HERMITE'S IDENTITY

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Hermite's identity is the following distribution relation for the floor function:

Lemma 1. *For all real values $x \ge 0$ and all natural numbers $m \ge 1$ we have*

$$\lfloor x \rfloor + \Big\lfloor x + \frac{1}{m} \Big\rfloor + \ldots + \Big\lfloor x + \frac{m-1}{m} \Big\rfloor = \lfloor mx \rfloor. \tag{1}$$

In this note we collect a few proofs of Hermite's identity.

Hermite's techniques from [3] were studied by Giulini [2] and Basoco [1].

1. Hermite

Hermite [3] proved (1) using generating functions. Assume that a function f is given as a power series

$$f(x) = A_0 + A_1 x + A_2 x^2 + \ldots;$$

then

$$\frac{f(x)}{1-x} = A_0 + (A_0 + A_1)x + (A_0 + A_1 + A_2)x^2 + \ldots$$

as can be verified easily by multiplying through by $1 - x$.

Next

$$f(x^a) = A_0 + A_1 x^a + A_2 x^{2a} + \ldots,$$

hence

$$\frac{f(x^a)}{1-x} = \sum_{n \ge 0} (A_0 + A_1 + \ldots + A_\nu) x^n,$$

where $\nu = \lfloor \frac{n}{a} \rfloor$. For $f(x) = \frac{1}{1-x}$ this implies

$$\frac{1}{(1-x)(1-x^a)} = \sum_{n \ge 0} \Big[1 + \Big\lfloor \frac{n}{a} \Big\rfloor \Big] x^n. \tag{2}$$

This implies

$$\frac{1}{(1-x)(1-x^{ma})} = \sum_{n \ge 0} \Big\lfloor \frac{n+ma}{ma} \Big\rfloor x^n$$

since $1 + \lfloor \frac{n}{ma} \rfloor = \lfloor \frac{n+ma}{ma} \rfloor$.

Multiplying this last equation through by x^{ka} we obtain

$$\frac{x^{ka}}{(1-x)(1-x^{ma})} = \sum_{n \ge 0} \Big\lfloor \frac{n+ma}{ma} \Big\rfloor x^{n+ka} = \sum_{n \ge ka} \Big\lfloor \frac{n+(m-k)a}{ma} \Big\rfloor x^n.$$

Observe that (1) is true for very small values of x; thus it is sufficient to prove the identity for *rational* values of x.

Now the identity

$$\frac{1-x^{ma}}{1-x^a} = 1 + x^a + x^{2a} + \ldots + x^{(m-1)a}$$

implies

$$\begin{aligned}\frac{x^a}{(1-x)(1-x^a)} &= \frac{x^a(1+x^a+a^{2a}+\ldots+x^{m-1})}{(1-x)(1-x^{ma})} \\ &= \sum_{k=1}^{m} \frac{x^{ka}}{(1-x)(1-x^{ma})} \\ &= \sum_{k=1}^{m} \sum_{n\ge ka} \Big\lfloor \frac{n+(m-k)a}{ma} \Big\rfloor x^n.\end{aligned}$$

Replacing the summation index k by $m-k$ we find

$$\frac{x^a}{(1-x)(1-x^a)} = \sum_{k=0}^{m-1} \sum_{n\ge ka} \Big\lfloor \frac{n+ka}{ma} \Big\rfloor x^n.$$

On the other hand we know

$$\frac{x^a}{(1-x)(1-x^a)} = \sum_{n\ge 0} \Big\lfloor \frac{n+a}{a} \Big\rfloor x^{n+a} = \sum_{n\ge a} \Big\lfloor \frac{n}{a} \Big\rfloor x^n.$$

Comparing the coefficient of x^n and setting $z = \frac{n}{ma}$ we find

$$\lfloor mz \rfloor = \Big\lfloor \frac{n}{a} \Big\rfloor = \sum_{k=0}^{m-1} \Big\lfloor \frac{n+ka}{ma} \Big\rfloor = \sum_{k=0}^{m-1} \Big\lfloor z + \frac{k}{m} \Big\rfloor.$$

2. Weber

In his proof of the Theorem of Kronecker and Weber given in [9], Weber has to deal with the class numbers of cyclotomic number fields, and he uses a result on the greatest integer function that generalizes Hermite's identity.

For $0 < x < 1$ (and $0 \le t < m$) we have

$$\Big\lfloor x + \frac{t}{m} \Big\rfloor = \begin{cases} \lfloor \frac{t}{m} \rfloor & = 0 \qquad \text{or} \\ \lfloor \frac{t}{m} \rfloor + 1 & = 1 \end{cases}$$

according as there is an integer between $\frac{t}{m}$ and $x + \frac{t}{m}$ or not. The first case holds if and only if $1 = \lfloor \frac{t}{m} \rfloor + 1 \le x + \frac{t}{m}$, i.e., if and only if

$$m - t \le mx. \tag{3}$$

The number of values of $t \ge 0$ satisfying this inequality is clearly $\lfloor mx \rfloor$. Thus there are $\lfloor mx \rfloor$ values of t for which $\lfloor x + \frac{t}{m} \rfloor = 1$, and Hermite's identity follows.

3. Stern

Stern [8] determines an integer k satisfying

$$\lfloor x \rfloor + \frac{k}{m} \le x < \lfloor x \rfloor + \frac{k+1}{m}.$$

Multiplying through by m he finds

$$m\lfloor x \rfloor + k \le mx < m\lfloor x \rfloor + k + 1,$$

thus $\lfloor mx \rfloor = m\lfloor x \rfloor + k$. On the other hand we have

$$x + \frac{m-k-1}{m} < \lfloor x \rfloor + \frac{k+1}{m} + \frac{m-k-1}{m} = \lfloor x \rfloor + \frac{k}{m} + \frac{m-k}{m} \le x + \frac{m-k}{m},$$

and this implies

$$x + \frac{m-k-1}{m} < \lfloor x \rfloor \le x + \frac{m-k}{m}.$$

Thus each term

$$\lfloor x \rfloor, \ \Big\lfloor x + \frac{1}{m} \Big\rfloor, \ldots, \Big\lfloor x + \frac{m-k-1}{m} \Big\rfloor$$

has the value $\lfloor x \rfloor$, whereas each term

$$\Big\lfloor x + \frac{m-k}{m} \Big\rfloor, \ldots, \Big\lfloor x + \frac{m-1}{m} \Big\rfloor$$

has the value $\lfloor x \rfloor + 1$. The sum of these terms thus equals

$$(m-k)\lfloor x \rfloor + k(\lfloor x \rfloor + 1) = m\lfloor x \rfloor + k = \lfloor mx \rfloor.$$

4. Polya & Szegö

The proof by Polya & Szegö [6] is very short. Clearly it suffices to consider the case $0 \le x < 1$. Determine k such that

$$x + \frac{k-1}{m} < 1 \le x + \frac{k}{m},$$

i.e., $-k = [mx - m] = [mx] - m$. Both sides are $= m - k$.

5. Matsuoka

The elementary proof given by Matsuoka [5] was included (without attribution) in [7, Ch. 12].

Proof. Consider the function

$$f(x) = \lfloor x \rfloor + \Big\lfloor x + \frac{1}{m} \Big\rfloor + \ldots + \Big\lfloor x + \frac{m-1}{m} \Big\rfloor - \lfloor mx \rfloor.$$

It is immediately seen that $f(x + \frac{1}{m}) = f(x)$ and that $f(x) = 0$ for $0 \le x < \frac{1}{m}$. Thus $f(x) = 0$ for all real values of x, and this proves the claim. □

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