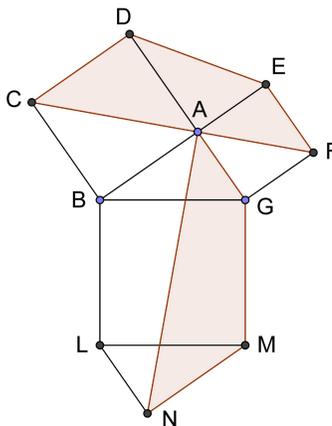


LEONARDO DA VINCI'S PROOF OF THE THEOREM OF PYTHAGORAS

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While collecting various proofs of the Pythagorean Theorem for presenting them in my class (see [12]) I discovered a beautiful proof credited to Leonardo da Vinci. It is based on the diagram on the right, and I leave the pleasure of reconstructing the simple proof from this diagram to the reader (see, however, the proof given at the end of this article).

Since I had decided to give correct references to as many results as possible (if only to set an example) I started looking at the many presentations of da Vinci's proofs on the internet for finding out where da Vinci had published his proof. It turned out that although this was a very well known proof, none of the many sources was able to point me to a book with a sound reference let alone to one of da Vinci's original publications. For this reason I gave up and simply remarked



As a rule, one has to be very careful in such situations,

meaning that claims that cannot be verified often turn out to be wrong. A simple search (with the keywords “da Vinci” and Pythagoras or Pitagoras) will reveal dozens of books that credit the proof above to da Vinci, including Maor's book [14] or the otherwise very accurate and highly readable book [17] by Ostermann & Wanner, where the proof is credited to da Vinci in Exercise 21 on p. 26.

Tracking down References. Eventually I developed an interest in finding the precise reference, and I began hunting for the origin of da Vinci's proof. The first thing I discovered, more or less by accident while browsing through old geometry texts on the internet, was that a proof identical to da Vinci's proof was found by Terquem in 1838. In [22, S. 103–104] he writes

The theorem of Pythagoras being very important, we will give here a new proof based only on the superposition of figures.

Proofs of the Pythagorean Theorem have been rediscovered over and over again, so the fact that Terquem had found a proof credited to da Vinci does not mean that da Vinci did not find it first. Terquem's proof was republished in 1893 by M. Balitrand [1], without any reference to da Vinci.

Next I found a remark in Eli Maor's wonderful book on the Pythagorean theorem; in [14, p.104] he writes

Loomis, on the authority of F.C. Boon, A.C. (Miscellaneous Mathematics, 1924) attributes this proof to Leonardo da Vinci (1452–1519).

Here the reference is to the first edition of Loomis [13]. I was unable to find the publication by Boon. Boon did publish a book in 1924, namely “A companion to elementary school mathematics”, which is also referred to by Loomis. As Brian Hopkins informed me, the pages of this book actually do carry the title “Miscellaneous Mathematics”, and it does credit the proof to da Vinci, suggesting that these two books actually are one and the same. Then I discovered [4], whose authors also give the reference to Boon but mention in addition that the credit to da Vinci may be found in Heath's book [6] (Boon also refers to Heath). Heath, who was extremely well versed in classical mathematics, writes in [6, S. 365]

It appears to come from one of the scientific papers of Leonardo da Vinci (1452–1519).

In addition he remarks that this proof may be found in compilations of proofs of the Pythagorean Theorem by J.W. Müller and Ign. Hoffmann. Another search for these names led me to the report on elementary geometry written by Max Simon [18], where the reference to da Vinci is also given. Indeed Heath quotes Simon very often, mainly his edition of the first six books of the Elements, but also (on pp. 202, 328) his report [18].

Using these references I was finally able to track down what I think is the correct historical development behind “da Vinci's proof”.

First Appearance of “da Vinci's” Proof. The story begins in 1790; in his book [21, p. 124–126] on “Geometry for soldiers, and those who are not”, Tempelhof (some sources, including wikipedia, spell his name Tempelhoff; in his book, Tempelhof is used) gives the proof in question and remarks

This proof is somewhat roundabout; but it has the advantage that the truth of the theorem can be made clearly visible.

Tempelhof does not give any information about who discovered the proof.

In 1819, Hoffmann [7] published a compilation of 32 proofs of the theorem of Pythagoras, using as his main sources earlier compilations, in particular the dissertations of Scherz & Stöber [19] (Strasbourg 1743; Scherz was Stöber's supervisor – his name figures prominently on the cover of this dissertation) and by Lange & Jetze [11] (Halle 1752; again, Lange was Jetze's supervisor). In the same year, Müller [15, p. 64] published, as a response to Hoffmann's publications, his own compilation, which contained “da Vinci's” proof as no. 15 (out of 18 different proofs). In this regard he wrote ([15, p. 64])

I have known this construction only for a few years now since it has been communicated to me orally.

This proof was one out of three that Hoffmann added to his 32 proofs in the second edition [8] of [7] along with the following comment:

In this appendix three proofs are given. The first seems to have been known for quite a while since it can be found in several older writings. Its discoverer is not named.

I do not know to which books the “older writings” refer; all of the ancient books and dissertations in the bibliography are available through google books. It is a pity that the list of figures, which are usually presented on the last couple of pages in these geometry books, have not been reproduced properly. If these figures could be accessed, searching for geometry textbooks containing the proof in question would be a lot easier.

The discoverer was finally revealed in [16, p. 70–71] (1826) by Müller, a professor of mathematics at the gymnasium in Nuremberg:

In this regard I remark that Mr. Hofrath Joh. Tobias Mayer from Göttingen is the discoverer of the proof that I have given as the fifteenth in my book mentioned above, “systematic compilation” etc. page 62 until 64. He has found this proof already in 1772 and has repeatedly presented it in his lectures in Altdorf given in the years 1779–1785 and has disseminated in this way. Therefore Tempelhof could include it in his Geometry for Soldiers published in 1790.

This sounds as if Mayer had seen the proof in Müller’s book and told him that this proof was due to himself.

I still do not know how Tempelhof learned about Mayer’s proof. Tempelhof studied in Frankfurt (Oder) and Halle, and joined the Prussian army in 1756. He taught officers in Berlin as well as the King’s son and his brother, but apparently did not leave Berlin except for taking part in various military campaigns. Altdorf, on the other hand, is a small town outside of Nuremberg; its university was closed in 1809.

Tobias Mayer. For most of today’s mathematicians, the name Tobias Mayer will be pretty much unknown. Johann Tobias Mayer was born in Göttingen in 1752 and died there in 1830. His father, Tobias Mayer (Marbach 1723 – Göttingen 1762), published his first book on mathematics at the age of 18. He died very young, and was famous for his accomplishments in astronomy – he even received a part of the Longitude Prize (see Forbes [5]) for his contribution to improving navigation on sea through his lunar tables. One of his inventions also was used in the measurements of the arc of the meridian in connection with determining the metre. Mayer’s correspondence with Euler was translated into English and published by Forbes.

Actually Tobias Mayer is mentioned in Gauss’s letter to Olbers from October 26, 1802:

I do not know any professor who has done much for science except the great Tobias Mayer, and he was regarded as a bad professor in his times.

There is, by the way, also a connection between Gauß and Tempelhof: Gauß had read both of Tempelhof’s books on analysis, and even sent him a copy of his dissertation. In a letter to Bolyai from December 16, 1799, he writes that, in his opinion, General von Tempelhof is one of the best German mathematicians (in 1799, France was the leading nation in mathematics, as is testified by mathematicians such as Fourier, Laplace, Lagrange, Legendre, Monge, Poisson, Poincaré, and Poncelet. On the German side there was Gauss).

Mayer’s son Johann Tobias Mayer began lecturing in Göttingen in 1773, and moved to Altdorf in 1780; in 1786 he went to Erlangen, and in 1799 he returned to Göttingen. He is the author of various textbooks on practical geometry and

differential calculus, and published many articles on physics. There is a very active Tobias-Mayer-Society and a Tobias-Mayer museum in Marbach.

Subsequent Development. The authorship of Johann Tobias Mayer is also mentioned subsequently by Hoffmann in various of his writings; in his comments on Euclid's Elements [9, p. 284] he writes

Another highly simple and astute proof, whose discovery is credited to Joh. Tob. Mayer in Göttingen, is the following.

A similar remark can be found in Hoffmann [10, p. 8]:

The Pythagorean Theorem according to Johann Tobias Mayer and a variation of this proof.

1) In my memoir "The Pythagorean Theorem, equipped with two and thirty proofs that are partially known and partially new", Mainz 1821, 2nd. ed. p. 36, and in my memoir "The Elements of Euclides etc. Mainz 1829, p. 284, I have mentioned the astute proof whose discovery is credited (by Joh. Wolfg. Müller "New contributions to the theory of parallels" etc., Augsburg and Leipzig 1826, p. 70–71) to the very sagacious geometer Joh. Tobias Mayer in Göttingen.

Enter da Vinci. Thus the proof of the theorem of Pythagoras based on the diagram above is due to Mayer (1772) and was rediscovered by Terquem (1838). The question remains where da Vinci enters the picture. He does so in Max Simon's report [18] "On the development of elementary geometry in the 19th century" published by the German Union of Mathematicians in 1906, in the same series in which e.g. Hilbert's report on algebraic numbers had appeared. Originally, Simon's report had been written for the encyclopedia of mathematics, but eventually Felix Klein rejected the submission. In fact, in his preface Simon writes

Thus when Mr. Klein finally rejected the review in its present form then this happened mainly because he did not have any scientific assistants at his disposal who would give all references with bibliographic precision at each place where some work is mentioned. In fact, the condition of the slips of papers necessitated an extremely time-consuming correction.

And on p. 111 we can finally read

The proof of the hexagon, which was included in very many textbooks, such as Mehler, is not due to Tédénat (Manuel), but may be found in the second edition of Hoffmann as no. 33 from older writings (Lionardo da Vinci).

The need for correcting the references is clearly visible here: the textbook by Mehler is probably the one by Müller we have already cited, and my guess is that the "manuel" by "Tédénat" is actually Terquem's book [22]; Lionardo, of course, should read Leonardo. There is a geometry textbook written by Tédénat in the year 7 of the revolution (1799), namely [20]; there Tédénat gives two proofs of the theorem of Pythagoras, the first in art. 176 (the proof given by Euclid) and the second, a variation of Euclid's proof, in art. 215.

At the end of Simon's memoir there are two pages listing all kinds of misprints, but his list of errata is far from being complete. Actually it is difficult to explain

Klein’s suggestion of publishing Simon’s memoir “in the present form” as a report in the same series of the *Jahresberichte* as Hilbert’s report.

Simon takes the attribution for this proof from [8], and his expression “older writings” (“ältere Schriften” in the German original) is exactly the expression used by Hoffmann. Apparently Simon was quoting from memory and must have mixed up the “older writings” with another reference involving Leonardo da Vinci (this is the fourth out of four references to da Vinci in his report); another explanation might be that some “scientific assistant” trying to make sense of Simon’s collection of slips of papers is responsible for this mix-up.

In any case my conclusion is that the legend of Leonardo da Vinci’s proof was given birth by Max Simon’s report from 1906, from where it was copied by Heath, Boon and Loomis and then was spread throughout the mathematical literature; nowadays the claim that Mayer’s proof was found by da Vinci can be found in dozens of books, hundreds of articles and thousands of web pages.

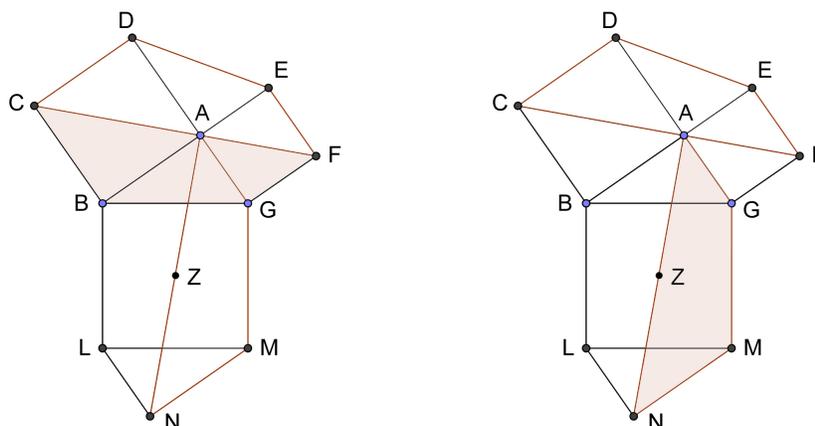
Summary. As a conclusion, here is a short summary of these developments:

1790	Tempelhof gives “da Vinci’s” proof in [21, p. 124–126] without attribution.
1819	Müller [15, p. 64] gives the proof without attribution.
1821	Hoffmann [8, S. 36] gives the proof and writes that its discoverer is unknown.
1826	Müller [16, p. 70–71] credits the proof to Johannes Tobias Mayer from Göttingen.
1838	Terquem [22] rediscovers Mayer’s proof.
1848	Hoffmann [10, p. 8] also credits the proof to Mayer, referring to Müller.
1906	Max Simon [18] credits the proof to da Vinci for the first time and refers to Hoffmann [8].
1908	Heath [6] reads about “da Vinci’s proof” in Simon’s report, but also refers to Müller and Hoffmann.
1924	Boon [2] credits the proof to da Vinci; in the bibliography, Heath [6] is cited.
1940	Loomis [13] credits the proof to da Vinci on the authority of Boon.

Despite my findings I still would love to hear from an expert on da Vinci’s mathematical writings that a proof of the Pythagorean theorem by da Vinci’s hand is not known.

A VARIATION OF MAYER’S PROOF

The following version of Mayer’s proof is perhaps a little bit simpler than the proofs given elsewhere. Take the diagram with the right-angled triangle and the squares on its sides, and rotate the square on the hypotenus along with the triangle by 180° around the center Z of the square. By construction, the area of the quadrilateral $AGMN$ is $\frac{1}{2}c^2 + F$, where F is the area of the right-angled triangle.



On the other hand, the area of the quadrilateral BCFG is equal to that of CFED since they are symmetric with respect to the line CF. Observe that A is on this line since $\angle CAF = \angle CAB + \angle BAG + \angle GAF = 45^\circ + 90^\circ + 45^\circ = 180^\circ$. Thus BCFG has area $\frac{1}{2}a^2 + \frac{1}{2}b^2 + F$.

Finally, rotating BCFG by 90° about G moves BCFG into AGMN, so their areas are equal, and Pythagoras follows.

ACKNOWLEDGEMENTS

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