A simple proof of the infinitude of Primes

Franz Lemmermeyer

March 30, 2020

The following is a simplification of the proof of the infinitude of primes using continued fractions given by Barnes [1]. Assume that there are only finitely many prime numbers, namely 2, $p_1 = 3, \ldots, p_n$. Let $q = p_1 \cdots p_n$ be the product of all odd primes; then $q^2 + 1$ is not divisible by any odd prime, hence must be a power of 2. Since $q^2 + 1 \equiv 2 \bmod 4$, we must have $q^2 + 1 = 2$ and therefore $q = 1$: contradiction.

Since no odd prime $p \equiv 3 \bmod 4$ can divide $q^2 + 1$, the proof actually shows that there are infinitely many primes $p \equiv 1 \bmod 4$.

References

[1] C.W. Barnes, *The infinitude of primes; a proof using continued fractions*, L'Ens. Math. **22** (1976), 313–316