Birational invariants of modular varieties and singular modular forms

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1 Two results about singularities of modular varieties

Let $D \subset \mathbb{C}^n$ be a bounded symmetric domain and

$$\Gamma \subset \mathrm{Bihol}(D)$$

an arithmetic group of biholomorphic mappings of D onto itselves. We assume that (D, Γ_0) is indecomposable (as direct product) for each subgroup $\Gamma_0 \subset \Gamma$ of finite index. We also assume

$$n = \dim D > 1.$$

We denote by

$$X = X_\Gamma = \overline{D/\Gamma}$$

the BAILY-BOREL compactification of D/Γ, which is a normal projective variety containing D/Γ as a ZARISKI-open subvariety. From our assumption follows that the field $K(\Gamma)$ of rational functions on X_Γ agrees with the field of meromorphic functions on D, which are Γ-invariant. The elements of $K(\Gamma)$ are called *modular functions*.

We consider a desingularization

$$\begin{array}{l} \pi : \widetilde{X}_\Gamma \longrightarrow X_\Gamma \\ \quad\ \ \cup \\ \quad\ \ X_{\mathrm{reg}} \end{array}$$

which contains the regular locus X_{reg} as ZARISKI-open subset.

An extension theorem

1.1 Therorem (BAUERMANN, [Ba]). *Let $U \subset X$ be any open subset, $\widetilde{U} = \pi^{-1}(U) \subset \widetilde{X}$ its inverse image and $U_{\mathrm{reg}} = U \cap X_{\mathrm{reg}}$ the regular locus.*

Every holomorphic alternating differential form of degree $p < n$ on U_{reg} extends holomorphically to $\widetilde{U}$, i.e. the natural restriction map

$$(\Lambda^p\Omega)(\widetilde{U}) \longrightarrow (\Lambda^p\Omega)(U_{\mathrm{reg}}), \quad p < n,$$

is an isomorphism.

1.2 Corollary (Pommerening, [Po]). *There is a natural isomorphism*

$$(\Lambda^p\Omega)(\widetilde{X}_\Gamma) \cong (\Lambda^p\Omega(D))^\Gamma.$$

(In the case of the Siegel modular group the latter result already has been proved in [FP]*.)*

Theorem 1.1 is a special case of a more general theorem. Instead of alternating differential forms one may consider arbitrary holomorphic tensors, i.e. elements

$$T \in \Omega^{\otimes p}(U_{\mathrm{reg}}).$$

Bauermann [Ba] worked out the conditions for the holomorphic extendability for sufficiently small (so called „neat“) groups Γ. Those condition are formulated in terms of the Fourier-Jacobi expension of T. It generalizes results of Tai [AMRT] and the author [Fr3].

Theorem 1.1 says that – in some sense – the singularites of X are harmless. The next theorem will show that they are not harmless at all.

Let $Y \subset X$ be an irreducible subvariety, for example a point. We consider the local ring

$$R = O_{X,Y}$$

of X at the general point of Y. By definition it consists of all modular functions which are regular on an open subset of X whose intersection with Y is not empty. We have a natural homomorphism

$$O_{X,Y} \longrightarrow K(Y)$$

onto the the field of rational function on Y. The kernel of this homomorphism is the maximal ideal $\mathbf{m}$.

The (Krull-)dimension of R is

$$\dim R = \dim X - \dim Y.$$

The *depth* of R is the maximal length of a chain of elements $(r_1, \dots, r_m)$ in R, such that image of r_i is a non zero divisior in $R/(r_1, \dots, r_{i-1})$ $(1 \le i \le m)$. It is well known that

$$\operatorname{depth}(R) \le \dim(R).$$

We only consider the case $\dim R \ge 2$. Ti follows from the normality of R that

$$2 \le \operatorname{depth}(R) \le \dim(R).$$

The *depth* can also be expressed by means of the cohomology groups

$$M_i := H^i(\operatorname{Spec}(R) - \mathbf{m}, \mathcal{O}) \qquad (\mathcal{O} = \text{ the structure sheaf }).$$

Those groups are finitely generated R-modules if $i \le \dim R - 2$. The depth is the maximal number m, such that

$$M_i = \begin{cases} R & \text{if } i = 0, \\ 0 & \text{if } 1 \le i < m. \end{cases}$$

The determination of those cohomology groups is an unsolved problem even in the case of points Y. Only the case of the HILBERT modular group has been settled ([Fr1], [FK]). The situation is better if one takes for Y an irreducible component of the boundary X-D/Γ. In this case the cohomology groups can be calculated in full generality [Bal]. The case of the SIEGEL modular group has been treated earlier ([Ca]).

The idea is as follows. One considers a small STEIN neighbourhood $U \subset X$ of a general point a of Y.

Standard comparison theorems show

$$M_i \cong H^i(U - (U \cap Y), \mathcal{O}) \ \text{ for } \quad i \le \dim R - 2,$$

where the right land side are analytic cohomology groups.

The choice of U can be made as follows:

There is an open STEIN subset

$$\mathcal{S} \subset D$$

which is invariant under a certain subgroup

$$\Gamma_0 \subset \Gamma$$

(Γ_0 is the stabilizer of a representative of a in the HARISH-CHANDRA boundary of D), such that the natural projetion induces a biholomorphic map

$$\mathcal{S}/\Gamma_0 \xrightarrow{\sim} U - (U \cap Y).$$

After that one is reduced to consider cohomology groups of Γ_0 acting on holomorphic functions on $\mathcal{S}$.

The structure of Γ_0 is rather simple, nevertheless the computation of this group cohomology is difficult. We do not go into the details but formulate only the result in an important special case:

We consider for D the SIEGEL upper half plane

$$\mathbf{H}_n := Z = \{Z^{(n)} = Z' = X + iY, \quad Y > 0\}$$

of all symmetric $n \times n$-matrices with positive definit imaginary part and for Γ the principal congruence subgroup

$$\Gamma_n[q] = \text{ kernel } (\mathrm{Sp}(n, \mathbb{Z}) \longrightarrow \mathrm{Sp}(n, \mathbb{Z}/q\mathbb{Z}))$$

of the SIEGEL modular group

$$\Gamma_n = \mathrm{Sp}(n, \mathbb{Z}).$$

This group acts on $\mathbf{H}_n$ by the formula

$$Z \mapsto M\langle Z \rangle = (AZ + B)(CZ + D)^{-1}; \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

In this case the „stabilizer“ Γ_0 consists of all elements of the form

$$M = \begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} E & S \\ 0 & E \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & & & a_1 \\ & \ddots & & \vdots \\ & & 1 & a_{n-1} \\ & & & \pm 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & \dots & s_1 \\ \vdots & & \vdots \\ s_1 & \dots & s_n \end{pmatrix}$$

(all the other entries of U and S are 0).

It is worth while the notice that the -1 only occurs if $q \le 2$. The subgroup $\Delta \subset \Gamma_0$ of all elements of the form

$$\begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix}$$

regulates the cohomology groups. This subgroup is isomorphic to $\mathbb{Z}^{n-1}$ or to a semidirect product $\mathbb{Z}^{n-1} \propto \mathbb{Z}/2\mathbb{Z}$, where the non trivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}^{n-1}$ by means of $a \mapsto -a$.

We have to consider the cohomology group of Δ acting trivially on $\mathbb{C}$. We obtain

$$H^i(\Delta, \mathbb{C}) = \begin{cases} \Lambda^i \mathbb{C}^{n-1}, \\ (\Lambda^i \mathbb{C}^{n-1})^{\mathbb{Z}/2\mathbb{Z}}. \end{cases}$$

The second case occurs if

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}$$

is in Δ. Now we can formulate the result in the Siegel case.

1.3 Theorem([Ca]). *Let R_0 denote the ring of holomorphic functions on $U \cap Y$. Then*

$$H^i(U - (U \cap Y), O) \cong H^i(\Delta, \mathbb{C}) \otimes_{\mathbb{C}} R_0, \quad \textit{for} \ \ 1 \le i \le n-2.$$

(The natural restriction

$$\mathcal{O}(U) \longrightarrow \mathcal{O}(U - (U \cap Y))$$

is an isomorphism. Hence R_0 can be considered as a factor ring of $\mathcal{O}(U - (U \cap Y))$ especially as a module over this ring.)

It is easy to see that

$$\begin{aligned} H^2(\mathbb{Z}^{n-1}, \mathbb{C})^{\mathbb{Z}/2\mathbb{Z}} &\neq 0 \ \text{ if } \ n > 3, \\ H^1(\mathbb{Z}^{n-1}, \mathbb{C}) \qquad &\neq 0 \ \text{ if } \ n \ge 3, \\ H^1(\mathbb{Z}^{n-1}, \mathbb{C})^{\mathbb{Z}/2\mathbb{Z}} &= 0. \end{aligned}$$

Hence we obtain

1.4 Corollary. *If R is the local ring at an irreducible component of the boundary*

$$X_n - \mathbf{H}_n / \Gamma_n[q],$$

then

$$\dim R = n$$

and

$$\operatorname{depth}(R) = \begin{cases} 2, & \text{if} \quad n = 2, \\ 2, & \text{if} \quad n \geq 3, \quad q \leq 2, \\ 3, & \text{if} \quad n \geq 3, \quad q > 2. \end{cases}$$

A ring is called Cohen-Macauley if dimension and depth agree. We obtain

1.5 Corollary. *The ring R is Cohen-Macauley if and only if*

$$n \leq 2 \quad \text{or} \quad (n = 3 \quad \text{and} \quad q \leq 2).$$

In this connection we should mention that by a result of Runge [Ru] not only R but the whole variaty $\overline{\mathbf{H}_3/\Gamma_3[q]}$ is Cohen-Macauley if $q \leq 2$. This contradicts to a statement of Tsugumine [Ts]. Actually the proof of Tsugumine is false.

2 Holomorphic tensors

Let X be an irreducible variety and $\widetilde{X}$ a nonsingular projective model of X. We consider tensors

$$T \in \Omega^{\otimes p}(\widetilde{X}),$$

which can be expressed locally (in analytic coordinates) as

$$T = \sum f_{i_1,\dots,i_p} dz_i \otimes \ldots \otimes dz_{i_p}$$

Let

$$\pi : \mathrm{GL}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(V)$$

be an irreducible polynomial representation of $\mathrm{GL}(n, \mathbb{C})$ on a finite dimensional vector space. We can consider tensors of type π and the subspaces

$$\Omega_\pi^{\otimes p}(\widetilde{X}).$$

(After the choice of a local coordinate system in a neighbourhood of a point $a \in \widetilde{X}$, we may consider $T(a)$ as element

$$T(a) \in (\mathbb{C}_n)^{\otimes p}.$$

The π-isotopic component in $(\mathbb{C}^n)^{\otimes p}$ does not depend on the choice of the coordinate system).

The dimensions

$$d_\pi := \dim \Omega_\pi^{\otimes p}(\widetilde{X})$$

are birational invariants of X, i.e. they depend only on the field of rational functions $K(X)$ on X,

$$d_\pi = d_\pi(K(X)).$$

If π is the one dimensional representation

$$\pi(A) = (\det A)^k,$$

the elements of $\Omega_\pi^{\otimes k}(\widetilde{X})$ are called *multicanonical*, the invariants

$$p_k = d_{(\det A)^k}; \quad k = 0, 1, 2, \dots$$

are the so called *plurigenera*. If

$$\pi_p : \mathrm{GL}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(\Lambda^p(\mathbb{C}^n))$$

is the natural representation on the p^{th} extension power then the elements of $\Omega_\pi^{\otimes k}(\widetilde{X})$ are alterning differential forms, the invariant are denoted by

$$g_p = d_{\pi_p}.$$

If $0 < p < n$ they sometimes are called *irregularities*. Obviously $g_n = p_1$, $g_0 = p_0 = 1$.

The arithmetic genus

$$g = \sum (-1)^p g_p$$

is of great importance.

The spaces $\Omega_\pi^{\otimes k}(\widetilde{X})$ are connected with the correspondending spaces of Γ-invariant tensors on the domain D, i.e. we have to compare

$$\Omega_\pi^{\otimes k}(\widetilde{X}) \qquad \text{and} \qquad \Omega_\pi^{\otimes k}(D)^\Gamma.$$

If Γ acts freely on D, one has a natural inclusion

$$\Omega_\pi^{\otimes k}(\widetilde{X}) \hookrightarrow \Omega_\pi^{\otimes k}(D)^\Gamma.$$

The elements of $\Omega_\pi^{\otimes k}(D)^\Gamma$ can be interpreted as vector valued automorphic forms. A vector valued holomorphic automorphic form is a holomorphic map

$$f : D \longrightarrow \mathcal{Z}$$

an D with values in a finite dimensional complex vector space $\mathcal{Z}$ and will the properties

$$f(\gamma z) = \rho(I(\gamma, z)) f(z), \quad \gamma \in \Gamma.$$

Here $I(\gamma, z) = (\partial\gamma_i / \partial z_j)$ denote the JACOBI matrix and

$$\rho : G \longrightarrow \mathrm{GL}(\mathcal{Z})$$

a representation of the complex LIE-group G generated by all

$$I(g, z); \quad g \in \mathrm{Aut}(D), \quad z \in D.$$

We denote the space of these modular forms by

$$[\Gamma, \rho].$$

From our assumption this is a finite dimensional vector space. We have

$$\Omega_\pi^{\otimes k}(D)^\Gamma = [\Gamma, \rho], \quad \rho := \pi | G.$$

The automorphic forms $T \in [\mathit{\Gamma}, \rho]$ define holomorphic tensors on

$$D'/\mathit{\Gamma},$$

where $D' \subset D$ is the biggest open subset of D on which $\mathit{\Gamma}$ acts freely. It follows from the results of [Ba] and Tai [AMRT]:

2.1 Proposition. *Let*

$$T \in [\mathit{\Gamma}, \rho], \quad \rho = \pi | G,$$

be a cusp form. Then there exists a subgroup $\mathit{\Gamma}_0 \subset \mathit{\Gamma}$ of finite index such that T extends holomorphically to the nonsingular model $\widetilde{X}_{\mathit{\Gamma}_0}$.

2.2 Corollary. *Each arithmetic subgroup $\mathit{\Gamma}$ admits a subgroup $\mathit{\Gamma}_0 \subset \mathit{\Gamma}$ of finite index such that the field $K(\mathit{\Gamma}_0)$ of modular functions is of general type.*

In the case of the Siegel modular group better results are known:

First we recall that the Jacobian „matrix“ of a symplectic substitution is given by the linear map

$$W \mapsto (CZ + D)'^{-1} W (CZ + D)^{-1},$$

where W varies in the tangent space of $\mathbf{H}_n$, i.e. in the space

$$\mathcal{Z}_n = \{W = W' = W^{(n)}\}$$

of all symmetric $(n \times n)$-matrices. The group G generated by the Jacobians is

$$G = \mathrm{GL}(n, \mathbb{C})/\{\pm E\}.$$

This means that Siegel modular forms belong to (rational) representations

$$\rho : \mathrm{GL}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(\mathcal{Z}), \quad \rho(-E) = id_{\mathcal{Z}}.$$

The rational representations are completely decomposable. Therefore we may assume that ρ is irreducible.

The *weight* k of ρ is the biggest integer such that

$$\rho_0(A) = (\det A)^{-k} \rho(A)$$

is a *reduced* representation, i.e. ρ_0 is polynomial but $\rho_0(A)/\det A$ is not polynomial. Obviousely k is nothing else but the last component of the highest weight vector. Good existence theorems for modular forms are known for big k (which means $k > 2n$) and for small k (which means $2k < n$).

Existence theorems for big weight

Multicanonial tensors are of type

$$(\Lambda^N \Omega)^{\otimes k} \quad (N = \frac{1}{2}(n(n+1)).$$

They correspond to *scalar valued* modular forms of weight $(n+1)k$. Using RIEMANN-ROCH theorems one obtains good estimates for the asymptotic behaviour of the dimension. Also explicit constructions using ϑ-series and POINCARÉ series give good results:

2.3 Theorem. *The field of modular functions $K(\Gamma_n)$ $(\Gamma_n = \mathrm{Sp}(n, \mathbb{Z}))$ is of general type if $n \geq 7$.*

For proofs and more comments we refer to [Fr3], [Mu], [Ta].

We mentioned already that $K(\Gamma_n)$ is unirational if $n \leq 5$. The case $n = 6$ remains open.

Using more complicated types of tensors one obtains results for subvarieties of the SIEGEL modular variety [Fr], [We]. The best results are due to WEISSAUER.

2.4 Theorem. *Let k be a natural number. There exists a constant $n(k)$ with the following property. Assume $n \geq n(k)$. Then every irreducible subvariaty of $\mathbf{H}_n/\Gamma_n$ (Γ_n = the full SIEGEL modular group) of codimension $\leq k$ is of general type. The bound $n(k)$ can be made explicite.*

2.5 Corollary. *Assume $n \geq 13$. The group of automorphism of the function field $K(\Gamma_n)$ is trivial.*

As we already mentioned, in the case of a large k the results describe mostly asymptotic behaviour. In the case of small weights we will obtain *explicit formulae* for some invariants.

3 Singular modular forms

In this section we restrict to the case of congruence groups of the SIEGEL modular group.

Such modular forms admits a FOURIER expansion

$$f(Z) = \sum a(T)e(TZ),$$
$$e(A) := e^{2\pi i\sigma(A)} \quad (\sigma = \text{ trace }),$$

where T runs through a lattice of rational symmetric matrices.

3.1 Definition. *A modular form is called **singular**, if*

$$a(T) \neq 0 \Longrightarrow \det T = 0.$$

3.2 Proposition. *Assume that*

$$\rho : \mathrm{GL}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(\mathcal{Z})$$

is a finite dimensional irreducible rational representation of weight k. A non-vanishing modular form with respect to ρ is singular if and only if $2k < n$. In that case one has

$$2k = \max\{\operatorname{rank} T;\ a(T) \neq 0\}$$

We have to introduce ϑ-series with harmonic coefficients. The ingredients are

1) A positive definite rational symmetric matrix S. We restrict to the case where the degree r of S is even,
$$r = 2k.$$

2) A harmonic polynomial
$$P : \mathbb{C}^{(r,n)} \longrightarrow \mathcal{Z}$$
with the properties
$$P(XA) = \rho_0(A')P(X)$$

Such a polynomial is called a harmonic form with respect to ρ_0.
More generally we consider functions

$$\varphi : \mathbb{Z}^{(r,n)} \longrightarrow \mathcal{Z},$$

which can be written as finite sum

$$\varphi(G) = \sum \varphi_\nu(G) P_\nu(S^{\frac{1}{2}}G),$$

where

$$\varphi_\nu : \mathbb{Z}^{(r,n)} \longrightarrow \mathcal{Z}$$

are functions which are periodic with respect to a suitable rational number q. We may consider them as functions

$$\varphi_\nu : R^{(r,n)} \longrightarrow \mathbb{C},$$

where R denotes the finite ring $\mathbb{Z}/q\mathbb{Z}$. A function φ of this type we call a *harmonic coefficient function* (with respect to (S, q)).

3.3 Lemma. *Assume that $S = S^{(2r)}$ is a positive rational matrix and*

$$\varphi : \mathbb{Z}^{(r,n)} \longrightarrow \mathcal{Z}$$

a harmonic coefficient function. The ϑ-series

$$\vartheta_\varphi(S; Z) = \sum_{G \in \mathbb{Z}^{(r,n)}} \varphi(G) e(S[G]Z)$$

is a modular form with respect to

$$\rho(A) = \rho_0(A)(\det A)^k$$

on a suitable congruence subgroup. (Obviously $\vartheta_\varphi(S;Z)$ is singular if $r < n$ in accordance with 3.2.)

We want to describe conditions under which ϑ-series belong to a given level. i.e. is an element of $[\Gamma_n[q], q]$.

The natural number q is fixed now.

3.4 Definition. *Let $S = S^{(2r)}$ be a rational symmetric positive matrix such that*

$$qS \quad \textit{and} \quad qS^{-1}$$

both are integral. A matrix

$$V \in \frac{1}{q}\mathbb{Z}^{(2r,n)}$$

is called isotropic (with respect to (S,q)) if

$$S^{-1}[V + X]$$

is integral for all integral X.

We have to consider very distinguished coefficient functions:

3.5 Definition. *Let (S,q) be as in 3.4 An isotropic coefficient functions (with respect to (S,q) and ρ_0) is a function*

$$\varphi : \mathbb{Z}^{(r,n)} \longrightarrow \mathcal{Z}$$

which can be written as finite sum

$$\varphi(G) = \sum_\nu e^{2\pi i\sigma(V_\nu' G)} P_\nu(G)$$

where
a) *V_ν is isotropic in the sense of 3.4.*
b) *P_ν is a harmonic form with respect to ρ_0.*

3.6 Proposition. *Assume that*

$$qS \quad \textit{and} \quad qS^{-1}$$

are even positive matrices (i.e. they are integral and have even diagonal).

We assume furthermore

$$\operatorname{sgn}(\det D)^r \left(\frac{(-1)^r \det(qS)}{|\det D|}\right) = 1 \quad \textit{for} \quad M \in \Gamma_n[q],$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the generalized Legendre symbol. Then for every harmonic coefficient function φ with respect to (S,q) and ρ_0 we have

$$\vartheta_\varphi(S;Z) \in [\Gamma_n[q], \varrho].$$

We denote by
$$\vartheta(S) = \vartheta(S, q, \rho_0)$$
the subspace of all ϑ-series $\vartheta_\varphi(S; Z)$, where φ is isotropic.

One of the main results of the theory of singular forms states th following.

3.7 Theorem. *Assume $2r \le n$. Then*
$$[\Gamma_n[q], \varrho] = \sum_S \vartheta(S),$$
where S runs over a (finite) system of unimodular classes of all rational positive S, such that S and qS^{-1} are integral and with the correct multiplier system.

Actually the theory gives more. It exhibits a basis of $[\Gamma_n[q], \varrho]$, which means that one can control the linear relations among the generating ϑ-series. This means that we can get control the linear relations among the ϑ-series. The classical Riemann ϑ-relations are of this type.

In the singular case, harmonic forms are something very trivial. From the formula
$$P(GA) = \rho_0(A')P(G)$$
one obtains immediately th following lemma.

3.8 Lemma. *Assume $r \le n$. Let $\mathcal{Z}_0 \subset \mathcal{Z}$ be the subspace of all vectors $a \in \mathcal{Z}$ with the following two properties:*

a) $\rho_0 \begin{pmatrix} E^{(r)} & 0 \\ 0 & 0 \end{pmatrix} a = a$

b) $P_a(X) := \rho_0(X', 0)a$ *is harmonic. The map*
$$a \mapsto P_a$$
defines an isomorphism of $\mathcal{Z}_0$ onto the space of all harmonic forms.

This means that an isotropic coefficient function may be written as
$$\varphi(G) = \rho_0(G', 0)\varphi_0(G)$$
where
$$\varphi_0 : \mathbb{Z}^{(r,n)} \longrightarrow \mathcal{Z}_0$$
is a function which may be written as linear combination of functions
$$e^{2\pi i \sigma(V'G)}a; \qquad V \text{ isotropic}, \ a \in \mathcal{Z}_0.$$
The function is periodic with period q. We consider it as a function
$$\varphi_0 : R^{(r,n)} \longrightarrow \mathcal{Z}_0,$$
where R is the finite ring
$$R = \mathbb{Z}/q\mathbb{Z}.$$
The function φ_0 is not determinated by its ϑ-series. One reason is the existence of non trivial units of S. The unit group

$$\mathcal{E}(S) = \{U \in \mathrm{GL}(n, \mathbb{Z}); \quad S[U] = S\}$$

acts on the space of coefficient functions by

$$\varphi(G) \mapsto \varphi(UG)$$

For φ_0 instead of φ this means

$$\varphi_0(G) \mapsto \rho_0 \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \varphi_0(UG)$$

Replaycing in the ϑ-series the summation index G by UG we see that the ϑ-series depend only on the orbit of φ_0 under this action. It is hence sufficient to consider $E(S)$-invariant φ_0. We denote the space of all $E(S)$-invariant Φ_0 by

$$A(S)$$

Hence $A(S)$ consists of all function $\varphi_0 : R^{(r,n)} \longrightarrow \mathcal{Z}_0$ with the property

$$\Phi_0(UG) = \rho_0 \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \varphi_0(G).$$

But there are still new hidden linear relations in $A(S)$ and – even more complicated – these are relations among different spaces $A(S)$. The only fact which can be seen immediately is

3.9 Lemma. *Asssume that the ϑ-series*

$$\vartheta_\varphi(S; Z), \quad \varphi_0 \in A(S)$$

vanishes identically. Then the restriction of φ_0 to the subfield of primitive matrices

$$R^{(r,n)} \subset R^{(r,n)}$$

vanishes identically.

(A matrix in $R^{(r,n)}$ is called primitive if it is a part of a matrix in $\mathrm{GL}(r, n)$).

Lemma 3.9 leads us to consider the space $B(S)$ of all functions

$$R^{(r,n)}_{\mathrm{prim}} \longrightarrow \mathcal{Z}_0$$

which are restrictions of function from $A(S)$, that is

$$B(S) = \{\varphi_0 | R^{(r,n)}_{\mathrm{prim}}; \quad \varphi_0 \in A(S)\}.$$

ϑ relations come in because the maps $A(S) \longrightarrow B(S)$ is not injective in general.

We now choose a subspace

$$A_0(S) \subset A(S)$$

which is isomorphic to $B(S)$,

$$A_0(S) \xrightarrow{\sim} B(S).$$

We denote by

$$\Theta_0(S) \subset \Theta(S)$$

the subspace of all ϑ-series coming from functions $\varphi_0 \in A_0(S)$. Of course $\Theta_0(S)$ depends on the choice of $A_0 = (S)$. From Lemma follows

3.10 Remark. *The map*

$$\begin{aligned} A_0(S) &\longrightarrow \Theta_0(S) \\ \varphi_0 &\mapsto \vartheta_\varphi(S;\cdot) \end{aligned}$$

is an isomorphism.

Now we can formulate a refined version which describes in principle the ϑ-relations :

3.11 Theorem. *One has*

$$[\Gamma_n[q], r] = \oplus\Theta_0(S),$$

where S runs through a system of representations of unimodular classes of matrices with properties 3.6

3.12 Corollary.

$$\dim[\Gamma_n[q], r] = \sum_S \dim B(S).$$

The calculation of $\dim B(S)$ is a finite problem for given (S, n, q).

4 Application at invariants

We consider alterning holomorphic differential forms of degree ν on a non-singular model of $K(\Gamma)$, $\Gamma \subset \mathrm{Sp}(n, \mathbb{Z})$. If $0 < \nu < N = \frac{1}{2}n(n+1)$ they correspond to modular forms with respect the natural representations

$$\mathrm{GL}(n, \mathbb{C}) \longrightarrow \mathrm{GL}\big(\Lambda^p(\mathrm{Sym}^2(\mathbb{C}^n))\big).$$

These representations are not irreducible but their decomposition into irreducible components can be described.

4.1 Proposition. *Let $\Gamma \subset \mathrm{Sp}(n, \mathbb{Z})$ be a congruence subgroup and $g_p(\Gamma)$ the dimension of the space of holomorphic p-forms on a non singular model of $K(\Gamma)$. We have*

$$g_p(\Gamma) \neq 0 \Longrightarrow p = \frac{n(n+1)}{2} - \frac{k(k+1)}{2}, \quad k \in \{0, \dots, n\}.$$

For every $k \in \{1, \dots, n-1\}$ these exists a distinguished irreducible subrepresentation

$$\rho_p \ \text{ of } \ \Lambda^p(\mathrm{Sym}^2(\mathbb{C}^n)),$$

such that

$$g_p(\Gamma) = \dim[\Gamma, \rho_p], \quad p = \frac{n(n+1)}{2} - \frac{k(k+1)}{2}$$

The weight of ρ_p *is* $n-p$

4.2 Corollary. *The space* $[\Gamma, \rho_p]$ *is a singular space if and only if* $k \geq \frac{n}{2}$.

In the first occuring case

$$k = n-1 \quad (p = n)$$

we need binary quadratic forms for the ϑ-series.

4.3 Theorem. *Let* p *be a prime and* $\Gamma_{n,0}[p]$ *the Hecke subgroup of* $\mathrm{Sp}(n, \mathbb{Z})$ *defined by* $C = 0 \bmod p$. *Assume that* $n \geq 3$ *is even. Then*

$$g_n(\Gamma_{n,0}[p]) = \begin{cases} 0 & \text{if } p \equiv 1 \bmod 4 \\ h(-p) & \text{otherwise,} \end{cases}$$

where $h(-p)$ *denotes the class number of* $\mathbb{Q}(\sqrt{-p})$.

Stable modular forms

For sake of simplicity we restrict to the case of the full Siegel modular group $\Gamma_n = \mathrm{Sp}(n, \mathbb{Z})$. we consider the Siegel-Φ-operator

$$[\Gamma_n, k] \longrightarrow [\Gamma_{n-1}, k],$$
$$f \longmapsto f|\Phi, \quad f|\Phi(Z) = \lim_{t \to \infty} f \begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix}.$$

We may consider the direct limit

$$[\Gamma_\infty, k] := \lim_n [\Gamma_n, k].$$

The elements of this limit are sequences $f_n \in [\Gamma_n, k]$ with the property

$$f_{n+1}|\Phi = f_n.$$

It follows from the theroy of singular modular forms that the natural homomorphism

$$[\Gamma_\infty, k] \longrightarrow [\Gamma_n, k]$$

is an isomorphism if $n > 2k$. Therefore the elements of $[\Gamma_\infty, k]$ may be considered as „stable modular forms“. We also consider the graded algebras

$$A(\Gamma_n) := \bigoplus_k [\Gamma_n, k], \quad (0 \leq n \leq \infty).$$

The geometric counterpart of this algebra is th direct limit

$$X_\infty = \lim_{\rightarrow} X_n,$$

where

$$X_n = \mathcal{H}_n/\Gamma_n \cup \ldots \cup \mathcal{H}_0/\Gamma_0$$

denotes the Satake compactification of $\mathcal{H}_n/\Gamma_n$. The Siegel operator

$$\Phi : A(\Gamma_n) \longrightarrow A(\Gamma_{n-1}$$

corresponds to the natural inclusion. In this sense we may write

$$X_\infty = \operatorname{proj} A(\Gamma_\infty).$$

We describe certain elements of $A(\Gamma_\infty)$. Let $S = S^{(m)}$ be a unimodular even positive matrix. It is well known that such a matrix exists if and only if $m \equiv 0 \bmod 8$. For every n we consider the theta series

$$\vartheta(S; Z^{(n)}) = \sum_{G \text{ integral}} e^{\pi \mathrm{i} S[G] Z}.$$

This sequence is an element of $A(\Gamma_\infty)$. The matrix S is called *irreducible* if it is not unimodular equivalent with a matrix of the type

$$\begin{pmatrix} S^{(m_1)} & 0 \\ 0 & S^{(m_2)} \end{pmatrix}, \quad m_1, m_2 > 0.$$

4.4 Theorem [Fr3]. *The algebra $A(\Gamma_\infty)$ is a polynomial ring generated by the systems $(\vartheta(S; Z^{(n)}))_n$, where S runs through a set of representatives of unimodular classes of unimodular even positive irreducible matrices.*

Corollary. *The homogenous field of fractions of $A(\Gamma_\infty)$ which consists of quotients of elements of the same weight is a rational function field in countable many variables*

$$\left(\frac{\vartheta(S^m; Z^{(n)})}{\vartheta(S^{(8)}; Z^{(n)})^{m/8}} \right)_n .$$

(In the case $m = 8$ there is precisely one unimodular class).

A similar result is true for the HECKE group $\Gamma_{n,0}[q]$, which is defined by the condition $C \equiv 0 \bmod q$, more generally for all congruence groups which contain all unimodular substitutions $Z \to Z[U]$, $U \in \mathrm{GL}(n, \mathbb{Z})$ [En]. For more general groups, for example the principal congruence group, the situation is more complicated because the isotropic structures come in.

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