SOME REMARKS ON SELBERG'S ZETA FUNCTIONS

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In 1959 Atle Selberg reported for the first time [1] on a new type of zeta functions of several complex variables, the so-called Eisenstein series of the unimodular group Γ_n of degree n. He succeded, by a number theoretical approach, in solving the problem of analytical continuation and proving a system of functional equations. Although a second report [3] contains further information on the subject, it seemed to me rather difficult to reproduce Selberg's proofs on the basis of his very short sketches. So I started with some work just at the point where Selberg remarked [1]:

"If one tries to establish the analytical properties and functional equations of the function ..., it seems that it is not practicable to follow the usual pattern of finding some integral representation of the function which at once gives the analytic continuation and sets into evidence the functional equation. I was able to find such a representation in the case $n = 3$, but it was already extremely complicated and by the way did *not* involve theta series."

I was wondering whether theta series could not be useful for the treatment of a function so strongly connected with quadratic forms. Approaching from a general theory of Dirichlet series attached to modular forms of higher degree, I got actually a proof for the analytic continuation and some of the functional equations by means of theta series. In order to eliminate disturbing terms of the theta series, one has to apply certain invariant differential operators. In this way one avoids the unpleasant computation of the residues. One gets rid of them. This method was originally conceived by

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A. Selberg already in 1960 but never published. He also succeeded in 1961 in obtain g the analytic continuation by means of theta functions, by using the differential operators. I got this information when I had finished the investigation which is the subject of this report. Complete proofs are given in the notes of my lectures on "Siegel's modular forms and Dirichlet series" delivered at the University of Maryland during the academic year 1969/70 (to appear).

We use the following notations:

$X = X^{(m,n)}$, a matrix of m rows and n columns; $X^{(n)} = X^{(n,n)}$;

U', the transpose of the matrix U; $Y[U] = U'YU$ if $Y' = Y$;

$\overset{n}{Y} = Y^{(n)}$ if $Y' = Y$; $|X|$, the determinant of $X = X^{(n)}$;

$\sigma(X)$, the trace of $X = X^{(n)}$;

$dY = (dy_{\mu\nu})$ if $Y = (y_{\mu\nu})$;

$\mathcal{R} = \mathcal{R}_n$, Minkowski's domain of reduced matrices $Y = Y^{(n)} > 0$;

$\Gamma_n = \{U \mid U = U^{(n)}$ integral, $|U| = \pm 1\}$;

$B_n = \{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \in \Gamma_n\}$;

$\Delta_n = \{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$, n-rowed, integral, non-singular$\}$;

$\mathbb{R}$: the field of real numbers;

$\mathbb{C}$: the field of complex numbers.

1. A sequence of positive matrices $\overset{n}{Y}, \overset{n-1}{Y}, \ldots, \overset{1}{Y}$, defines a descending chain if each link can be respresented by the preceding: $\overset{h}{Y} = \overset{h+1}{Y}[G_h]$ $(1 \le h < n)$ with an integral matrix G_h. The chain is called primitive if all matrices G_h are primitive. A second chain $\overset{n}{Y}{}^*, \overset{n-1}{Y}{}^*, \ldots, \overset{1}{Y}{}^*$ is said to be equivalent to the given chain if

$\overset{h}{Y}{}^* = \overset{h}{Y}[U_h]$, with $U_h \in \Gamma_h$ $(1 \le h \le n)$. Equivalence relations must be considered always as identities in $\overset{n}{Y}$. We introduce the class $\langle \overset{n}{Y},\dots,\overset{1}{Y} \rangle$ of all chains equivalent to $\overset{n}{Y},\dots,\overset{1}{Y}$ and use $\langle\ \rangle_p$ instead of $\langle\ \rangle$ if all chains of the class are primitive. Selberg's zeta functions are now defined by

$$(1)\quad \zeta(Y,s) = \sum_{\langle \overset{n}{Y},\dots,\overset{1}{Y} \rangle} \prod_{h=1}^{n-1} |\overset{h}{Y}|^{-z_h}, \qquad \zeta^*(Y,s) = \sum_{\langle \overset{n}{Y},\dots,\overset{1}{Y} \rangle_p} \prod_{h=1}^{n-1} |\overset{h}{Y}|^{-z_h}$$

where $Y = \overset{n}{Y}$, $s = (s_1,\dots,s_n)$ a system of n complex variables and $z_h = s_{h+1} - s_h + \frac{1}{2}$ $(1 \le h < n)$. They are related by

$$(2)\quad \begin{gathered} \zeta(Y,s) = Z(s)\zeta^*(Y,s), \\ Z(s) = \prod_{1\le\nu<\mu\le n} \zeta(2(s_\mu - s_\nu)+1), \quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s} . \end{gathered}$$

2. By means of the function

$$(3)\quad f_s(Y) = \prod_{\nu=1}^{n} t_{\nu\nu}^{2s_\nu + \nu - \frac{n+1}{2}},$$

where

$$Y = TT', \quad T = T^{(n)} = (t_{\mu\nu}), \quad t_{\mu\nu} = 0 \quad \text{for} \quad \mu < \nu, \quad t_{\nu\nu} > 0,$$

in the notation of [2], we represent $\zeta^*(Y,s)$ as Eisenstein series as follows

$$(4)\quad \zeta^*(Y,s) = |Y|^{-s_n - \frac{n-1}{4}} \sum_{U:\Gamma_n/B_n} f_s(Y[U]).$$

The convergence of these series for $\operatorname{Re} z_h > 1$ $(1 \le h < n)$ can be proved by a geometrical method. Moreover, one can show that $\zeta^*(Y,s)$

on $|Y| = 1$ does not grow faster to infinity than the "main term", so that in the domain of convergence

$$|\zeta^*(Y,s)| < C_1 |f_s(Y)| < C_2(\sigma(Y))^{\kappa} \quad (|Y| = 1)$$

holds with certain positive constants C_1, C_2, κ depending only on s. With $\tilde{s} = (s_n, s_{n-1}, \ldots, s_1)$, $\tilde{U} = (\delta_{n+1-\mu,\nu})$, $\delta_{\mu\nu}$ the Kronecker symbol, we have

$$f_s(Y^{-1}) = f_{-\tilde{s}}(Y[\tilde{U}]),$$

a relation which yields

$$\zeta^*(Y^{-1}, s) = |Y|^{s_n - s_1 + \frac{n-1}{2}} \zeta^*(Y, -\tilde{s})$$

and, because of $Z(-\tilde{s}) = Z(s)$, also

(5) $$\zeta(Y^{-1}, s) = |Y|^{s_n - s_1 + \frac{n-1}{2}} \zeta(Y, -\tilde{s}).$$

This transformation formula, in connection with a similar formula, obtained from an integral representation, will lead to one of the functional equations of $\zeta(Y,s)$. We note that the transformation $s \to -\tilde{s}$ leaves the domain of convergence of the zeta functions fixed.

3. The space $\mathcal{Y}$ of all positive matrices $Y = Y^{(n)} = (y_{\mu\nu})$ is a weakly symmetric Riemannian space in the sense of Selberg [2] with respect to the metric defined by $ds^2 = \sigma(Y^{-1}dY)^2$, the group G of isometries $Y \to Y[V]$ $(|V| \neq 0)$ and the involution $\mu(Y) = Y^{-1}$ which also is an isometry. Thus the ring L of the differential operators invariant under G, is commutative. Denote by $\hat{L}$ the image of $L \in L$ under $Y \to Y^{-1}$. If L is real, $\hat{L}$ is the adjoint operator of L with respect to the invariant volume element

$dv = |Y|^{-\frac{n+1}{2}} \prod_{\mu\le\nu} dy_{\mu\nu}$. With $\frac{\partial}{\partial Y} = (e_{\mu\nu}\frac{\partial}{\partial y_{\mu\nu}})$, $e_{\mu\nu} - \frac{1}{2}(1+\delta_{\mu\nu})$, we introduce the special operator $M = |Y|\,|\frac{\partial}{\partial Y}| \in L$ and define

(6)
$$P_k = |Y|^{-k}\hat{M}|Y|^k M.$$

Since L is commutative we get

(7)
$$\hat{P}_k = |Y|^k P_k |Y|^{-k}.$$

It is known that $f_s(Y)$ is an eigenfunction of L ; a computation yields in particular

(8)
$$Mf_s(Y) = \prod_{\nu=1}^{n} (s_\nu + \tfrac{n-1}{4}) f_s(Y).$$

We add here the important relation [2]

(9)
$$\int_{Y>0} e^{-\sigma(YX^{-1})} f_s(Y)dv = \pi^{\frac{n(n-1)}{4}} \prod_{\nu=1}^{n} \Gamma(s_\nu - \tfrac{n-1}{4}) f_s(X),$$

valid for $X > 0$, $Re s_\nu > \frac{n-1}{4}$ $(1 \le \nu \le n)$, and mention finally

(10)
$$|Y|^t f_s(Y) = f_{s+t}(Y) \quad \text{with} \quad (s+t)_\nu = s_\nu + t .$$

These are the main tools for getting the desired integral representation.

4. Our investigations aim at the following

Theorem (Selberg). *The function* $\prod_{1\le\nu<\mu\le n} (s_\mu - s_\nu - \frac{1}{2})\zeta(Y; s_1,\dots,s_n)$ *is holomorphic in* $\mathbb{C}^n$ *and*

$$\pi^{-2\sum_{\nu=1}^{n} \nu s_\nu} \prod_{1\le\nu<\mu\le n} \Gamma(s_\mu - s_\nu + \tfrac{1}{2}) |Y|^{s_n} \zeta(Y; s_1,\dots,s_n)$$

is invariant under the group of all cyclic permutations of the variables.

Proof: omit the well-known case $n = 2$ and assume $n \geq 3$. But then for plain technical reasons we replace in our theorem $Y^{(n)}$ by $S^{(n+1)}$ and $s = (s_1, \ldots, s_n)$ by $(s, s_{n+1}) = (s_1, \ldots, s_{n+1})$, now under the assumption $n \geq 2$, and introduce the theta series

$$(11) \qquad \theta(Y,S) = \sum_{G} e^{-\pi\sigma(S[G]Y)},$$

where G runs over all integral matrices of type $G^{(n+1,n)}$. It is known that

$$(12) \qquad \theta(Y^{-1},S^{-1}) = |Y|^{\frac{n+1}{2}} |S|^{\frac{n}{2}} \theta(Y,S)$$

and, because of (7),

$$(13) \qquad \Theta(Y,S) = P_{\frac{n+1}{2}}\theta(Y,S)$$

obviously satisfies the same transformation formula

$$(14) \qquad \Theta(Y^{-1},S^{-1}) = |Y|^{\frac{n+1}{2}} |S|^{\frac{n}{2}} \Theta(Y,S).$$

Since

$$(15) \qquad |\tfrac{\partial}{\partial Y}| e^{-\pi\sigma(S[G]Y)} = |-\pi S[G]| e^{-\pi\sigma(S[G]Y)}$$

vanishes if rank $G < n$, it suffices to sum in

$$(16) \qquad \Theta(Y,S) = \sum_{G} P_{\frac{n+1}{2}} e^{-\pi\sigma(S[G]Y)}$$

over all integral G of rank n. But then the integral

$$(17) \qquad \xi(S;s,s_{n+1}) = \int_{\mathcal{R}} \Theta(Y,S)|Y|^{s_{n+1} - s_1 + \frac{n}{2}} \zeta(Y,-\tilde{s})dv$$

exists, provided that the real part of the complex variable s_{n+1} is sufficiently large. With the help of (2), (10), (15) and $Z(-\tilde{s}) = Z(s)$ we rewrite (17) as follows

$$\xi(S;s,s_{n+1}) = (-\pi)^n Z(s) \sum_{U:\Gamma_n/B_n} \sum_{G} |S[G]| \cdot$$

$$\cdot \int_{\mathcal{R}} f_{-\tilde{s}+s_{n+1}-\frac{n+1}{4}}(Y[U])\hat{M}|Y|^{\frac{n+3}{2}} e^{-\pi\sigma(S[G]Y)}dv.$$

Denote by A the set of all integral G of rank n and replace in (18) the matrix G by $GW'U'$, where the new G runs over a complete set of representatives of the cosets A/B_n' and W over all matrices in B_n, while U' is fixed. Observing the invariance of $f_s(Y)$ under $Y \to Y[W]$ $(W \in B_n)$ we get by means of (8), (9)

(19) $$\xi(S;s,s_{n+1}) = (-\pi)^n Z(s) \sum_{G:A/B_n'} |S[G]| \cdot$$

$$\cdot \sum_{U:\Gamma_n/B_n} \sum_{W \in B_n} \int_{\mathcal{R}} f_{-\tilde{s}+s_{n+1}-\frac{n+1}{4}}(Y[UW])\hat{M}|Y|^{\frac{n+3}{2}} e^{-\pi\sigma(S[G]Y[UW])}dv$$

$$= 2(-\pi)^n Z(s) \sum_{G:A/B_n'} |S[G]| \int_{Y>0} f_{-\tilde{s}+s_{n+1}-\frac{n+1}{4}}(Y)\hat{M}|Y|^{\frac{n+3}{2}} e^{-\pi\sigma(S[G]Y)}dv$$

$$= 2(-\pi)^n Z(s) \sum_{G:A/B_n'} |S[G]| \int_{Y>0} e^{-\pi\sigma(S[G]Y)}|Y|^{\frac{n+3}{2}} M f_{-\tilde{s}+s_{n+1}-\frac{n+1}{4}}(Y)dv$$

$$= 2(-\pi)^n \prod_{\nu=1}^{n} (s_{n+1}-s_\nu-\tfrac{1}{2})Z(s) \cdot$$

$$\cdot \sum_{G:A/B_n'} \int_{Y>0} e^{-\pi\sigma(S[G]Y)} f_{-\tilde{s}+s_{n+1}+1+\frac{n+1}{4}}(Y)dv$$

$$= 2\pi^{n+\frac{n(n-1)}{4}} \prod_{\nu=1}^{n} \{(s_\nu - s_{n+1}+\tfrac{1}{2})\Gamma(s_{n+1}-s_\nu+\tfrac{3}{2})\}Z(s) \cdot$$

$$\cdot \sum_{G:A/B_n'} |S[G]| f_{-\tilde{s}+s_{n+1}+1+\frac{n+1}{4}}(\tfrac{1}{\pi}(S[G])^{-1}) =$$

$$= \frac{2\phi(s,s_{n+1})}{\phi(s)} Z(s) \sum_{G:A/B_n} f_{s-s_{n+1}-\frac{n+1}{4}}(S[G]),$$

where in general

(20) $\phi(s) =$

$$\prod_{1\le\nu<\mu\le n} \{\pi^{-(s_\mu - s_\nu + \frac{1}{2})}(s_\mu - s_\nu + \tfrac{1}{2})(s_\nu - s_\mu + \tfrac{1}{2})\Gamma(s_\mu - s_\nu + \tfrac{1}{2})\} = \phi(-\tilde{s}).$$

Let $G = U\begin{pmatrix} D_n \\ 0 \end{pmatrix}$. Then we can replace the summation $G:A/B_n$ by $U:\Gamma_{n+1}/B_{n+1}$ and $D_n:\Delta_n/B_n$. A complete set of coset representatives D_n is for instance given by

$$D_n = (d_{\mu\nu}): \quad \begin{array}{ll} 0 \le d_{\mu\nu} < d_{\mu\mu} & \text{for } 1 \le \mu < \nu \le n, \\ d_{\mu\nu} = 0 & \text{for } 1 \le \nu < \mu \le n. \end{array}$$

This yields in a similar way as one gets the relation (2)

(21) $$\sum_{G:A/B_n} f_{s-s_{n+1}-\frac{n+1}{4}}(S[G]) = \zeta^*(S;s,s_{n+1}) \prod_{\nu=1}^{n} \zeta(2(s_{n+1}-s_\nu)+1)$$

and finally

(22) $$\xi(S;s,s_{n+1}) = \frac{2\phi(s,s_{n+1})}{\phi(s)} Z(s,s_{n+1})\zeta^*(S;s,s_{n+1})$$

$$= \frac{2\phi(s,s_{n+1})}{\phi(s)} \zeta(S;s,s_{n+1}).$$

Starting again from (17) we get now by means of (5) and (14) the integral representation

(23) $\xi(S;s,s_{n+1}) =$

$$\int_{\substack{\mathcal{R} \\ |Y|\ge 1}} \{\Theta(Y,S)|Y|^{s_{n+1}-s_1+\frac{n}{2}}\zeta(Y,-\tilde{s}) + |S|^{-\frac{n}{2}}\Theta(Y,S^{-1})|Y|^{s_n-s_{n+1}+\frac{n}{2}}\zeta(Y,s)\}dv,$$

/hich shows that

24) $$\eta(S;s,s_{n+1}) = 2\phi(s,s_{n+1})|S|^{s_{n+1}}\,\zeta(S;s,s_{n+1})$$

$$= \phi(s)|S|^{s_{n+1}}\,\xi(S;s,s_{n+1})$$

s holomorphic in the domain

25) $$\mathfrak{D} = \{(s,s_{n+1}) \mid Re(s_{\nu+1}-s_\nu+\tfrac{1}{2}) > 1 \quad (1 \le \nu < n), \quad s_{n+1} \text{ arbitrary}\}.$$

oreover (23) makes evident the functional equation

26) $$|S|^{\frac{n}{2}}\,\xi(S;s,s_{n+1}) = \xi(S^{-1};-\tilde{s},-s_{n+1}).$$

e express both sides of (26) by η using again (5), now for $n+1$ in-
tead of n,

$$s|^{\frac{n}{2}}\,\xi(S;s,s_{n+1}) = \frac{1}{\phi(s)}\,|S|^{\frac{n}{2}-s_{n+1}}\,\eta(S;s,s_{n+1}),$$

$$(S^{-1};-\tilde{s},-s_{n+1}) = \frac{2\phi(-\tilde{s},-s_{n+1})}{\phi(-\tilde{s})}\,\zeta(S^{-1};-\tilde{s},-s_{n+1}) =$$

$$\frac{2\phi(s_{n+1},s)}{\phi(s)}|S|^{-s_{n+1}+s_n+\frac{n}{2}}\,\zeta(S;s_{n+1},s) = \frac{1}{\phi(s)}|S|^{\frac{n}{2}-s_{n+1}}\,\eta(S;s_{n+1},s).$$

mparison yields

7) $$\eta(S;s,s_{n+1}) = \eta(S;s_{n+1},s).$$

e invariance of $\eta(S;s,s_{n+1})$ under the cyclic permutation
$s,s_{n+1} \to s_{n+1},s$ implies that $\eta(S;s,s_{n+1})$ defines a holomorphic
nction in the domain

$$\mathfrak{G} = \bigcup_{\nu=0}^{n} \pi^{\nu}(\mathfrak{D}).$$

$\mathfrak{Y}$ be the image of $\mathfrak{D}$ under the projection $(s,s_{n+1}) \to (Res,Res_{n+1})$
en

$$\mathcal{J} = \mathcal{T} + i\mathbb{R}^{n+1},$$

i.e., $\mathcal{J}$ is a so-called tube domain. The holomorphy envelope $\mathcal{J}^*$ of $\mathcal{J}$ is again a tube domain:

$$\mathcal{J}^* = \mathcal{T}^* + i\mathbb{R}^{n+1},$$

where $\mathcal{T}^*$ denotes the convex hull of $\mathcal{T}$ in the sense of euclidean geometry. This is a well-known fact. One can easily see that $\mathcal{T}$ contains a system of straight lines $l_1, l_2, \ldots, l_{n+1}$ such that l_ν is parallel to the ν-th axis of $\mathbb{R}^{n+1}$. But then we have already $\mathcal{T}^* = \mathbb{R}^{n+1}$ and it turns out that $\eta(S; s, s_{n+1})$ is holomorphic in $\mathcal{J}^* = \mathbb{C}^{n+1}$.

The statements of our theorem (in the changed notation) now can be obtained from the proved results without any difficulty.

References

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