1 Two results about singularities of modular varieties

Let $D \subset \mathbb{C}^n$ be a bounded symmetric domain and

$$\Gamma \subset \text{Bihol}(D)$$

an arithmetic group of biholomorphic mappings of $D$ onto itself. We assume that $(D, \Gamma_0)$ is indecomposable (as direct product) for each subgroup $\Gamma_0 \subset \Gamma$ of finite index. We also assume

$$n = \dim D > 1.$$ 

We denote by

$$X = X_\Gamma = \overline{D/\Gamma}$$

the Baily-Borel compactification of $D/\Gamma$, which is a normal projective variety containing $D/\Gamma$ as a Zariski-open subvariety. From our assumption follows that the field $K(\Gamma)$ of rational functions on $X_\Gamma$ agrees with the field of meromorphic functions on $D$, which are $\Gamma$-invariant. The elements of $K(\Gamma)$ are called modular functions.

We consider a desingularization

$$\pi : \tilde{X}_\Gamma \to X_\Gamma$$

$$\cup$$

$$X_{\text{reg}}$$

which contains the regular locus $X_{\text{reg}}$ as Zariski-open subset.

An extension theorem

1.1 Theorem (Bauermann, [Ba]). Let $U \subset X$ be any open subset, $\tilde{U} = \pi^{-1}(U) \subset \tilde{X}$ its inverse image and $U_{\text{reg}} = U \cap X_{\text{reg}}$ the regular locus.

Every holomorphic alternating differential form of degree \( p < n \) on $U_{\text{reg}}$ extends holomorphically to $\tilde{U}$, i.e. the natural restriction map
is an isomorphism.

1.2 Corollary (Pommerening, [Po]). There is a natural isomorphism

\[(A^p\Omega)(\tilde{U}) \rightarrow (A^p\Omega(U_{\text{reg}})), \quad p < n,\]

(1) \(p < n\) is an isomorphism.

In the case of the Siegel modular group the latter result already has been proved in [FP].

Theorem 1.1 is a special case of a more general theorem. Instead of alternating differential forms one may consider arbitrary holomorphic tensors, i.e. elements

\[T \in \Omega^{\otimes p}(U_{\text{reg}}).\]

Bauermann [Ba] worked out the conditions for the holomorphic extendability for sufficiently small (so called "neat") groups \(\Gamma\). Those condition are formulated in terms of the Fourier-Jacobi expansion of \(T\). It generalizes results of Tai [AMRT] and the author [Fr3].

Theorem 1.1 says that – in some sense – the singularites of \(X\) are harmless. The next theorem will show that they are not harmless at all.

Let \(Y \subset X\) be an irreducible subvariety, for example a point. We consider the local ring

\[R = O_{X,Y}\]

of \(X\) at the general point of \(Y\). By definition it consists of all modular functions which are regular on an open subset of \(X\) whose intersection with \(Y\) is not empty. We have a natural homomorphism

\[O_{X,Y} \rightarrow K(Y)\]

onto the the field of rational function on \(Y\). The kernel of this homomorphism is the maximal ideal \(m\).

The (Krull-)dimension of \(R\) is

\[\dim R = \dim X - \dim Y.\]

The depth of \(R\) is the maximal length of a chain of elements \((r_1, \ldots, r_m)\) in \(R\), such that image of \(r_i\) is a non zero divisor in \(R/(r_1, \ldots, r_{i-1})\) \((1 \leq i \leq m)\). It is well known that

\[\text{depth}(R) \leq \dim(R).\]

We only consider the case \(\dim R \geq 2\). Ti follows from the normality of \(R\) that

\[2 \leq \text{depth}(R) \leq \dim(R).\]

The depth can also be expressed by means of the cohomology groups

\[M_i := H^i(\text{Spec}(R) - m, \mathcal{O}) \quad (\mathcal{O} = \text{ the structure sheaf }).\]

Those groups are finitely generated \(R\)-modules if \(i \leq \dim R - 2\). The depth is the maximal number \(m\), such that
Two results about singularities of modular varieties

\[ M_i = \begin{cases} R & \text{if } i = 0, \\ 0 & \text{if } 1 \leq i < m. \end{cases} \]

The determination of those cohomology groups is an unsolved problem even in the case of points \( Y \). Only the case of the HILBERT modular group has been settled ([Fr1], [FK]). The situation is better if one takes for \( Y \) an irreducible component of the boundary \( X-D/\Gamma \). In this case the cohomology groups can be calculated in full generality [Bal]. The case of the SIEGEL modular group has been treated earlier ([Ca]).

The idea is as follows. One considers a small STEIN neighbourhood \( U \subset X \) of a general point \( a \) of \( Y \).

Standard comparison theorems show

\[ M_i \cong H^i(U - (U \cap Y), \mathcal{O}) \text{ for } i \leq \dim R - 2, \]

where the right land side are analytic cohomology groups.

The choice of \( U \) can be made as follows:

There is an open STEIN subset

\[ S \subset D \]

which is invariant under a certain subgroup

\[ \Gamma_0 \subset \Gamma \]

(\( \Gamma_0 \) is the stabilizer of a representative of \( a \) in the HARISH-CHANDRA boundary of \( D \)), such that the natural projection induces a biholomorphic map

\[ S/\Gamma_0 \sim U - (U \cap Y). \]

After that one is reduced to consider cohomology groups of \( \Gamma_0 \) acting on holomorphic functions on \( S \).

The structure of \( \Gamma_0 \) is rather simple, nevertheless the computation of this group cohomology is difficult. We do not go into the details but formulate only the result in an important special case:

We consider for \( D \) the SIEGEL upper half plane

\[ \mathbf{H}_n := Z = \{ Z^{(n)} = Z' = X + iY, \ Y > 0 \} \]

of all symmetric \( n \times n \)-matrices with positive definite imaginary part and for \( \Gamma \) the principal congruence subgroup

\[ \Gamma_n[q] = \text{kernel } (\text{Sp}(n, \mathbb{Z}) \rightarrow \text{Sp}(n, \mathbb{Z}/q\mathbb{Z})) \]

of the SIEGEL modular group

\[ \Gamma_n = \text{Sp}(n, \mathbb{Z}). \]

This group acts on \( \mathbf{H}_n \) by the formula

\[ Z \mapsto M(Z) = (AZ + B)(CZ + D)^{-1}; \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \]

In this case the „stabilizer“ \( \Gamma_0 \) consists of all elements of the form
\[ M = \begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, \]

\[ U = \begin{pmatrix} 1 & a_1 \\ \vdots & \vdots \\ 1 & a_{n-1} \pm 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & \cdots & s_1 \\ \vdots & \ddots & \vdots \\ s_1 & \cdots & s_n \end{pmatrix}, \]

(all the other entries of \( U \) and \( S \) are 0).

It is worth while the notice that the \(-1\) only occurs if \( q \leq 2 \). The subgroup \( \Delta \subset \Gamma_0 \) of all elements of the form

\[ \begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix} \]

regulates the cohomology groups. This subgroup is isomorphic to \( \mathbb{Z}^{n-1} \) or to a semi-direct product \( \mathbb{Z}^{n-1} \rtimes \mathbb{Z}/2\mathbb{Z} \), where the non trivial element of \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{Z}^{n-1} \) by means of \( a \mapsto -a \).

We have to consider the cohomology group of \( \Delta \) acting trivially on \( \mathbb{C} \). We obtain

\[ H^i(\Delta, \mathbb{C}) = \left\{ \begin{array}{ll} A^i\mathbb{C}^{n-1} & \\ (A^i\mathbb{C}^{n-1}) \mathbb{Z}/2\mathbb{Z} & \end{array} \right. \]

The second case occurs if

\[ \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 0 & -1 \end{pmatrix} \]

is in \( \Delta \). Now we can formulate the result in the Siegel case.

**1.3 Theorem ([Ca]).** Let \( R_0 \) denote the ring of holomorphic functions on \( U \cap Y \). Then

\[ H^i(U - (U \cap Y), O) \cong H^i(\Delta, \mathbb{C}) \otimes_{\mathbb{C}} R_0, \quad \text{for } 1 \leq i \leq n - 2. \]

(The natural restriction

\[ O(U) \longrightarrow O(U - (U \cap Y)) \]

is an isomorphism. Hence \( R_0 \) can be considered as a factor ring of \( O(U - (U \cap Y)) \) especially as a module over this ring.)

It is easy to see that

\[ H^2(\mathbb{Z}^{n-1}, \mathbb{C})^{\mathbb{Z}/2\mathbb{Z}} \neq 0 \text{ if } n > 3, \]

\[ H^1(\mathbb{Z}^{n-1}, \mathbb{C}) \neq 0 \text{ if } n \geq 3, \]

\[ H^1(\mathbb{Z}^{n-1}, \mathbb{C})^{\mathbb{Z}/2\mathbb{Z}} = 0. \]

Hence we obtain

**1.4 Corollary.** If \( R \) is the local ring at an irreducible component of the boundary

\[ X_n - \mathbf{H}_n/\Gamma_n[q], \]

then
$\dim R = n$ 

and

$$\text{depth}(R) = \begin{cases} 
2, & \text{if } n = 2, \\
2, & \text{if } n \geq 3, \quad q \leq 2, \\
3, & \text{if } n \geq 3, \quad q > 2.
\end{cases}$$

A ring is called COHEN-MACAULEY if dimension and depth agree. We obtain

1.5 Corollary. The ring $R$ is Cohen-Macauley if and only if

$$n \leq 2 \quad \text{or} \quad (n = 3 \quad \text{and} \quad q \leq 2).$$

In this connection we should mention that by a result of RUNGE [Ru] not only $R$ but the whole varietiy $\overline{H_3/T_3[q]}$ is COHEN-MACAULEY if $q \leq 2$. This contradicts to a statement of TSUGUMINE [Ts]. Actually the proof of TSUGUMINE is false.

2 Holomorphic tensors

Let $X$ be an irreducible variety and $\tilde{X}$ a nonsingular projective model of $X$. We consider tensors

$$T \in \Omega^{\otimes p}(\tilde{X}),$$

which can be expressed locally (in analytic coordinates) as

$$T = \sum f_{i_1,\ldots,i_p} dz_{i_1} \otimes \ldots \otimes dz_{i_p}$$

Let

$$\pi : GL(n, \mathbb{C}) \longrightarrow GL(V)$$

be an irreducible polynomial representation of $GL(n, \mathbb{C})$ on a finite dimensional vector space. We can consider tensors of type $\pi$ and the subspaces

$$\Omega^{\otimes p}_\pi(\tilde{X}).$$

(After the choice of a local coordinate system in a neighbourhood of a point $a \in \tilde{X}$, we may consider $T(a)$ as element

$$T(a) \in (\mathfrak{g}_n^{\otimes p}).$$

The $\pi$-isotopic component in $(\mathfrak{g}_n^{\otimes p}$ does not depend on the choice of the coordinate system).

The dimensions

$$d_\pi := \dim \Omega^{\otimes p}_\pi(\tilde{X})$$

are birational invariants of $X$, i.e. they depend only on the field of rational functions $K(X)$ on $X$,

$$d_\pi = d_\pi(K(X)).$$
If $\pi$ is the one dimensional representation
\[ \pi(A) = (\det A)^k, \]
the elements of $\Omega_{\pi}^{\otimes k}(\tilde{X})$ are called \textit{multicanonical}, the invariants
\[ p_k = d(\det A)^k; \quad k = 0, 1, 2, \ldots \]
are the so called \textit{plurigenera}. If
\[ \pi_p : \text{GL}(n, \mathbb{C}) \longrightarrow \text{GL}(A^p(\mathbb{C}^n)) \]
is the natural representation on the $p^{th}$ extension power then the elements of $\Omega_{\pi}^{\otimes k}(\tilde{X})$ are alternating differential forms, the invariant are denoted by
\[ g_p = d_{\pi_p}. \]
If $0 < p < n$ they sometimes are called \textit{irregularities}. Obviously $g_n = p_1, g_0 = p_0 = 1$.

The arithmetic genus
\[ g = \sum (-1)^p g_p \]
is of great importance.

The spaces $\Omega_{\pi}^{\otimes k}(\tilde{X})$ are connected with the corresponding spaces of $\Gamma$-invariant tensors on the domain $D$, i.e. we have to compare
\[ \Omega_{\pi}^{\otimes k}(\tilde{X}) \quad \text{and} \quad \Omega_{\pi}^{\otimes k}(D)^\Gamma. \]
If $\Gamma$ acts freely on $D$, one has a natural inclusion
\[ \Omega_{\pi}^{\otimes k}(\tilde{X}) \hookrightarrow \Omega_{\pi}^{\otimes k}(D)^\Gamma. \]
The elements of $\Omega_{\pi}^{\otimes k}(D)^\Gamma$ can be interpreted as vector valued automorphic forms. A vector valued holomorphic automorphic form is a holomorphic map
\[ f : D \longrightarrow \mathcal{Z} \]
an $D$ with values in a finite dimensional complex vector space $\mathcal{Z}$ and will the properties
\[ f(\gamma z) = \rho(I(\gamma, z))f(z), \quad \gamma \in \Gamma. \]
Here $I(\gamma, z) = (\partial \gamma_i/\partial z_j)$ denote the JACOBI matrix and
\[ \rho : G \longrightarrow \text{GL}(\mathcal{Z}) \]
a representation of the complex LIE-group $G$ generated by all
\[ I(g, z); \quad g \in \text{Aut}(D), \quad z \in D. \]
We denote the space of these modular forms by
\[ [\Gamma, \rho]. \]
From our assumption this is a finite dimensional vector space. We have
\[ \Omega_{\pi}^{\otimes k}(D)^\Gamma = [\Gamma, \rho], \quad \rho := \pi|G. \]
The automorphic forms $T \in [\Gamma, \rho]$ define holomorphic tensors on $D'/\Gamma$, where $D' \subset D$ is the biggest open subset of $D$ on which $\Gamma$ acts freely. It follows from the results of [Ba] and Tai [AMRT]:

2.1 Proposition. Let $T \in [\Gamma, \rho]$, $\rho = \pi|G$, be a cusp form. Then there exists a subgroup $\Gamma_0 \subset \Gamma$ of finite index such that $T$ extends holomorphically to the nonsingular model $\tilde{X}_{\Gamma_0}$.

2.2 Corollary. Each arithmetic subgroup $\Gamma$ admits a subgroup $\Gamma_0 \subset \Gamma$ of finite index such that the field $K(\Gamma_0)$ of modular functions is of general type.

In the case of the SIEGEL modular group better results are known:

First we recall that the Jacobian „matrix“ of a symplectic substitution is given by the linear map

$$W \rightarrow (CZ + D)^{-1}W(CZ + D)^{-1},$$

where $W$ varies in the tangent space of $H_n$, i.e. in the space

$$Z_n = \{W = W' = W^{(n)}\}$$

of all symmetric $(n \times n)$-matrices. The group $G$ generated by the Jacobians is

$$G = \text{GL}(n, \mathbb{C})/\{\pm E\}.$$ 

This means that SIEGEL modular forms belong to (rational) representations

$$\rho : \text{GL}(n, \mathbb{C}) \longrightarrow \text{GL}(Z), \quad \rho(-E) = \text{id}_Z.$$ 

The rational representations are completely decomposable. Therefore we may assume that $\rho$ is irreducible.

The weight $k$ of $\rho$ is the biggest integer such that

$$\rho_0(A) = (\det A)^{-k}\rho(A)$$

is a reduced representation, i.e. $\rho_0$ is polynomial but $\rho_0(A)/\det A$ is not polynomial. Obviously $k$ is nothing else but the last component of the highest weight vector. Good existence theorems for modular forms are known for big $k$ (which means $k > 2n$) and for small $k$ (which means $2k < n$).
Existence theorems for big weight

Multicanonial tensors are of type

$$(A^N \Omega)^{\otimes k} \quad (N = \frac{1}{2}(n(n+1))).$$

They correspond to scalar valued modular forms of weight $(n+1)k$. Using Riemann-Roch theorems one obtains good estimates for the asymptotic behaviour of the dimension. Also explicit constructions using \(\vartheta\)-series and Poincaré series give good results:

2.3 Theorem. The field of modular functions $K(\Gamma_n) \ (\Gamma_n = \text{Sp}(n, \mathbb{Z}))$ is of general type if $n \geq 7$.

For proofs and more comments we refer to [Fr3], [Mu], [Ta].

We mentioned already that $K(\Gamma_n)$ is unirational if $n \leq 5$. The case $n = 6$ remains open.

Using more complicated types of tensors one obtains results for subvarieties of the Siegel modular variety [Fr], [We]. The best results are due to Weissauer.

2.4 Theorem. Let $k$ be a natural number. There exists a constant $n(k)$ with the following property. Assume $n \geq n(k)$. Then every irreducible subvariety of $H_n/\Gamma_n \ (\Gamma_n = \text{the full Siegel modular group})$ of codimension $\leq k$ is of general type. The bound $n(k)$ can be made explicite.

2.5 Corollary. Assume $n \geq 13$. The group of automorphism of the function field $K(\Gamma_n)$ is trivial.

As we already mentioned, in the case of a large $k$ the results describe mostly asymptotic behaviour. In the case of small weights we will obtain explicit formulae for some invariants.

3 Singular modular forms

In this section we restrict to the case of congruence groups of the Siegel modular group.

Such modular forms admits a Fourier expansion

$$f(Z) = \sum a(T)e(TZ),$$
$$e(A) := e^{2\pi i \sigma(A)} \quad (\sigma = \text{trace}),$$

where $T$ runs through a lattice of rational symmetric matrices.

3.1 Definition. A modular form is called singular, if

$$a(T) \neq 0 \implies \det T = 0.$$
3.2 Proposition. Assume that 
\[ \rho : \GL(n, \CC) \longrightarrow \GL(\ZZ) \]
is a finite dimensional irreducible rational representation of weight \( k \). A non-vanishing modular form with respect to \( \rho \) is singular if and only if \( 2k < n \). In that case one has 
\[ 2k = \max \{ \text{rank} T; \ a(T) \neq 0 \} \]

We have to introduce \( \vartheta \)-series with harmonic coefficients. The ingredients are
1) A positive definite rational symmetric matrix \( S \). We restrict to the case where the degree \( r \) of \( S \) is even,
\[ r = 2k. \]
2) A harmonic polynomial
\[ P : \CC((r,n)) \longrightarrow \ZZ \]
with the properties
\[ P(XA) = \rho_0(A')P(X) \]
Such a polynomial is called a harmonic form with respect to \( \rho_0 \).
More generally we consider functions
\[ \varphi : \ZZ((r,n)) \longrightarrow \ZZ, \]
which can be written as finite sum
\[ \varphi(G) = \sum \varphi_\nu(G)P_\nu(S^{1/2}G), \]
where
\[ \varphi_\nu : \ZZ((r,n)) \longrightarrow \ZZ \]
are functions which are periodic with respect to a suitable rational number \( q \). We may consider them as functions
\[ \varphi_\nu : R^{(r,n)} \longrightarrow \CC, \]
where \( R \) denotes the finite ring \( \ZZ/q\ZZ \). A function \( \varphi \) of this type we call a harmonic coefficient function (with respect to \( (S, q) \)).

3.3 Lemma. Assume that \( S = S^{(2r)} \) is a positive rational matrix and
\[ \varphi : \ZZ((r,n)) \longrightarrow \ZZ \]
a harmonic coefficient function. The \( \vartheta \)-series
\[ \vartheta_\varphi(S; Z) = \sum_{G \in \ZZ((r,n))} \varphi(G)e(S[G]Z) \]
is a modular form with respect to
\[ \rho \].
\( \rho(A) = \rho_0(A)(\det A)^k \)

on a suitable congruence subgroup. (Obviously \( \vartheta_\varphi(S; Z) \) is singular if \( r < n \) in accordance with 3.2.)

We want to describe conditions under which \( \vartheta \)-series belong to a given level, i.e. is an element of \([\Gamma_n[q], \varrho]\).

The natural number \( q \) is fixed now.

**3.4 Definition.** Let \( S = S^{(2r)} \) be a rational symmetric positive matrix such that

\[ qS \quad \text{and} \quad qS^{-1} \]

both are integral. A matrix

\[ V \in \frac{1}{q} \mathbb{Z}^{(2r,n)} \]

is called isotropic (with respect to \((S, q)\)) if

\[ S^{-1}[V + X] \]

is integral for all integral \( X \).

We have to consider very distinguished coefficient functions:

**3.5 Definition.** Let \((S, q)\) be as in 3.4 An isotropic coefficient functions (with respect to \((S, q)\) and \(\rho_0\)) is a function

\[ \varphi : \mathbb{Z}^{(r,n)} \longrightarrow \mathbb{Z} \]

which can be written as finite sum

\[ \varphi(G) = \sum_\nu e^{2\pi i\sigma(V_\nu G)} P_\nu(G) \]

where

a) \( V_\nu \) is isotropic in the sense of 3.4.

b) \( P_\nu \) is a harmonic form with respect to \( \rho_0 \).

**3.6 Proposition.** Assume that

\[ qS \quad \text{and} \quad qS^{-1} \]

are even positive matrices (i.e. they are integral and have even diagonal).

We assume furthermore

\[ \text{sgn}(\det D)^r \left( \frac{(-1)^r \det(qS)}{|\det D|} \right) = 1 \text{ for } M \in \Gamma_n[q], \]

where \((\cdot)\) denotes the generalized Legendre symbol. Then for every harmonic coefficient function \( \varphi \) with respect to \((S, q)\) and \(\rho_0\) we have

\[ \vartheta_\varphi(S; Z) \in [\Gamma_n[q], \varrho]. \]
§3 Singular modular forms

We denote by
\[ \vartheta(S) = \vartheta(S, q, \rho_0) \]
the subspace of all \( \vartheta \)-series \( \vartheta_\varphi(S; Z) \), where \( \varphi \) is isotropic.

One of the main results of the theory of singular forms states the following.

3.7 Theorem. Assume \( 2r \leq n \). Then
\[ [\Gamma_n[q], \vartheta] = \sum_S \vartheta(S), \]
where \( S \) runs over a (finite) system of unimodular classes of all rational positive \( S \), such that \( S \) and \( qS^{-1} \) are integral and with the correct multiplier system.

Actually the theory gives more. It exhibits a basis of \([\Gamma_n[q], \vartheta]\), which means that one can control the linear relations among the generating \( \vartheta \)-series. This means that we can get control the linear relations among the \( \vartheta \)-series. The classical RIEMANN \( \vartheta \)-relations are of this type.

In the singular case, harmonic forms are something very trivial. From the formula
\[ P(GA) = \rho_0(A')P(G) \]
one obtains immediately the following lemma.

3.8 Lemma. Assume \( r \leq n \). Let \( Z_0 \subset Z \) be the subspace of all vectors \( a \in Z \) with the following two properties:

a) \( \rho_0 \left( \begin{array}{cc} E(r) & 0 \\ 0 & 0 \end{array} \right) a = a \)

b) \( P_a(X) := \rho_0(X', 0)a \) is harmonic. The map
\[ a \mapsto P_a \]
defines an isomorphism of \( Z_0 \) onto the space of all harmonic forms.

This means that an isotropic coefficient function may be written as
\[ \varphi(G) = \rho_0(G', 0)\varphi_0(G) \]
where
\[ \varphi_0 : \mathbb{Z}^{(r,n)} \rightarrow Z_0 \]
is a function which may be written as linear combination of functions
\[ e^{2\pi i \sigma(V'G)}a; \quad V \ \text{isotropic,} \quad a \in Z_0. \]
The function is periodic with period \( q \). We consider it as a function
\[ \varphi_0 : R^{(r,n)} \rightarrow Z_0, \]
where \( R \) is the finite ring
\[ R = \mathbb{Z}/q\mathbb{Z}. \]
The function \( \varphi_0 \) is not determinated by its \( \vartheta \)-series. One reason is the existence of non trivial units of \( S \). The unit group
acts on the space of coefficient functions by
\[ \varphi(G) \mapsto \varphi(UG) \]
For \( \varphi_0 \) instead of \( \varphi \) this means
\[ \varphi_0(G) \mapsto \rho_0 \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \varphi_0(UG) \]
Replaying in the \( \vartheta \)-series the summation index \( G \) by \( UG \) we see that the \( \vartheta \)-series depend only on the orbit of \( \varphi_0 \) under this action. It is hence sufficient to consider \( E(S) \)-invariant \( \varphi_0 \). We denote the space of all \( E(S) \)-invariant \( \Phi_0 \) by
\[ A(S) \]
Hence \( A(S) \) consists of all function \( \varphi_0 : R^{(r,n)} \longrightarrow \mathbb{Z}_0 \) with the property
\[ \Phi_0(UG) = \rho_0 \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \varphi_0(G). \]
But there are still new hidden linear relations in \( A(S) \) and – even more complicated – these are relations among different spaces \( A(S) \). The only fact which can be seen immediately is

**3.9 Lemma.** Assume that the \( \vartheta \)-series
\[ \vartheta_\varphi(S;Z), \quad \varphi_0 \in A(S) \]
vanishes identically. Then the restriction of \( \varphi_0 \) to the subfield of primitive matrices
\[ R^{(r,n)} \subset R^{(r,n)} \]
vanishes identically.

( A matrix in \( R^{(r,n)} \) is called primitive if it is a part of a matrix in \( GL(r,n) \)).

Lemma 3.9 leads us to consider the space \( B(S) \) of all functions
\[ R^{(r,n)}_{\text{prim}} \longrightarrow \mathbb{Z}_0 \]
which are restrictions of function from \( A(S) \), that is
\[ B(S) = \{ \varphi_0|_{R^{(r,n)}_{\text{prim}}}; \quad \varphi_0 \in A(S) \}. \]
\( \vartheta \) relations come in because the maps \( A(S) \longrightarrow B(S) \) is not injective in general.

We now choose a subspace
\[ A_0(S) \subset A(S) \]
which is isomorphic to \( B(S) \),
\[ A_0(S) \sim B(S). \]
We denote by
Application at invariants

$$\Theta_0(S) \subset \Theta(S)$$

the subspace of all \(\vartheta\)-series coming from functions \(\varphi_0 \in A_0(S)\). Of course \(\Theta_0(S)\) depends on the choice of \(A_0 = (S)\). From Lemma follows

**3.10 Remark.** The map

\[
A_0(S) \longrightarrow \Theta_0(S) \\
\varphi_0 \mapsto \vartheta_{\varphi} (S; \cdot)
\]

is an isomorphism.

Now we can formulate a refined version which describes in principle the \(\vartheta\)-relations:

**3.11 Theorem.** One has

\[
[\Gamma_n[q], r] = \oplus \Theta_0(S),
\]

where \(S\) runs through a system of representations of unimodular classes of matrices with properties 3.6

**3.12 Corollary.**

\[
\dim[\Gamma_n[q], r] = \sum S \dim B(S).
\]

The calculation of \(\dim B(S)\) is a finite problem for given \((S, n, q)\).

**4 Application at invariants**

We consider alternating holomorphic differential forms of degree \(\nu\) on a non-singular model of \(K(\Gamma)\), \(\Gamma \subset \text{Sp}(n, \mathbb{Z})\). If \(0 < \nu < N = \frac{1}{2}n(n+1)\) they correspond to modular forms with respect the natural representations

\[
\text{GL}(n, \mathbb{C}) \longrightarrow \text{GL}(\Lambda^p(\text{Sym}^2(\mathbb{C}^n))).
\]

These representations are not irreducible but their decomposition into irreducible components can be described.

**4.1 Proposition.** Let \(\Gamma \subset \text{Sp}(n, \mathbb{Z})\) be a congruence subgroup and \(g_p(\Gamma)\) the dimension of the space of holomorphic \(p\)-forms on a non-singular model of \(K(\Gamma)\). We have

\[
g_p(\Gamma) \neq 0 \Rightarrow p = \frac{n(n+1)}{2} - \frac{k(k+1)}{2}, \quad k \in \{0, \ldots, n\}.
\]

For every \(k \in \{1, \ldots, n-1\}\) there exists a distinguished irreducible subrepresentation

\[
\rho_p \text{ of } \Lambda^p(\text{Sym}^2(\mathbb{C}^n)),
\]

such that

\[
g_p(\Gamma) = \dim[\Gamma, \rho_p], \quad p = \frac{n(n+1)}{2} - \frac{k(k+1)}{2}
\]
The weight of $\rho_p$ is $n - p$

4.2 Corollary. The space $[\Gamma, \rho_p]$ is a singular space if and only if $k \geq \frac{n}{2}$.

In the first occurring case

$$k = n - 1 \quad (p = n)$$

we need binary quadratic forms for the $\vartheta$-series.

4.3 Theorem. Let $p$ be a prime and $\Gamma_{n,0}[p]$ the Hecke subgroup of $\text{Sp}(n, \mathbb{Z})$ defined by $C = 0 \mod p$. Assume that $n \geq 3$ is even. Then

$$g_n(\Gamma_{n,0}[p]) = \begin{cases} 0 & \text{if } p \equiv 1 \mod 4 \\ h(-p) & \text{otherwise}, \end{cases}$$

where $h(-p)$ denotes the class number of $\mathbb{Q}(\sqrt{-p})$.

Stable modular forms

For sake of simplicity we restrict to the case of the full SIEGEL modular group $\Gamma_n = \text{Sp}(n, \mathbb{Z})$. We consider the SIEGEL-$\Phi$-operator

$$[\Gamma_n, k] \rightarrow [\Gamma_{n-1}, k],$$

$$f \mapsto f|\Phi, \quad f|\Phi(Z) = \lim_{t \rightarrow \infty} f\left( \begin{array}{cc} Z & 0 \\ 0 & it \end{array} \right).$$

We may consider the direct limit

$$[\Gamma_\infty, k] := \lim_{n} [\Gamma_n, k].$$

The elements of this limit are sequences $f_n \in [\Gamma_n, k]$ with the property

$$f_{n+1}|\Phi = f_n.$$  

It follows from the theory of singular modular forms that the natural homomorphism

$$[\Gamma_\infty, k] \rightarrow [\Gamma_n, k]$$

is an isomorphism if $n > 2k$. Therefore the elements of $[\Gamma_\infty, k]$ may be considered as "stable modular forms". We also consider the graded algebras

$$A(\Gamma_n) := \bigoplus_{k} [\Gamma_n, k], \quad (0 \leq n \leq \infty).$$

The geometric counterpart of this algebra is the direct limit

$$X_\infty = \lim_{\rightarrow} X_n,$$

where

$$X_n = \mathcal{H}_n/\Gamma_n \cup \ldots \cup \mathcal{H}_0/\Gamma_0$$

denotes the Satake compactification of $\mathcal{H}_n/\Gamma_n$. The Siegel operator
\[ \Phi : A(\Gamma_n) \to A(\Gamma_{n-1}) \]
corresponds to the natural inclusion. In this sense we may write
\[ X_\infty = \text{proj} A(\Gamma_\infty). \]

We describe certain elements of \( A(\Gamma_\infty) \). Let \( S = S^{(m)} \) be a unimodular even positive matrix. It is well known that such a matrix exists if and only if \( m \equiv 0 \mod 8 \). For every \( n \) we consider the theta series
\[ \vartheta(S; Z^{(n)}) = \sum_{G \text{ integral}} e^{\pi i S[G]Z}. \]
This sequence is an element of \( A(\Gamma_\infty) \). The matrix \( S \) is called \textit{irreducible} if it is not unimodular equivalent with a matrix of the type
\[ \begin{pmatrix} S^{(m_1)} & 0 \\ 0 & S^{(m_2)} \end{pmatrix}, \quad m_1, m_2 > 0. \]

\section*{4.4 Theorem [Fr3].}
The algebra \( A(\Gamma_\infty) \) is a polynomial ring generated by the systems \((\vartheta(S; Z^{(n)}))_n\), where \( S \) runs through a set of representatives of unimodular classes of unimodular even positive irreducible matrices.

\textbf{Corollary.} The homogenous field of fractions of \( A(\Gamma_\infty) \) which consists of quotients of elements of the same weight is a rational function field in countable many variables
\[ \left( \frac{\vartheta(S^m; Z^{(n)})}{\vartheta(S^{(8)}; Z^{(n)})^{m/8}} \right)_n. \]
(In the case \( m = 8 \) there is precisely one unimodular class).

A similar result is true for the HECKE group \( I_{n,0}[q] \), which is defined by the condition \( C \equiv 0 \mod q \), more generally for all congruence groups which contain all unimodular substitutions \( Z \to Z[U], \ U \in \text{GL}(n, \mathbb{Z}) \) [En]. For more general groups, for example the principal congruence group, the situation is more complicated because the isotropic structures come in.
Literature


Literatur
