

A three dimensional ball quotient

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Introduction

Modular varieties (compactified quotients of Hermitian domains by arithmetic subgroups) provide interesting examples of algebraic varieties. Up to finitely many cases they are expected to be of general type, but there are particular examples where this is not the case. So, for example, there are finitely many elliptic modular curves which are elliptic curves. Generalizing these results, we constructed in a recent paper [FS] many Siegel threefolds which admit a Calabi–Yau model. Some of them seem to be new. All Euler numbers which we obtained have been non-negative. This was a motivation for us to look for other kinds of examples. A promising class are the ball quotients which belong to the unitary group $U(1, 3)$. Its arithmetic subgroups are called Picard modular groups. The theory of Siegel modular varieties is far developed. In particular the classical theory of theta functions of very interesting examples. We just mention Igusa’s result that the modular variety related to the congruence group $\Gamma_2[4, 8]$ is embedded in $P^9\mathbb{C}$ where the equations are given by the quartic Riemann relations. This is a variety of general type whose desingularization admits many holomorphic differential forms of top-degree. There seems to be no comparable result in the case of Picard modular varieties. In this paper we determine a very particular example of a Picard modular variety of general type. On its non-singular models there exist many holomorphic differential forms. In a forthcoming paper we will show that one can construct Calabi-Yau manifolds by considering quotients of this variety and resolving singularities.

We describe the particular modular group that is considered in this paper. Let $U(1, 3) \subset GL(4, \mathbb{C})$ be the unitary group of the hermitian form

$$\bar{z}_0 w_0 - \bar{z}_1 w_1 - \bar{z}_2 w_2 - \bar{z}_3 w_3.$$

It acts on the ball

$$\mathcal{B}_3 = \{(z_1, z_2, z_3); |z_1|^2 + |z_2|^2 + |z_3|^2 < 1\}.$$

Let \mathcal{E} be the ring of Eisenstein numbers. We consider the arithmetic group

$$G_3 = \mathrm{U}(1, 3) \cap \mathrm{GL}(4, \mathcal{E})$$

and its congruence subgroups

$$G_3[a] = \mathrm{kernel}(G_3 \longrightarrow \mathrm{GL}(4, \mathcal{E}/a)) \quad (a \in \mathcal{E}, a \neq 0).$$

We are especially interested in the group $G_3[3]$. We shall determine the structure of the ring of modular forms $A(G_3[3])$. This algebra has 25 generators, 15 modular forms B_i of weight one and ten modular forms C_j of weight 2. Both will appear as Borcherds products. The zeros are located on certain 2-subballs of \mathcal{B}_3 . The forms C_i are cuspidal. Their squares define holomorphic differential forms on the non-singular models. We shall determine all relations between these forms and we shall obtain the dimension formula for the spaces of modular forms and the subspaces of cusp forms in all weights.

The subring $A(G_3[\sqrt{-3}])$ is related to the Segre cubic. This can be derived from the paper [F] which is an extension of the paper [AF]. It has also been worked out in detail by Kondo [Ko]. We will obtain the structure of $A(G_3[\sqrt{-3}])$ as a by-product. This ring is generated by 6 forms T_i of weight 3 that satisfy the relations

$$\sum_{i=1}^6 T_i = 0, \quad \sum_{i=1}^6 T_i^3 = 0$$

of the Segre cubic in a standard representation. In this way the modular variety associated to $G_3[3]$ appears as covering of the Segre cubic of degree 3^9 .

The proof uses the theory of Borcherds products. This theory has been established for the orthogonal group $\mathrm{O}(2, n)$. We use the natural embedding of $\mathrm{U}(1, n)$ into $\mathrm{O}(2, 2n)$ to carry over this theory to the case of the unitary group.

The proof uses also rather involved computer calculations. We used the computer algebra systems MAGMA and SINGULAR to perform these calculations. We feel that it is useless to publish any programs since hard- and software are changing rapidly. Instead of this we describe in some detail the way how the calculations can be done such that an interested reader can control them by writing own programs.

We are very grateful to G. Pfister who explained us some fine points of the SINGULAR computer algebra system. We have to thank D. Allcock who explained to us the structure of certain unitary groups. Finally we thank S. Kondo for his preprint [Ko].

1. Orthogonal modular forms

We recall some basic facts about automorphic forms on $O(2, n)$. A real quadratic space V is a finite dimensional real vector space that has been equipped with a non-degenerate bilinear form (\cdot, \cdot) . We denote the associated quadratic form by $q(a) = (a, a)/2$. We call it also the *norm*. We extend the bilinear form to $V(\mathbb{C}) = V \otimes_{\mathbb{R}} \mathbb{C}$ as \mathbb{C} -bilinear form.

We assume that the signature of V is $(2, n)$. The zero quadric in $V(\mathbb{C})$ is the complex submanifold defined by $(z, z) = 0$. We consider the set

$$\{z \in V(\mathbb{C}); \quad (z, z) = 0, \quad (z, \bar{z}) > 0\}.$$

This is an open subset of the zero quadric which has two connected components. We choose one of them and denote it by $\tilde{\mathcal{H}}$. Let $O'(V)$ be the subgroup of index two of the orthogonal group $O(V)$ that preserves the two connected components. It contains the reflections along vectors a with negative $q(a) < 0$ and is generated by them.

Now, let $M \subset V$ be an even lattice (i.e. $q(a) \in \mathbb{Z}$ for all $a \in M$.) We denote the dual lattice of M by M' . The discriminant group of M is the finite group M'/M . The quadratic form q induces a finite quadratic form $\bar{q}: M'/M \rightarrow \mathbb{Q}/\mathbb{Z}$.

The integral orthogonal group $O(M)$ consists of all $g \in O(V)$ that preserve M . We use the notation $O'(M) = O(M) \cap O'(V)$. The discriminant kernel is the group

$$\Gamma_M := \text{kernel}(O'(M) \longrightarrow \text{Aut}(M'/M)).$$

We denote by \mathcal{H} the image of $\tilde{\mathcal{H}}$ in the projective space $P^n(V(\mathbb{C}))$. It is a smooth subset and the group $O'(V)$ acts on it. Let $\Gamma \subset O(M')$ be a subgroup of finite index, then \mathcal{H}/Γ carries a structure as quasi-projective variety due to the theory of Baily–Borel.

An automorphic form of weight $k \in \mathbb{Z}$ with respect to Γ and with respect to a character v on Γ is a holomorphic function f on $\tilde{\mathcal{H}}$ with the properties

- 1) $f(\gamma z) = v(\gamma)f(z)$ for all $\gamma \in \Gamma$,
- 2) $f(tz) = t^{-k}f(z)$ for all $t \in \mathbb{C}^*$,
- 3) the form is regular at the cusps.

We omit the definition of 3). We denote the space of automorphic forms by $[\Gamma, k, v]$ or simply by $[\Gamma, k]$ if the character v is trivial.

2. Borcherds products

Let M be an even lattice with bilinear form (\cdot, \cdot) and associated quadratic form $q(x) = (x, x)/2$ of signature $(2, n)$. Borcherds' space of obstructions consists of all modular forms $f : \mathbb{H} \rightarrow \mathbb{C}[M'/M]$ with the transformation law $(f_\alpha)_{\alpha \in M'/M}$

- 1) $f_\alpha(\tau + 1) = e^{-2\pi i q(\alpha)} f_\alpha(\tau),$
- 2) $f_\alpha\left(-\frac{1}{\tau}\right) = -\sqrt{\frac{\tau}{i}}^{2+n} \frac{1}{\sqrt{\#M^*/M}} \sum_{\beta \in M^*/M} e^{2\pi i(\alpha, \beta)} f_\beta(\tau).$
- 3) f is holomorphic at the cusp infinity.

Let $\alpha \in M'$ be an element of the dual lattice and $n < 0$ a negative number. The Heegner divisor $H(\alpha, n) \subset \mathcal{H}$ is the union of all

$$v^\perp \cap \mathcal{H} \quad (v^\perp \text{ orthogonal complement of } v \text{ in } P(V(\mathbb{C})),$$

where v runs through all elements $\alpha \in M'$ with

$$v \equiv \alpha \pmod{M} \quad \text{and} \quad q(v) = n.$$

We consider $H(\alpha, n)$ as a divisor on \mathcal{H} by attaching multiplicity 1 to all components. We have $H(\alpha, n) = H(-\alpha, n)$, more precisely, this divisor depends only on the image of α in $(M'/M)/\pm 1$. It is invariant under the discriminant kernel Γ_M and its image in \mathcal{H}/Γ_M is a closed algebraic subvariety.

A fundamental result of Borcherds states (see [AF], Theorem 5.2 for this version of Borcherds theorem):

2.1 Theorem. *Assume $n > 2$. A finite linear combination*

$$\sum_{\alpha \in (L'/L)/\pm 1, n < 0} C(\alpha, n) H(\alpha, n) \quad (C(\alpha, n) \in \mathbb{N}_{\geq 0})$$

*is the divisor of an automorphic form of weight k with respect to Γ_M if for every **cuspidal form** f in the space of obstructions,*

$$f_\alpha(\tau) = \sum_{n \in \mathbb{Q}} a_\alpha(n) \exp(2\pi i n \tau),$$

the relation

$$\sum_{n < 0, \alpha \in L'/L} a_\alpha(-n) C(\alpha, n) = 0$$

holds. The weight of this modular form is

$$k = \sum_{n \in \mathbb{Q}, \alpha \in L'/L} b_\alpha(n) C(\alpha, -n),$$

where $b_\alpha(n)$ denotes the Fourier coefficients of the Eisenstein series with the constant term

$$b_\alpha(0) = \begin{cases} -1/2 & \text{if } \alpha = 0, \\ 0 & \text{else.} \end{cases}$$

Corollary. *An individual Heegner divisor $H(\alpha, n)$ is the divisor of an automorphic form if and only if $a_\alpha(n) = 0$ for every cusp form in the space of obstructions.*

3. A special case

We consider a special lattice which can be defined by means of the ring of Eisenstein integers.

$$\mathcal{E} := \mathbb{Z}[\zeta] \quad (\zeta = -1/2 + \sqrt{-3}/2).$$

We consider the isomorphism

$$\mathcal{E}/\sqrt{-3}\mathcal{E} \xrightarrow{\sim} \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}, \quad 1 \mapsto 1.$$

We mention that $\zeta \equiv 1 \pmod{\sqrt{-3}}$, hence the image of ζ in \mathbb{F}_3 is 1.

The quadratic form $\bar{x}x$ equips \mathcal{E} with a structure as even lattice. The associated bilinear form is $2 \operatorname{Re} \bar{x}y = \operatorname{tr}(\bar{x}y)$. (This is isomorphic to the root lattice A_2 .) The dual lattice is

$$\mathcal{E}' := \frac{1}{\sqrt{-3}}\mathcal{E}.$$

The discriminant group

$$\mathcal{E}'/\mathcal{E} \cong \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$$

has order three. The elements $1/\sqrt{-3}$ and $\zeta/\sqrt{-3}$ from \mathcal{E}' both give 1 in \mathbb{F}_3 .

We have to consider the lattice of signature $(2, 6)$

$$M := \mathcal{E}^{1,3}, \quad M' = \frac{1}{\sqrt{-3}}M,$$

whose underlying group is \mathcal{E}^4 which is equipped with

$$(x, y) = 2 \operatorname{Re}(\bar{x}_0y_0 - \bar{x}_1y_1 - \bar{x}_2x_2 - \bar{x}_3y_3), \quad q(x) = \bar{x}_0x_0 - \bar{x}_2x_2 - \bar{x}_3x_3 - \bar{x}_4x_4.$$

The discriminant group M'/M has order 3^4 , hence $(M'/M)/\pm 1$ has order 41.

We want to determine the cuspidal part of the space of obstructions. The transformation law is

$$\begin{aligned} 1) \quad & f_\alpha(\tau + 1) = e^{-2\pi i q(\alpha)} f_\alpha(\tau), \\ 2) \quad & f_\alpha\left(-\frac{1}{\tau}\right) = -\tau^4 \frac{1}{\sqrt{\#M^*/M}} \sum_{\beta \in M^*/M} e^{2\pi i(\alpha, \beta)} f_\beta(\tau). \end{aligned}$$

The functions f_α have the period 3. They admit an expansion

$$f_\alpha(\tau) = \sum_{\nu=1}^{\infty} a_\nu q^{\nu/3}, \quad q^{1/3} := e^{2\pi i \tau/3}.$$

The smallest power of q that can occur is $q^{1/3}$. This is also the smallest power that occurs in η^8 , where

$$\eta(\tau) = e^{2\pi i \tau/24} \prod_{\nu=1}^{\infty} (1 - e^{2\pi i \tau \nu}).$$

We recall the transformation formula $\eta(-1/\tau) = \sqrt{\tau/i} \eta(\tau)$. This shows that f_α/η^8 is a modular form of weight 0 and hence constant. This shows.

3.1 Lemma. *The space of obstructions for $M = \mathcal{E}^{1,3}$ consists of all vector valued functions of the form $f_\alpha = C_\alpha \eta^8$, where C_α are constants that satisfy*

$$C_\alpha \neq 0 \implies q(\alpha) \equiv 1/3 \pmod{1}$$

and

$$C_\alpha = -\frac{1}{\sqrt{\#M^*/M}} \sum_{\beta \in M^*/M} e^{2\pi i(\alpha, \beta)} C_\beta.$$

Corollary. *If $f = \sum a_\nu q^\nu$ is a cuspidal element of the space of obstructions then*

$$a_\nu \neq 0 \implies \nu \equiv 1/3 \pmod{1}.$$

Corollary. *An individual Heegner divisor $H(\alpha, n)$ is the divisor of an automorphic form if $n \not\equiv -1/3 \pmod{1}$.*

In Theorem 2.1 we gave the formula

$$k = \sum_{n \in \mathbb{Q}, \alpha \in L'/L} b_\alpha(n) C(\alpha, -n),$$

for the weight of a Heegner divisor

$$\sum_{\alpha \in (L'/L)/\pm 1, n < 0} C(\alpha, n) H(\alpha, n) \quad (C(\alpha, n) \in \mathbb{Z})$$

in the case that the obstruction condition is satisfied. But we can consider this number in all cases, and call it then the virtual weight of the Heegner divisor. We compute the virtual weight in some cases. The Fourier coefficients of the corresponding Eisenstein series can be computed by means of the formulae in [BK].

3.2 Lemma. *The virtual weight of the Heegner divisors $H(\alpha, n)$ the three cases (α, n)*

$$\left(\frac{1}{\sqrt{-3}}(0, 1, 0, 0), -\frac{1}{3} \right), \quad \left(\frac{1}{\sqrt{-3}}(0, 1, 1, 0), -\frac{2}{3} \right), \quad \left((0, 1, 0, 0), -1 \right)$$

are 1, 9 and 30. In the last two cases the Borcherds products exist, but not in the first case.

(There is an analogous result in the $\mathcal{E}^{1,4}$ case [AF]: the virtual weights there are $1/2, 5, 27$). The reason that in the first case a Borcherds product cannot exist, can explained by means of the fact that the smallest weight of a non-vanishing automorphic form is the singular weight, which is 2 in our case.

We recall that the quadratic form on M' induces a finite quadratic form $\bar{q} : M'/M \rightarrow \mathbb{Q}/\mathbb{Z}$ and of course also a \mathbb{Q}/\mathbb{Z} -valued bilinear form $\bar{q}(\alpha + \beta) - \bar{q}(\alpha) - \bar{q}(\beta)$. So we can talk about orthogonal elements in M'/M .

3.3 Proposition. *Let $\alpha_1, \alpha_2, \alpha_3$ be three pairwise orthogonal elements of M'/M that are images of vectors of M' of norm $-1/3$. Then there exists a Borcherds product on Γ_M with divisor $H(\alpha_1, -1/3) + H(\alpha_2, -1/3) + H(\alpha_3, -1/3)$. The weight of this modular form is 3.*

Proof. The proof rests on the explicit computation of the space of obstructions. The weight can be taken from Proposition 3.3.

4. Ball quotients

We recall some basic facts about ball quotients. Let V be a finite dimensional complex vector space which is equipped with a hermitian form $\langle \cdot, \cdot \rangle$ of signature $(1, n)$. This means that there exists an isomorphism $V \cong \mathbb{C}^{n+1}$ such that

$$\langle z, w \rangle = \bar{z}_0 w_0 - \bar{z}_1 w_1 - \cdots - \bar{z}_n w_n.$$

A line (=one dimensional sub-vector-space of V) is called positive, if it is represented by an element of positive norm ($\langle z, z \rangle > 0$). We denote by $\mathcal{B} = \mathcal{B}(V) \subset P(V)$ the set of all positive lines. (As usual $P(V)$ denotes the projective space of V , i.e. the set of all lines in V .) In the above model the component z_0 of an element $z \in \mathbb{C}^{n+1}$ with positive norm is different from 0. Hence each positive line contains a unique z with $z_0 = 1$. This identifies $\mathcal{B}(V)$ with the standard n -ball

$$\mathcal{B}_n := \{ z \in \mathbb{C}^n; \quad |z_1|^2 + \cdots + |z_n|^2 < 1 \}.$$

The unitary group $U(V) \cong U(1, n)$ acts on \mathcal{B} as group of biholomorphic automorphisms.

We need an arithmetic structure and consider therefore an lattice $M \subset V$. First of all, this is a discrete additive subgroup such that V/M is compact (which means that M generates V as real vector space). We assume that there exists a non-real multiplier $a \in \mathbb{C}$, $aM \subset M$. Then the set of all these multipliers is an order \mathfrak{o} in an imaginary quadratic field K . This gives as a rational structure on V . An element of V is called *rational* if it is contained in $\mathbb{Q}M$ which is actually a K -vector space. An element of $P(V)$ is called rational if it can be represented by a rational element. Then it can be represented also by an element of M . An element of $P(V)$ is called isotropic if it can be represented by an isotropic element $\langle a, a \rangle = 0$. A *rational boundary point* is a rational isotropic element of $P(V)$. The extended ball \mathcal{B}^* is the union of \mathcal{B} with the set of all rational boundary points.

The modular group $U(M)$ with respect to the lattice M is the subgroup of $U(M)$ which preserves M . More generally one admits a subgroup of finite index $G \subset U(M)$. By the theory of Baily-Borel the quotient

$$X_G := \mathcal{B}^*/G$$

carries a structure in the form of a projective algebraic variety. Hence the uncompactified quotient

$$X_G^0 := \mathcal{B}/G$$

is a quasi projective variety. The number of classes of rational boundary points is finite.

We recall the notion of a modular form of integral weight k . For this we consider the inverse image $\tilde{\mathcal{B}}$ of \mathcal{B} in $V - \{0\}$. It consists of all $z \in V$ with $\langle z, z \rangle > 0$. This is a connected open subset.

Let $G \subset U(M)$ be a subgroup of finite index and let $v : G \rightarrow \mathbb{C}^*$ be a character. A modular form of weight k on G with respect to a v is a holomorphic function $f : \tilde{\mathcal{B}} \rightarrow \mathbb{C}$ with the properties

- 1) $f(tz) = t^{-k}f(z)$ for all $t \in \mathbb{C}^*$,
- 2) $f(\gamma z) = v(\gamma)f(z)$ for all $\gamma \in G$,
- 3) f is regular at the cusps.

We explain the meaning of 3). We choose a non-zero isotropic vector $\alpha \in M \otimes \mathbb{Q}$ and a vector $\alpha' \in M \otimes \mathbb{Q}$ such that $\langle \alpha, \alpha' \rangle = 1$. We consider

$$f_{\alpha, \alpha'}(\tau) := f(\tau\alpha + i\alpha').$$

The positivity condition $\langle \tau\alpha + i\alpha', \tau\alpha + i\alpha' \rangle > 0$ means that τ varies in an upper half plane $\text{Im } \tau > C \geq 0$. It is easy to see that $f_{\alpha, \alpha'}$ has some period N . (This follows from the description of the stabilizer of a cusp which will be given in Section 8.) We obtain a Fourier expansion

$$f_{\alpha, \alpha'}(\tau) = \sum a_n e^{2\pi i n \tau / N}.$$

Regularity at the cusps means that a_n vanishes for $n < 0$ (for all choices of α, α'). Then we can define

$$f(\alpha) := a_0.$$

4.1 Lemma. *The value $f(\alpha)$ does not depend on the choice of α' . Moreover it depends only on the G -orbit of α if the multipliers of f are trivial. Finally $f(C\alpha) = \bar{C}^k f(\alpha)$.*

Proof. One considers more generally

$$f_{\alpha, \alpha'}(\tau, w) := f(\tau\alpha + i\alpha' + w)$$

where w is orthogonal to α, α' . This function admits a Fourier expansion

$$f_{\alpha, \alpha'}(\tau, w) = \sum a_n(w) e^{2\pi i n \tau / N}.$$

The functions $a_n(w)$ are holomorphic on the whole vector space (orthogonal complement of $\mathbb{C}\alpha + \mathbb{C}\beta$). The function $a_0(w)$ is an abelian function and hence constant. (This follows also from the description of the stabilizer of a cusp given in Section 8.)

Now we can prove the independence of the choice of α' . Let $\langle \alpha, \alpha'' \rangle = 1$. Then $i(\alpha'' - \alpha')$ is orthogonal to α and hence of the form $C\alpha + w$ where w is orthogonal to α, α' . We have $f(\tau\alpha + i\alpha'') = f((\tau + C)\alpha + i\alpha' + w)$. The limits for $\text{Im } \tau \rightarrow \infty$ are the same.

To get the value $f(C\alpha)$ we choose α'/\bar{C} as complementary vector. We get

$$\begin{aligned} f(C\alpha) &:= \lim_{\text{Im } \tau \rightarrow \infty} f(\tau C\alpha + i\alpha'/\bar{C}) = \\ &\bar{C}^k \lim_{\text{Im } \tau \rightarrow \infty} f(\tau |C|^2 \alpha + i\alpha') = \bar{C}^k \lim_{\text{Im } \tau \rightarrow \infty} f(\tau\alpha + i\alpha') = \bar{C}^k f(\alpha). \end{aligned}$$

This finishes the proof of Lemma 4.1. □

4.2 Definition. *Assume that the order \mathfrak{o} of multipliers is a principal ideal ring. A cusp is a primitive isotropic vector in M .*

Hence the zero dimensional boundary points are the images of the cusps in the projective space. The number of cusps over a zero dimensional boundary point is the number of units in \mathfrak{o} .

So we defined the value $f(\alpha)$ of a modular form at each cusp. In the case that f has trivial multipliers it depends only on the G -orbit of the cusp. The number of these G -orbits (cusp-classes) is finite.

5. Restriction of orthogonal to unitary automorphic forms

We use the notation of the previous section, so V is a hermitian space. We can consider the complex vector space V also as a real vector space and equip it with the real bilinear form

$$(a, b) = \langle a, b \rangle + \langle b, a \rangle.$$

The signature of this form is $(2, 2n)$. This gives us an embedding

$$\mathrm{U}(V, \langle \cdot, \cdot \rangle) \subset \mathrm{O}(V, (\cdot, \cdot)).$$

We complexify the vector space V . This means that we consider $V \otimes_{\mathbb{R}} \mathbb{C}$ where the complex multiplication is given by

$$i(v \otimes C) := v \otimes (iC).$$

We extend the action of $\mathrm{O}(V)$ by \mathbb{C} -linearity to $V \otimes_{\mathbb{R}} \mathbb{C}$. Now we consider the map

$$V \longrightarrow V \otimes \mathbb{C}, \quad v \longmapsto (iv) \otimes 1 + v \otimes i.$$

This map is \mathbb{C} -linear, as can be seen from the following computation:

$$\begin{aligned} iv &\longmapsto (-v) \otimes 1 + (iv) \otimes i, \\ i((iv) \otimes 1 + v \otimes i) &= iv \otimes i + v \otimes (-1). \end{aligned}$$

The expressions on the right-hand-side are the same.

This \mathbb{C} -linear map is also compatible with the actions of $\mathrm{U}(V) \subset \mathrm{O}(V)$: The image of $g(v)$ for $g \in \mathrm{U}(V)$ and $v \in V$ in $V \otimes_{\mathbb{R}} \mathbb{C}$ is $ig(v) \otimes 1 + g(v) \otimes i$. Because of $ig(v) = g(iv)$ this equals $g(iv \otimes 1 + v \otimes i)$. Recall that g acts on $V \otimes_{\mathbb{R}} \mathbb{C}$ by \mathbb{C} -linear extension.

The image of $\tilde{\mathcal{B}}$ is contained in the set defined by $(z, z) = 0$ and $(z, \bar{z}) > 0$. We can choose the connected component $\tilde{\mathcal{H}}$ such that it contains the image of $\tilde{\mathcal{B}}$. Since the group $\mathrm{U}(V)$ is connected, we have that $\mathrm{U}(V)$ is contained in $\mathrm{SO}'(V) = \mathrm{SO}(V) \cap \mathrm{O}'(V)$.

We can use the holomorphic embedding $\tilde{\mathcal{B}} \hookrightarrow \tilde{\mathcal{H}}$ to restrict an orthogonal modular form F for some group $\Gamma \subset \mathrm{O}(M)$ to a holomorphic function f on $\tilde{\mathcal{B}}$ that is given by the formula $f(z) = F(iz \otimes 1 + z \otimes i)$. The equation $F(tz) = t^{-k}F(z)$ for real t gives $f(tz) = t^{-k}f(z)$ for real t . Since f is holomorphic this must hold also for complex t . This shows that f is a unitary automorphic form of the same weight with respect to the group $\Gamma \cap \mathrm{U}(M)$ and the restricted character.

We apply this to the lattice $M = \mathcal{E}^{1,3}$. We use the notation $G_3 = \mathrm{U}(M)$. The discriminant kernel is the kernel of the homomorphism

$$\mathrm{U}(\mathcal{E}^{1,3}) \longrightarrow \mathrm{Aut}((\mathcal{E}/\sqrt{-3}\mathcal{E})^4).$$

This is the $G_3[\sqrt{-3}]$ which we introduced above.

Basic elements of G_3 are reflections along vectors b of norm $\langle b, b \rangle = -1$. They are defined by

$$a \longmapsto a - (1 - \eta) \frac{\langle b, a \rangle}{\langle b, b \rangle} b$$

where η is a 6th root of unity (hence a power of $-\zeta$). They transform b to ηb and act as identity on the orthogonal complement of b . Their order equals the order of η . We call them biflections, triflections or hexflections corresponding to the order of η . Let G^+ be the subgroup of G generated by the hexflections.

For a given vector b of norm -1 we always have 2 possible triflections along b , since there are two primitive third roots of unity. When we talk in the following about “the” triflection along b , we mean the triflection with factor ζ . Similarly we understand by “the” hexflexion along b always the hexflexion with the factor $-\zeta$.

We mention some results that can be taken from [ATC]. There they are formulated and proved for $\mathcal{E}^{1,4}$. As Allcock explained to us, the proofs there also work for $\mathcal{E}^{1,3}$. In the proof one has to replace the root lattice E_6 by A_5 .

5.1 Proposition [ACT]. *The group G_3^+ has index two in G_3 . The negative of the identity is not contained in G_3^+ . Generators of G_3^+ are the 5 hexflections along the following vectors:*

$$(0, 0, 1, 0), \quad (0, 1, 1, 0), \quad (-1, -\zeta, 0, 0) \quad (0, 1, 0, 1), \quad (0, 0, 0, 1).$$

We need another important result.

5.2 Proposition [ACT]. *The group G_3^+ acts transitively on the set of primitive isotropic vectors.*

Now we consider triflections. Because of

$$\zeta \equiv 1 \pmod{\sqrt{-3}}$$

they are elements of $G_3[\sqrt{-3}]$.

5.3 Proposition [ACT]. *The group $G_3[\sqrt{-3}]$ is generated by triflections.*

Finally we need the following result.

5.4 Proposition [ACT]. *Let $a, b \in \mathcal{E}^{1,3}$ be two vectors with the property $a \equiv b \pmod{\sqrt{-3}}$. Assume that either both are primitive isotropic or that both are of norm -1. Then they are equivalent mod $G_3[\sqrt{-3}]$.*

These theorems have geometric applications. First we treat the full modular group G_3 .

5.5 Proposition. *The space $\overline{\mathcal{B}_3/G_3}$ has one boundary point. The images of orthogonal complements of integral vectors of norm -1 define an irreducible divisor.*

Similarly we get for the congruence group of level $\sqrt{-3}$.

5.6 Proposition. *The space $\overline{\mathcal{B}_3/G_3[\sqrt{-3}]}$ has 10 boundary points. The images of the orthogonal complements of integral vectors of norm -1 define a divisor with 15 components.*

Let $a \in \mathcal{E}^{1,3}$ be a vector of norm -1 . The intersection of its orthogonal complement with $\tilde{\mathcal{B}}_3$ is the fixed point set of the triflection along the the norm -1 vector $\sqrt{-3}a$. By a short mirror in $\tilde{\mathcal{B}}$ we understand the fixed point set of a triflection.

We observe that ζa and $\zeta^2 a$ have different orthogonal complements in $\tilde{\mathcal{H}}_3$. But their intersections with $\tilde{\mathcal{B}}_3$ agree. Now we consider in $\mathcal{E}^{1,3}$ three vectors a_1, a_2, a_3 of norm -1 which are orthogonal in the sense that $\text{tr}(\bar{a}_i, a_j) = 0 \equiv 3 \pmod{3}$ for $i \neq j$. We restrict the corresponding Borcherds product (see Proposition 3.3) to the ball \mathcal{B}_3 . What we have seen is that all multiplicities of the zeros are three. Hence there exists a holomorphic cube root of the restriction. This gives the following result.

5.7 Theorem. *Let a_1, a_2, a_3 be integral vectors of norm -1 with the property $\text{tr}(\bar{a}_i, a_j) \equiv 0 \pmod{3}$ for $i \neq j$. There exists a modular form of weight 1 on the group $G_3[\sqrt{-3}]$ such that the zero divisor in $\overline{\mathcal{B}_3/G[\sqrt{-3}]}$ consists of three of the 15 divisors described in Proposition 5.6, namely the images of the orthogonal complements of the a_1, a_2, a_3 . There are 15 triples mod $\sqrt{-3}$. Hence we get 15 well-defined one-dimensional spaces of modular forms. The group G_3 permutes them transitively.*

As an example we can take the three pairwise orthogonal vectors

$$(0, 1, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1).$$

5.8 Theorem. *There exists a modular form of weight 1 on the group $G_3[\sqrt{-3}]$. The zero divisor of this form consists of three $G_3[\sqrt{-3}]$ -orbits. As representatives in the ball \mathcal{B}_3 one can take the three divisors*

$$w_1 = 0, \quad w_2 = 0, \quad w_3 = 0.$$

The multiplicities are one.

We want to determine the cusps where the 15 forms vanish. So let a be a cusp and b be an integral vector of norm -1 . The image of the orthogonal complement of b in $\overline{\mathcal{B}_3/G_3[\sqrt{-3}]}$ contains the cusp a if and only if there exists an $g \in G_3[\sqrt{-3}]$ such that $\langle a, g(b) \rangle = 0$. This condition implies $\langle a, b \rangle \equiv 0 \pmod{\sqrt{-3}}$. The converse is also true.

5.9 Lemma. *Assume that a is a cusp and b an integral vector of norm -1 such that*

$$\langle a, b \rangle \equiv 0 \pmod{\sqrt{-3}}.$$

Then the image of a in $\overline{\mathcal{B}_3/G_3[\sqrt{-3}]}$ is contained in the image of the orthogonal complement of b .

Proof. Since we can replace a, b by $g(a), g(b)$ with some $g \in G_3$, we have to solve the problem only for one distinguished a . After that we still may replace b by some $h(b)$ where now $h \in G_3[\sqrt{-3}]$. This reduces the problem to finitely many cases (namely six). Each of them can be settled by hand. \square

One can use this Lemma to determine the cuspidal zeros of our Borcherds products.

5.10 Proposition. *Each of the 15 Borcherds products (Theorem 5.7) vanishes at 6 of the 10 boundary points of $\overline{\mathcal{B}_3/G_3[\sqrt{-3}]}$.*

So far we do not have any information about the multiplier systems. We will determine them in the next section.

6. The multiplier system

As we have seen there exists a modular form of weight 1 on the three ball \mathcal{B}_3 with respect to the kernel $G_3[\sqrt{-3}]$ of the homomorphism

$$U(\mathcal{E}^{1,3}) \longrightarrow \text{Aut}((\mathcal{E}/\sqrt{-3}\mathcal{E})^4).$$

We have to determine the multiplier system. Since $G_3[\sqrt{-3}]$ is generated by the reflections along vectors a of norm -1 , we need the multipliers only for them. If a maps to a vector different from the vectors

$$\pm(0, 1, 0, 0), \quad \pm(0, 0, 1, 0), \quad \pm(0, 0, 0, 1)$$

then the multiplier is 1 since otherwise the form would vanish along the corresponding mirror. Hence it is sufficient to determine the multiplier for the 3 reflections R_1, R_2, R_3 along the three vectors above (now considered in $\mathcal{E}^{1,3}$).

Our method will be to restrict the form to a one-dimensional ball where it can be identified. For this we consider a pair of cusps α, β with the property $\langle \alpha, \beta \rangle = 1$. We consider the sub-lattice

$$M = \mathcal{E}\alpha + \mathcal{E}\beta.$$

The hermitian form is

$$\langle a_1\alpha + b_1\beta, a_2\alpha + b_2\beta \rangle = \bar{a}_1b_2 + \bar{a}_2b_1.$$

The lattice M is isometric equivalent to $\mathcal{E}^{1,1}$. Using the basis α, β , one can identify $U(M)$ with the set of all matrices $M \in \text{GL}(2, \mathcal{E})$ with the property

$$\bar{M}' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The subgroup

$$G_1[\sqrt{-3}] := \text{kernel}(U(M) \longrightarrow \text{Aut}(M/\sqrt{-3}M)).$$

can be considered as subgroup of $G_3[\sqrt{-3}]$ by acting trivial on the orthogonal complement of M . The set $\tilde{\mathcal{B}}_1$ corresponds in this model to the set of all (z_1, z_2) with the property $\text{Re}(\bar{z}_1 z_2) > 0$. The matrix M acts on (z_1, z_2) by multiplication from the left (considering it as column): If $F(z_0, z_1, z_2, z_3)$ is a modular form for $G_3[\sqrt{-3}]$ then

$$f(z_1, z_2) := F(z_1\alpha + z_2\beta)$$

is a modular form on $G_1[\sqrt{-3}]$ of the same weight and the restricted character. We want to relate $G_1[\sqrt{-3}]$ to an elliptic modular group acting on the upper half plane. The modular group $G_1[\sqrt{-3}]$ corresponds to the group of all matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathcal{E})$ with the properties

$$a \equiv d \equiv 1 \pmod{\sqrt{-3}}, \quad b \equiv c \equiv 0 \pmod{\sqrt{-3}}, \quad \text{tr}(\bar{a}c) = \text{tr}(\bar{b}d) = 0, \quad \bar{a}d + \bar{c}d = 1.$$

The transformation formula for f is

$$f(az_1 + bz_2, cz_1 + dz_2) = v(\gamma)f(z_1, z_2) \quad \text{and} \quad f(tz_1, tz_2) = t^{-1}f(z_1, z_2).$$

We consider

$$f_0(\tau) := f(-1/\sqrt{-3}, \tau).$$

Then τ varies in the usual upper half plane and the transformation formula reads as

$$f_0\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\gamma\tau + \delta)f_0(\tau) \quad \text{where} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} d & -c/\sqrt{-3} \\ -b\sqrt{-3} & a \end{pmatrix}.$$

If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is an integral matrix from the Hecke group $\Gamma_1[3] \subset \text{SL}(2, \mathbb{Z})$ (defined by $\alpha \equiv \delta \equiv 1 \pmod{3}$ and $\gamma \equiv 0 \pmod{3}$), then the $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to the group in question. Hence we see that $f_0(\tau)$ is a modular form for the Hecke group $\Gamma_1[3]$. So far we have no information about the multipliers of f_0 . We will determine them under a certain assumption on F . The vector $\alpha - \bar{\zeta}\beta$ has norm 1. It is orthogonal to the vector $\alpha + \zeta\beta$ which has norm -1. Hence $\alpha - \bar{\zeta}\beta$ is a point in

\tilde{B}_1 which is fixed under the triflection along $\alpha + \zeta\beta$. This triflection is contained in $G_1[\sqrt{-3}]$. Its matrix is

$$\begin{pmatrix} \bar{\zeta} & 1 - \zeta \\ 1 - \zeta & -2\bar{\zeta} \end{pmatrix} = \bar{\zeta}A \quad \text{where} \quad A = \begin{pmatrix} 1 & \sqrt{-3} \\ \sqrt{-3} & -2 \end{pmatrix}.$$

The matrix that corresponds to A in the Hecke group is

$$B = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}.$$

Now we assume that F vanishes at the fixed point $\alpha - \bar{\zeta}\beta$. Then f_0 vanishes at the fixed point of B . This is enough to identify f_0 up to a constant factor. We recall that there is an Eisenstein series of weight 1 on the group $\Gamma_1[3]$ whose q -expansion is given by

$$E := -\frac{1}{6} - \frac{q}{1-q} + \sum_{\nu=1}^{\infty} \left[\frac{q^{3\nu-1}}{1-q^{3\nu-1}} - \frac{q^{3\nu+1}}{1-q^{3\nu+1}} \right].$$

It has also an expression as theta series, namely $\vartheta(S, \tau) = -6G(\tau)$ where

$$\vartheta(S, \tau) = \sum_{g \in \mathbb{Z}^2} e^{2\pi i S[g]\tau}, \quad S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The Eisenstein series has trivial multiplier system on $\Gamma_1[3]$. It has no zero at the cusps but it also vanishes at the fixed point. We claim that this zero is of first order and that the zeros of E are equivalent even with respect to the principal congruence subgroup of level 3. To prove this we count the zeros of the discriminant (cusp form of weight 12). It has a zero of third order at each of the 4 cusps of the principal congruence subgroup of level 3. Hence it has 12 zeros and a form of weight 1 must have one zero. It follows that f_0/E is a modular form of weight 0 and hence constant. So we have identified the modular form and we can determine the multiplier of the triflection A . Recall that A is the product of multiplication by $\bar{\zeta}$ and of a transformation with trivial multiplier. From the rule $f(tz_1, tz_2) = t^{-1}f(z_1, z_2)$ we can see that the multiplier of the triflection is ζ .

6.1 Proposition. *Let F be a modular form on $G_3[\sqrt{-3}]$ with an arbitrary multiplier system. Let α, β be two cusps with the property $\langle \alpha, \beta \rangle = 1$. Assume that $F(\alpha - \bar{\zeta}\beta) = 0$. Then the multiplier of the triflection along $\alpha + \zeta\beta$ is ζ . Moreover the values of F at the cusps α, β are non-zero and related by*

$$F(\beta) = F(\alpha).$$

Proof. It only remains to compare the values at the cusps α, β . We can assume that $f_0(\tau) = \vartheta(S, \tau)$. By definition of f_0 we have

$$F(-\alpha/\sqrt{-3} + \tau\beta) = \vartheta(S, \tau).$$

First we compute the value of F at the isotropic vector β . By definition

$$F(\beta) = \lim_{\text{Im } \tau \rightarrow \infty} F(\tau\beta + i\beta'), \quad \langle \beta, \beta' \rangle = 1.$$

We can rewrite this as

$$F(\beta) = \sqrt{3}^{-1} \lim_{\text{Im } \tau \rightarrow \infty} F(\tau\beta - \beta'/\sqrt{-3}).$$

We choose $\beta' = \alpha$ to get

$$F(\beta) = \sqrt{3}^{-1} \lim_{\text{Im } \tau \rightarrow \infty} F(\beta\tau - \alpha/\sqrt{-3}) = \sqrt{3}^{-1} \lim_{\text{Im } \tau \rightarrow \infty} \vartheta(S, \tau) = \sqrt{3}^{-1}.$$

Next we compute $F(\alpha)$,

$$F(\alpha) = \lim_{\text{Im } \tau \rightarrow \infty} F(\tau\alpha + i\beta) = \sqrt{3} \lim_{\text{Im } \tau \rightarrow \infty} F(\tau\alpha + \sqrt{-3}\beta).$$

Now we use

$$F(\alpha\tau + \sqrt{-3}\beta) = -\frac{1}{\tau\sqrt{-3}} F\left(-\frac{\alpha}{\sqrt{-3}} + \left(-\frac{1}{\tau}\right)\beta\right) = -\frac{1}{\tau\sqrt{-3}} \vartheta\left(S, -\frac{1}{\tau}\right).$$

The theta inversion formula states

$$\vartheta\left(S, -\frac{1}{\tau}\right) = \frac{\tau}{i} \sqrt{\det S}^{-1} \vartheta(S^{-1}, \tau) = \frac{\tau}{\sqrt{-3}} \vartheta(S^{-1}, \tau).$$

We get $F(\alpha\tau + \sqrt{-3}\beta) = \frac{1}{3} \vartheta(S^{-1}, \tau)$ and finally

$$F(\alpha) = \frac{\sqrt{3}}{3} = \sqrt{3}^{-1} = F(\beta). \quad \square$$

Now we come back to the question of determining the characters of the 15 Borcherds products of weight 1. We consider the special case of Theorem 5.8. We treat the case R_1 (triflection along $(0, 1, 0, 0)$). We consider the two vectors

$$\alpha = (-\zeta\sqrt{-3}, -\bar{\zeta}, -\zeta, -\zeta), \quad \beta = (\sqrt{-3}, -1, 1, 1).$$

They are isotropic and have the property $\langle \alpha, \beta \rangle = 1$. We have

$$\alpha + \zeta\beta = (0, 1, 0, 0).$$

Since the Borcherds product vanishes along the orthogonal complement of this vector, we get $F(\alpha - \bar{\zeta}\beta) = 0$. Now we can apply Proposition 6.1 and obtain that the multiplier of R_1 is ζ .

6.2 Theorem. *Let F be the modular form on $G_3[\sqrt{-3}]$ of weight one that vanishes along the three mirrors $w_1 = 0$, $w_2 = 0$, $w_3 = 0$. Then the multiplier $v(\gamma)$ of the triflection along a vector $a \in \mathcal{E}^{1,3}$ of norm -1 is ζ if a is congruent mod $\sqrt{-3}$ to one of the vectors $\pm(0, 1, 0, 0)$, $\pm(0, 0, 1, 0)$, $\pm(0, 0, 0, 1)$ and 1 otherwise.*

The character can be described in more detail. We consider the subgroup $G_3[3]$ of $G_3[\sqrt{-3}]$ that acts trivially on $(\mathcal{E}/3\mathcal{E})^3$. One can check that

$$G_3[\sqrt{-3}]/G_3[3] \cong (\mathbb{Z}/3\mathbb{Z})^{10}.$$

It also can be checked that there exists a character on this group which has the same effect on triflections as described in Theorem 6.2. Hence both characters must agree.

6.3 Lemma. *The 15 forms F described in Theorem 5.7 have trivial character on the group $G_3[3]$.*

7. The congruence group of level three

In this section we use the model of $\mathcal{E}^{1,3}$ given by the hermitian form

$$\langle a, b \rangle = \bar{a}_1 b_2 + \bar{a}_2 b_1 - \bar{a}_3 b_3 - \bar{a}_4 b_4.$$

This is equivalent to the form we used in the previous section. The corresponding Gram matrix is

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The group G can be identified with the set of all $g \in \text{GL}(4, \mathcal{E})$ such that $\bar{g}' H g = H$. We know (Proposition 5.2) that every cusp is of the form $g(e)$, $g \in G$, where

$$e := (1, 0, 0, 0)$$

denotes our standard cusp. We have to study the congruence group of level three, $G_3[3]$, especially the action of this group on the cusps. The problem is to decide when two cusps a, b are in the same $G_3[3]$ -orbit. Of course then $a \equiv b \pmod{3}$. But it will turn out that the converse is not true.

We have to determine the stabilizer Stab of the standard cusp $e = (1, 0, 0, 0)$ in G . There are three types of elements in the stabilizer.

1) The transformation

$$(a_1, a_2, a_3, a_4) \mapsto (a_1 + a_2\sqrt{-3}, a_2, a_3, a_4).$$

2) The transvections

$$(a_1, a_2, a_3, a_4) \mapsto (a_1 + (|b_1|^2 + |b_2|^2)a_2/2 + \bar{b}_1 a_3 + \bar{b}_2 a_4, a_2, a_3 + a_2 b_1, a_4 + a_2 b_2).$$

3) Unitary transformations in the variables a_3, a_4 . They are generated by

$$(a_3, a_4) \mapsto (a_4, a_3) \quad \text{and} \quad (-\zeta a_3, a_4).$$

It is not difficult to show that Stab is generated by the transformations 1),2),3).

Due to Proposition 5.2, the cusp classes of the group $G_3[3]$ are in one-to-one correspondence with the double cosets in

$$G_3[3] \backslash G_3 / \text{Stab}.$$

The group $G_3[3] \backslash G_3$ can be considered as a subgroup of $\text{GL}(4, \mathcal{E}/3\mathcal{E})$. Using Proposition 5.1 one can compute this group and also the image of Stab in this group. We implemented the groups in **MAGMA** and obtained the following result.

7.1 Proposition. *The order of $G_3[3] \backslash G_3$ is $2^5 \cdot 3^{12} \cdot 5$. The order of the image of Stab in this group is $2^3 \cdot 3^7$. Hence the number of cusp classes of $G_3[3]$ is $4860 = 2^2 \cdot 3^5 \cdot 5$.*

We recall that in the Baily–Borel compactification $\overline{\mathcal{B}_3/G_3[3]}$ the cusps have to be counted projectively. Hence we obtain the following result.

7.2 Corollary of Proposition 7.1. *The number of boundary points of $\overline{\mathcal{B}_3/G_3[3]}$ is 810.*

A vector in $a \in (\mathcal{E}/3\mathcal{E})^4$ is called primitive if it is not of the form $a = \sqrt{-3}b$. It is called isotropic if $\langle b, b \rangle \equiv 0 \pmod{3}$ where $b \in \mathcal{E}^4$ is an inverse image of a . We denote the set of all such primitive isotropic vectors by \mathcal{P} . The group $G_3[3] \backslash G_3$ acts on \mathcal{P} . One can show (for example by computation) that G_3 acts transitively on \mathcal{P} . As a consequence every vector of \mathcal{P} is the image of a cusp. There is a natural equivariant map

$$G_3[3] \backslash G / \text{Stab} \longrightarrow \mathcal{P}, \quad g \mapsto g(e).$$

The number of elements of \mathcal{P} computes as $2^2 \cdot 3^4 \cdot 5$. This shows that over each point of \mathcal{P} there are three cusp classes. This can be also understood as follows. Consider the group

$$\text{Stab}' := \{g \in G; g(e) \equiv e \pmod{0}\}.$$

Then we have

$$G_3[3] \backslash G / \text{Stab}' \xrightarrow{\sim} \mathcal{P}.$$

We have the following result.

7.3 Proposition. *The group Stab is a normal subgroup of index three of Stab' . The element*

$$A := \begin{pmatrix} 6\zeta + 4 & 6\zeta + 18 & 3 & -5\zeta + 5 \\ 3\zeta - 3 & 18\zeta + 10 & 3\zeta + 3 & 6\zeta + 9 \\ -3\zeta - 3 & -9 & \zeta - 2 & 4\zeta - 1 \\ -3\zeta - 3 & \zeta - 10 & \zeta - 1 & 5\zeta \end{pmatrix}$$

is contained in Stab' but not in Stab .

Proof. The first column of A is mod 3 congruent to the standard cusp $e = (1, 0, 0, 0)$ (considered as column). Hence $A \in \text{Stab}'$. We prove indirectly that A is not in Stab . If A is in Stab then there exists an $B \in G[3]$ such that $C := BA$ has first column e . Since C is unitary, the first and third column are orthogonal. This gives $c_{23} = 0$. The norm of the third column is -1 . This gives

$$|c_{33}|^2 + |c_{43}|^2 = 1.$$

Hence either $c_{33} = 0$ or $c_{43} = 0$. Since $A \equiv C \pmod{3}$ we get $a_{33} \equiv 0 \pmod{3}$ or $a_{43} \equiv 0 \pmod{3}$. But this is obviously false. \square

Computational Access to Cusp Classes

We have a program that selects for any two cusps a, b an element $g \in G_3$ with $b = g(a)$.

Hence we can find for any cusp a the corresponding double coset in $G[3] \backslash G / \text{Stab}$. One can compute a system of representatives. In this way one can construct an explicit system \mathcal{C} of representatives of the 4860 cusp classes and we have an explicit bijective map

$$\mathcal{C} \xrightarrow{\sim} G[3] \backslash G / \text{Stab}.$$

This makes it possible to find for each cusp a its representative in \mathcal{C} . One just writes a in the form $a = g(e)$ to obtain an element of $G[3] \backslash G / \text{Stab}$ and takes its inverse image in \mathcal{C} .

In this way we can describe the action of G on the set of representatives \mathcal{C} . (This factors through $G[3]$.)

8. First relations

As in the previous section we use the hermitian form

$$\langle a, b \rangle = \bar{a}_1 b_2 + \bar{a}_2 b_1 - \bar{a}_2 b_3 - \bar{a}_4 b_4.$$

We give a list of representatives of $G_3[\sqrt{-3}]$ -orbits of pairs $\pm a$ where a is an integral vector of norm -1 . They correspond to the 15 short mirrors in $\overline{\mathcal{B}_3}/G_3[\sqrt{-3}]$.

1	2	3	4	5
(0, 0, 1, 0)	(0, 0, 0, 1)	(1, 0, 1, 0)	(1, 0, -1, 0)	(1, 0, 0, 1)
6	7	8	9	10
(1, 0, 0, -1)	(0, 1, 1, 0)	(0, 1, -1, 0)	(0, 1, 0, 1)	(0, 1, 0, -1)
11	12	13	14	15
(ζ , -1, 1, 1)	(ζ , -1, 1, -1)	(ζ , -1, -1, 1)	(ζ , -1, -1, -1)	(ζ , 1, 0, 0)

We give the list of triples of short mirrors which are pairwise orthogonal in the sense of Theorem 5.7:

8.1 Definition. *The modular forms of weight one which vanish along the following triples of short mirrors*

(1,2,15), (2,4,8), (2,3,7), (1,6,10), (1,5,9), (12,13,15), (11,14,15), (4,6,11), (4,5,12), (8,10,14), (8,9,13), (3,6,13), (3,5,14), (7,10,12), (7,9,11)

are denoted by B_1, \dots, B_{15} in this ordering.

The forms B_i are of course only determined up to constant factors. Later we will normalize them in a suitable way.

It is possible to determine the values of these 15 modular forms at the 4860 cusp classes. Let F be one of the 15 forms. First one can determine the set of cusps at which F does not vanish. Then one can decompose this set into orbits under the group $G_3[\sqrt{-3}]$. It turns out that there are 4 orbits. It is enough to compute the value for one element in each orbit. For this we constructed for each pair of orbits O_1 and O_2 a pair of vectors $a \in O_1$, $b \in O_2$ that satisfies the conditions in Proposition 6.1. Then we know $F(a) = F(b)$. Now all values of F at the cusps are determined up to a constant factor.

Using the values of the B_i at the cusps, we can describe the action of the group G_3 on them. The group G_3 acts on modular forms through $(f, \gamma) \mapsto f^\gamma$, where

$$f^\gamma(z) := f(\gamma z).$$

This is an action from the right. Up to constant factors the functions B_i are permuted under this action. We describe the action of an element $g \in G_3$ by a list

$$\begin{pmatrix} \sigma_1 & \cdots & \sigma_{15} \\ \varepsilon_1 & \cdots & \varepsilon_{15} \end{pmatrix}.$$

This list has to be read as follows:

$$B[i]^g = \varepsilon(i)B[\sigma(i)].$$

So the image of $B[i]$ is described by the i th column of this list.

8.2 Lemma. *The forms B_i can be normalized in such a way that the transformation group corresponding to G_3 is generated by the following three transformations.*

8.3 Lemma. *The transformation group corresponding to Γ_3 on the forms B_i is generated by the following three transformations.*

$$\begin{pmatrix} 8 & 15 & 7 & 9 & 2 & 12 & 4 & 14 & 3 & 5 & 11 & 6 & 1 & 13 & 10 \\ \zeta & -\zeta & 1 & 1 & \bar{\zeta} & -\zeta & \bar{\zeta} & 1 & \zeta & -\bar{\zeta} & \bar{\zeta} & -\zeta & 1 & \bar{\zeta} & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 15 & 14 & 2 & 1 & 13 & 12 & 8 & 7 & 11 & 5 & 9 & 6 & 10 & 4 \\ \zeta & \bar{\zeta} & \bar{\zeta} & -1 & \bar{\zeta} & \zeta & 1 & -\zeta & -\bar{\zeta} & 1 & -\bar{\zeta} & 1 & 1 & -\bar{\zeta} & 1 \end{pmatrix},$$

$$\begin{pmatrix} 12 & 6 & 11 & 3 & 13 & 8 & 4 & 14 & 9 & 1 & 7 & 15 & 5 & 2 & 10 \\ -\zeta & -\zeta & -\zeta & -\zeta & \zeta & \bar{\zeta} & -1 & -1 & -1 & -\zeta & -1 & -\zeta & \bar{\zeta} & \bar{\zeta} & -\bar{\zeta} \end{pmatrix}.$$

Using Definition 8.1 and the list just before it, one can verify that, for example, $B_1B_{13}B_{15}$ and $B_3B_5B_7$ have the same zeros (9 short mirrors). Hence they agree up to a constant factor. The constant factor can be determined since they are no cusp forms and since we know the values at the cusps. Using the normalization described in Lemma 8.2 one gets the following relation:

$$B_1B_{13}B_{15} = B_3B_5B_7.$$

If one applies Lemma 8.2 one can produce the following 10 relations

$$\begin{aligned} B_1B_{13}B_{15} &= B_3B_5B_7, & B_4B_{11}B_{13} &= B_5B_{10}B_{12}, & B_1B_{12}B_{14} &= B_3B_4B_6, \\ B_6B_{10}B_{15} &= B_7B_{11}B_{14}, & B_4B_9B_{15} &= B_5B_8B_{14}, & B_2B_{12}B_{15} &= B_3B_8B_{11}, \\ B_1B_9B_{11} &= B_2B_5B_6, & B_6B_8B_{13} &= B_7B_9B_{12}, & B_2B_{13}B_{14} &= B_3B_9B_{10}, \\ B_1B_8B_{10} &= B_2B_4B_7. \end{aligned}$$

An essential point is that all these relations are defined over \mathbb{Q} . This is not clear from advance since the transformations described in Lemma 8.2 are not defined over \mathbb{Q} .

In the same way one can produce 15 quaternary relations. In this way we can prove the following result.

8.4 Proposition. *There are 10 relations of the form $B_iB_jB_k = B_\alpha B_\beta B_\gamma$ where the six indices are pairwise different. One of them is $B_1B_{13}B_{15} = B_3B_5B_7$. The others can be obtained by means of the action of G_3 . There are also 15 relations of the type $B_iB_jB_kB_l = B_\alpha B_\beta B_\gamma B_\delta$ with 8 pairwise different indices. One of them is $B_4B_6B_{13}B_{15} = B_5B_7B_{12}B_{14}$. The others can be obtained from the action of G_3 . These relations are all defined over \mathbb{Q} .*

There are also linear relations for the third powers B_i^3 . They are modular forms for $G_3[\sqrt{-3}]$ with trivial character. The ring of these modular forms can be determined from the paper [F]. In this paper the ring of modular forms in the 4-dimensional case $G_4[\sqrt{-3}]$ has been determined. Our three dimensional case occurs as a factor of this ring. This can be deduced from lemma 3.7 in [F]. All what we have to know here is that the space of modular forms of weight 3 with trivial character with respect to $G_3[\sqrt{-3}]$

We denote by

$$A(G_3[3]) = \bigoplus_{k=0}^{\infty} [G_3[3], k]$$

the ring of modular forms with respect to the principal congruence subgroup of level 3. Here $[G_3[3], k]$ is the space of modular forms of weight k with trivial character and similarly

$$A(G_3[\sqrt{-3}]) = \bigoplus_{k=0}^{\infty} [G_3[\sqrt{-3}], k].$$

We consider the subring generated by the third powers B_i^3 . There is a relation to the Segre cubic. Recall that it is defined as follows. One considers in the polynomial ring $\mathbb{C}[T_1, \dots, T_6]$ the ideal generated by $T_1 + \dots + T_6$ and $T_1^3 + \dots + T_6^3$. Then the Segre cubic is the projective threefold associated to the graded algebra

$$\mathbb{C}[T_1, \dots, T_6] / \langle T_1 + \dots + T_6, T_1^3 + \dots + T_6^3 \rangle.$$

This algebra is normal.

8.5 Proposition. *The assignments*

$$\begin{aligned} T_1 &\longmapsto X_1^3 + X_{13}^3 - X_{15}^3, \\ T_2 &\longmapsto X_1^3 - X_{13}^3 + X_{15}^3, \\ T_3 &\longmapsto -X_2^3 - X_{13}^3 + X_{14}^3, \\ T_4 &\longmapsto -X_2^3 + X_{13}^3 - X_{14}^3, \\ T_5 &\longmapsto -X_4^3 - X_{11}^3 - X_{13}^3, \\ T_6 &\longmapsto -X_6^3 + X_{10}^3 - X_{15}^3. \end{aligned}$$

defines an isomorphism

$$\mathbb{C}[T_1, \dots, T_6] / \langle T_1 + \dots + T_6, T_1^3 + \dots + T_6^3 \rangle \xrightarrow{\sim} \mathbb{C}[B_1^3, \dots, B_{15}^3].$$

Moreover, the algebra $A(G_3[\sqrt{-3}])$ is generated by the B_i^3 .

Proof. This follows from the results in [F] and has also been worked out in a self contained way by Kondo [Ko].

On the Segre cubic the symmetric group S_6 acts in an obvious way. We can see this action also in the modular picture. Up to constant factors it acts on the 15 modular forms B_i up to constant factors by permutation (Lemma 8.2). One can compute the action of G_3 on the 6 expressions T_1, \dots, T_6 . To describe it we need a certain sign character on G_3 .

8.6 Lemma. *The group G_3 has a sign character $\varepsilon : G_3 \rightarrow \{1, -1\}$ that associates -1 to each hexflexion and to the transformation $a \mapsto -a$.*

Now we can describe the action of G_3 on the T_i .

8.7 Proposition. *There exists a surjective homomorphism $\phi : G_3 \rightarrow S_6$ with the property*

$$T_i(g(z)) = \varepsilon(g)T_{\phi(g)(i)}.$$

The kernel of this homomorphism is the subgroup of G_3 generated by $G_3[\sqrt{-3}]$ and the negative of the identity.

Hence we have

$$G_3 / \pm (G_3[\sqrt{-3}]) \cong S_6.$$

Now we obtain the following Lemma.

8.8 Lemma. *The isomorphism*

$$\mathbb{C}[T_1, \dots, T_6] / \langle T_1 + \dots + T_6, T_1^3 + \dots + T_6^3 \rangle \xrightarrow{\sim} \mathbb{C}[B_1^3, \dots, B_{15}^3]$$

is equivariant with respect to a surjective homomorphism $\Gamma_3 \rightarrow S_6 \times \{\pm 1\}$. Here $S_6 \times \{\pm 1\}$ acts on the variables T_i by permutation in combination with the sign. This means that (σ, ε) acts by $T_i \mapsto \varepsilon T_{\sigma(i)}$. The three transformations in Lemma 8.2 map to the three pairs

$$(1, 6, 4, 2, 5, 3), \quad \varepsilon = 1; \quad (6, 5, 2, 4, 3, 1), \quad \varepsilon = -1; \quad (3, 2, 4, 6, 5, 1), \quad \varepsilon = -1.$$

There is a subgroup of index two of Γ_3 such that ε is the sign of σ .

Proof. We just mention that the three transformations in Lemma 8.2 map to the three permutations

$$(1, 6, 4, 2, 5, 3), \quad (6, 5, 2, 4, 3, 1), \quad (3, 2, 4, 6, 5, 1).$$

(Here (a_1, \dots, a_6) stands for the permutation $i \mapsto a_i$.) □

In the remaining three sections we describe the proof of the main results.

9. The algebra of modular forms

We consider the following 10 functions:

$$\begin{aligned}
C_1 &= B_2 B_4 B_{15} / B_8, \\
C_2 &= B_2 B_{13} B_{15} / B_3, \\
C_3 &= B_3 B_6 B_{10} / B_{14}, \\
C_4 &= B_3 B_5 B_8 / B_{15}, \\
C_5 &= B_8 B_{13} B_{14} / B_9, \\
C_6 &= B_5 B_7 B_{14} / B_{15}, \\
C_7 &= B_2 B_6 B_{15} / B_{11}, \\
C_8 &= B_1 B_8 B_{11} / B_2, \\
C_9 &= B_6 B_{13} B_{15} / B_7, \\
C_{10} &= B_2 B_4 B_6 / B_1.
\end{aligned}$$

Looking at the divisors (s. Definition 8.1) we see that they are holomorphic. Hence they are modular forms of weight 2.

We want to determine the algebraic relations between the 25 forms B_i, C_j . For this we consider variables B_i, Y_j and the natural homomorphism

$$\mathbb{C}[X_1, \dots, X_{15}, Y_1, \dots, Y_{10}] \longrightarrow \mathbb{C}[B_1, \dots, B_{15}, C_1, \dots, C_{10}].$$

Then consider the action of G_3 on the variables X_i given by the formulae described in Lemma 8.2. They induce obvious transformations of the variables Y_i . In this way we get an action of the group G_3 on the polynomial ring $\mathbb{C}[X_1, \dots, Y_{10}]$ such that the homomorphism above is G_3 -equivariant.

We describe some explicit relations:

9.1 Proposition. *The following tables contains explicit relations of weight 3, 4, 5 and 6 between the forms B_i, C_j . The last entry in each line gives the order of the G_3 -orbit of the relation where two relations are considered to be equal if they coincide up to a constant factor.*

Relations of weight 3:

<i>type I:</i>	$X_1X_{13}X_{15} - X_3X_5X_7$	10
<i>type II:</i>	$X_2^3 - X_{10}^3 + X_{11}^3$	10
<i>type III:</i>	$Y_2X_1 - X_2X_5X_7$	60

Relations of weight 4:

<i>type I:</i>	$X_4X_6X_{13}X_{15} - X_5X_7X_{12}X_{14}$	15
<i>type II:</i>	$Y_1X_1^2 - X_6^2X_{11}X_{14} - X_7^2X_{10}X_{15}$	90
<i>type III:</i>	$Y_1Y_2 - X_2X_5X_{10}X_{15}$	45

Relations of weight 5:

<i>type I:</i>	$Y_1^2X_1 - X_6X_{11}^2X_{14}^2 - X_7X_{10}^2X_{15}^2$	90
<i>type II:</i>	$Y_4X_3X_9X_{13} - Y_7X_1X_7X_8 + X_2^2X_4X_{10}X_{14}$	180
<i>type III:</i>	$-Y_1Y_6X_4 + Y_2Y_7X_8 + Y_3Y_9X_{12}$	15

Relations of weight 6:

<i>type I:</i>	$Y_4^3 - X_3^3X_8^3 + X_2^3X_{12}^3$	10
<i>type II:</i>	$Y_9X_5X_6X_{12}X_{15} + X_1X_3^2X_8X_9^2 - X_4X_7X_{13}^2X_{14}^2$	90
<i>type III:</i>	$-Y_5X_4X_8X_{14}X_{15} + Y_8X_6X_7X_8X_{12} + X_5X_8^3X_9X_{13}$	75

Supplement. *If one applies an element of G_3 to one of these relations then one gets up to a constant factor a relation with rational coefficients.*

Proof. We start with the supplement. Since we know the action of G_3 on the X_i (and as a consequence on the Y_i) this can be checked using generators of G_3 .

The proof of the relations uses the ring $\mathbb{Q}[X_1, \dots, X_{15}]$. We consider the ideal \mathcal{I} that is generated by the relations which involve only the X_i (relations of weight 3, type I and II and relations of weight 4 type I and their transformed under G_3 multiplied by constants such that they are rational). This ideal is rather simple and it is no problem to get a **Gröbner** basis for it using a computer algebra system as SINGULAR. So one can decide whether a given polynomial from $\mathbb{Q}[X_1, \dots, X_{15}]$ is contained in this ideal. The proof of the relations – take for example the relation $C_2B_1 = B_2B_5B_7$ (weight 3, type III) – can be given as follows. Let $\Delta = B_1 \cdots B_{15}$. It is enough to show that the modular form $\Delta(C_2B_1 - B_2B_5B_7)$ vanishes. But ΔC_2 can be expressed as monomial in the B_i by definition of C_2 . Hence we have to verify a relation in the ring $\mathbb{Q}[X_1, \dots, X_{15}]$. It turns out that this relation comes already from the ideal \mathcal{I} . In this way all the listed relations can be verified. \square

Now we can formulate the main result of this paper.

9.2 Theorem. *The algebra of modular forms of $A(G_3[3])$ is generated by the forms B_1, \dots, B_{15} and C_1, \dots, C_{10} . Defining relations are the G_3 -orbits of the relations described in Proposition 9.1. The following dimension formula holds.*

$$\dim[G_3[3], k] = \begin{cases} 0 & \text{for } k < 0, \\ 1 & \text{for } k = 0, \\ 10 & \text{for } k = 1, \\ 130 & \text{for } k = 2, \\ 750 & \text{for } k = 3, \\ 3115 & \text{for } k = 4, \\ -1377 + (8019/2)k - 2187tk^2 + (729/2)k^3 & \text{for } k > 4. \end{cases}$$

This algebra is defined over \mathbb{Q} , where the \mathbb{Q} -structure is generated by these generators.

In weight ≥ 7 the algebra is generated by the forms B_i alone.

We also can determine the dimensions of the spaces of cusp forms.

9.3 Theorem. *The forms C_i are cusp forms. Let $[G_3[3], k]_0$ be the space of cusp forms. Then*

$$\dim[G_3[3], k] - \dim[G_3[3], k]_0 = \begin{cases} 15 & \text{for } k = 1, \\ 120 & \text{for } k = 2, \\ 405 & \text{for } k = 3, \\ 765 & \text{for } k = 4, \\ 810 & \text{for } k > 4. \end{cases}$$

10. The proof of the main result, Gröbner bases

The proof is a mixture of commutative algebra, computer algebra and theory of modular forms. We denote by

$$\mathcal{J} \subset \mathbb{Q}[X_1, \dots, X_{15}, Y_1, \dots, Y_{10}]$$

the ideal generated by the relations described in Proposition 9.1. The ideal

$$\mathbb{C}\mathcal{J} \subset \mathbb{C}[X_1, \dots, X_{15}, Y_1, \dots, Y_{10}]$$

is invariant under the group G_3 . Recall that we defined also the ideal

$$\mathcal{I} \subset \mathbb{Q}[X_1, \dots, X_{10}]$$

that is generated by relations of weight 3 and 4. The ideal $\mathbb{C}\mathcal{I}$ is also invariant under G_3 . The ideal \mathcal{I} is contained in $\mathcal{J} \cap \mathbb{Q}[X_1, \dots, X_{10}]$ but both ideals are different. The reason is as follows. If we consider the relations of weight 5 and type III and replace in them $Y_i Y_j$ by means of the relations of weight 4 and type III, we get relations in $\mathbb{Q}[X_1, \dots, X_{15}]$ of weight 5 that are not contained in \mathcal{I} . The precise picture can be obtained with the help of computer algebra. SINGULAR computes for both ideals Gröbner bases which enable a proof of the following statement.

10.1 Lemma. *The ideal*

$$\mathcal{I}_{\text{sat}} := \mathcal{J} \cap \mathbb{Q}[X_1, \dots, X_{15}]$$

has the following property. It consists of all polynomials P such that

$$P \cdot X_1 \cdots X_{15} \in \mathcal{I}.$$

Moreover it has the following property. Let $P \in \mathbb{Q}[X_1, \dots, X_{15}]$ be a polynomial such that there exists a monomial $M = X_1^{\nu_1} \cdots X_{15}^{\nu_{15}}$ with the property $MP \in \mathcal{I}_{\text{sat}}$, then $P \in \mathcal{I}_{\text{sat}}$.

Gröbner bases also give the dimensions of the ideals.

10.2 Lemma. *The Krull dimensions of the rings*

$$\mathbb{Q}[X_1, \dots, X_{15}, Y_1, \dots, Y_{10}]/\mathcal{J}, \quad \mathbb{Q}[X_1, \dots, X_{15}]/\mathcal{I}, \quad \mathbb{Q}[X_1, \dots, X_{15}]/\mathcal{I}_{\text{sat}}$$

are four.

We also can get information about the Hilbert polynomials. Let $A = \bigoplus A_k$ be a finitely generated graded algebra over a field A_0 . The Hilbert series is

$$\sum_k \dim A_k t^k.$$

There exist a polynomial $H(t)$, the Hilbert polynomial, such that for a suitable natural number k_0 one has

$$H(k) = \dim A_k, \quad k \equiv 0 \pmod{k_0}, \quad k \gg 0.$$

In the case that A is generated by A_1 , one can take $k_0 = 1$. In our situation the gradings have to be defined such that the weight of X_i is 1 and the weights of Y_j is 2.

10.3 Lemma. *The Hilbert polynomial of $\mathbb{Q}[X_1, \dots, X_{15}]/\mathcal{I}_{\text{sat}}$ is*

$$-1377 + (8019/2)k - 2187k^2 + (729/2)k^3.$$

The Hilbert series is

$$1 + 15t + 120t^2 + 660t^3 + 2745t^4 + 8898t^5 + 22665t^6 + 44550t^7 + 77355t^8 + \dots$$

In the case $k > 6$ the value $H(k)$ of the Hilbert polynomial gives the correct value.

As a consequence we get also the Hilbert polynomial of the ideal \mathcal{J} .

10.4 Lemma. *The algebras*

$$\mathbb{Q}[X_1, \dots, X_{15}, Y_1, \dots, Y_{10}]/\mathcal{J}, \quad \mathbb{Q}[X_1, \dots, X_{15}]/\mathcal{I}_{\text{sat}}$$

agree in weight > 6 . Hence they have the same Hilbert polynomial.

Our next goal is to prove that \mathcal{I}_{sat} is a prime ideal. Since this ideal is very involved it seems to impossible to do this by straightforward computer algebra. Instead of this we make use of rather deep results from the theory of modular forms.

11. The proof of the main result, modular forms

One has to investigate the relations between the modular forms B_i, C_j in more detail.

11.1 Lemma. *The algebra of modular forms*

$$A(\Gamma_3[3]) = \bigoplus [G_3[3], k]$$

is the normalization of the subalgebra

$$\mathbb{C}[B_1, \dots, B_{15}].$$

Proof. The modular forms B_i have no common zero in the Baily-Borel compactification. This can be deduced from the concrete description of their zero divisors. It is also an immediate consequence of the result about Segre cubic. A general result of Hilbert implies that $A(G_3[3])$ is integral over the image of the above homomorphism. We want to have more, namely that it is the normalization of the image. For this we have to show that the quotient fields agree.

The analogous statement for the group $G_3[\sqrt{-3}]$ is true. The general statement follows by means of a Galois argument. One has to show that $G_3[\sqrt{-3}]/G_3[3]$ acts faithfully on the image ring. This follows from the transformation formulae above. \square

The dimension formula for space of ball-modular forms has been computed by Kato using the Selberg trace formula [Ka]. His result gives that for $k > 6$ the dimension of the space of cusp forms is $C(k-1)(k-2)(k-3)$. The constant C can be expressed by the volume of the fundamental domain. In our case we can obtain it as follows. We compare it with the Hilbert polynomial of the ring $A(G_3[\sqrt{-3}])$ which can be deduced from the known structure of this ring. We make use of the fact that the highest coefficient of the Hilbert polynomial of $A(G_3[3])$ is just the product of the highest coefficient of the Hilbert polynomial of $A(G_3[\sqrt{-3}])$ and the covering degree which is 3^9 . In this way one can prove $C = 729/2$. To get the dimension of the space of all modular forms, we have to add the number of cusps 810. So we get

$$(729/2)(k-1)(k-2)(k-3) + 810.$$

This agrees with our Hilbert polynomial $H(k)$ which we defined in Lemma 10.3. Hence we obtain the following lemma.

11.2 Lemma. *The algebras*

$$A(G_3[3]), \quad \mathbb{Q}[X_1, \dots, X_{15}]/\mathcal{I}_{\text{sat}}$$

have the same Hilbert polynomial $H(k)$.

We consider the natural homomorphism

$$\mathbb{Q}[X_1, \dots, X_{15}] \longrightarrow A(G_3[3]).$$

Its kernel is a prime ideal \mathfrak{p} which contains \mathcal{I}_{sat} . We have the inequalities

$$\dim(\mathbb{Q}[X_1, \dots, X_{15}]/\mathcal{I}_{\text{sat}})_k \geq \dim(\mathbb{Q}[X_1, \dots, X_{15}]/\mathfrak{p})_k \geq \mathbb{C}[B_1, \dots, B_{15}]_k.$$

The highest coefficient of the Hilbert polynomial of $\mathbb{C}[B_1, \dots, B_{15}]$ equals the highest coefficient of the Hilbert polynomial of its normalization. Hence the above inequalities must produce equality for the highest coefficients. This shows that the algebras

$$\mathbb{Q}[X_1, \dots, X_{15}]/\mathcal{I}_{\text{sat}} \quad \text{and} \quad \mathbb{Q}[X_1, \dots, X_{15}]/\mathfrak{p}$$

have the same highest coefficients. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal prime ideals that contain \mathcal{I}_{sat} and such that $\mathbb{Q}[X_1, \dots, X_{15}]/\mathfrak{p}_i$ has Krull dimension 4. The ideal \mathfrak{p} is one of them. It is easy to see that the algebras

$$\mathbb{Q}[X_1, \dots, X_{15}]/\mathcal{I}_{\text{sat}} \quad \text{and} \quad \prod_{i=1}^n \mathbb{Q}[X_1, \dots, X_{15}]/\mathfrak{p}_i$$

have the same highest coefficient. This implies $n = 1$. So we have proved the following lemma.

11.3 Lemma. *There is only one minimal prime ideal \mathfrak{p} containing \mathcal{I}_{sat} and which has the property that $\mathbb{Q}[X_1, \dots, X_{15}]/\mathfrak{p}$ has dimension 4.*

Geometrically this means that the associated projective variety has only one irreducible 4-dimensional component. But there might be irreducible components of smaller dimension. We want to exclude this. For this we need some results of commutative algebra which have been developed to get a computational access to problems as computing the primary decomposition of a polynomial ideal. We refer to the book [GP], especially to Chapt. 4. Following the ideas which are developed there, we choose 4 independent variables.

11.4 Lemma. *The variables X_1, X_3, X_5, X_6 are independent variables for \mathcal{I}_{sat} in the sense that*

$$\mathbb{Q}[X_1, X_3, X_5, X_6] \cap \mathcal{I}_{\text{sat}} = 0.$$

We checked this with the help of computer algebra. We want to extend the ground field \mathbb{Q} to the field of rational functions $K = \mathbb{Q}(X_1, X_3, X_5, X_6)$ over \mathbb{Q} in the variables X_1, X_3, X_5, X_6 . We consider over K the polynomial ring in the remaining 11 variables,

$$\mathcal{R} = K[X_2, X_4, X_7, \dots, X_{15}].$$

Then we extend \mathcal{I}_{sat} to this ring. We mention some general facts:

- 1) The Krull dimension of $\mathcal{R}/\mathcal{I}_{\text{sat}}\mathcal{R}$ is zero.
- 2) The minimal prime ideals of \mathcal{R} containing $\mathcal{I}_{\text{sat}}\mathcal{R}$ are in one-to-one correspondence with the minimal prime ideals of $\mathbb{C}[X_1, \dots, X_{15}]$ containing \mathcal{I}_{sat} and with the additional property that the Krull dimension of their quotient is 4.

We know that there is only one prime ideal with this property (Lemma 11.3). Hence we see that there is only one minimal prime ideal containing $\mathcal{I}_{\text{sat}}\mathcal{R}$. This must agree with the radical of $\mathcal{I}_{\text{sat}}\mathcal{R}$. So we have seen that the radical is a prime ideal. Making again use from computer algebra, one can check that $\mathcal{I}_{\text{sat}}\mathcal{R}$ agrees with its radical. In this way we see that $\mathcal{I}_{\text{sat}}\mathcal{R}$ is a prime ideal. We obtain that

$$\tilde{\mathcal{I}} = \mathbb{Q}[X_1, \dots, X_{15}] \cap \mathcal{I}_{\text{sat}}\mathcal{R}$$

is also a prime ideal. We want to prove $\tilde{\mathcal{I}} = \mathcal{I}_{\text{sat}}$.

The proof again uses computer algebra. The fact is that we can construct a Gröbner basis of the ideal $\mathcal{I}_{\text{sat}}\mathcal{R}$ with respect to the lexicographical ordering

of the variables such that $X_i > X_j$ for $i > j$.

$$\begin{aligned}
& (X_1^3)X_{15}^6 + (-X_1^3X_3^3 - X_1^3X_5^3 + X_1^3X_6^3)X_{15}^3 + (X_1^3X_3^3X_5^3 - X_3^3X_5^3X_6^3), \\
& X_{14}^3 + X_{15}^3 + (-X_3^3), \\
& X_{13}^3 - X_{14}^3 + (-X_5^3 + X_6^3), \\
& X_{12}^3 + X_{13}^3 - X_{14}^3 - X_{15}^3, \\
& X_{11}^3 - X_{15}^3 + (X_5^3), \\
& (-X_3X_5X_6)X_{10} + (X_1)X_{11}X_{13}X_{14}, \\
& X_9^3 - X_{11}^3 - X_{13}^3 + X_{15}^3, \\
& (-X_3X_5X_6)X_8 + (X_1)X_9X_{12}X_{15}, \\
& (-X_3X_5)X_7 + (X_1)X_{13}X_{15}, \\
& (-X_3X_6)X_4 + (X_1)X_{12}X_{14}, \\
& (-X_5X_6)X_2 + (X_1)X_9X_{11}.
\end{aligned}$$

The leading terms (first term in each line) play a fundamental role in the theory of Gröbner bases. We denote by h the product of the coefficients of the leading terms,

$$h = -X_1^3X_3^4X_5^4X_6^4.$$

From the general theory of Gröbner bases one knows ([GP], Proposition 4.3.1) that $\tilde{\mathcal{I}}$ consists of all polynomials P such the product with a suitable power of h is in \mathcal{I}_{sat} . From Lemma 10.1 we obtain that $P \in \mathcal{I}_{\text{sat}}$. So we obtained the following result.

11.5 Lemma. *The ideal \mathcal{I}_{sat} is a prime ideal. It is the kernel of the natural homomorphism*

$$\mathbb{Q}[X_1, \dots, X_{15}] \longrightarrow A(G_3[3]).$$

Our next goal is to prove that the ideal

$$\mathcal{J} \subset \mathbb{Q}[X_1, \dots, X_{15}, Y_1, \dots, Y_{10}]$$

is a prime ideal, or equivalently, the natural homomorphism

$$\mathbb{Q}[X_1, \dots, X_{15}, Y_1, \dots, Y_{10}]/\mathcal{J} \longrightarrow A(G_3[3])$$

is injective. For this we need a detailed investigation of the relations between the modular forms B_i, C_j . We make use of the fact that monomials in them are modular forms with respect to $G_3[\sqrt{-3}]$ and that we can determine their characters. Since modular forms with different characters are linearly independent, we have a method to prove that all relations com from the ideal \mathcal{J} . On

this way we could prove that all relations ≤ 6 in the weights are contained in the ideal \mathcal{J} . Moreover we got the following dimensions

$$15, \quad 130, \quad 750, \quad 3115, \quad 9558, \quad 22680.$$

The last two values are exactly the values $H(5)$ and $H(6)$ of the Hilbert polynomial.

In the weight $k \geq 7$ all monomials in the C_i, B_i can be expressed as polynomials in the B_i as consequence of the relations in weight ≤ 6 . This detailed investigation of relations implies the following result.

11.6 Lemma. *The ideal \mathcal{J} (see Proposition 9.1) is a prime ideal. It is the kernel of the natural homomorphism*

$$\mathbb{Q}[X_1, \dots, X_{15}, Y_1, \dots, Y_{15}] \longrightarrow A(G_3[3]).$$

12. The proof of the main result, normality

We have to study the algebra

$$R = \mathbb{Q}[B_1, \dots, B_{15}, C_1, \dots, C_{10}] \otimes_{\mathbb{Q}} \mathbb{C}.$$

12.1 Theorem. *The ring*

$$R = \mathbb{Q}[B_1, \dots, B_{15}, C_1, \dots, C_{10}] \otimes_{\mathbb{Q}} \mathbb{C}$$

is an integral domain. The natural homomorphism

$$\mathbb{Q}[B_1, \dots, B_{15}, C_1, \dots, C_{10}] \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow A(G_3[3])$$

induces an isomorphism in weights $k \equiv 0 \pmod{k_0}$ for suitable k_0 . (The same is true already for the algebra $\mathbb{Q}[B_1, \dots, B_{15}]$).

Proof. We only have to show that R is an integral domain. (Then a dimension argument shows that $R \rightarrow A(G_3[3])$ is injective and the Theorem follows from the comparison of the Hilbert polynomials, Lemma 11.2.) Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal prime ideals of R . Their intersection is the radical. Since \mathbb{C} is a separable field extension of \mathbb{Q} the algebra R is reduced. Hence it suffices to show that $n = 1$. The primes \mathfrak{p}_i are conjugate under the automorphism group of \mathbb{C} . We want to compare highest coefficients of Hilbert polynomials. The Hilbert polynomial $H(k)$ of R is the same as described in Lemma 10.3

and Lemma 10.4. We denote its highest coefficient by d ($= 729/2$). The highest coefficient of the Hilbert polynomials of R/\mathfrak{p}_i are d/n . The kernel of the homomorphism $R \rightarrow A(G_3[3])$ is one of the primes \mathfrak{p}_i . Hence the highest coefficient of the Hilbert polynomial of the image is d/n . The highest coefficient of the Hilbert polynomial of the normalization doesn't change. We know that the normalization is the full ring of modular forms. We know that the highest coefficient of the Hilbert polynomial of $A(G_3[3])$ is d . This shows $n = 1$. \square

Let R be a noetherian local ring. The *depth* of R is largest number n such there exists a regular sequence a_1, \dots, a_n . This means that the a_i are elements of the maximal ideal and that a_{i+1} is a non-zero divisor in $R/(a_1, \dots, a_i)$ for all $i < n$. Recall that a noetherian ring R satisfies Serre's condition S_m if all localizations by prime ideal $R_{\mathfrak{p}}$ with the property $\dim R_{\mathfrak{p}} \geq m$ have depth $\geq m$. Serre's normality criterion says that a noetherian ring R is normal if it satisfies S_2 and if $R_{\mathfrak{p}}$ is regular for all prime ideals with the property $\dim R_{\mathfrak{p}} \leq 1$.

12.2 Lemma. *Let \mathfrak{p} be a prime ideal of a ring R and $f \in \mathfrak{p}$ a non-zero divisor in R . Then its image $f/1$ in $R_{\mathfrak{p}}$ is a non-zero divisor.*

The proof is simple and can be omitted. \square

We are interested in finitely generated graded algebras $R = \bigoplus R_n$ over a field R_0 . If f is a homogenous element of R than, besides the usual localization R_f , one considers $R_{(f)}$ which is the subring of all homogenous elements of degree 0. There is a natural open embedding $\text{Spec } R_{(f)} \rightarrow \text{proj } R$. Let \mathfrak{p} be a prime ideal of R . We assume that \mathfrak{p} is different from $R_{>0}$. We denote by \mathfrak{p}_0 the ideal generated by the homogenous elements of \mathfrak{p} . This is an element of $\text{proj } R$.

12.3 Lemma. *Let \mathfrak{p} be a prime ideal in R which is different from $R_{>0}$. Assume that his homogenous part \mathfrak{p}_0 is a regular point of $\text{proj } R$. Then \mathfrak{p} is a regular point of $\text{Spec } R$.*

Proof. By assumption there exists an element f of positive degree which is not contained in \mathfrak{p} . Then $\text{Spec } R_{(f)}$ is an open neighborhood of \mathfrak{p}_0 . Since the regular locus is open we can choose f such that $R_{(f)}$ is a regular ring. Since $R_f = R_{(f)}[1/f]$ is isomorphic to the polynomial ring in one variable, it is regular too. This shows that $R_{\mathfrak{p}}$ is regular. \square

We want to apply this to the ring

$$R = \mathbb{Q}[B_1, \dots, B_{15}, C_1, \dots, C_{10}] = \mathbb{Q}[X_1, \dots, X_{15}, Y_1, \dots, Y_{10}]/\mathcal{J}.$$

We know that this is an integral domain. For each cusp s we can consider the homogenous ideal \mathfrak{m} in R generated by all homogeneous elements that vanish at the cusp. This gives a finite system S of ideals. They are prime ideals which are maximal in the system of graded ideals. We claim that the scheme

$\text{proj } R - S$ is regular. To prove this we can extend the ground field \mathbb{Q} to \mathbb{C} . So we obtain the variety $\mathcal{B}_3/G_3[3]$ (without cusps) which is smooth since $G_3[3]$ acts fixed-point-free.

We want to apply Serre's criterion to the ring R . We have already seen that $R_{\mathfrak{p}}$ is regular for all \mathfrak{p} with $\dim R_{\mathfrak{p}} \leq 1$. Hence we have to check the condition S_2 . We first check it for the ideals $\mathfrak{m} \in S$.

12.4 Lemma. *Let s be a cusp. There exist two homogenous elements $f, g \in R$ vanishing at the cusp s such that g is a non-zero divisor in R/f .*

Proof. We consider $f = B_1$ and $g = B_8$. There are cusps in which both vanish. One has to prove that B_8 is a non-zero divisor in R/B_1 . To express this in a computable ideal theoretic way we recall the definition of the ideal quotient

$$\mathfrak{a} : \mathfrak{b} := \{x \in R; x\mathfrak{b} \subset \mathfrak{a}\}.$$

The statement above means

$$(X_1, \mathcal{J}) : (X_8) = (X_1, \mathcal{J}).$$

This ideal quotient can be computed by SINGULAR. The same method works for all other cusps. In this way we could prove Lemma 12.4. \square

It is worthwhile to mention that this result involves the variables Y_i . The analogous statement in the ring $\mathbb{Q}[X_1, \dots, X_{15}]$ is false.

Now we can verify S_2 for R . Let \mathfrak{p} be a prime ideal of R . We can assume that $R_{\mathfrak{p}}$ is not regular. We know then that \mathfrak{p} contains one of the ideals $\mathfrak{m} \in S$. (This is also true for the ideal $R_{>0}$). Using Lemma 12.4 in connection with Lemma 12.2 we get that the depth of $R_{\mathfrak{p}}$ is ≥ 2 . This completes the proof of the main result (Theorem 9.2) of this paper. Theorem 9.3 follows by an easy computation in weights < 7 , since we know the values of the forms B_i at the cusps. We recall that the forms C_i are cuspidal.

Final Remark. The canonical weight of the 3-ball is 4. This means that G -invariant differential forms of top-degree correspond to modular forms of weight 4 (with respect to a certain character). By a general result, such a cusp form of weight 4 defines a *holomorphic* differential form on any non-singular model of X_G . Hence groups $G \supset G_3[3]$ have some chance to produce Calabi–Yau manifold such that the associated Calabi-Yau form corresponds to one of the C_i^2 . We will give examples in a forthcoming paper.

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