ON THE NORMALITY OF HILBERT CLASS FIELDS

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Abstract. We will provide an example of a non-normal extension of the rationals whose Hilbert class field is normal.

It is known that if $K/\mathbb{Q}$ is a normal extension, then the Hilbert class field $H = K^1$ of K is als normal over the rationals. In a question (163131) on math overflow, an anonymous user asked whether the following converse is true: if $H/\mathbb{Q}$ is normal, must $K/\mathbb{Q}$ also be normal? The answer is negative: in this note we will show that the Hilbert class field H of $K = \mathbb{Q}(\sqrt[4]{-5}\,)$ is a Galois extension of the rationals.

Before we give our example we will show that it is, in a certain sense, the simplest possible example. More exactly we will show

Proposition 1. *The Hilbert class field of a cubic non-normal extension $K/\mathbb{Q}$ is not normal.*

Proof. Assume that the Hilbert class field H of K is normal over the rationals. Let d denote the discriminant of K; then it is well known that the normal closure of K is given by $K(\sqrt{d}\,)$. Thus if $H/\mathbb{Q}$ is normal, then H must contain $\sqrt{d}$, which implies that $K(\sqrt{d}\,)/K$ is unramified.

The primes dividing d all must have even ramification index in $K(\sqrt{d}\,)/\mathbb{Q}$; since $K(\sqrt{d}\,)/K$ is unramified, the ramification index is at most 3. Thus all primes dividing d have ramification index 2 in $K(\sqrt{d}\,)/\mathbb{Q}$. Since $K(\sqrt{d}\,)/K$ is unramified, all ramified primes must have ramification index 2 already in K. This contradicts the decomposition law: ramified primes split as $p\mathfrak{O}_K = \mathfrak{p}^3$ or $p\mathfrak{O}_K = \mathfrak{p}\mathfrak{q}^2$, and in the first case the ramification index of p is divisible by 3, in the second case $\mathfrak{p}$ must ramifiy in the normal closure of K. $\square$

For non-normal quartic field, the situation is different:

Proposition 2. *The Hilbert class field H of $K = \mathbb{Q}(\sqrt[4]{-5}\,)$ is normal over $\mathbb{Q}$.*

Proof. Let $F = \mathbb{Q}(\zeta_5)$ denote the field of 5th roots of unity. We claim that KF/K is a cyclic quartic unramified extension. Since only the infinite prime and 5 ramify in $F/\mathbb{Q}$, it is sufficient to show that the infinitey primes and the primes above 5 are unramified in KF/K. Since K is totally complex, no infinite prime can ramify in any extension of K. The fact that the primes above 5 are unramified in KF/K is a simple consquence of Abhyankar's Lemma, and may also be proved directly.

Since the quadratic subextension of F is $\mathbb{Q}(\sqrt{5}\,)$, the field KF contains $\sqrt{5}$ as well as $\sqrt{-5}$, hence it contains $i = \sqrt{-1}$. Thus KF is the compositum of the normal closure $K(i)$ of K and of F, hence is normal over $\mathbb{Q}$. $\square$

Submission History

I submitted this note to `http://arxiv.org` at the beginning of May 2014. The submission was removed, and as an explanation I received the following email:

> *Your submission did not appear to be complete, in that the article appeared to contain no references, citations or they are not appearing correctly.*

I then replied

> *The article does not have any references. Of course I could include one if necessary*

but did not receive any reply within a week. Thus I included a reference to the article itself and resubmitted the manuscript on May 12. Two hours later I received the following mail:

> *It is not appropriate to add a single reference to the same paper you're writing as a reference. You must either provide a normal set of references/citations for your work, or post it in another forum.*
>
> *Your submission privileges have been revoked until you agree to follow our policies.*