SPHERICAL FUNCTIONS AND QUADRATIC FORMS

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INTRODUCTION. An analytical treatment of the problem of representation of quadratic forms $T[\mathfrak{x}]$ by a given positive form $S[\mathfrak{x}]$ seems to be possible in the following general shape: Let $S = S^{(m)}$ and $T = T^{(n)}$ with $m > n$ be positive real matrices of m and n rows respectively. In the set of all real matrices $X = X^{(m,n)}$, having m rows and n columns, we denote by $\mathfrak{B}$ a domain of homogeneity, i.e. a subset which contains with X also XV, $V = V^{(n)}$ being an arbitrary non-singular real matrix of n rows. Further let $\mathfrak{C}$ be a subset of the set of all reduced positive real matrices $Y = Y^{(n)}$ in the sense of Minkowski, such that with Y, $\mathfrak{C}$ also contains λY, λ being an arbitrary positive real number. Then the number $a_t(\mathfrak{B}, \mathfrak{C})$ of all integral matrices $G = G^{(m,n)}$ which yield a representation

$$S[G] = G' S G = T, \tag{1}$$

with $G \in \mathfrak{B}$, $T \in \mathfrak{C}$, $|T| = t$ or at least the mean value

$$A_t(\mathfrak{B}, \mathfrak{C}) = \frac{1}{t} \sum_{\tau \leq t} a_\tau (\mathfrak{B}, \mathfrak{C}) \tag{2}$$

allows an asymptotic computation provided that $\mathfrak{B}$ and $\mathfrak{C}$ are measurable in a certain sense.

A method which is fitted for an analogous problem in algebraic number fields was developed by E. Hecke [2]. This method will probably work also in our case. It is based on the approximation of

$$\phi(s\,; \mathfrak{B}, \mathfrak{C}) = \sum_{t=1}^{\infty} a_t (\mathfrak{B}, \mathfrak{C})\, t^{-s} \tag{3}$$

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by a finite or infinite linear combination of certain zeta functions, i.e. by functions having a well-known behaviour on the strength of a Dirichlet series development and a functional equation of Riemannian type. We introduce $S = Q'Q$, $Q' = Q > 0$,

$$f(X) = \begin{cases} 1, & \text{for } Q^{-1}X \in \mathfrak{B} \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

and

$$g(Y) = \begin{cases} 1, & \text{for } Y \in \mathfrak{C}, \\ 0, & \text{for } Y \notin \mathfrak{C},\ Y \text{ reduced}; \end{cases} \tag{5}$$

$$g(Y[U]) = g(Y) \text{ for unimodular } U.$$

Then we have obviously

$$\phi(s; \mathfrak{B}, \mathfrak{C}) = \sum_{G} f(QG)\, g(S[G])\, |S[G]|^{-s}, \tag{6}$$

the summation taken over a complete set of integral matrices $G = G^{(m,n)}$ of rank n, such that each two do not differ by a unimodular right factor. The approximation of $\phi(s; \mathfrak{B}, \mathfrak{C})$ amounts to one of the functions $f(X)$ and $g(Y)$. Here we have to make use of the angular characters of quadratic forms [5] in so far as it concerns the function $g(Y)$. The theory of these angular characters is at the present sufficiently developed [6] only in the case $n = 2$ so that number-theoretical investigations of the desired kind are possible. Provided that $\mathfrak{B}$ is the full space of all real matrices $X = X^{(m,n)}$ of rank n, an asymptotic computation of $A_t(\mathfrak{B}, \mathfrak{C})$ with the method I have in mind could be carried out indeed in the case $n = 2$ [7]. For the approximation of $f(X)$ we need in the case $n = 1$ the spherical harmonics of m variables [1]. Apparently nobody has so far observed the significance of the spherical harmonics for this number theoretical problem.

The aim of this paper is to introduce a generalized class of spherical functions which are useful for the approximation of $f(X)$ for arbitrary n. One obtains a reasonable theory if one replaces the special but discontinuous functions $f(X)$ by the class of all functions $g(X)$ continuous in $X'X > 0$ which satisfy, just as $f(X)$, the relation of homogeneity

$$g(XV) = g(X) \text{ for non-singular } V = V^{(n)}. \tag{7}$$

Then we can ask for the uniform approximation of these functions by elementary functions with a certain typical behaviour. Applying Weierstrass's well-known approximation theorem for continuous functions to $g(X)$, and using a certain positive hermitian metric in the space of the functions $g(X)$, we obtain by a straight-forward conclusion the following result: Let $X = (x_{\mu\nu})$, $\frac{\partial}{\partial X} = \left(\frac{\partial}{\partial x_{\mu\nu}}\right)$, $\Lambda = X\frac{\partial}{\partial X'} - \left(X\frac{\partial}{\partial X'}\right)'$ and denote by $\sigma(W)$ the trace of the square matrix W. Then we can find a finite set of polynomials $u_{ij}(X)$ with the properties

$$\left.\begin{array}{l} 1.\ u_{ij}(XV) = |V|^{2i}\, u_{ij}(X) \quad \text{for non-singular } V = V^{(n)}, \\ 2.\ \sigma(\Lambda^{2h})\, u_{ij}(X) = \lambda_{ij}^{(h)}\, u_{ij}(X) \text{ with constant eigenvalues} \\ \quad \lambda_{ij}^{(h)} \text{ for } h = 1, 2, \ldots, n, \\ 3.\ \left|\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right| u_{ij}(X) = 0, \end{array}\right\} \tag{8}$$

such that

$$\left|g(X) - \sum_{i,j} |X'X|^{-i}\, u_{ij}(X)\right| < \epsilon \tag{9}$$

for all X of rank n, where ϵ denotes a given positive real number.

All functions of X we are taking into consideration depend only upon the equivalence class $\widetilde{X}$ of X which consists of all matrices XV with arbitrary real $V = V^{(n)}$ of determinant $|V| = 1$. Thus it is obvious to introduce the Plücker coordinates

$$\xi_\alpha = \xi_{\alpha_1\alpha_2\ldots\alpha_n} = |x_{\alpha_\mu\nu}|, \quad (\mu, \nu = 1, 2, \ldots, n) \tag{10}$$

of $\widetilde{X}$. For brevity we shall call these coordinates also Plücker co-ordinates of X. We denote by ξ the set of all ξ_α's. The first of the characteristic properties (8) says that $u_{ij}(x)$ is representable as an algebraic form in ξ of degree $2i$:

$$u_{ij}(X) = v_{ij}(\xi).$$

In particular we have

$$|X'X| = \sum_{\alpha} \xi_{\alpha}^2, \tag{11}$$

where the sum must be extended over all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Thus $|X'X|^{-1}\, u_{ij}(X)$ defines a function on the sphere

$$\sum_{\alpha} \xi_{\alpha}^2 = 1,$$

and it seems to be justified to speak of $u_{ij}(x)$ as a generalized spherical function. However we have to observe that the Plücker coordinates are not independent so that $u_{ij}(X)$ *de facto* is only defined on the Grassmannian manifold represented by the ξ_{α}'s.

The set of all differential operators $\sigma(\Lambda^{2h})$, $(h = 1, 2, \ldots, n)$ completed by $\sigma(X'\partial/\partial X)$ has a remarkable basic property which can be described in the following way. We define a linear differential operator Ω as a polynomial in the elements of $\partial/\partial X$ with functions of X as coefficients which have derivatives of arbitrary high order. We call Ω simply 'invariant' if Ω is invariant relative to the group of substitutions $X \to UXV$ where $U = U^{(m)}$ is an arbitrary orthogonal matrix and $V = V^{(n)}$ an arbitrary non-singular one. Two invariant linear operators are said to be equal if they are of the same effect on all functions $f(X)$ which are invariant relative to $X \to XV$, $|V| = 1$. The invariant linear differential operators form obviously a ring $\mathfrak{R}$. We shall prove that $\mathfrak{R}$ is generated by the operators $\sigma\left(X' \dfrac{\partial}{\partial X}\right)$, $\sigma(\Lambda^{2h})$, $(h = 1, 2, \ldots, n)$. Thus the first two of the conditions (8) say that $u_{ij}(X)$ is a polynomial in the Plücker coordinates and also an eigenfunction of the ring $\mathfrak{R}$.

Our further interest is now concentrated on the series

$$\phi(s, S; u, v) = \sum_G u(QG)\, v(S[G])\, |S[G]|^{-s}, \tag{12}$$

$u(X)$ being an arbitrary spherical function of degree $2kn$ and $v(Y)$ an arbitrary angular character. The sum must be extended over the same set of matrices G as in (6). Now the question arises whether the functions (12) are zeta functions in the described sense, i.e. whether these functions satisfy a functional equation which expresses a simple transformation property relative to the substitution $s \to k' - s$ with a suitable $k' > 0$. Let $v(Y)$ run over all angular characters, then we obtain in $\phi(s, S; u, v)$ a set of functions which is supposed to be linear equivalent with the single series

$$\vartheta(Y, S; u) = \sum_G u(Q\, G)\, e^{-\pi\sigma(YS[G])},\ (Y = Y^{(n)} > 0), \tag{13}$$

(see [5]). At the present, this fact is provable only for $n = 2$. W. Roelcke investigated this case by using Mellin's integral-transformation [6]. In (13) G runs over all integral matrices of the type $G^{(m,n)}$.

We can probably expect that $\vartheta(Y, S; u)$ has a simple transformation property relative to the substitution $Y \to Y^{-1}$ if the functions $\phi(s, S; u, v)$ satisfy a functional equation of Riemannian type at all. In this respect we meet the following situation. Applying Poisson summation method to the theta-series

$$\vartheta(X, Y, S; u) = \sum_G u(Q(G + X))\, e^{-\pi\sigma(YS[G+X])},$$

we obtain with regard to $u(XV) = |V|^{2k} u(X)$ for

$$\vartheta(Y, S; u) = \vartheta(0, Y, S; u),$$

the representation

$$\vartheta(Y, S; u) = |S|^{-n/2}\, |Y|^{-m/2-k} \sum_G u^*(-i\, Q^{-1} G\, R^{-1})\, e^{-\pi\sigma(Y^{-1}S^{-1}[G])} \tag{14}$$

with $Y = R'R$, $R = R' > 0$ and

$$u^*(X) = \int_{\mathfrak{X}} u(X + T)\, e^{-\pi\sigma(T'T)}\, [dT], \tag{15}$$

where $\mathfrak{T}$ denotes the full space of all real matrices $T = T^{(m,n)} = (t_{\mu\nu})$ and $[dT]$ the product of all differentials $dt_{\mu\nu}$ [1]. According to

$$\int_{\mathfrak{T}} e^{-\pi\sigma(T'T)}\,[d\,T] = 1$$

one can state that $u^*(X) - u(X)$ is a polynomial in the elements of X with a degree less than that of $u(X)$ provided $u(X) \neq 0$. In general however it is

$$u^*(X) - u(X) \neq 0,$$

as examples show, and even $u^*(X)$ no algebraic form in the Plücker coordinates of X. Therefore it is also impossible to split off the factor R^{-1} in $u^*(-i\,Q^{-1}\,G\,R^{-1})$. If we assume however

$$\sigma\left(\frac{\partial}{\partial X'}\,\frac{\partial}{\partial X}\right) u(X) = 0, \tag{16}$$

it follows, as it was proved recently also by C. S. Herz [4],

$$u^*(X) = u(X),$$

i.e. $u(X)$ is an eigenfunction of Gauss integral-transformation (15). Moreover it can be shown that (16) is not only sufficient but also necessary for $u(X)$ being an eigenfunction of this kind. Assuming (16) we now obtain

$$u^*(-i\,Q^{-1}\,G\,R^{-1}) = (-1)^{kn}\,|Y|^{-k}\,u(Q^{-1}\,G),$$

and thus we see that (14) can be rewritten in the form

$$\vartheta(Y, S; u) = (-1)^{kn}\,|S|^{-n/2}\,|Y|^{-m/2-2k}\,\vartheta(Y^{-1}, S^{-1}; u). \tag{17}$$

The relation of homogeneity $u(X\,V) = |V|^{2k}\,u(X)$ effects a decomposition of the differential equation (16) into the system

$$\frac{\partial}{\partial X'}\,\frac{\partial}{\partial X}\,u(X) = 0. \tag{18}$$

A consequence of this is $\sigma(\Lambda^{2h})\,u(X) = \lambda^{(h)}\,u(X)$ for $h = 1, 2, \ldots, n$ with certain constant eigenvalues $\lambda^{(h)}$. In the case $n = 1$ our supposition (16) does not go beyond (8). Thus the general transformation formula (17) corresponds with the results of

Schoeneberg [8]. Maybe it is sufficient also in the case $n > 1$ to take into consideration only those spherical functions $u(x)$ which are solutions of the Laplacian differential equation (16) in order to approximate the functions defined by (4). Solutions of (18) which also satisfy the relation of homogeneity are given by

$$u(x) = |L'X|^{2k}, \tag{19}$$

$L = L^{(m,n)}$ being an arbitrary complex solution of $L'L = 0$, [3].

Now we assume the generalized spherical function $u(X)$ to be a non-constant eigenfunction of Gauss integral-transformation. Further let $v(Y)$ be a bounded angular character, i.e. we have

$$\left.\begin{aligned} \left(\sigma\left(Y\frac{\partial}{\partial Y}\right)^h + \lambda_h\right) v(Y) = 0, \text{ for } h = 1, 2, \ldots, n, \\ v(Y[U]) = v(Y) \text{ for unimodular } U, \end{aligned}\right\} \tag{20}$$

with the notation

$$Y = (y_{\mu\nu}), \frac{\partial}{\partial Y} = \left(e_{\mu\nu}\frac{\partial}{\partial y_{\mu\nu}}\right), \; e_{\mu\nu} = \begin{cases} 1, \text{ for } \mu = \nu, \\ \tfrac{1}{2}, \text{ for } \mu \neq \nu. \end{cases} \tag{21}$$

$\lambda_1, \lambda_2, \ldots, \lambda_\mu$ are constant eigenvalues; in particular we have $\lambda_1 = 0$. It is easy to show that $v^*(Y) = v(Y^{-1})$ also defines an angular character which in general however belongs to another system of eigenvalues λ_h. Since now rank $X < n$ implies $u(X) = 0$, it is sufficient to extend the sum in

$$\vartheta(Y, S; u) = \sum_G u(Q\,G)\, e^{-\pi\sigma(YS[G])}, \tag{22}$$

over all integral matrices G of rank n so that always $S[G] > 0$. This is important because at present we can prove a functional equation for $\phi(s, S; u, v)$ only if the theta-series shows the behaviour of a so-called cusp form. By means obtained in [5] we shall prove that the function defined by the Dirichlet series (12) is an entire function of s which satisfies the functional equation

$$\xi(\tfrac{1}{2}m + 2k - s, S; u, v) = (-1)^{kn} |S|^{-n/2} \xi(s, S^{-1}; u, v^*), \tag{23}$$

where

$$\xi(s, S; u, v) = \pi^{-ns}\Gamma(s-\beta_1)\Gamma(s-\beta_2)\ldots\Gamma(s-\beta_n)\,\phi(s, S; u, v), \quad (24)$$

with certain constants $\beta_1, \beta_2, \ldots, \beta_n$ which depend upon the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ only. This is the main result of the present paper.

1. Plücker coordinates. We denote by $\delta_{\mu\nu}$ the Kronecker symbol and introduce

$$I_\alpha = I(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\delta_{\alpha_\nu \mu}), \quad (\mu = 1, 2, \ldots, m; \nu = 1, 2, \ldots, n), \quad (25)$$

$\alpha_1, \alpha_2, \ldots, \alpha_n$ being an arbitrary system of integers in the interval from 1 to m, so that

$$I'_\alpha X = (x_{\alpha_\mu \nu}), \quad (\mu, \nu = 1, 2, \ldots, n).$$

The Plücker coordinates ξ_α of X are given by

$$\xi_\alpha = \xi_{\alpha_1\alpha_2\ldots\alpha_n} = |I'_\alpha X|. \quad (26)$$

Any summation over α is to extend always over the full system of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_n \leqslant m$. If λ_α are the Plücker coordinates of $L = L^{(m,n)}$ then we have as is well known

$$|L' X| = \sum_\alpha \lambda_\alpha \xi_\alpha, \text{ particularly } |X' X| = \sum_\alpha \xi_\alpha^2.$$

We compute the effect of some differential operators on functions of the type $f(\xi) = f(\ldots, \xi_\alpha, \ldots)$. First we state

$$\frac{\partial}{\partial X}\xi_\alpha = I_\alpha (X' I_\alpha)^{-1}\xi_\alpha. \quad (27)$$

Denoting by $A^\alpha_{\mu\nu}$ the algebraic complement of $x_{\alpha_\mu \nu}$ in $|I'_\alpha X|$ we obtain indeed

$$\frac{\partial}{\partial X}\xi_\alpha = \left(\frac{\partial}{\partial x_{\mu\nu}}\xi_\alpha\right) = \left(\sum_\sigma \delta_{\alpha_\sigma \mu} A^\alpha_{\sigma\nu}\right) = (\delta_{\alpha_\nu \mu})(A^\alpha_{\mu\nu}) = I_\alpha (X' I_\alpha)^{-1}\xi_\alpha.$$

(27) yields in particular

$$X' \frac{\partial}{\partial X}\xi_\alpha = \xi_\alpha E, \; E = \text{unit matrix}. \quad (28)$$

Consequently

$$X' \frac{\partial}{\partial X} f(\xi) = \sum_{\alpha} \frac{\partial f(\xi)}{\partial \xi_\alpha} X' \frac{\partial}{\partial X} \xi_\alpha = \sum_{\alpha} \xi_\alpha \frac{\partial}{\partial \xi_\alpha} f(\xi) E. \tag{29}$$

Let $L = L^{(m,n)} = (l_{\mu\nu})$ be a constant matrix of rank n and $(\zeta_{\mu\nu}) = (X'L)^{-1}$, then it is

$$\sum_{\rho,\sigma} \zeta_{\mu\rho} x_{\sigma\rho} l_{\sigma\nu} = \delta_{\mu\nu}.$$

Differentiation yields

$$\sum_{\rho,\sigma} \frac{\partial \zeta_{\mu\rho}}{\partial x_{\kappa\lambda}} x_{\sigma\rho} l_{\sigma\nu} + \zeta_{\mu\lambda} l_{\kappa\nu} = 0,$$

from which

$$\frac{\partial \zeta_{\sigma\nu}}{\partial x_{\rho\mu}} = - \sum_{\kappa} \zeta_{\sigma\mu} l_{\rho\kappa} \xi_{\kappa\nu}$$

follows. Thus we find

$$\frac{\partial}{\partial X'} L(X'L)^{-1} = \Big(\sum_{\rho,\sigma} \frac{\partial}{\partial x_{\rho\mu}} l_{\rho\sigma} \zeta_{\sigma\nu} \Big) = - \Big(\sum_{\rho,\sigma} l_{\rho\sigma} \zeta_{\sigma\mu} l_{\rho\kappa} \zeta_{\kappa\nu} \Big)$$
$$= - (\zeta_{\mu\nu})' L' L(\zeta_{\mu\nu}),$$

particularly for $L = I_\alpha$

$$\frac{\partial}{\partial X'} I_\alpha (X' I_\alpha)^{-1} = - (I'_\alpha X)^{-1} (X' I_\alpha)^{-1}, \tag{30}$$

according to $I'_\alpha I_\alpha = E$. Applying the operator $\partial/\partial X'$ to

$$\frac{\partial}{\partial X} f(\xi) = \sum_{\alpha} \frac{\partial f(\xi)}{\partial \xi_\alpha} I_\alpha (X' I_\alpha)^{-1} \xi_\alpha,$$

we obtain

$$\frac{\partial}{\partial X'} \frac{\partial}{\partial X} f(\xi) = \sum_{\alpha,\beta} \xi_\beta \frac{\partial}{\partial \xi_\beta} \Big(\xi_\alpha \frac{\partial f(\xi)}{\partial \xi_\alpha} \Big) (I'_\beta X)^{-1} I'_\beta I_\alpha (X' I_\alpha)^{-1} -$$
$$- \sum_{\alpha} \xi_\alpha \frac{\partial f(\xi)}{\partial \xi_\alpha} (I'_\alpha X)^{-1} (X' I_\alpha)^{-1}$$
$$= \sum_{\alpha,\beta} (I'_\beta X)^{-1} I'_\beta I_\alpha (X' I_\alpha)^{-1} \xi_\beta \xi_\alpha \frac{\partial^2 f(\xi)}{\partial \xi_\beta \partial \xi_\alpha}.$$

Therefore

$$X\left(X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)' f(\xi) = \sum_{\alpha,\beta} \Xi_{\alpha\beta}\frac{\partial}{\partial \xi_\alpha}\frac{\partial}{\partial \xi_\beta} f(\xi), \tag{31}$$

with

$$\Xi_{\alpha\beta} = \xi_\alpha\xi_\beta X(I'_\alpha X)^{-1} I'_\alpha I_\beta (X' I_\beta)^{-1} X' = \Xi'_{\beta\alpha} \tag{32}$$

holds. The operators

$$\frac{\partial}{\partial X'}\frac{\partial}{\partial X} \text{ and } X\left(X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)'$$

annihilate the same functions. From

$$X\left(X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)' g(X) = X\left[\frac{\partial}{\partial X'}\frac{\partial}{\partial X} g(X)\right] X' = 0,$$

follows, by left and right-hand multiplication with X' and X respectively, since $X'X > 0$, indeed that

$$\frac{\partial}{\partial X'}\frac{\partial}{\partial X} g(X) = 0.$$

We compute the elements of the matrix

$$\Xi_{\alpha\beta} = (\xi^{\alpha\beta}_{\mu\nu}),\ (\mu, \nu = 1, 2, \ldots, m), \tag{33}$$

as functions of the Plücker coordinates. Since $I'_\alpha I_\beta = (\delta_{\alpha_\mu\beta_\nu})$ we obtain

$$\xi^{\alpha\beta}_{\mu\nu} = \sum_{\rho,\sigma,\tau,\kappa} x_{\mu\rho} A^\alpha_{\sigma\rho}\,\delta_{\alpha_\sigma\beta_\tau} A^\beta_{\tau\kappa}\, x_{\nu\kappa}, \tag{34}$$

with $A^\alpha_{\sigma\rho}$ in the significance already introduced. The replacement of the σth row in $|I'_\alpha X|$ by $(x_{\mu 1}, x_{\mu 2}, \ldots x_{\mu n})$ leads obviously to $\sum_\rho x_{\mu\rho} A^\alpha_{\sigma\rho}$. Thus we obtain

$$\sum_\rho x_{\mu\rho} A^\alpha_{\sigma\rho} = |I'(\alpha_1, \ldots, \alpha_{\sigma-1}, \mu, \alpha_{\sigma+1}, \ldots, \alpha_n) X|$$
$$= \xi_{\alpha_1\ldots\alpha_{\sigma-1}\mu\alpha_{\sigma+1}\ldots\alpha_n}.$$

Now we introduce the notation

$$\xi^{\mu\to\nu}_\alpha = \xi_{\alpha_1\ldots\alpha_{\sigma-1}\nu\alpha_{\sigma+1}\ldots\alpha_n} \text{ or } 0, \tag{35}$$

according as $\mu = \alpha_\sigma$ or $\mu \neq \alpha_1, \alpha_2, \ldots \alpha_n$. So we can rewrite (34) as

$$\xi^{\alpha\beta}_{\mu\nu} = \sum_{\sigma,\tau} \xi^{\alpha_\sigma \to \mu}_{\alpha} \, \delta_{\alpha_\sigma \beta_\tau} \, \xi^{\beta_\tau \to \nu}_{\beta} = \sum_{\rho} \xi^{\rho \to \mu}_{\alpha} \, \xi^{\rho \to \nu}_{\beta}. \tag{36}$$

According to the signification of the symbol $\xi^{\mu \to \nu}_{\alpha}$ the sum in (36) can be extended over all integers ρ from 1 to m. A special consequence of (32) is also

$$\sigma\Big(X' X \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \Big) f(\xi) = \sigma\Big(X \Big(X \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \Big)' \Big) f(\xi)$$

$$= \sum_{\alpha,\beta} \xi^{\alpha\beta} \frac{\partial}{\partial \xi_\alpha} \frac{\partial}{\partial \xi_\beta} f(\xi), \tag{37}$$

with

$$\xi^{\alpha\beta} = \sigma(\Xi_{\alpha\beta}) = \sum_{\rho,\mu} \xi^{\rho \to \mu}_{\alpha} \, \xi^{\rho \to \mu}_{\beta}. \tag{38}$$

The remaining formulae of this section apply to the special case $m = n + 1$. Now we note

$$\xi_\alpha = \eta_\kappa, \quad \xi^{\alpha\beta}_{\mu\nu} = \eta^{\kappa\lambda}_{\mu\nu}, \quad \xi^{\alpha\beta} = \eta^{\kappa\lambda}$$

for

$$\alpha = (1, \ldots, \kappa - 1, \kappa + 1, \ldots, m), \beta = (1, \ldots, \lambda - 1, \lambda + 1, \ldots, m).$$

It is easy to see that

$$\xi^{\rho \to \mu}_{1 \ldots \kappa - 1 \kappa + 1 \ldots m} = \delta_{\mu\kappa}(-1)^{\kappa + \rho + 1} \eta_\rho + \delta_{\mu\rho} \eta_\kappa.$$

Thus we obtain

$$\begin{aligned}\eta^{\kappa\lambda}_{\mu\nu} &= \sum_{\rho} \xi^{\rho \to \mu}_{1 \ldots \kappa - 1 \kappa + 1 \ldots m} \, \xi^{\rho \to \nu}_{1 \ldots \lambda - 1 \lambda + 1 \ldots m} \\ &= \sum_{\rho} \Big(\delta_{\mu\kappa}(-1)^{\kappa + \rho + 1} \eta_\rho + \delta_{\mu\rho} \eta_\kappa) (\delta_{\nu\lambda}(-1)^{\lambda + \rho + 1} \eta_\rho + \delta_{\nu\rho} \eta_\lambda \Big) \\ &= \delta_{\mu\kappa} \delta_{\nu\lambda}(-1)^{\kappa + \lambda} \sum_{\rho} \eta_\rho^2 + \delta_{\mu\nu} \eta_\kappa \eta_\lambda + \\ &\qquad + \delta_{\mu\kappa}(-1)^{\kappa + \nu + 1} \eta_\nu \eta_\lambda + \delta_{\nu\lambda}(-1)^{\lambda + \mu + 1} \eta_\mu \eta_\kappa\end{aligned}$$

and

$$\eta^{\kappa\lambda} = \sum_{\mu} \eta^{\kappa\lambda}_{\mu\mu} = \delta_{\kappa\lambda} \sum_{\rho} \eta_\rho^2 + (m - 2) \eta_\kappa \eta_\lambda.$$

Now we find for (37) the expression

$$\sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)f(\eta)$$
$$= \left\{\left(\sum_\rho \eta_\rho^2\right)\left(\sum_\kappa \frac{\partial^2}{\partial \eta_\kappa^2}\right) + (m-2)\left(\sum_\kappa \eta_\kappa \frac{\partial}{\partial \eta_\kappa}\right)^2 - \right.$$
$$\left. - (m-2)\left(\sum_\kappa \eta_\kappa \frac{\partial}{\partial \eta_\kappa}\right)\right\} f(\eta). \quad (39)$$

2. Invariant differential operators. Let Ω be a linear differential operator, i.e. a polynomial in the elements of X, and let us assume that the coefficients which are functions of X have derivatives of arbitrary high order. Ω as a polynomial in the elements of $\partial/\partial X$ has a certain degree; this we call plainly the degree of Ω. All linear operators Ω of degree $\leqslant h$ constitute a module which we denote by $\mathfrak{M}_h$. Obviously $\mathfrak{M}_h \subset \mathfrak{M}_{h+1}$ for all h. The module of all linear differential operators which is indentical with $\mathfrak{M} = \bigcup_h \mathfrak{M}_h$ defines a non-commutative ring. It is easy to see that

$$\Omega_1 \Omega_2 \equiv \Omega_2 \Omega_1 \ (\mathrm{mod}\ \mathfrak{M}_{h-1}), \quad (40)$$

provided that the product $\Omega_1 \Omega_2$ lies in $\mathfrak{M}_h$. The aim of the following considerations is the determination of a basis for the subring $\mathfrak{R}$ of $\mathfrak{M}$ consisting of all linear operators which are invariant relative to the substitution

$$X \to UXV, \frac{\partial}{\partial X} \to U\frac{\partial}{\partial X}V'^{-1} \text{ with } U'U = E, |V| \neq 0. \quad (41)$$

In the sequel we use the notation $\mathfrak{R}_h$ for the intersection $\mathfrak{R} \cap \mathfrak{M}_h$.

Let $\Omega = F(X, \partial/\partial X)$ be a given operator in $\mathfrak{R}_h$. We choose a matrix $T = T^{(m,n)}$ with variable elements which are commutable with those of X. Then we have in particular

$$F(UX, UT) = F(X, T) \text{ for } U'U = E.$$

Thus, according to well-known theorems of the theory of algebraic invariants, we see that $F(X, T)$ is a polynomial in the elements of $X'T$ and $T'T$ with functions of $X'X$ as coefficients. Then there exists also a representation

$$F(X, T) = \sum_{\nu} G_\nu(X'X,\, T'T)\, H_\nu(X'T),$$

G_ν being a polynomial in the elements of $T'T$ and H_ν a polynomial in the elements of $X'T$. Now we observe the invariance of $F(X, T)$ relative to the substitutions $X \to XV$, $T \to TV'^{-1}$. We set $V = V_0 V_1$ with V_0 determined by $(X'X)[V_0] = E$ and an arbitrary orthogonal matrix V_1. Using the notation $W = (T'T)\ [V_0'^{-1}]$ we obtain

$$F(X, T) = \sum_{\nu} G_\nu(E, W[V_1])\, H_\nu(V'_1(V_0^{-1}(V'_0 X'\, T)')' V_1).$$

The argument of H_ν is of course, since we are still moving in commutative domains, with $V'_1 V'_0 X'T V_0'^{-1} V_1 = V'X'T\,V'^{-1}$ identical. Because of (40) all products performed in $F\left(X, \dfrac{\partial}{\partial X}\right)$ admit commutations if we carry out the computations modulo $\mathfrak{M}_{h-1}$. So it turns out

$$\Omega \equiv \sum_{\nu} G_\nu\left(E, \left(\left(\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)[V_0'^{-1}]\right)[V_1]\right)\times$$
$$\times H_\nu\left(V_1'\left(V_0^{-1}\left(V_0'\, X'\, \frac{\partial}{\partial X'}\right)'\right)' V_1\right) \pmod{\mathfrak{M}_{h-1}}.$$

Applying (29) we see that

$$V_1'\left(V_0^{-1}\left(V_0'\, X'\, \frac{\partial}{\partial X}\right)'\right)' V_1 f(\xi) = \sum_{\alpha} \xi_\alpha \frac{\partial}{\partial \xi_\alpha} f(\xi)\ E$$
$$= \frac{1}{n}\,\sigma\left(X'\,\frac{\partial}{\partial X}\right) f(\xi)\ E$$

holds for an arbitrary function $f(\xi)$. Thus we obtain

$$H_\nu\left(V_1'\left(V_0^{-1}\left(V_0'\, X'\, \frac{\partial}{\partial X}\right)'\right)' V_1\right) f(\xi) = H_\nu\left(\frac{1}{n}\,\sigma\left(X'\,\frac{\partial}{\partial X}\right) E\right) f(\xi)$$
$$= h_\nu\left(\sigma\left(X'\,\frac{\partial}{\partial X}\right)\right) f(\xi),$$

where $h_\nu(z)$ denotes a polynomial of z. This leads to

$$\Omega f(\xi) \equiv \sum_{\nu} G_{\nu}\Big(E, \Big(\Big(\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\Big)[V_0'^{-1}]\Big)[V_1]\Big) \times$$

$$\times h_{\nu}\Big(\sigma\Big(X'\frac{\partial}{\partial X}\Big)\Big) f(\xi) \Big(\operatorname{mod} \mathfrak{M}_{h-1} f(\xi)\Big). \quad (42)$$

$G_{\nu}(E, W[V_1])$ represents a continuous function on the compact group of all orthogonal matrices V_1. Thus the mean value

$$M_{V_1}\{G_{\nu}(E, W[V_1])\} = g_{\nu}(W)$$

(in the sense of the theory of almost periodic functions) exists. It is a polynomial in the elements of W which is invariant relative to orthogonal substitutions :

$$g_{\nu}(W[V_1]) = g_{\nu}(W), \ (V_1' V_1 = E).$$

Accordingly $g_{\nu}(W)$ is a symmetric polynomial in the characteristic roots of W, thus a polynomial in $\sigma(W^h)$, $(h = 1, 2, \ldots, n)$:

$$g_{\nu}(W) = p_{\nu}(\sigma(W), \sigma(W^2), \ldots, \sigma(W^n)).$$

With regard to the signification of W and V_1 we state easily

$$\sigma(W^h) = \sigma(X'XT'T)^h.$$

If we compute the mean value with respect to V_1 on the right hand side of (42) we obtain by means of the deduced relations

$$\Omega f(\xi) \equiv \Omega^* f(\xi) \ (\operatorname{mod} \mathfrak{M}_{h-1} f(\xi)),$$

with

$$\Omega^* = \sum_{\nu} p_{\nu}\Big(\ldots, \sigma\Big(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\Big)^h, \ldots\Big) h_{\nu}\Big(\sigma\Big(X'\frac{\partial}{\partial X}\Big)\Big). \quad (43)$$

It is obvious that this operation is invariant relative to the substitutions (41).

In the sequel we shall identify invariant operations which have the same effect on all functions of the kind $f(\xi)$. Then we can state the following facts: To a given operator $\Omega \in \mathfrak{R}_h$ there exists an operator $\Omega^* \in \mathfrak{R}_h$ of the special form (43) such that $\Omega - \Omega^* \in \mathfrak{R}_{h-1}$. Induction on h yields at once

THEOREM 1. *The invariant operators* $\sigma\left(X'\frac{\partial}{\partial X}\right)$, $\sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)^h$ $(h=1,2,\ldots,n)$ *form a basis for the ring* $\mathfrak{R}$ *of all invariant linear operators.*

Our argument shows moreover that we only need the basis elements $\sigma\left(X'\frac{\partial}{\partial X}\right)$, $\sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)^k$ $(k=1,2,\ldots,[h/2])$ for the representation of an invariant operator of degree $h \leqslant 2n$. Now it is easy to see that the invariant operators

$$\sigma\left(X'\frac{\partial}{\partial X}\right),\ \sigma\left(X\frac{\partial}{\partial X'}\left(X\frac{\partial}{\partial X'}\right)'\right)^h,\ (h=1,2,\ldots,n) \qquad (44)$$

also generate $\mathfrak{R}$. We have

$$\sigma\left(X\frac{\partial}{\partial X'}\left(X\frac{\partial}{\partial X'}\right)'\right)^h \equiv \sigma\left(X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}X'\right)^h \equiv \sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)^h \pmod{\mathfrak{M}_{2h-1}}.$$

Thus by induction on h we obtain

$$\begin{aligned}&\sigma\left(X\frac{\partial}{\partial X'}\left(X\frac{\partial}{\partial X'}\right)'\right)^h\\ &=\sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)^h+\\ &\quad+q_h\left(\sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right),\ldots,\sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)^{h-1},\sigma\left(X'\frac{\partial}{\partial X}\right)\right)\end{aligned}$$

where q_h denotes a certain polynomial. This proves the basis property for the system (44). We determine yet a third basis for the ring $\mathfrak{R}$.

THEOREM 2. *If* $\Lambda = X\frac{\partial}{\partial X} - \left(X\frac{\partial}{\partial X'}\right)'$ *then the invariant operators* $\sigma\left(X'\frac{\partial}{\partial X}\right)$, $\sigma(\Lambda^{2h})$ $(h=1,2,\ldots,n)$ *form a basis for the ring* $\mathfrak{R}$ *of all invariant linear operators.*

In order to prove this we compute

$$\sigma(\Lambda^{2h}) = \sigma\Big(X\,\frac{\partial}{\partial X'} - \Big(X\,\frac{\partial}{\partial X'}\Big)'\Big)^{2h}$$

$$= (-1)^h\,\sigma\Big(X\,\frac{\partial}{\partial X'}\Big(X\,\frac{\partial}{\partial X'}\Big)'\Big)^h +$$

$$+ (-1)^h\,\sigma\Big(\Big(X\,\frac{\partial}{\partial X'}\Big)'\, X\,\frac{\partial}{\partial X'}\Big)^h + \sum_\nu \pm\,\sigma(P_\nu).$$

Here P_ν are products of the form

$$\Big(X\,\frac{\partial}{\partial X'}\Big)^{\mu_1}\Big(\Big(X\,\frac{\partial}{\partial X'}\Big)'\Big)^{\nu_1}\cdots\Big(X\,\frac{\partial}{\partial X'}\Big)^{\mu_r}\Big(\Big(X\,\frac{\partial}{\partial X'}\Big)'\Big)^{\nu_r},$$

and it happens at least once that one of the exponents μ_i, ν_i is greater than 1. Hence it is

$$P_\nu = Q_\nu\Big(X\,\frac{\partial}{\partial X'}\Big)^2 R_\nu \quad\text{or}\quad P_\nu = Q_\nu\Big(\Big(X\,\frac{\partial}{\partial X'}\Big)'\Big)^2 R_\nu,$$

with certain products Q_ν and R_ν which are also of the given form. Now it follows in the first case (the second one can be treated analogously)

$$\sigma(P_\nu) = \sigma(P_\nu')$$

$$\equiv \sigma\Big(R_\nu'\,\frac{\partial}{\partial X}\, X'\,\frac{\partial}{\partial X}\, X'\, Q_\nu'\Big)$$

$$\equiv \sigma\Big(X'\, Q_\nu'\, R_\nu'\,\frac{\partial}{\partial X}\cdot X'\,\frac{\partial}{\partial X}\Big) \quad (\mathrm{mod}\ \mathfrak{M}_{2h-1}),$$

and therefore by means of (29)

$$\sigma(P_\nu)\, f(\xi) \equiv \sigma\Big(X'\, Q_\nu'\, R_\nu'\,\frac{\partial}{\partial X}\Big)\frac{1}{n}\,\sigma\Big(X'\,\frac{\partial}{\partial X}\Big) f(\xi)$$

$$\equiv \sigma\Big(\frac{\partial}{\partial X'}\, R_\nu\, Q_\nu\, X\Big)\frac{1}{n}\,\sigma\Big(X'\,\frac{\partial}{\partial X}\Big) f(\xi)$$

$$\equiv \sigma\Big(X\,\frac{\partial}{\partial X'}\, R_\nu\, Q_\nu\Big)\frac{1}{n}\,\sigma\Big(X'\,\frac{\partial}{\partial X}\Big) f(\xi) \quad (\mathrm{mod}\ \mathfrak{M}_{2h-1}\, f(\xi)).$$

So we obtain

$$\sigma(\Lambda^{2h})\, f(\xi) \equiv (-1)^h\, 2\, \sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big)^h f(\xi) +$$

$$+ \Omega_h\, \sigma\Big(X' \frac{\partial}{\partial X} \Big) f(\xi) \quad (\mathrm{mod}\ \mathfrak{M}_{2h-1} f(\xi)),$$

with a certain invariant operator Ω_h of degree at most $2h-1$ which can be represented as a polynomial of the form

$$\Omega_h = p_h\Big(\sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big), \ldots,$$

$$\sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big)^{h-1},\ \sigma\Big(X' \frac{\partial}{\partial X} \Big) \Big).$$

The same is true for the operator

$$\sigma(\Lambda^{2h}) - (-1)^h\, 2\, \sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big)^h - \Omega_h\, \sigma\Big(X' \frac{\partial}{\partial X} \Big).$$

So we see that

$$\sigma(\Lambda^{2h}) = (-1)^h\, 2\sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big)^h +$$

$$+ q_h\Big(\sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big), \ldots,$$

$$\sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big)^{h-1},\ \sigma\Big(X' \frac{\partial}{\partial X} \Big) \Big)$$

holds with a certain polynomial q_h. Theorem 2 now is an easy consequence of this.

3. Spherical functions. A polynomial $u(X)$ shall be called a spherical function of type (m, n) if the following conditions are satisfied:

$$\left.\begin{array}{l} 1.\quad u(XV) = u(X) \text{ for } |V| = 1, \\ 2.\quad u(X) \text{ is an eigenfunction of all invariant linear} \\ \qquad \text{differential operators,} \\ 3.\quad \left| \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \right| u(X) = 0, \end{array}\right\} \quad (45)$$

The first condition says that $u(X)$ is a polynomial in the Plücker coordinates ξ_α of X:

$$u(X) = f(\xi).$$

The second condition is, according to Theorem 2, equivalent to

$$\sigma\left(X' \frac{\partial}{\partial X} \right) u(X) = n \sum_\alpha \xi_\alpha \frac{\partial}{\partial \xi_\alpha} f(\xi) = k\, n\, f(\xi) = k\, n\, u(X) \quad (46)$$

and

$$\sigma(\Lambda^{2h})\, u(X) = \lambda^{(h)}\, u(X)\ (h = 1, 2, \ldots, n), \quad (47)$$

k, $\lambda^{(1)}$, $\lambda^{(2)}, \ldots, \lambda^{(n)}$ being certain constants. Thus $f(\xi)$ is an algebraic form of degree k so k is a non-negative integer. In order to understand the third condition we observe that $|X'X| \left| \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \right|$ is an invariant linear operator. So we have also

$$|X'X| \left| \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \right| u(X) = \lambda\, u(X) \quad (48)$$

with a certain constant λ. The third condition is obviously equivalent to $\lambda = 0$.

Let us assume that the polynomial $u(X)$ satisfies only the first two but not the third of the conditions (45). Then it is obvious that $|X'X|$ is a divisor of $u(X)$. We prove the existence of an integer $j \geqslant 1$ such that $u_j(X) = |X'X|^{-j}\, u(X)$ is still a polynomial which satisfies also the third of the conditions (45); in other words $u_j(X)$ represents a spherical function. First of all we observe that the elements $y_{\kappa\lambda}$ of the matrix $Y = X'X$ can be considered as constants with respect to the operator Λ. It is indeed

$$\begin{aligned}
\Lambda\, y_{\kappa\lambda} &= \left(\sum \left(x_{\mu\rho} \frac{\partial}{\partial x_{\nu\rho}} - x_{\nu\rho} \frac{\partial}{\partial x_{\mu\rho}} \right) \right) \sum_\sigma x_{\sigma\kappa}\, x_{\sigma\lambda} \\
&= \left(\sum_{\rho,\sigma} x_{\mu\rho} (\delta_{\nu\sigma}\, \delta_{\rho\kappa}\, x_{\sigma\lambda} + x_{\sigma\kappa}\, \delta_{\nu\sigma}\, \delta_{\rho\lambda}) \right) - \\
&\quad - \left(\sum_{\rho,\sigma} x_{\nu\rho} (\delta_{\mu\sigma}\, \delta_{\rho\kappa}\, x_{\sigma\lambda} + x_{\sigma\kappa}\, \delta_{\mu\sigma}\, \delta_{\rho\lambda}) \right) \\
&= (x_{\mu\kappa}\, x_{\nu\lambda} + x_{\mu\lambda}\, x_{\nu\kappa}) - (x_{\nu\kappa}\, x_{\mu\lambda} + x_{\nu\lambda}\, x_{\mu\kappa}) = 0,
\end{aligned}$$

so that each function $\phi(Y)$, in particular $|X'X|^j$, is commutable with the operators $\sigma(\Lambda^{2h})$. Therefore (46) and (47) imply

$$\sigma\left(X' \frac{\partial}{\partial X}\right) u_j(X) = (k-2j)\, n\, u_j(X),$$

$$\sigma(\Lambda^{2h})\, u_j(X) = \lambda^{(h)}\, u_j(X) \quad (h = 1, 2, \ldots, n),$$

consequently also

$$|X'X| \left|\frac{\partial}{\partial X'} \frac{\partial}{\partial X}\right| u_j(X) = \lambda_j\, u_j(X) \quad (j = 1, 2, 3, \ldots)$$

with certain constants λ_j. Now we deduce: If $\lambda_1 \neq 0$ then $|X'X|$ divides $u_1(X)$, i.e. $u_2(X)$ is a polynomial. If $\lambda_2 \neq 0$ the same conclusion shows that $u_3(X)$ is a polynomial. So it turns out that an integer $j \geqslant 1$ exists such that $u_j(X)$ is a polynomial but $\lambda_j = 0$.

We assert that a polynomial $u(X)$ with the three properties

$$\left.\begin{aligned} &1.\quad u(XV) = |V|^k\, u(X) \quad \text{for } |V| \neq 0, \\ &2.\quad \sigma\left(\frac{\partial}{\partial X'} \frac{\partial}{\partial X}\right) u(X) = 0, \\ &3.\quad |X'X| \text{ is no divisor of } u(X), \end{aligned}\right\} \tag{49}$$

is already a spherical function. First we state that

$$\sigma\left(\frac{\partial}{\partial X'} \frac{\partial}{\partial X}\right) u(XV) = |V|^k\, \sigma\left(\frac{\partial}{\partial X'} \frac{\partial}{\partial X}\right) u(X) = 0$$

is true for arbitrary non-singular V. Replacing $X \to XV^{-1}$, we obtain

$$\sigma\left(V \frac{\partial}{\partial X'} \frac{\partial}{\partial X} V'\right) u(X) = 0.$$

This implies

$$\frac{\partial}{\partial X'} \frac{\partial}{\partial X}\, u(X) = 0. \tag{50}$$

According to Theorem 1 it turns out that $u(X)$ is an eigenfunction of all invariant linear operators. The third of the conditions (45) is a consequence of the fact that $|X'X|$ does not divide $u(X)$. This proves our assertion.

A special class of spherical functions is given by

$$u(X) = |L'X|^k \text{ with } L = L^{(m,n)},\ L'L = 0. \tag{51}$$

We assume that L is of rank n. Then at least one of the Plücker coordinates λ_α of L differs from 0. According to a well-known formula we have

$$u(X) = \Big(\sum_\alpha \lambda_\alpha\, \xi_\alpha\Big)^k,$$

an expression which is obviously not divisible by $|X'X| = \sum_\alpha \xi_\alpha^2$. So it is sufficient to prove that the algebraic forms (51) are solutions of (50). This can be done in the following way. We set

$$X = (x_{\mu\nu}) = (\mathfrak{x}_1\, \mathfrak{x}_2 \cdots \mathfrak{x}_n),\ L = (l_{\mu\nu}) = (\mathfrak{l}_1\, \mathfrak{l}_2 \cdots \mathfrak{l}_n)$$

and denote by $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_m$ the columns of the m-rowed unit matrix. Then we have

$$\frac{\partial}{\partial X}|L'X|^k = k\,|L'X|^{k-1} M$$

with

$$M = \frac{\partial}{\partial X}|L'X| = \Big(\frac{\partial}{\partial x_{\mu\nu}}|L'(\mathfrak{x}_1 \cdots \mathfrak{x}_n)|\Big)$$
$$= (|L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\mu\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)|).$$

For $\mu \neq \nu$ we have

$$\frac{\partial}{\partial x_{\rho\mu}}|L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\rho\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)|$$
$$= |L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\mu-1}\, \mathfrak{e}_\rho\, \mathfrak{x}_{\mu+1} \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\rho\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)| = 0,$$

since two columns of this determinant are equal. For $\mu = \nu$ we have also

$$\frac{\partial}{\partial x_{\rho\mu}}|L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\rho\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)| = 0$$

since the elements of this determinant do not depend upon $x_{\rho\nu}$ at all. So we obtain

$$\frac{\partial}{\partial X'} M = \left(\sum_\rho \frac{\partial}{\partial x_{\rho\mu}} |L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1} \mathfrak{e}_\rho \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)| \right) = 0,$$

and it follows that

$$\frac{\partial}{\partial X'} \frac{\partial}{\partial X} |L' X|^k = k \frac{\partial}{\partial X'} |L' X|^{k-1} M$$

$$= k(k-1) |L' X|^{k-2} M' M + k |L' X|^{k-1} \frac{\partial}{\partial X'} M$$

$$= k(k-1) |L' X|^{k-2} M' M.$$

Now it remains to show that $L' L = (\mathfrak{l}'_\mu \mathfrak{l}_\nu) = 0$ implies $M' M = 0$. First we form with a variable matrix $Z = Z^{(m,n)} = (z_{\mu\nu})$ the product

$$Z' M = \left(\sum_\rho z_{\rho\mu} |L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1} \mathfrak{e}_\rho \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)| \right)$$

$$= (|L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1} \mathfrak{z}_\mu \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)|).$$

Here $\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_n$ denote the columns of Z. We choose $Z = M$ and prove that in this case $\mathfrak{z}_\mu$ is a linear combination of the columns $\mathfrak{l}_1, \mathfrak{l}_2, \ldots, \mathfrak{l}_n$ of L. Then it turns out that $L' \mathfrak{z}_\mu = 0$ and finally $M' M = 0$. We introduce $L' \mathfrak{x}_\nu = \mathfrak{a}_\nu$ and denote by $\mathfrak{e}^*_1, \mathfrak{e}^*_2, \ldots, \mathfrak{e}^*_n$ the columns of the n-rowed unit matrix. Because of $L' \mathfrak{e}_\mu = \sum l_{\mu\rho} \mathfrak{e}^*_\rho$ we now obtain indeed

$$\mathfrak{Z}_\nu = (|L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1} \mathfrak{e}_\mu \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)|)$$

$$= \left(\left| \left(\mathfrak{a}_1 \cdots \mathfrak{a}_{\nu-1} \sum_\rho l_{\mu\rho} \mathfrak{e}^*_\rho \mathfrak{a}_{\nu+1} \cdots \mathfrak{a}_n \right) \right| \right)$$

$$= \left(\sum l_{\mu\rho} |(\mathfrak{a}_1 \cdots \mathfrak{a}_{\nu-1} \mathfrak{e}^*_\rho \mathfrak{a}_{\nu+1} \cdots \mathfrak{a}_n)| \right)$$

$$= \sum_\rho |(\mathfrak{a}_1 \cdots \mathfrak{a}_{\nu-1} \mathfrak{n}^*_\rho \mathfrak{a}_{\nu+1} \cdots \mathfrak{a}_n)| \, l_\rho.$$

Vice versa we shall prove that the conditions $M' M = 0$, which is an identity in X, and rank $L = n$ imply also $L' L = 0$. Based on the deduced formulae we have

$$M'M = \Big(\sum_{\rho} | L(\mathfrak{x}_1 \dots \mathfrak{x}_{\nu-1}\, \mathfrak{l}_\rho\, \mathfrak{x}_{\nu+1} \dots \mathfrak{x}_n) | \times$$
$$\times | (\mathfrak{a}_1 \dots \mathfrak{a}_{\mu-1}\, \mathfrak{e}^*_\rho\, \mathfrak{a}_{\mu+1} \dots \mathfrak{a}_n) | \Big)$$

Since rank $L = n$, a column $\mathfrak{E}_\kappa$ exists which solves $L'\mathfrak{E}_\kappa = \mathfrak{Q}_\kappa$, if $\mathfrak{Q}_\kappa$ is given. So we can choose in particular $\mathfrak{Q}_\kappa = \mathfrak{u}^*_\kappa$. The identical vanishing of $M'M$ now implies

$$| (\mathfrak{e}^*_1 \dots \mathfrak{e}^*_{\nu-1}\, L'\, \mathfrak{l}_\mu\, \mathfrak{e}^*_{\nu+1} \dots \mathfrak{e}^*_n) | = 0,$$

thus $\mathfrak{l}'_\nu\, \mathfrak{l}_\mu = 0$ or $L'L = 0$ as we asserted.

The above argument also shows that $|L'X|$ is a solution of the system

$$\frac{\partial}{\partial X'} \frac{\partial}{\partial X} | L'X |^k = 0, \tag{52}$$

if and only if either

$$k = 0, 1 \text{ or rank } L < n \text{ or } L'L = 0. \tag{53}$$

Without proof it may be mentioned that

$$\left| L' \frac{\partial}{\partial X} \right|^k e^{-\sigma(X'X)} = (-2)^{nk} | L'X |^k e^{-\sigma(X'X)} \text{ for } L'L = 0.$$

By means of (31), (33), (36) it can be proved in the particular case $m = 3$, $n = 2$ that the special function

$$u(X) = \xi_{23}\, \xi_{13} = \eta_1\, \eta_2 \tag{54}$$

satisfies the differential equations

$$\sigma\Big(X' \frac{\partial}{\partial X} \Big) u(X) = 4\, u(X),\ \sigma\Big(X \Big(X \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \Big)' \Big) u(X) = 2\, u(X),$$
$$\sigma\Big(X \Big(X \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \Big)' \Big)^2 u(X) = 8\, u(X),\ \left| \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \right| u(X) = 0. \tag{55}$$

Observing that in general the operators

$$\sigma\Big(X' \frac{\partial}{\partial X} \Big),\ \sigma\Big(X \Big(X \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \Big)' \Big)^h,\ (h = 1, 2, \dots, n)$$

generate the ring of all invariant linear operators, this can be proved easily with the given methods, we see that $u(X)$ is a spherical function of type (3, 2). However, we have

$$\sigma\left(\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right) u(X) = 2\,(x_{11}\,x_{21} + x_{12}\,x_{22}) \neq 0. \tag{56}$$

4. **The approximation theorem.** Let $F(X)$ be a complex-valued function defined and continuous in the domain $X'X > 0$. Let $M_U\{F(UX)\}$ denote the mean value of $F(UX)$ relative to the compact group of all orthogonal matrices U in the sense of the theory of almost periodic functions. This mean value is a function of X which is invariant relative to the substitutions $X \to UX$ $(U'U = E)$, therefore it depends only upon $X'X$. If $F(X)$ is invariant relative to the substitutions $X \to XV(|V| \neq 0)$ then the mean value is obviously independent of X and therefore is a constant.

For two complex-valued functions $\phi(X)$ and $\psi(X)$, defined and continuous in $X'X > 0$ with the transformation invariance

$$\phi(XV) = |V|^k\,\phi(X), \quad \psi(XV) = |V|^k\,\psi(X) \text{ for } |V| \neq 0, \tag{57}$$

we define a scalar product by

$$(\phi(X), \psi(X))_k = M_U\{\phi(UX)\,\overline{\psi(UX)}\,|X'X|^{-k}\}. \tag{58}$$

It has the property of the translation invariance:

$$(\phi(UX), \psi(UX))_k = (\phi(X), \psi(X))_k \text{ for } U'U = E \tag{59}$$

and determines a positive hermitian metric, i.e.

$$(\phi(X), \phi(X))_k = 0 \text{ implies } \phi(X) = 0.$$

This metric and Weierstrass's approximation theorem for continuous functions are the essential means for the proof of the following approximation theorem.

THEOREM 3. *Let $g(X)$ be a complex-valued function, defined and continuous in $X'X > 0$, which is invariant relative to the substitutions $X \to XV$ with $|V| \neq 0$. Then there exists a finite set of spherical functions $u_{ij}(X)$ of degree $2in$ such that*

$$\left| g(X) - \sum_{i,j} |X'X|^{-i} u_{ij}(X) \right| < \epsilon \tag{60}$$

holds in the whole domain $X'X > 0$ where ϵ denotes a given positive number.

According to the Weierstrass approximation theorem there exist algebraic forms $p_h(X)$ of degree $h (h = 1, 2, \ldots, 2k)$ such that

$$\left| g(X) - \sum_{h=0}^{2k} p_h(X) \right| < \epsilon \tag{61}$$

for all X of the compact domain $X'X = E$. We introduce the mean values

$$q_h(X) = M_V \{p_h(XV)\}$$

relative to the compact group of all orthogonal matrices V. Observing that the compact domain defined by $X'X = E$ is mapped onto itself by the substitutions $X \to XV (V'V = E)$ and besides also $g(XV) = g(X)$ is valid, we obtain from (61), by computing the mean values,

$$\left| g(X) - \sum_{h=0}^{2k} q_h(X) \right| < \epsilon \text{ for } X'X = E. \tag{62}$$

According to well-known theorems of the theory of algebraic invariants, the algebraic form $q_h(X)$, being invariant relative to the substitutions $X \to XV (V'V = E)$, is representable as an algebraic form in the elements of the matrix XX':

$$q_h(X) = q^*_h(XX').$$

This shows in particular that h is even if $q_h(X) \neq 0$.

Let X be an arbitrary matrix of rank n. Then we can determine a non-singular matrix $R = R^{(h)}$ such that

$$X'X = R'R.$$

Replacing X in (62) by XR^{-1} and observing that $g(XR^{-1}) = g(X)$ we obtain

$$\left| g(X) - \sum_{h=0}^{k} q_{2h}(XR^{-1}) \right| < \epsilon, \text{ for } X'X > 0. \tag{63}$$

It is obvious that

$$q_{2h}(XR^{-1}) = q^*_{2h}(XR^{-1}R'^{-1}X') = q^*_{2h}(X(X'X)^{-1}X')$$

is independent of the choice of R, and thus represents a one-valued function of X. It is easy to see that

$$u_h(X) = |X'X|^h q_{2h}(XR^{-1}) = |X'X|^h q^*_{2h}(X(X'X)^{-1}X')$$

is an algebraic form with the invariance property

$$u_h(XV) = |V|^{2h} u_h(X) \text{ for } |V| \neq 0. \tag{64}$$

In place of (63) we obtain now

$$\left| g(X) - \sum_{h=0}^{k} |X'X|^{-h} u_h(X) \right| < \epsilon, \text{ for } X'X > 0. \tag{65}$$

The following considerations apply to the linear space consisting of all algebraic forms of degree $2hn$ with the invariance property (64). With $u(X)$ also $u(UX)$ belongs to this space, U being an arbitrary orthogonal matrix. For an arbitrary subspace $\mathfrak{L}$ which also has these two properties we prove a lemma of which the approximation theorem is an easy consequence.

Lemma. *Let $\mathfrak{L}$ be a linear space of algebraic forms $v(X)$ which satisfy the transformation formula*

$$v(XV) = |V|^{2h} v(X) \textit{ for } |V| \neq 0. \tag{66}$$

Assume that with $v(X)$, $\mathfrak{L}$ also contains $v(UX)$, U being an arbitrary orthogonal matrix. Further let k be a given integer $\geqslant 0$. Then there exists a basis $v_1(X), v_2(X), \ldots, v_s(X)$ in $\mathfrak{L}$ such that

$$\sigma(\Lambda^{2j})\, v_i(X) = \lambda_{ij}\, v_i(X) \textit{ for } i = 1, 2, \ldots, s,\ j = 1, 2, \ldots, k, \tag{67}$$

with non-negative real eigenvalues λ_{ij}.

The proof will be based on induction on k. For $k = 0$ our assertion is only that $\mathfrak{L}$ has a finite dimension. This is trivial. So we can assume that the lemma is valid for a given value of k ($\geqslant 0$). Then we prove it for $k+1$ in place of k. First of all we distribute the basis functions $v_i(X)$ into classes $\mathfrak{K}_\nu$ $(\nu = 1, 2, \ldots, l)$ so that two functions $v_i(X)$ have the same system of eigenvalues $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik}$ if and only if they belong to the same class. Let us assume perhaps $v_\nu(X) \in \mathfrak{K}_\nu$ $(\nu = 1, 2, \ldots, l)$. $\mathfrak{L}_\nu$ denote the linear space generated by all $v_i(X) \in \mathfrak{K}_\nu$. Then we have

$$\mathfrak{L} = \mathfrak{L}_1 + \mathfrak{L}_2 + \ldots + \mathfrak{L}_l.$$

Let $v(X) \in \mathfrak{L}$ be an arbitrary eigenfunction of the operators $\sigma(\Lambda^{2j})$ $(j = 1, 2, \ldots, k)$:

$$\sigma(\Lambda^{2j})\, v(X) = \lambda_j\, v(X), \; (j = 1, 2, \ldots, k).$$

Then there exists a unique decomposition

$$v(X) = \sum_{\nu=1}^{l} w_\nu(X) \quad \text{with } w_\nu(X) \in \mathfrak{L}_\nu.$$

On account of

$$\sigma(\Lambda^{2j})\, w_\nu(X) = \lambda_{\nu j}\, w_\nu(X), \; (\nu = 1, 2, \ldots, l)$$

it follows that

$$\sigma(\Lambda^{2j})\, v(X) = \sum_{\nu=1}^{l} \lambda_{\nu j}\, w_\nu(X) = \sum_{\nu=1}^{l} \lambda_j\, w_\nu(X),$$

therefore

$$\lambda_{\nu j}\, w_\nu(X) = \lambda_j\, w_\nu(X)$$

for all ν and j. $w_\nu(X) \neq 0$ implies

$$\lambda_{\nu j} = \lambda_j \text{ for } j = 1, 2, \ldots, k.$$

This of course is impossible for two different $\nu \leqslant l$. Thus only one $w_\nu(X)$ differs from 0 and $v(X) \in \mathfrak{L}_\nu$ is proved. Because of the invariance properties of the operators $\sigma(\Lambda^{2j})$ it is obvious that with $v(X)$ also $v(UX)$ is an eigenfunction of the operators $\sigma(\Lambda^{2j})$ $(j = 1, 2, \ldots, k)$ if U denotes an orthogonal matrix. $v(X)$ and $v(UX)$ even belong to the same system of eigenvalues. This proves that, with $v(X)$, $\mathfrak{L}_\nu$ contains also $v(UX)$. In other words, each subspace $\mathfrak{L}_\nu$ itself has the characteristic properties of $\mathfrak{L}$. So it suffices to consider the subspaces $\mathfrak{L}_\nu$ individually. Without loss of generality we can identify such $\mathfrak{L}_\nu$ with $\mathfrak{L}$, i.e. we can assume

$$\lambda_{ij} = \lambda_j, \ (j = 1, 2, \dots k), \tag{68}$$

for all i.

For a given orthogonal matrix U the mapping $v(X) \to v(UX)$ defines a linear transformation of $\mathfrak{L}$ into itself. Thus we have

$$v_\mu(UX) = \sum_{\nu=1}^{s} D_{\mu\nu}(U)\, v_\nu(X), \ (\mu = 1, 2, \dots s),$$

with certain coefficients $D_{\mu\nu}(U)$. We assume that the $v_\nu(X)$'s form an orthogonal and normalized basis, i.e.

$$(v_\mu(X),\ v_\nu(X))_{2h} = \delta_{\mu\nu}.$$

Because of the translation invariance of the scalar product the $v_\nu(UX)$'s are also orthogonal and normalized. This proves that

$$D(U) = (D_{\mu\nu}(U))$$

is a unitary matrix and it turns out that the function

$$F(X, X^*) = \sum_{\nu=1}^{s} v_\nu(X)\, \overline{v_\nu(X^*)} \tag{69}$$

is invariant relative to the simultaneous substitutions

$$X \to UX, X^* \to UX^*, (U'U = E).$$

Here a general conclusion applied already by E. Hecke [8] again gets importance. On the strength of the invariance property of $F(X, X^*)$ we deduce now the differential equations for the basis functions $v_\nu(X)$. It is well known that

$$U = (E + S)\ (E - S)^{-1} \text{ with } S' + S = 0,$$

defines a parameter representation of the orthogonal matrices. Since $F(UX, UX^*)$ is independent from U and so from S, the partial derivatives of this function relative to the elements $s_{\rho\sigma}(\rho < \sigma)$ of the matrix S vanish necessarily, i.e. we have

$$\frac{\partial}{\partial s_{\rho\sigma}} F(UX,\ UX^*) = 0.$$

It suffices to discuss these conditions for $S = 0$. A development into powers of S yields

$$U = E + 2S + \text{higher powers of } S,$$

therefore

$$\begin{aligned}\frac{\partial}{\partial s_{\rho\sigma}} UX &= 2\frac{\partial}{\partial s_{\rho\sigma}} SX = 2\left(\sum_{\alpha} \frac{\partial}{\partial s_{\rho\sigma}} s_{\mu\alpha}\, x_{\alpha\nu}\right)\\ &= 2(\delta_{\rho\mu}\, x_{\sigma\nu} - \delta_{\sigma\mu}\, x_{\rho\nu}), \text{ for } S = 0.\end{aligned}$$

This leads in the case $S = 0$ to

$$\begin{aligned}\frac{\partial}{\partial s_{\rho\sigma}} F(UX,\, UX^*) = {} & 2\sum_{\mu,\nu} (\delta_{\rho\mu}\, x_{\sigma\nu} - \delta_{\sigma\mu}\, x_{\rho\nu})\, \frac{\partial F(X, X^*)}{\partial x_{\mu\nu}}\\ & + 2\sum_{\mu,\nu} (\delta_{\rho\mu}\, x^*_{\sigma\nu} - \delta_{\sigma\mu}\, x^*_{\rho\nu})\, \frac{\partial F(X, X^*)}{\partial x^*_{\mu\nu}}\\ = {} & 2\sum_{\nu} x_{\sigma\nu} \frac{\partial F(X,\ X^*)}{\partial x_{\rho\nu}} - 2\sum_{\nu} x_{\rho\nu} \frac{\partial F(X, X^*)}{\partial x_{\sigma\nu}}\\ & + 2\sum_{\nu} x^*_{\sigma\nu} \frac{\partial F(X,\ X^*)}{\partial x^*_{\rho\nu}} - 2\sum_{\nu} x^*_{\rho\nu} \frac{\partial F(X, X^*)}{\partial x^*_{\sigma\nu}}\\ = {} & 0,\end{aligned}$$

or, rewritten by means of the matrix calculus,

$$\Lambda\, F(X,\ X^*) + \Lambda^*\, F(X, X^*) = 0. \tag{70}$$

Here Λ^* denotes the operator which arises from Λ if we replace X by X^*. According to (69) we obtain

$$\sum_{\nu} \Lambda\, v_\nu(X)\, \overline{v_\nu(X^*)} + \sum_{\nu} v_\nu(X)\, \overline{\Lambda^*\, v_\nu(X^*)} = 0. \tag{71}$$

This shows, since the functions $v_\nu(X)$ are independent, that relations of the kind

$$\Lambda\, v_\nu(X) = \sum_{\mu} v_\mu(X)\, C_{\mu\nu}, \quad (\nu = 1, 2, \ldots, s), \tag{72}$$

with certain constant matrices $C_{\mu\nu}$ are valid. Λ and so all $C_{\mu\nu}$'s are skew-symmetric:

$$C'_{\mu\nu} = -C_{\mu\nu} \tag{73}$$

In (72) now we replace X by X^* and also turn over to conjugate complex expressions. By means of the relations, besides (72), which we obtain in this way, (71) can be rewritten as follows:

$$\sum_{\mu,\nu} \overline{v_\nu(X^*)}\, v_\mu(X)\, C_{\mu\nu} + \sum_{\mu,\nu} v_\nu(X)\, \overline{v_\mu(X^*)}\, \bar{C}_{\mu\nu} = 0.$$

This implies, according to the independence of the functions $v_\nu(X)$,

$$\bar{C}_{\mu\nu} = -C_{\nu\mu} = C'_{\nu\mu}. \tag{74}$$

Thus particularly the elements of $C_{\mu\mu}$ are pure imaginary numbers.

The repeated application of Λ to the basis functions $v_\nu(X)$ yields

$$\Lambda^j\, v_\nu(X) = \sum_\mu v_\mu(X)\, C^{(j)}_{\mu\nu}, \tag{75}$$

with certain constant matrices $C^{(j)}_{\mu\nu}$ which obviously satisfy the relations

$$C^{(i+j)}_{\mu\nu} = \sum_\rho C^{(i)}_{\mu\rho}\, C^{(j)}_{\rho\nu}, \quad C^{(1)}_{\mu\nu} = C_{\mu\nu}.$$

By means of induction on j it is easy to see that

$$\overline{C^{(j)}_{\mu\nu}} = C^{(j)\prime}_{\nu\mu}$$

holds for all $j \geqslant 1$. This shows

$$C^{(2j)}_{\mu\mu} = \sum_\rho C^{(j)}_{\mu\rho}\, C^{(j)}_{\rho\mu} = \sum_\rho C^{(j)}_{\mu\rho}\, \overline{C^{(j)}_{\mu\rho}}{}' \geqslant 0. \tag{76}$$

Taking the trace we obtain from (75) with $2j$ in place of j

$$\sigma(\Lambda^{2j})\, v_\nu(X) = \sum_\mu v_\mu(X)\, c^{(j)}_{\mu\nu}, \tag{77}$$

where

$$c^{(j)}_{\mu\nu} = \sigma(C^{(2j)}_{\mu\nu}) = \sigma(\overline{C^{(2j)}_{\nu\mu}}) = \overline{c^{(j)}_{\nu\mu}}.$$

This is to say $(c^{(j)}_{\mu\nu})$ is a hermitian matrix. So we can find a unitary matrix U which transforms $(c^{(k+1)}_{\mu\nu})$ into a diagonal matrix. We express this fact in the form

$$(c_{\mu\nu}^{(k+1)})\, U = U(\delta_{\mu\nu}\, \lambda_{k+1}^{(\nu)}). \tag{78}$$

The functions of the new basis

$$(w_1(X), w_2(X), \ldots, w_s(X)) = (v_1(X), v_2(X), \ldots, v_s(X))\, U$$

now turn out to be eigenfunctions of the operator $\sigma(\Lambda^{2k+2})$:

$$\sigma(\Lambda^{2k+2})\, w_\nu(X) = \lambda_{k+1}^{(\nu)}\, w_\nu(X),\ (v = 1, 2, \ldots, s). \tag{79}$$

The relations

$$\sigma(\Lambda^{2j})\, v_\nu(X) = \lambda_j\, v_\nu(X),\ (j = 1, 2, \ldots, k;\ \nu = 1, 2, \ldots, s)$$

of course can be carried over at once to the functions $w_\nu(X)$:

$$\sigma(\Lambda^{2j})\, w_\nu(X) = \lambda_j\, w_\nu(X),\ (j = 1, 2, \ldots, k;\ \nu = 1, 2, \ldots, s). \tag{80}$$

Finally we have to observe that for an arbitrary system of complex numbers $z_1, z_2, \ldots, z_s$ the hermitian form

$$\sum_{\mu,\nu} c_{\mu\nu}^{(j)}\, z_\mu\, \bar{z}_\nu = \sum_{\mu,\nu} \sigma(C_{\mu\nu}^{(2j)}\, z_\mu\, \bar{z}_\nu) = \sum_{\mu,\nu,\rho} \sigma(C_{\mu\rho}^{(j)}\, C_{\rho\nu}^{(j)}\, z_\mu\, \bar{z}_\nu)$$

$$= \sum_{\mu,\nu,\rho} \sigma(C_{\mu\rho}^{(j)}\, \overline{C_{\nu\rho}^{(j)\prime}}\, z_\mu\, \bar{z}) = \sum_{\rho} \sigma\Big(\sum_{\mu,\nu} C_{\mu\rho}^{(j)}\, \overline{C_{\nu\rho}^{(j)\prime}}\, z_\mu\, \bar{z}_\nu\Big)$$

$$= \sum_{\rho} \sigma\Big(\sum_{\mu} C_{\mu\rho}^{(j)}\, z_\mu\Big)\Big(\sum_{\nu} C_{\nu\rho}^{(j)}\, z_\nu\Big)' \geqslant 0,$$

so that in particular for $j = k + 1$

$$\lambda_{k+1}^{(\nu)} \geqslant 0,\ (\nu = 1, 2, \ldots, s). \tag{81}$$

By this our lemma is proved.

On the strength of the lemma, applied to the special case $k = n$, one can see that a decomposition of the algebraic form $u_h(X)$, appearing in (65), into eigenfunctions $u_{hj}(X)$ of the operators $\sigma(\Lambda^{2\nu})$ $(\nu = 1, 2, \ldots, n)$ is possible:

$$u_h(X) = \sum_j u_{hj}(X).$$

As we have seen before, $|X'X|^{-a}\, u_{hj}(X)$ represents a spherical function of the type (m, n) for a suitable exponent $a \geqslant 0$ which may depend upon h and j. This completes the proof of Theorem 3.

5. Theta series. The Gauss-transform $u^*(X)$ of a function $u(X)$ is defined by

$$u^*(X) = \int_{\mathfrak{T}} u(X+T)\, e^{-\pi\sigma(T'T)}\, [dT]. \tag{82}$$

Here $\mathfrak{T}$ denotes the full space of all real matrices $T = T^{(m,n)} = (t_{\mu\nu})$ and $[dT]$ the product of all differentials $dt_{\mu\nu}$. We call $u(X)$ an eigenfunction of the Gauss-transformation if $u(X) \neq 0$ and $u^*(X) = \lambda\, u(X)$ holds. Assume that $u(X)$ is a polynomial different from 0, then $u^*(X)$ is also one and according to

$$\int_{\mathfrak{T}} e^{-\pi\sigma(T'T)}\, [dT] = 1,$$

we have

$$\text{degree}\,(u^*(X) - u(X)) < \text{degree}\, u(X), \tag{83}$$

so that necessarily $\lambda = 1$ if $u(X)$ is an eigenfunction.

In the sequel we denote by $Y = Y^{(n)}$ a positive real matrix having variable elements and by $S = S^{(m)}$ a positive matrix having arbitrary but fixed chosen elements.

THEOREM 4. *Let $u(X)$ be a polynomial with the invariance property*

$$u(XV) = |V|^{2k}\, u(X) \text{ for } |V| \neq 0, \tag{84}$$

assume also that $u(X)$ is an eigenfunction of the Gauss-transformation. Further on we introduce Q by $S = Q'Q$, $Q' = Q > 0$. Then the theta series

$$\vartheta(Y, S; u) = \sum_G u(QG)\, e^{-\pi\sigma(YS[G])}, \tag{85}$$

where G has to run over all integral matrices of the type $G^{(m,n)}$, satisfies the transformation formula

$$\vartheta(Y^{-1}, S; u) = (-1)^{kn}\, |S|^{-n/2}\, |Y|^{m/2+2k}\, \vartheta(Y, S^{-1}; u). \tag{86}$$

In order to prove this we develop the theta series

$$\vartheta(X, Y, S; u) = \sum_G u(Q(G+X))\, e^{-\pi\sigma(YS[G+X])}, \tag{87}$$

which is a periodic function of X, into a Fourier series. We obtain

$$\vartheta(X, Y, S; u) = \sum_G \alpha(G, Y, S; u)\, e^{2\pi i \sigma(G'X)} \tag{88}$$

with

$$\alpha(G, Y, S; u) = \int_{\mathfrak{X}} u(QX)\, e^{-\pi\sigma(YS[X]) - 2\pi i\sigma(G'X)}\, [dX],$$

$\mathfrak{X}$ denoting the full X-space. In this integral we substitute $X_1 = QXR$ for X, R being determined by $Y = R'R$, $R' = R > 0$. Using (84) we find by a simple computation

$$\alpha(G, Y, S; u) = |S|^{-n/2}\, |Y|^{-m/2-k}\, u^*(-i\, Q^{-1}\, G\, R^{-1})\, e^{-\pi\sigma(Y^{-1}S^{-1}[G])},$$

$u^*(X)$ denoting the Gauss-transform of $u(X)$. We can split off the factor R^{-1} from $u^*(-i\, Q^{-1}\, G\, R^{-1})$ if and only if $u^*(X) = u(X)$. In the case $u^*(X) \neq u(X)$, $u^*(X)$ is not even homogeneous as (83) shows. In order to obtain a reasonable transformation formula for our theta series it is necessary to assume $u^*(X) = u(X)$. Now it is easy to state that

$$\alpha(G, Y, S; u) = (-1)^{kn}\, |S|^{-n/2}\, |Y|^{-m/2-2k}\, u(Q^{-1}\, G)\, e^{-\pi\sigma(Y^{-1}S^{-1}[G])}.$$

Thus we can carry over (88) with $X = 0$ and Y^{-1} instead of Y immediately into (86), the asserted formula.

The question whether for spherical functions $u(X)$, $u^*(X) = u(X)$ always holds is to be answered in the negative. For instance in the special case $m = 3$, $n = 2$ the spherical function

$$\begin{aligned} u(X) &= \xi_{23}\, \xi_{13} \\ &= x_{11}\, x_{21}\, x_{32}^2 + x_{12}\, x_{22}\, x_{31}^2 - x_{12}\, x_{21}\, x_{31}\, x_{32} - x_{11}\, x_{22}\, x_{31}\, x_{32}, \end{aligned}$$

we considered already, differs from $u^*(X)$. A simple computation yields

$$u^*(X) = u(X) + \frac{1}{2\pi}\,(x_{11}\, x_{21} + x_{12}\, x_{22}).$$

It is however remarkable that generally with $u(X)$ also $u^*(X)$ is an eigenfunction of the operators $\sigma(\Lambda^{2\nu})$, $(\nu = 1, 2, \ldots, n)$, and that with $u(X)$, $\left|\frac{\partial}{\partial X'}\,\frac{\partial}{\partial X}\right|$ also annihilates $u^*(X)$. We formulate these facts more precisely in

THEOREM 5. *If a polynomial $u(X)$ satisfies the relations*

$$\left| \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \right| u(X) = 0,\ \sigma(\Lambda^{2\nu})\, u(X) = \lambda_\nu\, u(X),\ (\nu = 1, 2, \dots, n), \quad (89)$$

then they are valid also for the Gauss-transform $u^*(X)$ of $u(X)$.

The proof is based on a general integral transformation which can be considered as a generalization of the method of partial integration. Now we denote by $[dX]$ the exterior product of all differentials $dx_{\mu\nu}$. Let $\omega_{\rho\sigma}$ be the exterior product of all $dx_{\mu\nu}$ with the exception of $dx_{\rho\sigma}$. We intend to choose the order of the factors such that

$$[dX] = dx_{\rho\sigma} \cdot \omega_{\rho\sigma}$$

holds. We set $\Omega = (\omega_{\mu\nu})$. Further on let $\mathfrak{G}$ be an oriented compact and measurable domain in the X-space with a measurable boundary $\mathfrak{R}$. We carry over the orientation from $\mathfrak{G}$ *to* $\mathfrak{R}$. On these premisses we prove the following

LEMMA. *Let $A = A^{(m)} = (a_{\mu\nu})$ and $B = B^{(m)} = (b_{\mu\nu})$ be matrices with functions of X as elements which have derivatives of sufficiently high order. Then*

$$\int_{\mathfrak{G}} A \Lambda^k B[dX] = \int_{\mathfrak{G}} (B' \Lambda^k A')' [dX] +$$

$$+ \sum_{\nu=0}^{k-1} \int_{\mathfrak{R}} (\Lambda^\nu A')'(X\Omega' - \Omega X')\, \Lambda^{k-1-\nu} B \quad (90)$$

holds for an arbitrary natural number k.

First of all we establish by means of Stokes' integral-formula

$$\int_{\mathfrak{G}} A X \frac{\partial}{\partial X'} B[dX] = \int_{\mathfrak{R}} A X \Omega' B -$$

$$- \int_{\mathfrak{G}} \left(B' \left(X \frac{\partial}{\partial X'} \right)' A' \right)' [dX] - n \int_{\mathfrak{G}} A\, B[dX]. \quad (91)$$

A straightforward computation yields indeed

$$\int_{\mathfrak{G}} AX \frac{\partial}{\partial X'} B[dX]$$

$$= \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} a_{\mu\rho}\, x_{\rho\sigma} \frac{\partial}{\partial x_{\tau\sigma}} b_{\tau\nu} \Big) [dX]$$

$$= \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} \frac{\partial}{\partial x_{\tau\sigma}} a_{\mu\rho}\, x_{\rho\sigma}\, b_{\tau\nu} \Big) [dX] - \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} b_{\tau\nu} \frac{\partial}{\partial x_{\tau\sigma}} a_{\mu\rho}\, x_{\rho\sigma} \Big) [dX]$$

$$= \int_{\mathfrak{G}} \Big(d \sum_{\rho,\sigma,\tau} a_{\mu\rho}\, x_{\rho\sigma}\, b_{\tau\nu}\, \omega_{\tau\sigma} \Big) -$$

$$- \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} b_{\tau\nu}\, x_{\rho\sigma} \frac{\partial}{\partial x_{\tau\sigma}} a_{\mu\sigma} \Big) [dX] - \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} b_{\tau\nu}\, a_{\mu\rho}\, \delta_{\tau\rho} \Big) [dX]$$

$$= \int_{\mathfrak{R}} \Big(\sum_{\rho,\sigma,\tau} a_{\mu\rho}\, x_{\rho\sigma}\, \omega_{\tau\sigma}\, b_{\tau\nu} \Big) -$$

$$- \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} b_{\tau\mu}\, x_{\rho\sigma} \frac{\partial}{\partial x_{\tau\sigma}} a_{\nu\rho} \Big)' [dX] - n \int_{\mathfrak{G}} \Big(\sum_{\rho} a_{\mu\rho}\, b_{\rho\nu} \Big) [dX]$$

$$= \int_{\mathfrak{R}} AX\Omega' B - \int_{\mathfrak{G}} \Big(B' \Big(X \frac{\partial}{\partial X'} \Big)' A' \Big)' [dX] - n \int_{\mathfrak{G}} AB[dX].$$

The analogous formula

$$\int_{\mathfrak{G}} A \Big(X \frac{\partial}{\partial X'} \Big)' B[dX]$$

$$= \int_{\mathfrak{R}} A\Omega X'B - \int_{\mathfrak{G}} \Big(B'X \frac{\partial}{\partial X'} A' \Big)' [dX] - n \int_{\mathfrak{G}} AB[dX] \qquad (92)$$

can be obtained from (91) in a simple manner by transposition. Subtraction of (92) from (91) yields (90) in the special case $k = 1$. A general proof of (90) is now possible without difficulty by induction on k.

Now we choose $A = \phi(X)E$, $B = \psi(X)E$ and form the trace of (90). At the same time we extend $\mathfrak{G}$ to the full X-space which was designed with $\mathfrak{X}$. By a suitable choice of $\phi(X)$ we shall take care later that all limit processes remain legitimate, in particular also that all boundary integrals vanish. Our lemma then yields

$$\int_{\mathfrak{X}} \phi(X)\, \sigma(\Lambda^k)\, \psi(X)\, [dX] = \int_{\mathfrak{X}} \psi(X)\, \sigma(\Lambda^k)\, \phi(X)\, [X]. \tag{93}$$

In

$$u^*(T) = \int_{\mathfrak{X}} u(T+X)\, e^{-\pi\sigma(X'X)}\, [dX]$$

we substitute $X_1 = X - T$ and then replace T by $-iT$. So we obtain for

$$w(T) = u^*(-iT)\, e^{-\pi\sigma(T'T)}$$

the representation

$$w(T) = \int_{\mathfrak{X}} u(X)\, e^{-\pi\sigma(X'X) - 2\pi i\sigma(X'T)} [dX].$$

Since $\sigma(X'X)$ can be considered as a constant with respect to the operator Λ it is sufficient to prove that

$$\sigma(\Lambda^{2\nu})\, w(X) = \lambda_\nu\, w(X),\ (\nu = 1, 2, \ldots, n).$$

Then these differential equations are also valid for $u^*(X)$ in place of $w(X)$. We set more precisely $\Lambda = \Lambda_x$, and denote by Λ_t the operator which arises from Λ_x by the substitution $X \to T$. It is easy to see that

$$\Lambda_x\, e^{-2\pi i\sigma(X'T)} = \Lambda'_t\, e^{-2\pi i\sigma(X'T)},$$

and by induction on k

$$\Lambda_x^k\, e^{-2\pi i\sigma(X'T)} = (\Lambda_t^k)'\, e^{-2\pi i\sigma(X'T)}.$$

This leads indeed to

$$\begin{aligned}(\Lambda_t^{k+1})'\, e^{-2\pi i\sigma(X'T)} &= (\Lambda_t^k\, \Lambda'_x)'\, e^{-2\pi i\sigma(X'T)}\\ &= \Lambda_x(\Lambda_t^k)'\, e^{-2\pi i\sigma(X'T)}\\ &= \Lambda_x^{k+1}\, e^{-2\pi i\sigma(X'T)}.\end{aligned}$$

Taking the trace, we obtain

$$\sigma(\Lambda_x^k)\, e^{-2\pi i\sigma(X'T)} = \sigma(\Lambda_t^k)\, e^{-2\pi i\sigma(X'T)}.$$

In order to determine the effect of $\sigma(\Lambda_t^{2\nu})$ on $w(T)$ we apply (93) to $k = 2\nu$, $\phi(X) = u(X)\, e^{-\pi\sigma(X'X)}$, $\psi(X) = e^{-2\pi i\sigma(X'T)}$. So we get finally

$$\begin{aligned}\sigma(\Lambda_t^{2\nu})\, w(T) &= \int_{\mathfrak{X}} u(X)\, e^{-\pi\sigma(X'X)}\, \sigma(\Lambda_t^{2\nu})\, e^{-2\pi i\sigma(X'T)}[dX] \\ &= \int_{\mathfrak{X}} u(X)\, e^{-\pi\sigma(X'X)}\, \sigma(\Lambda_x^{2\nu})\, e^{-2\pi i\sigma(X'T)}[dX] \\ &= \int_{\mathfrak{X}} e^{-2\pi i\sigma(X'T)}\, \sigma(\Lambda_x^{2\nu})\, u(X)\, e^{-\pi\sigma(X'X)}[dX] \\ &= \lambda_\nu \int_{\mathfrak{X}} u(X)\, e^{-\pi\sigma(X'X) - 2\pi i\sigma(X'T)}[dX] = \lambda_\nu w(T),\end{aligned}$$

the asserted relations. It is trivial that also the first of the differential equations (89) can be carried over from $u(X)$ to $u^*(X)$. So Theorem 5 is proved.

A characterization of the algebraic forms which are eigenfunctions of the Gauss-transformation yields

THEOREM 6. *The Gauss-transform $u^*(X)$ of an algebraic form $u(X)$ is identical with $u(X)$ if and only if $u(X)$ is a solution of the Laplacian differential equation* $\sigma\left(\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right) u(X) = 0$.

In this connection the fact that X is a rectangular matrix of an arbitrary number of columns does not come into appearance. So we can assume without loss of generality that $n = 1$. Let $X = (x_\mu)$, then $\Lambda = \left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}\right)$. In the sequel we denote by $u(X)$ an algebraic form of degree k which does not vanish identically.

1. Assume $\sigma\left(\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right) u(X) = 0$. $u(X)$ then represents an ordinary spherical harmonic of m variables. Thus we have

$$\sigma(\Lambda^2)\; u(X) = \lambda_1 u(X),$$

and also, according to Theorem 5,

$$\sigma(\Lambda^2)\, u^*(X) = \lambda_1 u^*(X),$$

with a constant eigenvalue λ_1. We apply the well-known operator identity

$$\tfrac{1}{2}\,\sigma(\Lambda^2) = \Big(\sum_\nu x_\nu \frac{\partial}{\partial x_\nu}\Big)^2 + (m-2)\Big(\sum_\nu x_\nu \frac{\partial}{\partial x_\nu}\Big) - \\ - \Big(\sum_\nu x_\nu^2\Big)\Big(\sum_\nu \frac{\partial^2}{\partial x_\nu^2}\Big) \qquad (94)$$

to $u^*(X)$. Let

$$u^*(X) = \sum_{\nu=0}^{k} h_\nu(X) \text{ with } h_k(X) = u(X),$$

be the decomposition of $u^*(X)$ into homogeneous terms such that $h_\nu(X)$ has the degree ν. Then we obtain

$$\tfrac{1}{2}\,\lambda_1\, u^*(X) = \sum_{\nu=0}^{k} \nu(\nu+m-2)\, h_\nu(X),$$

therefore

$$\tfrac{1}{2}\,\lambda_1 = \nu(\nu+m-2) \text{ if } h_\nu(X) \neq 0.$$

This happens for $\nu = k$, so that $\tfrac{1}{2}\lambda_1 > \nu(\nu+m-2)$ for $\nu < k$, which proves $u^*(X) = u(X)$.

2. We apply the Gauss-transformation to the polynomial

$$u(X) = \sum a_{\nu_1\nu_2\ldots\nu_m} x_1^{\nu_1} x_2^{\nu_2} \ldots x_m^{\nu_m}.$$

A first computation yields

$$(x_1^{\nu_1}\ldots x_m^{\nu_m})^* = \int_{-\infty}^{\infty}\cdots\int e^{-\pi(t_1^2+\ldots+t_m^2)}(x_1+t_1)^{\nu_1}\ldots(x_m+t_m)^{\nu_m}\, dt_1\ldots dt_m$$

$$= \prod_{i=1}^{m}\int_{-\infty}^{\infty} e^{-\pi t^2}(x_i+t)^{\nu_i}\, dt$$

$$= \prod_{i=1}^{m}\Big(\sum_{\substack{\alpha_i,\beta_i\geq 0\\ \alpha_i+\beta_i=\nu_i}}\binom{\nu_i}{\alpha_i} x_i^{\alpha_i}\int_{-\infty}^{\infty}\Big(e^{-\pi t^2}\, t^{\beta_i}\, dt\Big).$$

Obviously it suffices to define the summation on even β_i. For such β_i

$$\int_{-\infty}^{\infty} e^{-\pi t^2}\, t^{\beta_i}\, dt = \pi^{-(\beta_i+1)/2}\, \Gamma\Big(\frac{\beta_i+1}{2}\Big)$$

holds. Consequently

$$(x_1^{\nu_1} \dots x_m^{\nu_m})^* = \prod_{i=1}^{m} \Big(\sum_{\substack{\alpha_i,\beta_i \geq 0 \\ \alpha_i + 2\beta_i = \nu_i}} \binom{\nu_i}{\alpha_i} \Gamma(\beta_i + \tfrac{1}{2})\, \pi^{-\beta_i - \frac{1}{2}}\, x_i^{\alpha_i} \Big)$$

$$= \sum_{i=1}^{m} \sum_{\substack{\alpha_i,\beta_i \geq 0 \\ \alpha_i + 2\beta_i = \nu_i}} \binom{\nu_1}{\alpha_1} \dots \binom{\nu_m}{\alpha_m} \Gamma(\beta_i + \tfrac{1}{2}) \dots \Gamma(\beta_m + \tfrac{1}{2}) \times$$

$$\times\, \pi^{-(\beta_1 + \dots + \beta_m) - \frac{1}{2}m}\, x_1^{\alpha_1} \dots x_m{}^{\alpha_m},$$

thus

$$u^*(X) = \sum a_{\nu_1 \nu_2 \dots \nu_m} (x_1^{\nu_1} x_2^{\nu_2} \dots x_m{}^{\nu_m})^*$$

$$= \sum b_{\alpha_1 \alpha_2 \dots \alpha_m} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m{}^{\alpha_m},$$

with

$$b_{\alpha_1 \dots \alpha_m} = \sum_{i=1}^{m} \sum_{\beta_i \geq 0} a_{\alpha_1 + 2\beta_1 \dots \alpha_m + 2\beta_m} \binom{\alpha_1 + 2\,\beta_1}{\alpha_1} \dots \binom{\alpha_m + 2\,\beta_m}{\alpha_m} \times$$

$$\times\, \Gamma(\beta_1 + \tfrac{1}{2}) \dots \Gamma(\beta_m + \tfrac{1}{2})\, \pi^{-(\beta_1 + \dots + \beta_m) - m/2}$$

Since $u(X)$ is assumed to be an algebraic form of degree k we need only to consider those systems of numbers $\beta_1, \beta_2, \dots \beta_m$ for which

$$\alpha_1 + \dots + \alpha_m + 2(\beta_1 + \dots + \beta_m) = k.$$

In particular we obtain, for $\alpha_1 + \dots + \alpha_m = k - 2$,

$$b_{\alpha_1 \dots \alpha_m} = \frac{1}{4\pi} \sum_{i=1}^{m} (\alpha_i + 2)(\alpha_i + 1)\, a_{\alpha_1 \dots \alpha_{i-1} \alpha_i + 2\, \alpha_{i+1} \dots \alpha_m}.$$

Now $u^*(X) = u(X)$ implies

$$b_{\alpha_1 \alpha_2 \dots \alpha_m} = 0, \text{ for } \alpha_1 + \alpha_2 + \dots + \alpha_m = k - 2.$$

These relations first mean that

$$\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} u(X)$$

$$= \sum \left(\sum_{i=1}^{m} (\alpha_i + 2)(\alpha_i + 1)\, a_{\alpha_1 \dots \alpha_{i-1} \alpha_i + 2 \alpha_{i+1} \dots \alpha_m} \right) x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m} = 0.$$

So Theorem 6 is proved.

6. Angular characters. In the space of all positive real matrices $Y = Y^{(n)} = (y_{\mu\nu})$ we can develop a theory of invariant linear differential operators analogously to that of the X-space. Since we meet a rather simpler situation in the Y-space we need now only brief considerations. Let $\frac{\partial}{\partial Y} = \left(e_{\mu\nu} \frac{\partial}{\partial y_{\mu\nu}} \right)$ with $e_{\mu\nu} = 1$ or $\frac{1}{2}$ according as $\mu = \nu$ or $\mu \neq \nu$. Ω denotes a linear differential operator, i.e. a polynomial in the elements of $\frac{\partial}{\partial Y}$ with functions of Y as coefficients. We call $\Omega = F\left(Y, \frac{\partial}{\partial Y} \right)$ invariant if

$$F\left(Y[U], \frac{\partial}{\partial Y} [U'^{-1}] \right) = F\left(Y, \frac{\partial}{\partial Y} \right) \text{ for } |U| \neq 0 \tag{95}$$

holds. Let $\mathfrak{M}$ be again the module of all linear operators, $\mathfrak{M}_h$ the module of all operators of degree $\leqslant h$ and $\mathfrak{R}$ the ring of all invariant linear operators. (40) is still valid now.

THEOREM 7. *The invariant operators* $\sigma\left(Y \frac{\partial}{\partial Y} \right)^h$ $(h = 1, 2 \dots, n)$ *and likewise* $\sigma\left(\left(Y \frac{\partial}{\partial Y} \right)' \right)^h$, $(h = 1, 2, \dots, n)$ *form a basis for the ring* $\mathfrak{R}$ *of all invariant linear operators.*

In order to prove this theorem, which was first announced by A. Selberg, we choose a symmetric matrix $W = W^{(n)}$ with variable elements. We assume that they are commutable with those of Y. Let $\Omega = F\left(Y, \frac{\partial}{\partial Y} \right)$ be invariant. Then we have with regard to (95)

$$F(Y, W) = F(E, (W[U_0'^{-1}])[V]), \tag{96}$$

if $Y[U_0] = E$ and V an arbitrary orthogonal matrix. The right side of (96) is a symmetric polynomial in the characteristic roots of $W[U'^{-1}_0]$, consequently also representable as a polynomial in

$$\sigma(W[U'^{-1}_0])^h = \sigma(YW)^h = \sigma((YW)')^h \ (h = 1, 2, \dots, n).$$

Let

$$F(Y, W) = p(\sigma(YW), \dots, \sigma(YW)^n) = p(\sigma(YW)', \dots, \sigma((YW)')^n)$$

be such a representation. If Ω is of degree k, we obtain on the strength of (40)

$$\Omega \equiv p\left(\sigma\left(Y\frac{\partial}{\partial Y}\right), \dots, \sigma\left(Y\frac{\partial}{\partial Y}\right)^n\right)$$

$$\equiv p\left(\sigma\left(Y\frac{\partial}{\partial Y}\right)', \dots, \sigma\left(\left(Y\frac{\partial}{\partial Y}\right)'\right)^n\right) \pmod{\mathfrak{M}_{k-}}.$$

Our assertion now follows in the usual manner by induction on k.

Let $Y_1 = Y^{-1}$. Introducing $dY_1 = -Y^{-1}\,dY\,Y^{-1}$ in

$$d\phi = \sigma\left(dY\frac{\partial}{\partial Y}\right)\phi = \sigma\left(dY_1\frac{\partial}{\partial Y_1}\right)\phi,$$

where ϕ denotes an arbitrary function, we obtain

$$Y\frac{\partial}{\partial Y} = -\left(Y_1\frac{\partial}{\partial Y_1}\right)'.$$

Thus the substitution $Y \to Y^{-1}$ maps the ring $\mathfrak{R}$ of all invariant linear operators into itself according to Theorem 7.

The space defined by $Y > 0$ can be considered as a Riemannian space relative to the metric introduced by the differential form

$$ds^2 = \sigma(Y^{-1}\,dY)^2.$$

Let ω denote the invariant volume element on the determinant surface $|Y| = 1$. Further let $\mathfrak{F}_n$ be a fundamental domain in $Y > 0$ relative to the group of transformations $Y \to Y[U]$ with unimodular U. For instance we can take for $\mathfrak{F}_n$ the domain of all reduced positive Y in the sense of Minkowski. Let $\mathfrak{B}_n$ denote the intersection of $\mathfrak{F}_n$ with the determinant surface $|Y| = 1$.

A function $v(Y)$ shall be called an *angular character* if

$$\left.\begin{array}{ll} 1. & v(Y) \text{ is holomorphic in } Y > 0 \text{ and homogeneous of degree } 0, \\ 2. & v(Y) \text{ is an eigenfunction of the ring } \mathfrak{R} \text{ of all invariant linear operators}, \\ 3. & v(Y[U]) = v(Y) \text{ is valid for unimodular } U, \\ 4. & v(Y) \text{ is square integrable over } \mathfrak{B}_n, \text{ so that} \\ & \displaystyle\int_{\mathfrak{B}_n} v(Y)\,\overline{v(Y)}\,\omega \text{ exists.} \end{array}\right\} \tag{97}$$

In applications we replace the fourth condition first of all by the sharper one saying that $v(Y)$ is bounded. Since the metric introduced in $Y > 0$ and also the ring $\mathfrak{R}$ are invariant relative to the substitution $Y \to Y^{-1}$, it is easy to see, that with $v(Y)$ also $v^*(Y) = v(Y^{-1})$ represents an angular character. But in general $v(Y)$ and $v(Y^*)$ belong to different eigenvalues. This may be mentioned without proof.

Here we note yet an integral formula, a generalization of the Euler gamma-integral, which was proved already elsewhere [5]:

$$\int_{Y>0} e^{-2\pi\sigma(TY)}\, v(Y)\, |Y|^{s-(n+1)/2}\, [dY]$$
$$= (2\pi)^{-ns}\, \Gamma(s-\alpha_1)\,\Gamma(s-\alpha_2)\ldots\Gamma(s-\alpha_n)\,\pi^{n(n-1)/4}\, v(T^{-1})\, |T|^{-s}. \tag{98}$$

Here it is $T = T^{(n)} > 0$, $v(Y)$ a bounded angular character, $[dY] = \prod_{\mu\leq\nu} dy_{\mu\nu}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ a set of constants which are uniquely determined up to the order by the eigenvalues of $v(Y)$. We have to assume $\operatorname{Re} s > (n-1)/2$.

7. Zeta-functions. In the sequel $u(X)$ denotes a spherical function of type (m, n) and degree $2kn$ and $v(Y)$ an angular character.

We assume that $v(Y)$ is bounded and $u(X)$ a non-constant eigenfunction of the Gauss-transformation so that in particular $k > 0$. The theta series (85) can be rewritten as

$$\vartheta(Y, S; u) = \sum_{T>0} a(T, S; u)\, e^{-\pi\sigma(YT)}, \tag{99}$$

with

$$a(T, S; u) = \sum_{\substack{G \\ S[G]=T}} u(QG), \tag{100}$$

where the finite sum must be extended over all integral G with $S[G] = T$. We introduce y and Y_1 by $Y = yY_1$, $y > 0$, $|Y_1| = 1$ and apply the Mellin transformation to the theta series, this being considered as a function of y, i.e. we form the function

$$\eta(s;\, Y_1,\, S;\, u) = \int_0^\infty \vartheta(y\,Y_1,\, S;\, u)\; y^{ns-1}\, dy.$$

For brevity we set $c_n = n^{-\frac{1}{2}}\, 2^{n(n-1)/4}$, denote by $\{T\}$ the class of all with $T(> 0)$ equivalent matrices $T[U]$, where U denotes an arbitrary unimodular matrix, and by $\epsilon(T)$ the number of units of T. We use $\mathfrak{H}_n$, $\mathfrak{B}_n$, ω in the introduced meaning. In anology with a computation carried out in [5] we obtain now

$$\begin{aligned}
&\xi_0(s;\, S;\, u,\, v) \\
&= \int_{\mathfrak{B}_n} \eta(s;\, Y_1,\, S;\, u)\, v(Y_1)\, \omega \\
&= \int_{\mathfrak{B}_n} \int_0^\infty \vartheta(yY_1,\, S;\, u)\, v(Y_1)\, y^{ns-1}\, dy\, \omega \\
&= c_n \int_{\mathfrak{H}_n} \vartheta(Y,\, S;\, u)\, v(Y)\, |\,Y\,|^{s-(n+1)/2}\, [dY] \\
&= c_n \sum_{T>0} a(T,\, S;\, u) \int_{\mathfrak{H}_n} e^{-\pi\sigma(YT)}\, v(Y)\, |\,Y\,|^{s-(n+1)/2}\, [dY]
\end{aligned}$$

$$= c_n \sum_{\{T\}} \frac{a(T, S; u)}{\epsilon(T)} \sum_{U} \int_{\mathfrak{F}_n} e^{-\pi\sigma(YT[u])} v(Y) \mid Y \mid^{s-(n+1)/2} [dY], \quad (101)$$

where U has to run over all unimodular matrices. The addition of all integrals and the application of the integral-formula (98) yield

$$\xi_0(s; S; u, v)$$
$$= 2 c_n \sum_{\{T\}} \frac{a(T, S; u)}{\epsilon(T)} \int_{Y>0} e^{-\pi\sigma(YT)} v(Y) \mid Y \mid^{s-(n+1)/2} [dY]$$
$$= \frac{2}{\sqrt{n}} (2\pi)^{n(n-1)/4} \pi^{-ns} \Gamma(s-\alpha_1) \Gamma(s-\alpha_2) \ldots \Gamma(s-\alpha_n) \phi(s, S; u, v^*)$$

with

$$\phi(s, S; u, v^*) = \sum_{\{T\}} \frac{a(T, S; u)\, v^*(T)}{\epsilon(T) \mid T \mid^s}, \quad v^*(T) = v(T^{-1}). \quad (102)$$

The summation over the classes $\{T\}$ can be replaced by a summation over $T \in \mathfrak{F}_n$. According to the signification of $a(T, S; u)$ we get

$$\phi(s, S; u, v^*) = \sum_{\substack{G \\ S[G] \in \mathfrak{F}_n}} \frac{u(Q\,G)\, v^*(S[G])}{\epsilon(S[G]) \mid S[G] \mid^s}, \quad (103)$$

where G has to run over all integral matrices such that $S[G] \in \mathfrak{F}_n$. A set of matrices of this kind can be obtained by forming the products $G = G^* U$, where G^* must run over a full set of integral matrices of rank n, such that each two are not right-associated, and U over a full set of units of $S[G^*]$ provided that G^* is given. Two matrices are called right-associated if they differ by a unimodular right factor only. Writing G^*U in place of G we see that the general term in (103) does not depend upon U, thus we obtain finally, after writing again G instead of G^*,

$$\phi(s, S; u, v^*) = \sum_{G} \frac{u(QG)\, v^*(S[G])}{\mid S[G] \mid^s}, \quad (104)$$

where the sum must be extended over a full set of integral matrices G of rank n, such that each two are not right-associated.

A functional equation for the functions $\phi(s, S; u, v)$ and $\phi(s, S; u, v^*)$ can be obtained by decomposing the integral over $\mathfrak{F}_n$ in

$$\xi_0(s, S; u, v) = c_n \int_{\mathfrak{F}_n} \vartheta(Y, S; u)\, v(Y)\, |Y|^{s-(n+1)/2} [dY] \quad (105)$$

into two parts corresponding to the decomposition of $\mathfrak{F}_n$ by the determinant surface $|Y| = 1$. We assume that $\mathfrak{F}_n$ is invariant relative to the substitution $Y \to Y^{-1}$. In that part of the integral which must be extended over the intersection of $\mathfrak{F}_n$ with $|Y| < 1$ we substitute $Y \to Y^{-1}$ and apply the transformation formula. Observing that $|Y|^{-(n+1)/2}[dY]$ is invariant relative to the substitution $Y \to Y^{-1}$ we obtain the following representations

$$\begin{aligned} &\xi_0(s, S; u, v) \\ &= c_n \int_{\substack{Y \in \mathfrak{F}_n \\ |Y| \geq 1}} \vartheta(Y, S; u)\, v(Y)\, |Y|^{s-(n+1)/2} [dY] + \\ &\quad + c_n \int_{\substack{Y \in \mathfrak{F}_n \\ |Y| \geq 1}} \vartheta(Y^{-1}, S; u)\, v^*(Y)\, |Y|^{-s-(n+1)/2} [dY] \\ &= c_n \int_{\substack{Y \in \mathfrak{F}_n \\ |Y| \geq 1}} \{\vartheta(Y, S; u)\, v(Y)\, |Y|^s + \\ &\quad + (-1)^{kn} |S|^{-n/2} \vartheta(Y, S^{-1}; u)\, v^*(Y)\, |Y|^{m/2+2k-s}\}\, |Y|^{-(n+1)/2} [dY]. \end{aligned}$$

All these expressions have first of all a meaning only if the real part of s is sufficiently large. The last integral however represents, as can be seen easily, an entire function of s. So the analytical continuation of $\phi(s, S; u, v^*)$ into the whole s-plane is performed. It is obvious that $\phi(s, S; u, v^*)$ is an entire function of s. The integral-representation of $\xi_0(s, S; u, v)$ yields directly

$$\xi_0(m/2 + 2k - s, S; u, v) = (-1)^{kn} |S|^{-n/2} \xi_0(s, S^{-1}; u, v^*). \quad (106)$$

The replacement of v by v^* and consequently v^* by v may carry over $\alpha_1, \alpha_2, \ldots \alpha_n$ into $\beta_1, \beta_2, \ldots \beta_n$. Introducing

$$\xi(s, S; u, v) = \frac{\sqrt{n}}{2} (2\pi)^{-n(n-1)/4} \xi_0(s, S; u, v^*)$$

we obtain finally the following result.

THEOREM 8. *Let $u(X)$ be a spherical function of type (m, n) and degree $2kn$, $v(Y)$ an angular character. Assume that $u(X)$ is a non-constant eigenfunction of the Gauss-transformation and that $v(Y)$ is bounded. Then the Dirichlet series*

$$\phi(s, S; u, v) = \sum_{G} \frac{u(QG)\, v(S[G])}{|S[G]|^s},$$

where the sum must be extended over a full set of integral matrices G of rank n such that each two are not right-associated, represents an entire function of s. It satisfies the functional equation

$$\xi(m/2 + 2k - s, S; u, v) = (-1)^{kn} |S|^{-n/2} \xi(s, S^{-1}; u, v^*)$$

where

$$\xi(s, S; u, v) = \pi^{-ns} \Gamma(s - \beta_1) \Gamma(s - \beta_2) \ldots \Gamma(s - \beta_n) \phi(s, S; u, v)$$

with certain constants $\beta_1, \beta_2, \ldots, \beta_n$ which depend only upon the eigenvalues of v.

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