_6$
48	288	A_5A_7	48	−480	$A_2A_3D_7$	48	1056	$A_2D_5^2$
48	1056	$A_3^2E_6$	49	1260	A_6^2	56	−728	$A_1A_6D_5$
60	432	$A_1A_2A_9$	60	432	$A_1^2A_4E_6$	60	432	$A_2A_4D_6$
64	1088	$A_1^2A_3D_7$	64	1088	$A_1^2D_4D_6$	64	1088	$A_1^2D_5^2$
64	1088	$A_1^4D_8$	64	1088	$A_1^5E_7$	64	1088	$A_3^2D_6$
64	4160	$A_3D_4D_5$	64	−3008	$A_1A_7D_4$	64	23616	D_4^3
72	990	$A_1A_3A_8$	72	−468	$A_1A_2A_3E_6$	72	−468	$A_2A_5D_5$
72	2448	$A_1A_2^2D_7$	80	2240	$A_1A_4A_7$	80	2752	$A_3A_4D_5$
80	6336	$A_1^3A_9$	80	−1856	$A_1^2A_4D_6$	80	−9024	$A_4D_4^2$
81	5886	$A_2^2A_8$	81	−7236	$A_2^3E_6$	84	−336	$A_1A_5A_6$
84	−3024	$A_2A_6D_4$	96	4320	$A_2A_3A_7$	96	−2592	$A_1A_2A_3D_6$
96	−2592	$A_1A_2D_4D_5$	96	−2592	$A_1^2A_5D_5$	96	−2592	$A_1^3A_2D_7$
96	−2592	$A_1^3A_3E_6$	96	−2592	$A_3A_5D_4$			

A Hecke eigenvalue.

Our cusp form is an eigenform of all Hecke operators because Hecke operators map cusp forms to cusp forms and preserve the space generated by standard theta series.

We use the definition of the Hecke operator $T(p)$ as given in [F]. The explicit formula of the action of $T(p)$ for a prime p on theta series ([F, IV.5.10]) states that

$$\vartheta_{\Lambda_\nu}|T(p) = \beta(p,m,n) \cdot \sum_{\mu=1}^{h} n(\Lambda_\mu(p), \Lambda_\nu)\vartheta_{\Lambda_\mu},$$

where $n(L(p), M)$ denotes the number of sublattices of M of type $L(p)$. The constants $\beta(p,m,n)$ depend on the normalization of $T(p)$. We refer to [F] for explicit expressions. For example

$$\beta(p, 24, 12) = p^{\frac{n(n+1)}{2} - 12n}.$$

It is well known and easy to prove that the matrix with entries $\#\mathrm{Aut}(\Lambda_\mu) \cdot n(\Lambda_\nu, \Lambda_\mu(p))$ is symmetric.

Because of the linear independence of the 24 theta series it is sufficient to know one row (or column) of this matrix if one wants to compute the eigenvalue $\lambda(p)$ of $T(p)$. We can compute one row of this matrix in the case $p = 2$. One easily derives

$$\mathrm{mass}(L) = \frac{n(L(2), \Lambda)}{\#\mathrm{Aut}(\Lambda)},$$

where $\mathrm{mass}(L)$ is the mass introduced above. As we already mentioned these masses have been computed in [DLMN] for the Leech lattice Λ. Using table I of this paper one obtains

$$\lambda(2) = 2^7 \cdot 3^{11} \cdot 5 \cdot 17 \cdot 901141 \cdot \beta(2, 24, 12).$$

We obtain now some information about the Satake parameters $x_1, \ldots x_{12}$ of our cusp form at the place 2. We recall briefly their definition [A]. The local Hecke algebra at a prime is isomorphic to the ring of invariants $\mathbf{C}[X_0^{\pm 1}, \ldots, X_n^{\pm 1}]^{W_n}$, where W_n is the symplectic Weyl group ([F], IV.3.19). Every homomorphism of this ring into the field of complex numbers is the restriction of a homomorphism of the whole $\mathbf{C}[X_0^{\pm 1}, \ldots, X_n^{\pm 1}]$. The images x_j of the variables X_j are the Satake parameters. They are determined up to the action of W_n. Every Siegel eigenform of the local Hecke algebra at a prime p defines such a homomorphism. The Ramanujan conjecture says that the Satake parameters $x_1, \ldots, x_n$ of an eigen cusp form have absolute value 1. It is known that in degree $n > 1$ this is not always true.

To obtain information about the Satake parameters (at the prime $p = 2$) we need the image of the operator $T(p)$ in the local Hecke algebra. This formula can be found in [F]. We choose a root $y_j = \sqrt{x_j}$ for each Satake parameter. A direct consequence of formula [F], IV.3.14, a) and b) is

$$\frac{\lambda(p)^2}{x_0^{-2} x_1 \cdots x_n} = \prod_{j=1}^{12} \left(y_j + y_j^{-1} \right)^2 \quad \text{and} \quad p^{\frac{n(n+1)}{2} - 12n} = x_0^{-2} x_1 \cdots x_n.$$

The computed value $\lambda(2)$ now gives:

Theorem 13. *The Satake parameters $x_i = y_i^2$ of our cusp form of degree 12 and weight 12 at the place $p = 2$ satisfy*

$$\left| \prod_{i=1}^{12} (y_i + y_i^{-1}) \right| = \frac{3^{11} \cdot 5 \cdot 17 \cdot 901141}{2^{26}}.$$

Corollary. *The Ramanujan conjecture $|x_i| = 1$ is violated for $p = 2$.*

Open problems.

We list a few questions about the Siegel cusp form f.

1. Are the coefficients of the cusp form of weight 12 and degree 12 all integers when normalized so that the coefficient of D_{12} is 1?

   One can prove that the coefficients of $f^{(m/2)}/\#\mathrm{Aut}(\Lambda)$ are contained in $\mathbf{Z}[1/2]$ for arbitrary Λ and in $(1/2)\mathbf{Z}$ in case of the Leech lattice. This means that the denominators of the normalized coefficients divide $2^7 \cdot 3^5 \cdot 5^2 \cdot 7$.

2. Why are the coefficients of f similar to those of the modular form above? Is there a similar relation for the coefficients of lattices of determinant 1 mod 8? Is it possible to write down some simple explicit formula for the coefficients of f?

   From [We] it follows that the standard L-function $L(f, s)$ of f has a pole at $s = 1$. This suggests that $L(f, s) = \zeta(s)L(s)$, where L belongs to a 24-dimensional l-adic Galois representation. This Galois representation cannot be pure (theorem 13) and therefore one might expect that its weight filtration sheds light on the relationship with $\eta(8\tau)^{12}\theta(\tau)$.

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