SPATIAL HOMOTOPY TRUNCATION FOR PATH CONNECTED CW-COMPLEXES

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ABSTRACT. This article is concerned with spatial homology truncation for path connected CW-complexes and the following question: Which continuous maps between two compact pseudomanifolds with isolated singularities induce continuous maps between the corresponding intersection spaces, and when is this assignment functorial? Chapter 1 deals with the construction of a spatial homology truncation functor for path connected CW-complexes, which extends existing results for simply connected CW-complexes. In Chapter 2 we partially use the results of the first chapter to present different approaches to the problem of inducing maps between intersection spaces. Finally, the induced maps between reduced homology groups of intersection spaces and the induced maps between intersection homology groups will be assembled in a morphism of reflective diagrams.

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Notation

Let **Top** be the category of topological spaces and continuous maps. By a *map* between topological spaces, we always mean a contiuous map (unless otherwise stated). Let **HoTop** be the homotopy category of topological spaces (i.e. the category of topological spaces and homotopy classes of continuous maps).

A *3-diagram* Γ *of spaces* (see [5, page 3f]) is a diagram in **Top** of the form

$$X \stackrel{f}{\leftarrow} A \stackrel{g}{\rightarrow} Y.$$

Its *realization* $|\Gamma|$ is defined by the pushout of f and g, namely

$$|\Gamma| = (X \bigsqcup Y)/(f(a) \sim g(a), \text{ for all } a \in A).$$

If $f : A \hookrightarrow X$ is an inclusion, then we will also write $|\Gamma| = Y \cup_g X$. A *morphism* $\Gamma \to \Gamma'$ of 3-diagrams of spaces is a commutative diagram in **Top** of the form

$$\begin{array}{ccccc} X & \stackrel{f}{\longleftarrow} & A & \stackrel{g}{\longrightarrow} & Y \\ \downarrow \xi & & \downarrow \alpha & & \downarrow \eta \\ X' & \stackrel{f'}{\longleftarrow} & A' & \stackrel{g'}{\longrightarrow} & Y'. \end{array}$$

By the universal property of the pushout, every morphism $\Gamma \to \Gamma'$ of 3-diagrams induces a map $|\Gamma| \to |\Gamma'|$ between realizations, such that the obvious diagrams commute. If $f : A \hookrightarrow X$ is an inclusion, $Z := X' = A' = Y'$ and $f' = g' = \mathrm{id}_Z$, then we will also write $\eta \cup_g \xi : Y \cup_g X \to Z$ for the induced map between realizations.

The unit interval $[0, 1]$ will be denoted by I. The *cone* of a topological space X is defined by $\mathrm{cone}(X) = (X \times I)/(X \times \{0\})$. Let $f : X \to Y$ be a morphism in **Top**. The *mapping cylinder* $\mathrm{cyl}(f)$ of f is defined as the realization of

$$X \times I \stackrel{\text{at } 1}{\hookleftarrow} X \stackrel{f}{\to} Y.$$

The *mapping cone* $\mathrm{cone}(f)$ of $f : X \to Y$ is defined as the realization of

$$\mathrm{cone}(X) \stackrel{\text{at } 1}{\hookleftarrow} X \stackrel{f}{\to} Y.$$

In the following, **CW** denotes the category of CW-complexes and cellular maps. $\mathbf{CW}^0$ denotes the full subcategory of path connected CW-complexes and $\mathbf{CW}^1$ denotes the full subcategory of simply connected CW-complexes. Let **HoCW** be the category of CW-complexes and homotopy classes of cellular maps. Finally, let $\mathbf{HoCW}_n$ be the category of CW-complexes and rel n-skeleton homotopy classes of cellular maps. If $f : K \to L$ is a cellular map between CW-complexes, then its restriction to n-skeletons is denoted by $f^n : K^n \to L^n$. The basepoint of the n-sphere S^n will be denoted by s_0.

Introduction

This article is concerned with spatial homology truncation for path connected CW-complexes and the following question: Which continuous maps between two compact pseudomanifolds with isolated singularities induce continuous maps between the corresponding intersection spaces, and when is this assignment functorial?

A *spatial homology truncation* (Moore approximation) of a given CW-complex K in degree $k > 0$ is a cellular map $e_k : K_{<k} \to K$ from a suitable CW-complex $K_{<k}$ to K, such that e_k induces an isomorphism on (cellular) homology groups in dimensions below k and the homology groups of $K_{<k}$ vanish in dimensions k and higher. In Chapter 1, we focus on the construction of a *spatial homology truncation functor* (see [1, page viii]). This is motivated by the necessity of spatial homology truncation for CW-complexes in the construction of intersection spaces and by the above question concerning functorial properties of the intersection space construction (see Chapter 2). For this purpose, let $p : \mathbf{Top} \to \mathbf{HoTop}$ denote the natural projection functor. (Thus, p is the identity on objects and sends a continuous map to its homotopy class.) Moreover, let $i : \mathbf{C} \to \mathbf{Top}$ be a functor from a category $\mathbf{C}$ to $\mathbf{Top}$. ($\mathbf{C}$ is called a category of spaces; in practice, $\mathbf{C}$ is a subcategory of $\mathbf{Top}$ and i is the inclusion functor, or objects in $\mathbf{C}$ are spaces equipped with some extra structure and i is the forgetful functor.) A *spatial homology truncation functor* is a covariant functor

$$t_{<k} : \mathbf{C} \to \mathbf{HoTop}$$

together with a natural transformation $\mathrm{emb}_k : t_{<k} \to pi$, such that for all objects L in $\mathbf{C}$, $\mathrm{emb}_k(L) : t_{<k}(L) \to pi(L)$ is (the homotopy class of) a spatial homology truncation of $pi(L)$ in degree k. (In our setting, all spaces will be CW-complexes.) If $t_{<k}$ is not a functor but only a covariant assignment of objects and morphisms, then we will refer to it as a *spatial homology truncation assignment*. In [1, Chapter 1], the construction of such an assignment is carried out for simply connected CW-complexes and $k \geq 3$. (The assumption of simple connectivity allows the application of the Hurewicz and the Whitehead theorem.) One might be tempted to choose $\mathbf{C} = \mathbf{CW}^1$ and i as the inclusion functor. However, [1, Section 1.1.1, page 3ff] gives an example of simply connected CW-complexes X and Y, such that for obvious choices of $t_{<3}X$ and $t_{<3}Y$ it is not possible to choose $\mathrm{emb}_3(X) : t_{<3}X \to X$ and $\mathrm{emb}_3(Y) : t_{<3}Y \to Y$ in such a way that $t_{<3}$ can be defined consistently on all cellular maps $X \to Y$ (such that emb_3 becomes a natural transformation). Note that these difficulties do not occur in the Eckmann-Hilton dual of the problem, which involves Postnikov approximations instead of Moore approximations (for more details, see [1, Section 1.1.1, page 3]). The "lack of functoriality" for Moore approximations is solved by the introduction of the category $\mathbf{CW}_{k\supset\partial}$ of k-boundary split CW-complexes. Its objects are pairs consisting of an object K in $\mathbf{CW}^1$ and a direct sum complement in $C_k(K)$ of the group $Z_k(K)$ of k-cycles of K. Its morphisms are morphisms in $\mathbf{CW}^1$ which preserve the chosen sum complements. The choice $\mathbf{C} = \mathbf{CW}_{k\supset\partial}$ (and i the forgetful functor) results in the construction of a spatial homology truncation assignment $t_{<k} : \mathbf{CW}_{k\supset\partial} \to \mathbf{HoCW}_{k-1}$, which is a spatial homology truncation functor on suitable subcategories of $\mathbf{CW}_{k\supset\partial}$

(see [1, Theorem 1.41, page 51]).
In Section 1.1, we show that spatial homology truncations $e_k : K_{<k} \to K$ exist for any given CW-complex K and $k > 0$. Section 1.2 carries over the essential steps of the spatial homology truncation machine presented in [1] to path connected CW-complexes. This will result in the construction of a spatial homology truncation assignment

$$t^0_{<k} : \mathbf{CW}^0_{k\supset\partial} \to \mathbf{HoCW}_{k-1},$$

which extends $t_{<k} : \mathbf{CW}_{k\supset\partial} \to \mathbf{HoCW}_{k-1}$ and is a spatial homology truncation functor on suitable subcategories of $\mathbf{CW}^0_{k\supset\partial}$ (see Theorem 1.20). For this purpose, the category $\mathbf{CW}^0_{k\supset\partial}$ is introduced as a suitable extension of $\mathbf{CW}_{k\supset\partial}$ to path connected spaces. Objects in $\mathbf{CW}^0_{k\supset\partial}$ are objects in $\mathbf{CW}^0$ equipped with some extra structure, which is preserved by the morphisms. As a byproduct, it will be shown in Section 1.3 that every path connected CW-complex is homotopy equivalent rel 2-skeleton to a CW-complex which has a cell-basis for its group of n-cycles for all $n \geq 3$.

Given an integer $n \geq 2$ and a perversity $\overline{p}$, the intersection space construction can be applied to an n-dimensional compact topological pseudomanifold X with isolated singularities after specification of a CW-structure and a homology truncation in dimension $k = n - 1 - \overline{p}(n)$ for every link of X. If we take these pseudomanifolds equipped with the required extra structure as the objects of a category $\mathbf{P}(n, \overline{p})$ whose morphisms are continuous maps (with some additional properties) between them, then the intersection space construction can be seen as an assignment on the object level:

$$\operatorname{Ob} \mathbf{P}(n, \overline{p}) \to \operatorname{Ob} \mathbf{HoTop}.$$

Chapter 2 focuses on the problem to extend this assignment to a covariant functor $\mathbf{P}_*(n, \overline{p}) \to \mathbf{HoTop}$ on suitable subcategories $\mathbf{P}_*(n, \overline{p})$ of $\mathbf{P}(n, \overline{p})$. Section 2.1 deals with the problem in cut-off degree $k = 1$. Section 2.2 uses canonical maps for an approach to the problem in the case of pseudomanifolds X with a single isolated singularity. In Section 2.3, we restrict our attention to pseudomanifolds in $\mathbf{P}(n, \overline{p})$, whose links are equipped with a CW-structure, such that the group of k-cycles has a cell-basis, and take advantage of the fact that in this case the required spatial homology truncation in dimension k can be taken to be an inclusion of a suitable subcomplex. We will see in Section 2.4 that the independence of choices of the homotopy type of an intersection space is connected with the existence of a functor $\mathbf{P}_*(n, \overline{p}) \to \mathbf{HoTop}$ with certain properties. This viewpoint will be applied to pseudomanifolds with links in the interleaf category. Section 2.5 uses the results on functoriality of spatial homology truncation (see Chapter 1) to define, for suitable continuous maps between pseudomanifolds, induced maps between the corresponding intersection spaces in a functorial way, if all involved links are completed to objects in a suitable subcategory of $\mathbf{CW}^0_{k\supset\partial}$ and have vanishing $(k+1)$st homotopy group. Finally, Section 2.6 uses some of the constructed functors $\mathbf{P}_*(n, \overline{p}) \to \mathbf{HoTop}$ to induce morphisms between k-reflective diagrams. This relates the homomorphisms induced on reduced homology groups of intersection spaces to the homomorphisms that can be induced on intersection homology groups.

1 Spatial Homology Truncation for Path Connected CW-Complexes

Recall that, by [1, Proposition 1.6, page 12], every simply connected CW-complex K can be completed to a homological n-truncation structure $(K, K/n, h_K, K_{<n})$ for all integers $n \geq 3$ (see [1, Definition 1.4, page 11] and Definition 1.5). This proposition is central for the spatial homology truncation machine presented in [1, Chapter 1.1]. In the proof, the assumption of simple connectivity is used twice. Firstly, the Hurewicz theorem is applied in order to identify $\pi_n(L^n, L^{n-1})$ and $C_n(L)$ for simply connected CW-complexes L and $n \geq 3$ via the Hurewicz map throughout the proof. Secondly, the homological version of the Whitehead theorem is used to conclude that a certain cellular map $h' : K^n \to K/n$ constructed in the proof is a homotopy equivalence (h_K is finally taken to be a suitable homotopy inverse of h').

The result of Section 1.1 is the existence of a Moore approximation $e_n : K_{<n} \to K$ for any CW-complex K and any integer $n > 0$ (see Corollary 1.4). If K is path connected and $n \geq 2$, then it can be achieved that $K_{<n}$ is n-dimensional and e_n restricts to the identity map on $(n-1)$-skeletons (compare Proposition 1.3). The proof will make use of Proposition 1.1. This proposition states that every choice of basis in $C_n(K)$ can be realized by an n-dimensional CW-complex L satisfying $L^{n-1} = K^{n-1}$ and a cellular map $h : L \to K^n$ which restricts to the identity map on K^{n-1} and induces an isomorphism $C_n(L) \xrightarrow{\cong} C_n(K)$ sending the cell-basis of $C_n(L)$ to the chosen basis of $C_n(K)$. Modifying the proof of [1, Proposition 1.6, page 12], the general form of the Hurewicz theorem will be applied for $n \geq 2$ to conclude that the Hurewicz homomorphism $\pi_n(K^n, K^{n-1}) \to C_n(K)$ is surjective. However, Example 1.2 shows that the map $h : L \to K^n$ constructed in the proof of Proposition 1.1 is in general not a homotopy equivalence.

In Section 1.2, we introduce categories $\mathbf{CW}^0_{n\supset\partial}$ for $n \geq 3$ (Definition 1.7), such that

- $\mathbf{CW}_{n\supset\partial}$ is a full subcategory of $\mathbf{CW}^0_{n\supset\partial}$ (see Example 1.8).
- every path connected CW-complex can be completed to an object in $\mathbf{CW}^0_{n\supset\partial}$ (see Remark 1.9). (Objects (K, Σ_K) in $\mathbf{CW}^0_{n\supset\partial}$ will be objects K in $\mathbf{CW}^0$ equipped with some extra structure Σ_K.)

In order to generalize [1, Theorem 1.41, page 51] to path connected spaces, we extend $t_{<n} : \mathbf{CW}_{n\supset\partial} \to \mathbf{HoCW}_{n-1}$ to a spatial homology truncation assignment

$$t^0_{<n} : \mathbf{CW}^0_{n\supset\partial} \to \mathbf{HoCW}_{n-1}$$

(see Theorem 1.20). In particular, we construct a natural transformation $\mathrm{emb}^0_n : t^0_{<n} \to t^0_{<\infty}$, where $t^0_{<\infty} : \mathbf{CW}^0_{n\supset\partial} \to \mathbf{HoCW}_{n-1}$ is the natural projection functor. (Define $t^0_{<\infty}$ as the composition of the forgetful functor $\mathbf{CW}^0_{n\supset\partial} \to \mathbf{CW}^0$ and the natural projection functor $\mathbf{CW}^0 \to \mathbf{HoCW}_{n-1}$.) For every object (K, Σ_K) in $\mathbf{CW}^0_{n\supset\partial}$,

$$\mathrm{emb}^0_n(K, \Sigma_K) : t^0_{<n}(K, \Sigma_K) \to t^0_{<\infty}(K, \Sigma_K) = K$$

is a spatial homology truncation of K in degree n. If $(K, \Sigma_K) = (K, Y_K)$ is an object in $\mathbf{CW}_{n\supset\partial}$, then $\mathrm{emb}^0_n(K, Y_K) = \mathrm{emb}_n(K, Y_K)$.

In Section 1.3, we will conclude from Proposition 1.13 and the Whitehead theorem that every path connected CW-complex is homotopy equivalent rel 2-skeleton to a CW-complex having a cell-basis for its group of n-cycles for all $n \geq 3$.

1.1 Existence of Moore Approximations for Path Connected CW-Complexes

A spatial homology truncation (Moore approximation) of a given CW-complex K in degree $k > 0$ is a pair $(K_{<k}, e_k)$, where $K_{<k}$ is a CW-complex and $e_k : K_{<k} \to K$ is a cellular map, which induces isomorphisms $H_r(K_{<k}) \xrightarrow{\cong} H_r(K)$ for $r < k$ and such that $H_r(K_{<k}) = 0$ for $r \geq k$ (compare [5, page 6]). The purpose of this section is to show that Moore approximations exist for all CW-complexes K and all integers $k > 0$.

First, recall some basic facts about the Hurewicz map. Let $n \geq 1$ be an integer. Given a pointed pair (X, A, x_0), the Hurewicz map is defined by

$$\mathrm{Hur} : \pi_n(X, A, x_0) \to H_n(X, A), \quad \mathrm{Hur}([f]) = f_*(\nu),$$

where $f_* : H_n(D^n, \partial D^n) \to H_n(X, A)$ is induced by $f : (D^n, \partial D^n, s_0) \to (X, A, x_0)$ and ν is a fixed generator of $H_n(D^n, \partial D^n) \cong \mathbb{Z}$. The Hurewicz map is natural: If $\varphi : (X, A, x_0) \to (Y, B, y_0)$ is a map of pointed pairs, then the following diagram commutes (note that $\varphi_* \,\mathrm{Hur}([f]) = \varphi_* f_*(\nu) = (\varphi \circ f)_*(\nu) = \mathrm{Hur}([\varphi \circ f]) = \mathrm{Hur}\, \varphi_*([f])$):

$$\begin{array}{ccc}
\pi_n(X, A, x_0) & \xrightarrow{\varphi_*} & \pi_n(Y, B, y_0) \\
\Big\downarrow{\scriptstyle \mathrm{Hur}} & & \Big\downarrow{\scriptstyle \mathrm{Hur}} \\
H_n(X, A) & \xrightarrow{\varphi_*} & H_n(Y, B).
\end{array}$$

The Hurewicz map is a group homomorphism for $n \geq 2$ by [2, Proposition 4.36, page 369]. If $n \geq 2$ and K is a CW-complex with basepoint $k_0 \in K^{n-1}$, then there is in particular the Hurewicz homomorphism

$$\mathrm{Hur} : \pi_n\left(K^n, K^{n-1}, k_0\right) \to H_n\left(K^n, K^{n-1}\right), \quad \mathrm{Hur}([f]) = f_*(\nu).$$

Let $\left\{e^n_\gamma\right\}$ be the set of n-cells of K. The characteristic maps

$$\chi\left(e^n_\gamma\right) : (D^n, \partial D^n) \to \left(K^n, K^{n-1}\right)$$

induce homomorphisms on homology groups:

$$\chi\left(e^n_\gamma\right)_* : H_n(D^n, \partial D^n) \to H_n\left(K^n, K^{n-1}\right).$$

By [3, Lemma 10.1, page 201], a basis of the free abelian group $H_n(K^n, K^{n-1})$ is given by $\{\chi\left(e^n_\gamma\right)_*(\nu)\}$. We make the following identification:

$$\begin{array}{rcl}
H_n\left(K^n, K^{n-1}\right) = \bigoplus_\gamma \mathbb{Z}\, \chi\left(e^n_\gamma\right)_*(\nu) & \xrightarrow{\cong} & \bigoplus_\gamma \mathbb{Z}\, e^n_\gamma = C_n(K), \\
\chi\left(e^n_\gamma\right)_*(\nu) & \mapsto & e^n_\gamma \qquad \forall\, \gamma.
\end{array}$$

The following proposition shows that for a path connected CW-complex, every choice of basis in the nth cellular chain group ($n \geq 2$) can be realized topologically.

1.1 Proposition. *Let $n \geq 2$ be an integer and let K be a path connected CW-complex. Given a basis $\{\theta_\alpha\}$ of $C_n(K)$, there exist*

- *an n-dimensional CW-complex L satisfying $L^{n-1} = K^{n-1}$ and*
- *a cellular map $h : L \to K^n$ which restricts to the identity map on K^{n-1} and such that h induces an isomorphism on the nth cellular chain groups:*

$$h_* : C_n(L) \xrightarrow{\cong} C_n(K^n)$$

which sends the cell-basis of $C_n(L)$ to the given basis $\{\theta_\alpha\}$ of $C_n(K^n)$.

In particular, the cellular map h induces a chain isomorphism. Therefore, it induces an isomorphism $H_r(L) \cong H_r(K^n)$ for all integers r.

Proof. Choose a basepoint $x_0 \in K^{n-1}$. The general form of the Hurewicz theorem [2, Theorem 4.37, page 371] is applied to the pointed CW-pair (K^n, K^{n-1}, x_0): Since $K^n - K^{n-1}$ has only cells of dimension $n > n-1$, the pair (K^n, K^{n-1}) is $(n-1)$-connected according to [2, Corollary 4.12, page 351]. Moreover, K^{n-1} and K^n are path connected, since $n \geq 2$. (Note that if X is a CW-complex and $m \geq 1$ is an integer, then X is path connected if and only if X^m is path connected.) Therefore, by the general form of the Hurewicz theorem, the following Hurewicz homomorphism is surjective:

$$\mathrm{Hur} : \pi_n\left(K^n, K^{n-1}, x_0\right) \twoheadrightarrow C_n(K).$$

For every α choose a preimage $[\vartheta_\alpha] \in \pi_n(K^n, K^{n-1}, x_0)$ of $\theta_\alpha \in C_n(K)$ represented by

$$\vartheta_\alpha : \left(D^n, S^{n-1}, s_0\right) \to \left(K^n, K^{n-1}, x_0\right), \quad \text{where } \mathrm{Hur}([\vartheta_\alpha]) = \theta_\alpha.$$

Using the restrictions $a_\alpha := \vartheta_\alpha|_{S^{n-1}} : S^{n-1} \to K^{n-1}$ and taking new n-cells w_α, define

$$L := K^{n-1} \cup \bigcup_{a_\alpha} w_\alpha, \quad \text{where } [\chi(w_\alpha)] \in \pi_n\left(L, K^{n-1}, x_0\right).$$

Again, there is a corresponding surjective Hurewicz homomorphism

$$\mathrm{Hur} : \pi_n\left(L, K^{n-1}, x_0\right) \twoheadrightarrow C_n(L), \quad \text{where } \mathrm{Hur}([\chi(w_\alpha)]) = \chi(w_\alpha)_*(\nu) = w_\alpha.$$

Since $\chi(w_\alpha)|_{S^{n-1}} = a_\alpha = \vartheta_\alpha|_{S^{n-1}}$, the morphism of 3-diagrams of spaces

$$\begin{array}{ccccc}
K^{n-1} & \xleftarrow{\coprod a_\alpha} & \coprod_\alpha S^{n-1} & \xrightarrow{\text{incl}} & \coprod_\alpha D^n \\
\downarrow{=} & & \downarrow{\coprod a_\alpha} & & \downarrow{\coprod \vartheta_\alpha} \\
K^{n-1} & \xleftarrow{=} & K^{n-1} & \xrightarrow{\text{incl}} & K^n
\end{array}$$

induces a map $h : (L, K^{n-1}, x_0) \to (K^n, K^{n-1}, x_0)$ with the following properties:

$$h(x) = x \ \ \forall\, x \in K^{n-1}, \qquad h \circ \chi(w_\alpha) = \vartheta_\alpha \ \ \forall\, \alpha.$$

The following diagram commutes by naturality of Hurewicz maps:

$$\begin{array}{ccc} \pi_n\left(L, K^{n-1}, x_0\right) & \xrightarrow{h_*} & \pi_n\left(K^n, K^{n-1}, x_0\right) \\ \Big\downarrow \text{Hur} & & \Big\downarrow \text{Hur} \\ C_n\left(L\right) & \xrightarrow{h_*} & C_n\left(K^n\right). \end{array}$$

All in all, every n-cell $w_\alpha \in C_n\left(L\right)$ satisfies

$$h_*\left(w_\alpha\right) = h_* \operatorname{Hur}([\chi\left(w_\alpha\right)]) = \operatorname{Hur} h_*([\chi\left(w_\alpha\right)]) = \operatorname{Hur}([h \circ \chi\left(w_\alpha\right)]) = \operatorname{Hur}([\vartheta_\alpha]) = \theta_\alpha.$$

■

Compared to the proof of the original proposition, see [1, Proposition 1.6, page 12], the proof of Proposition 1.1 is different in the following way. The homotopy extension property is used in the original proof to construct a map $h' : K^n \to K/n \ (\stackrel{\wedge}{=} L)$, which turns out to be a homotopy equivalence by application of the homological version of the Whitehead theorem, and h is taken to be its homotopy inverse. The present proof, however, yields a direct construction of h with the desired properties, which does not make use of the homotopy extension property and the homological version of the Whitehead theorem. The map h obtained in the proof of Proposition 1.1 is not a homotopy equivalence in general, as the following example shows. (Nonetheless, the proof of Proposition 1.13 shows that if h is constructed more carefully, then it can be achieved that h is a homotopy equivalence.) The following example serves as the leading example for Chapter 1.

Example. 1.2 Let $n \geq 2$ and consider the n-dimensional path connected CW-complex $K := S^1 \vee S^n$. It consists of a single 0-cell x_0, which is taken as a basepoint for K, a single 1-cell and a single n-cell θ. Thus, $\{\theta\}$ forms a basis of $C_n\left(K\right)$. Following the construction in the proof of Proposition 1.1, we will construct a space L and a map $h : L \to K$ with the desired properties, such that h is not a homotopy equivalence.

Consider the universal cover $\widetilde{K} \xrightarrow{p} K$ of K. By [2, Example 4.27, page 364] $\widetilde{K}$ consists of the real line $\mathbb{R}$ with a copy S^n_k of the n-sphere attached at every integer point $k \in \mathbb{R}$ and p is the obvious covering map, which maps all integers to x_0 and which restricts to identity maps $S^n_k = S^n$. Let e_k be the n-cell of the CW-complex $\widetilde{K}$ corresponding to the n-sphere S^n_k.

By naturality of Hurewicz maps, the map $p : (\widetilde{K}, \mathbb{R}, 0) \to (K, S^1, x_0)$ of pointed pairs induces the commutative diagram

$$\begin{array}{ccc} \pi_n(\widetilde{K}, \mathbb{R}, 0) & \xrightarrow{p_* \ \cong} & \pi_n\left(K, S^1, x_0\right) \\ \Big\downarrow \text{Hur} \ \cong & & \Big\downarrow \text{Hur} \\ C_n(\widetilde{K}) & \xrightarrow{p_*} & C_n\left(K\right) \end{array} \qquad (*).$$

Let us look at the marked isomorphisms. The left Hurewicz map is an isomorphism by the Hurewicz theorem [2, Theorem 4.37, page 371], since $\mathbb{R}$ is simply connected. Next, we show that $p_* : \pi_n(\widetilde{K}, \mathbb{R}, 0) \to \pi_n(K, S^1, x_0)$ is an isomorphism. This follows from the following commutative diagram with exact rows, which is obtained by naturality of the long exact sequence of homotopy groups:

$$\begin{array}{ccccccc}
\pi_n(\mathbb{R}, 0) & \longrightarrow & \pi_n(\widetilde{K}, 0) & \xrightarrow{\cong} & \pi_n(\widetilde{K}, \mathbb{R}, 0) & \longrightarrow & \pi_{n-1}(\mathbb{R}, 0) \\
\downarrow & & \downarrow {\scriptstyle p_* \ \cong} & & \downarrow {\scriptstyle p_*} & & \downarrow \\
\pi_n(S^1, x_0) & \longrightarrow & \pi_n(K, x_0) & \xrightarrow{\cong} & \pi_n(K, S^1, x_0) & \longrightarrow & \pi_{n-1}(S^1, x_0) .
\end{array}$$

The marked isomorphisms can be explained as follows. The covering map $p : (\widetilde{K}, 0) \to (K, x_0)$ induces an isomorphism $p_* : \pi_n(\widetilde{K}, 0) \xrightarrow{\cong} \pi_n(K, x_0)$ by [2, Proposition 4.1, page 342]. As $\mathbb{R}$ is contractible, the map $\pi_n(\widetilde{K}, 0) \to \pi_n(\widetilde{K}, \mathbb{R}, 0)$ is an isomorphism by exactness of the first row. By exactness of the second row, the map $\pi_n(K, x_0) \to \pi_n(K, S^1, x_0)$ is an isomorphism for $n \geq 3$, since the higher homotopy groups of S^1 vanish. In the case $n = 2$ it suffices to show that $\pi_2(K, S^1, x_0) \to \pi_1(S^1, x_0)$ is the zero map. This is true by exactness of the following portion of the long exact homotopy sequence of the pair (K, S^1, x_0),

$$\pi_2\left(K, S^1, x_0\right) \to \pi_1\left(S^1, x_0\right) \xrightarrow{\cong} \pi_1(K, x_0),$$

where the inclusion $S^1 \hookrightarrow K = S^1 \vee S^2$ induces an isomorphism on fundamental groups by the Seifert-Van Kampen theorem.

The element $T := 2e_0 - e_1 \in C_n(\widetilde{K})$ satisfies $p_*(T) = 2\theta - \theta = \theta \in C_n(K)$. Following the isomorphisms in diagram $(*)$, T corresponds to a homotopy class

$$[\vartheta] := p_* \operatorname{Hur}^{-1}(T) \in \pi_n\left(K, S^1, x_0\right), \quad \text{where } \operatorname{Hur}([\vartheta]) = p_*(T) = \theta.$$

Therefore, $[\vartheta]$ is a preimage of θ under the Hurewicz homomorphism

$$\operatorname{Hur} : \pi_n\left(K, S^1, x_0\right) \twoheadrightarrow C_n(K) = \mathbb{Z}\theta.$$

We may assume that $\vartheta|_{S^{n-1}}$ is the constant map mapping all points to x_0, because the representative $\vartheta : (D^n, S^{n-1}, s_0) \to (K, S^1, x_0)$ is homotopic to such a map through maps $(D^n, S^{n-1}, s_0) \to (K, S^1, x_0)$. The reason is that for $n \geq 3$, the restriction $\vartheta|_{S^{n-1}} : (S^{n-1}, s_0) \to (S^1, x_0)$ is nullhomotopic rel s_0, since $\pi_{n-1}(S^1, x_0)$ vanishes in this case. For $n = 2$, $\vartheta|_{S^1}$ is also nullhomotopic rel s_0, since it was shown above that the restriction map $\pi_2(K, S^1, x_0) \to \pi_1(S^1, x_0)$ is the zero map.

The choice of ϑ can be used in the proof of Proposition 1.1 to construct the desired cellular map $h : L \to K$. For the n-dimensional CW-complex L we get back the original CW-complex K by choice of ϑ. The cellular map $h : K \to K$ is the unique map which restricts to the identity map on S^1 and satisfies $h \circ \chi(\theta) = \vartheta$.

By the lifting criterion [2, Proposition 1.33, page 61], the composition $h \circ p : (\widetilde{K}, 0) \to (K, x_0)$ can be lifted under $p : (\widetilde{K}, 0) \to (K, x_0)$ to a map $\widetilde{h} : (\widetilde{K}, 0) \to (\widetilde{K}, 0)$:

$$\begin{array}{ccc} (\widetilde{K},0) & \xrightarrow{\widetilde{h}} & (\widetilde{K},0) \\ \downarrow p & & \downarrow p \\ (K,x_0) & \xrightarrow{h} & (K,x_0). \end{array}$$

Note that $\widetilde{h}$ restricts to the identity map on $\mathbb{R} \subset \widetilde{K}$. (This can be seen as follows: Take any $x \in \mathbb{R}$ and choose a path $\gamma : I \to \mathbb{R} \hookrightarrow \widetilde{K}$ between $\gamma(0) = 0$ and $\gamma(1) = x$. By the path lifting property [2, page 60], γ is the unique lift of $p \circ \gamma : I \to S^1 \hookrightarrow K$ which sends $0 \in I$ to $0 \in \widetilde{K}$. But $p \circ \widetilde{h} \circ \gamma = h \circ p \circ \gamma = p \circ \gamma$ (h restricts to the identity map on S^1). Thus, $\widetilde{h} \circ \gamma$ is also a lift of $p \circ \gamma$ sending 0 to 0. By uniqueness, $\widetilde{h} \circ \gamma = \gamma$. Evaluation at $1 \in I$ yields $\widetilde{h}(x) = x$.) In particular, $\widetilde{h}$ is cellular, and we claim that

$$\widetilde{h}_* : C_n(\widetilde{K}) \to C_n(\widetilde{K}), \quad \widetilde{h}_*(e_k) = 2e_k - e_{k+1} \quad \forall\, k \in \mathbb{Z} \qquad (**).$$

Let us prove $(**)$ for $k = 0$ first. Consider the following commutative diagram, which results from $p \circ \widetilde{h} = h \circ p$ and the naturality of Hurewicz maps (the marked isomorphisms have already been explained):

$$\begin{array}{ccccc} \pi_n(K, S^1, x_0) & \xleftarrow{p_*\ \cong} & \pi_n(\widetilde{K}, \mathbb{R}, 0) & \xrightarrow{\mathrm{Hur}\ \cong} & C_n(\widetilde{K}) \\ \downarrow h_* & & \downarrow \widetilde{h}_* & & \downarrow \widetilde{h}_* \\ \pi_n(K, S^1, x_0) & \xleftarrow{p_*\ \cong} & \pi_n(\widetilde{K}, \mathbb{R}, 0) & \xrightarrow{\mathrm{Hur}\ \cong} & C_n(\widetilde{K}). \end{array}$$

The element $[\chi(\theta)] \in \pi_n(K, S^1, x_0)$ satisfies $h_*[\chi(\theta)] = [h \circ \chi(\theta)] = [\vartheta]$. We show that the elements $[\chi(\theta)]$ and $[\vartheta]$ in $\pi_n(K, S^1, x_0)$ correspond to e_0 and T in $C_n(\widetilde{K})$ under the isomorphism $\mathrm{Hur} \circ p_*^{-1}$. The second corresesopondence is clear by definition of $[\vartheta]$. To see the first correspondence, note that the element $[\chi(e_0)] \in \pi_n(\widetilde{K}, \mathbb{R}, 0)$ is mapped by Hur to $e_0 \in C_n(\widetilde{K})$ and by p_* to $[\chi(\theta)] \in \pi_n(K, S^1, x_0)$. All in all, $\widetilde{h}_*(e_0) = T = 2e_0 - e_1$.

To prove $(**)$ for arbitrary $k \in \mathbb{Z}$, consider the cellular deck transformation $\tau_k : \widetilde{K} \to \widetilde{K}$, which is given by the shift $x \mapsto x + k$ for $x \in \mathbb{R} \subset \widetilde{K}$ and restricts to identity maps $S^n_m = S^n_{m+k}$ for all m. Hence, $\tau_k \circ \chi(e_m) = \chi(e_{m+k})$ for all m. Thus, τ_k induces the automorphism on $C_n(\widetilde{K})$, which is given by the shift $e_m \mapsto e_{m+k}$ for all m. Note that $(**)$ follows from $\widetilde{h} \circ \tau_k = \tau_k \circ \widetilde{h}$. This is clear on $\mathbb{R} \subset \widetilde{K}$, where $\widetilde{h}$ restricts to the identity map. It remains to show that $\widetilde{h} \circ \tau_k \circ \chi(e_m) = \tau_k \circ \widetilde{h} \circ \chi(e_m)$ for all m. In fact, both sides of the equation are lifts of $h \circ p \circ \chi(e_k) : S^n \to K$ under $p : \widetilde{K} \to K$ which send s_0 to k. Thus, they must agree by the unique lifting property [2, Proposition 1.34, page 62].

Finally, we use $(**)$ to show that h is not a homotopy equivalence. Otherwise, by [2, Exercise 2, page 358], h induces an isomorphism on $\pi_n(K, x_0)$. The covering map p induces an isomorphism $p_* : \pi_n(\widetilde{K}, 0) \xrightarrow{\cong} \pi_n(K, x_0)$ $(n \geq 2)$. Therefore, $\widetilde{h}$ induces an isomorphism on $\pi_n(\widetilde{K}, 0)$. This homotopy group can be identified with $H_n(\widetilde{K})$ via the Hurewicz isomorphism, because $\widetilde{K}$ is $(n-1)$-connected. Consequently, $\widetilde{h}$ induces an isomorphism on $H_n(\widetilde{K}) = C_n(\widetilde{K})$ (note that $\ker \partial_n^{(\widetilde{K})} = C_n\left(\widetilde{K}\right)$ and $\operatorname{im} \partial_{n+1}^{(\widetilde{K})} = 0$). But $\widetilde{h}_* : C_n(\widetilde{K}) \to C_n(\widetilde{K})$ cannot be surjective, since the cell e_0 are not in the image of $\widetilde{h}_*$. Otherwise, for suitable integers $a < b$, c_k and d_k (without loss of generality, $c_a \neq 0$),

$$e_0 = \widetilde{h}_*\left(\sum_{k=a}^{b} c_k e_k\right) \overset{(**)}{=} \sum_{k=a}^{b} c_k\left(2e_k - e_{k+1}\right) = 2c_a e_a + \sum_{k=a+1}^{b+1} d_k e_k.$$

Since $\{e_k\}_{k\in\mathbb{Z}}$ forms a basis of $C_n(\widetilde{K})$, it follows from $c_a \neq 0$ that $a = 0$ and $d_k = 0$ for all k. Hence, $2c_0 = 1$, which is impossible, since c_0 is an integer.

The following proposition shows the existence of Moore approximations for any path connected CW-complex and any integer ≥ 2.

1.3 Proposition. *Let K be a path connected CW-complex. Given an integer $n \geq 2$, there exists an n-dimensional CW-complex $K_{<n}$ such that $(K_{<n})^{n-1} = K^{n-1}$ and a cellular map $e_n : K_{<n} \to K$ which restricts to $\mathrm{id}_{K^{n-1}}$ and such that e_n induces an isomorphism $e_{n*} : H_r(K_{<n}) \xrightarrow{\cong} H_r(K)$ for $r < n$ and $H_r(K_{<n}) = 0$ for $r \geq n$.*

Proof. Since $\operatorname{im} \partial_n \,(\subset C_{n-1}(K))$ is free abelian, one can choose a splitting

$$s : \operatorname{im} \partial_n \to C_n(K)$$

of $\partial_n : C_n(K) \twoheadrightarrow \operatorname{im} \partial_n$. Writing $Z_n(K) = \ker \partial_n$ and $Y = \operatorname{im} s$, we have

$$C_n(K) = Z_n(K) \oplus Y.$$

Choose bases $\{\zeta_\beta\}$ of $Z_n(K)$ and $\{\eta_\alpha\}$ of Y. This yields a basis $\{\zeta_\beta\} \cup \{\eta_\alpha\}$ of $C_n(K)$. Application of Proposition 1.1 to K and to this basis of $C_n(K)$ yields

- an n-dimensional CW-complex L with $(n-1)$-skeleton K^{n-1} and
- a cellular map $h : L \to K^n$ which restricts to the identity map on K^{n-1} and which induces an isomorphism

$$h_* : C_n(L) \xrightarrow{\cong} C_n(K^n)$$

  sending the cell-basis of $C_n(L)$ to the given basis $\{\zeta_\beta\} \cup \{\eta_\alpha\}$ of $C_n(K^n)$.

Since $h : L \to K^n$ is cellular, it induces the commutative diagram

$$\begin{array}{ccc} C_n(L) & \xrightarrow{h_*} & C_n(K^n) \\ \downarrow \partial_n & & \downarrow \partial_n \\ C_{n-1}(L) & \xrightarrow{h_*} & C_{n-1}(K^n). \end{array}$$

The map h_* in the first line is an isomorphism. Since $h : L \to K^n$ restricts to the identity map on the common $(n-1)$-skeleton K^{n-1}, the map h_* in the second line is given by the identity map on $C_{n-1}(L) = C_{n-1}(K^n)$. Thus, commutativity implies that the isomorphism $h_* : C_n(L) \xrightarrow{\cong} C_n(K^n)$ restricts to an isomorphism

$$h_*| : Z_n(L) \xrightarrow{\cong} Z_n(K^n).$$

The inverse map $h_*|^{-1} = h_*^{-1}| : Z_n(K^n) \xrightarrow{\cong} Z_n(L)$ sends the basis $\{\zeta_\beta\}$ of $Z_n(K^n)$ to a basis $\{h_*^{-1}(\zeta_\beta)\}$ of $Z_n(L)$. This basis consists of n-cells of L by construction of h. Thus, the n-dimensional CW-complex L has a basis of cells for its group of n-cycles. By [1, Lemma 1.2, page 6], L is n-segmented (see [1, Definition 1.1, page 6]). By [1, Proposition 1.3, page 7], there is a unique subcomplex $K_{<n} \subset L$ satisfying the properties (1.1) and (1.2) of [1, Definition 1.1, page 6] and such that $(K_{<n})^{n-1} = K^{n-1}$. ($K_{<n}$ is obtained from L by taking away the n-cycle cells.) The cellular map

$$e_n : K_{<n} \overset{i}{\hookrightarrow} L \xrightarrow{h} K^n \overset{j}{\hookrightarrow} K$$

has the required properties, where $i : K_{<n} \hookrightarrow L$ and $j : K^n \hookrightarrow K$ are the inclusions:
For $r \geq n$ one has $H_r(K_{<n}) = 0$ by property (1.1) of [1, Definition 1.1, page 6].
For $r < n$ the induced map $e_{n*} : H_r(K_{<n}) \to H_r(K)$ factorizes as

$$e_{n*} : H_r(K_{<n}) \xrightarrow{i_*} H_r(L) \xrightarrow{h_*} H_r(K^n) \xrightarrow{j_*} H_r(K),$$

where i_* is an isomorphism by property (1.2) of [1, Definition 1.1, page 6] and j_* is an isomorphism, since cells of dimension $> n$ have no influence on $H_r(K)$ for $r < n$. Finally, h_* is an isomorphism, since h induces a chain isomorphism. ∎

1.4 Corollary. *Given a CW-complex K and an integer $n > 0$, there exists a Moore approximation $e_n : K_{<n} \to K$.*

Proof. If K is path connected and $n \geq 2$, then the claim follows from Proposition 1.3. If K is path connected and $n = 1$, then one can take e_1 to be the inclusion of a 0-cell $K_{<1} = k_0 \hookrightarrow K$. In the general case, write K as the disjoint union of its connected components $K^{(\alpha)}$. Then, $K^{(\alpha)}$ is a connected CW-complex for every α and in particular path connected. For every α, take a Moore approximation $e_n^{(\alpha)} : K_{<n}^{(\alpha)} \to K^{(\alpha)}$. Then, $\bigsqcup_\alpha e_n^{(\alpha)} : \bigsqcup_\alpha K_{<n}^{(\alpha)} \to \bigsqcup_\alpha K^{(\alpha)} = K$ is a valid Moore approximation of K by the additivity axiom for homology (see [3, Definition 6.1, page 183]). ∎

1.2 Spatial Homology Truncation for Path Connected CW-Complexes

Let $n \geq 3$ be an integer. The following definition extends the concept of a homological n-truncation structure (see [1, Definition 1.4, page 11]) to path connected spaces:

Definition. 1.5 A *(homological) n-truncation structure* is a quadruple $(K, K/n, h_K, K_{<n})$, where

1. K is a path connected CW-complex.
2. K/n is an n-dimensional CW-complex with $(K/n)^{n-1} = K^{n-1}$ and such that $Z_n(K/n)$ has a cell-basis.
3. $h_K : K/n \to K^n$ is a cellular map which restricts to the identity map on K^{n-1} and which is a homotopy equivalence rel K^{n-1}.
4. $K_{<n} \subset K/n$ is the uniquely determined subcomplex with properties (1.1) and (1.2) of [1, Definition 1.1, page 6] and such that $(K_{<n})^{n-1} = K^{n-1}$.

Next, we define the rel $(n-1)$-skeleton homotopy category $\mathbf{HoCW}^0_{\supset<n}$, which contains $\mathbf{HoCW}_{\supset<n}$ (see [1, page 26f]) as a full subcategory:

Definition. 1.6 The category $\mathbf{HoCW}^0_{\supset<n}$ consists of the following objects and morphisms:

- Objects in $\mathbf{HoCW}^0_{\supset<n}$ are n-truncation structures as in Definition 1.5.
- A morphism $F : (K, K/n, h_K, K_{<n}) \to (L, L/n, h_L, L_{<n})$ in $\mathbf{HoCW}^0_{\supset<n}$ is a quadruple $F = ([f], [f_n], [f/n], [f_{<n}])$ represented by a diagram

$$\begin{array}{ccccccc} K_{<n} & \xrightarrow{i_K\,=\,\mathrm{incl}} & K/n & \xrightarrow{h_K} & K^n & \xrightarrow{j_K\,=\,\mathrm{incl}} & K \\ \downarrow{\scriptstyle f_{<n}} & & \downarrow{\scriptstyle f/n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f} \\ L_{<n} & \xrightarrow{i_L\,=\,\mathrm{incl}} & L/n & \xrightarrow{h_L} & L^n & \xrightarrow{j_L\,=\,\mathrm{incl}} & L, \end{array}$$

  such that all squares commute up to homotopy rel K^{n-1}. (This agrees with the definition of morphisms in $\mathbf{HoCW}_{\supset<n}$. Note that $f_n \neq f^n$ in general.)

The main step in the proof of [1, Theorem 1.41, page 51] is the construction of a covariant assignment $\tau_{<n} : \mathbf{CW}_{n\supset\partial} \to \mathbf{HoCW}_{\supset<n}$ of objects and morphisms (see [1, page 29ff]). In order to generalize this theorem to path connected CW-complexes (see Theorem 1.20), we extend the category $\mathbf{CW}_{n\supset\partial}$ of n-boundary-split CW-complexes (compare [1, Definition 1.22, page 28]) to the category $\mathbf{CW}^0_{n\supset\partial}$ (see Definition 1.7 and Example 1.8). Objects in $\mathbf{CW}^0_{n\supset\partial}$ will be objects in $\mathbf{CW}^0$ equipped with some extra structure, such that every object in $\mathbf{CW}^0$ can be completed to an object in $\mathbf{CW}^0_{n\supset\partial}$ (see Remark 1.9). Afterwards, we will extend $\tau_{<n}$ to a covariant assignment

$$\tau^0_{<n} : \mathbf{CW}^0_{n\supset\partial} \to \mathbf{HoCW}^0_{\supset<n}$$

of objects and morphisms (see Corollary 1.19). The definition of $\tau^0_{<n}$ on objects will make use of Proposition 1.13, which generalizes [1, Proposition 1.6, page 12]. Its definition on morphisms will make use of Proposition 1.14, which is a generalized version of the compression theorem [1, Theorem 1.32, page 35].

First, recall some facts about covering spaces of CW-complexes.
Every CW-complex is Hausdorff by [2, Proposition A.3, page 522] and locally contractible by [2, Proposition A.4, page 523]. In particular, every CW-complex is locally path connected and semilocally 1-connected (see [3, Definition 8.3, page 155]).
Let $p : X \to Y$ be a covering map (see [3, Definition 3.1, page 139]). If Y is a CW-complex, then we will assume in the following that X is the CW-complex obtained by taking as characteristic maps all possible lifts of all characteristic maps of Y (compare [3, Theorem 8.10, page 198]). Note that p becomes a cellular map satisfying $p^{-1}(Y^m) = X^m$ for all $m \geq 0$. The restriction $p^m : X^m \to Y^m$ to m-skeletons is again a covering map for all $m \geq 1$ (X^m and Y^m are path connected for $m \geq 1$).
Now let $p : X \to Y$ and $p' : X' \to Y'$ be covering maps, where Y and Y' are CW-complexes. Assume that X is simply connected. If $f : Y \to Y'$ is a cellular map, then the composition $f \circ p$ has a lift $\widetilde{f}$ under p', which is unique after specifying the image of one point by [3, Corollary 4.2, page 144]:

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{f}} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow{f} & Y'. \end{array}$$

In other words, if $x \in X$ and $x' \in X'$ are points such that $f(p(x)) = p'(x')$, then there is a unique map $\widetilde{f} : X \to X'$ which sends x to x' and makes the previous diagram commute. Note that $\widetilde{f}$ is cellular, because for all $m \geq 0$ we have

$$\widetilde{f}(X^m) \subset (p'^{-1} \circ p' \circ \widetilde{f})(X^m) = (p'^{-1} \circ f \circ p)(X^m) \subset p'^{-1}(Y'^m) = X'^m.$$

If $g : X^m \to X'^m$ is a lift of $f^m \circ p^m$ under the covering map p'^m for some $m \geq 1$, then there exists a unique lift $\widetilde{f}$ of $f \circ p$ under p' which restricts to g on m-skeletons. (Choose $x \in X^m$ and let $\widetilde{f} : X \to X'$ be the unique lift of $f \circ p$ under p' which sends x to $g(x)$. Its restriction $\widetilde{f}^m : X^m \to X'^m$ to m-skeletons is a lift of $f^m \circ p^m$ under p'^m, which agrees with g at x. Hence, $\widetilde{f}^m = g$ by uniqueness.)
Finally, note that every path connected CW-complex K has a universal cover $p_K : \widetilde{K} \to K$ by [3, Theorem 8.4, page 155].

Definition. 1.7 The category $\mathbf{CW}^0_{n\supset\partial}$ consists of the following objects and morphisms:

- Objects in $\mathbf{CW}^0_{n\supset\partial}$ are quadruples $\left(K, Y_K, \overline{K}, q_K\right)$, where
  1. K is a path connected CW-complex.
  2. $Y_K \subset C_n(K)$ is a subgroup which arises as the image of some splitting of the boundary map $\partial_n : C_n(K) \twoheadrightarrow \operatorname{im} \partial_n \,(\subset C_{n-1}(K))$.
  3. $\overline{K}$ is an n-dimensional CW-complex, such that
     (i) $\overline{K}^{n-1} = \widetilde{K}^{n-1}$ and $\overline{K} \subset \widetilde{K}^n$, where $p_K : \widetilde{K} \to K$ is the universal cover of K. (Hence, $p_K^{n-1} : \widetilde{K}^{n-1} \to K^{n-1}$ is the universal cover of K^{n-1} and $p_K^n : \widetilde{K}^n \to K^n$ is the universal cover of K^n, since $n \geq 3$.)

(ii) $p_K^n \circ j_K \circ \chi\left(\overline{e}_\gamma^n\right) = \chi\left(e_\gamma^n\right)$ for all γ, where $\left\{e_\gamma^n\right\}$ are the n-cells of K, $\left\{\overline{e}_\gamma^n\right\}$ are the n-cells of $\overline{K}$ and $j_K : \overline{K} \hookrightarrow \widetilde{K}^n$ denotes the inclusion.

4. $q_K = p_K^n \circ j_K : \overline{K} \to K^n$. (Thus, q_K is uniquely determined by $\overline{K}$.)

Since $\overline{K}^{n-1} = \widetilde{K}^{n-1}$, the restriction of q_K to $(n-1)$-skeletons is $p_K^{n-1} : \widetilde{K}^{n-1} \to K^{n-1}$. By 3$(ii)$, we have $q_K \circ \chi\left(\overline{e}_\gamma^n\right) = \chi\left(e_\gamma^n\right)$ for all γ. Thus, q_K induces the isomorphism $q_{K*} : C_n\left(\overline{K}\right) \xrightarrow{\cong} C_n(K)$, $\overline{e}_\gamma^n \mapsto e_\gamma^n$. We define the composition

$$u_K : C_n(K) \xrightarrow{q_{K*}^{-1}} C_n\left(\overline{K}\right) \xhookrightarrow{j_{K*}} C_n(\widetilde{K}^n).$$

- Morphisms $\left(K, Y_K, \overline{K}, q_K\right) \to \left(L, Y_L, \overline{L}, q_L\right)$ in $\mathbf{CW}^0_{n\supset\partial}$ are pairs $(f, \widetilde{f})$, which consist of a cellular map $f : K \to L$ and a lift $\widetilde{f} : \widetilde{K}^n \to \widetilde{L}^n$ of $f^n \circ p_K^n$ under p_L^n,

$$\begin{array}{ccc} \widetilde{K}^n & \xrightarrow{\widetilde{f}} & \widetilde{L}^n \\ {\scriptstyle p_K^n}\downarrow & & \downarrow{\scriptstyle p_L^n} \\ K^n & \xrightarrow{f^n} & L^n, \end{array}$$

such that the induced homomorphism $\widetilde{f}_* : C_n(\widetilde{K}^n) \to C_n(\widetilde{L}^n)$ maps $u_K(Y_K)$ into $u_L(Y_L)$. The composition with a second morphism $(g, \widetilde{g}) : \left(L, Y_L, \overline{L}, q_L\right) \to \left(P, Y_P, \overline{P}, q_P\right)$ is defined by $(g, \widetilde{g}) \circ (f, \widetilde{f}) = (g \circ f, \widetilde{g} \circ \widetilde{f})$.

Let us motivate the definition of objects and morphisms in $\mathbf{CW}^0_{n\supset\partial}$.
If (K, Y_K) is an object in $\mathbf{CW}_{n\supset\partial}$, then the identification $\pi_n(K^n, K^{n-1}) \cong C_n(K)$ via the Hurewicz isomorphism allows us to think of any element of $C_n(K)$ as (the homotopy class of) a map $(D^n, S^{n-1}) \to (K^n, K^{n-1})$. This observation plays a central role in the proof of [1, Proposition 1.6, page 12]. If $\left(K, Y_K, \overline{K}, q_K\right)$ is an object in $\mathbf{CW}^0_{n\supset\partial}$, then $\overline{K}$ can be seen as the topological realization of a chosen splitting of the (now surjective) Hurewicz map $\mathrm{Hur} : \pi_n(K^n, K^{n-1}, k_0) \twoheadrightarrow C_n(K)$ (where $k_0 \in K^{n-1}$ is a fixed basepoint). Via this splitting we can still identify elements of $C_n(K)$ with elements of $\pi_n(K^n, K^{n-1}, k_0)$. (This was already done in the proof of Proposition 1.1.) The splitting which corresponds to $\overline{K}$ is explicitly given by $u_K : C_n(K) \to C_n(\widetilde{K}^n)$ after the identifications

$$C_n(\widetilde{K}^n) \xrightarrow{\mathrm{Hur}^{-1}\ \cong} \pi_n(\widetilde{K}^n, \widetilde{K}^{n-1}) \xrightarrow{p_{K*}^n\ \cong} \pi_n(K^n, K^{n-1}, k_0).$$

Given k_0, the second identification is only well-defined up to the choice of a lift $\widetilde{k}_0 \in \widetilde{K}^{n-1}$ of k_0 under p_K. Equivalently, it is well-defined up to an automorphism of $\pi_n(K^n, K^{n-1}, k_0)$ which comes from the free action of $\pi_1(K^{n-1}, k_0)$ on $\pi_n(K^n, K^{n-1}, k_0)$. (In fact, $\pi_n(K^n, K^{n-1}, k_0)$ is a free $\pi_1(K^{n-1}, k_0)$-module with basis the homotopy classes of the characteristic maps of the n-cells of K after application of change-of-basepoint isomorphisms, see [2, Lemma 4.38, page 371].) In the following, however, we will prefer the approach given in Definition 1.7. The reason is that in the proof of Proposition 1.13 (the counterpart of [1, Proposition 1.6, page 12]) we will apply the Whitehead theorem to the simply connected spaces $\overline{K}$ to construct the desired homotopy equivalence.

If $(f,\widetilde{f}):\big(K,Y_K,\overline{K},q_K\big)\to\big(L,Y_L,\overline{L},q_L\big)$ is a morphism in $\mathbf{CW}^0_{n\supset\partial}$, then the condition $\widetilde{f}_*u_K(Y_K)\subset u_L(Y_L)$ corresponds to the statement that $f^n_*:\pi_n(K^n,K^{n-1},k_0)\to\pi_n(L^n,L^{n-1},f(k_0))$ maps the image of Y_K under a suitable splitting u_K into the image of Y_L under a suitable u_L. This is exactly the condition on relative homotopy groups that is needed to generalize the compression theorem [1, Theorem 1.32, page 35] (see Proposition 1.14). Again, the splittings are only well-defined up to the choice of lifts of the basepoints. This corresponds to the fact that f^n can be lifted to $\widetilde{f}$ in several ways.

Example. 1.8 Note that $\mathbf{CW}_{n\supset\partial}$ is a full subcategory of $\mathbf{CW}^0_{n\supset\partial}$. In fact, if (K,Y_K) is an object in $\mathbf{CW}_{n\supset\partial}$, then K^{n-1} and K^n are simply connected. Therefore, $p^{n-1}_K=\mathrm{id}_{K^{n-1}}$, $p^n_K=j_K=q_K=\mathrm{id}_{K^n}$ and $u_K=\mathrm{id}_{C_n(K)}$. Thus, $(K,Y_K,K^n,\mathrm{id}_{K^n})$ is the unique completion of (K,Y_K) to an object in $\mathbf{CW}^0_{n\supset\partial}$. Moreover, if (K,Y_K) and (L,Y_L) are objects in $\mathbf{CW}_{n\supset\partial}$, then all morphisms $(K,Y_K,K^n,\mathrm{id}_{K^n})\to(L,Y_L,L^n,\mathrm{id}_{L^n})$ in $\mathbf{CW}^0_{n\supset\partial}$ are of the form (f,f^n), where $f:K\to L$ is a cellular map such that the induced homomorphism $\widetilde{f}_*=f_*:C_n(K)\to C_n(L)$ satisfies $f_*(Y_K)\subset Y_L$. Thus, (f,f^n) corresponds to the morphism $f:(K,Y_K)\to(L,Y_L)$ in $\mathbf{CW}_{n\supset\partial}$.

Remark. 1.9 Every path connected CW-complex K can be completed to an object $\big(K,Y_K,\overline{K},q_K\big)$ in $\mathbf{CW}^0_{n\supset\partial}$ by choosing Y_K and $\overline{K}$ with the required properties. The choice of Y_K as the image of a splitting of the boundary map $\partial_n:C_n(K)\twoheadrightarrow\operatorname{im}\partial_n\,(\subset C_{n-1}(K))$ is always possible, since $\operatorname{im}\partial_n$ is free abelian. In order to construct the desired $\overline{K}$, note that the characteristic maps of the n-cells of $\widetilde{K}$ are all possible lifts under $p^n_K:\widetilde{K}^n\to K^n$ of the characteristic maps of the n-cells $\{e^n_\gamma\}$ of K. For every γ, choose one lift of $\chi\left(e^n_\gamma\right)$ under p^n_K and denote the corresponding n-cell of $\widetilde{K}^n$ by $\overline{e}^n_\gamma$. Then $\overline{K}$ can be taken to be the subcomplex $\widetilde{K}^{n-1}\cup\bigcup_\gamma\overline{e}^n_\gamma$ of $\widetilde{K}^n$.

Example. 1.10 For $n\geq 3$ and $K:=S^1\vee S^n$, we have introduced the universal cover $p:\widetilde{K}\to K$ in Example 1.2. Choose any integer k and set $\overline{K}:=\mathbb{R}\vee S^n_k$ and $q_K:\overline{K}\hookrightarrow\widetilde{K}\xrightarrow{p}K$. Then, $\big(K,0,\overline{K},q_K\big)$ is an object in $\mathbf{CW}^0_{n\supset\partial}$. (Note that $Z_n(K)=C_n(K)=\mathbb{Z}\theta$, where θ is the single n-cell of K, and we have to choose $Y_K=0$.) Recall that in Example 1.2 we have constructed a cellular map $h:K\to K$ and a lift $\widetilde{h}:\widetilde{K}\to\widetilde{K}$ of $h\circ p$ under p. Since $Y_K=0$, this yields a morphism $(h,\widetilde{h}):\big(K,0,\overline{K},q_K\big)\to\big(K,0,\overline{K},q_K\big)$ in $\mathbf{CW}^0_{n\supset\partial}$. The homomorphism $j_{K*}:C_n\left(\overline{K}\right)\hookrightarrow C_n(\widetilde{K})$ induced by the inclusion $j_K:\overline{K}\hookrightarrow\widetilde{K}$ is explicitly given by $\mathbb{Z}e_k\hookrightarrow\bigoplus_{m\in\mathbb{Z}}\mathbb{Z}e_m$. The induced homomorphism $\widetilde{h}_*:C_n(\widetilde{K})\to C_n(\widetilde{K})$ does not map the image of j_{K*} into the image of j_{K*}, because it was shown that $\widetilde{h}_*\left(j_{K*}\left(e_k\right)\right)=\widetilde{h}_*\left(e_k\right)=2e_k-e_{k+1}\notin\mathbb{Z}e_k$ for the single n-cell e_k of $\overline{K}$. Equivalently, $\widetilde{h}_*$ does not map the image of $u_K:C_n(K)\hookrightarrow C_n(\widetilde{K})$ into the image of u_K. The following proposition deals with the question when this property is satisfied.

1.11 Proposition. *Let $\big(K,Y_K,\overline{K},q_K\big)$ and $\big(L,Y_L,\overline{L},q_L\big)$ be objects in $\mathbf{CW}^0_{n\supset\partial}$. Let $(f,\widetilde{f})$ be a pair, which consists of a cellular map $f:K\to L$ and a lift $\widetilde{f}:\widetilde{K}^n\to\widetilde{L}^n$ of $f^n\circ p^n_K$ under p^n_L. The following statements are equivalent:*

(i) There is an extension $\overline{f}:\overline{K}\to\overline{L}$ of $\widetilde{f}^{n-1}:\widetilde{K}^{n-1}\to\widetilde{L}^{n-1}$, such that the following diagram commutes up to homotopy rel $\widetilde{K}^{n-1}$:

$$\begin{array}{ccc} \overline{K} & \xrightarrow{\overline{f}} & \overline{L} \\ {\scriptstyle q_K}\downarrow & & \downarrow{\scriptstyle q_L} \\ K^n & \xrightarrow{f^n} & L^n. \end{array}$$

(ii) The following diagram commutes:

$$\begin{array}{ccc} C_n(K) & \xrightarrow{u_K} & C_n(\widetilde{K}^n) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \widetilde{f}_*} \\ C_n(L) & \xrightarrow{u_L} & C_n(\widetilde{L}^n). \end{array}$$

(iii) The induced homomorphism $\widetilde{f}_* : C_n(\widetilde{K}^n) \to C_n(\widetilde{L}^n)$ *maps the image of* $u_K : C_n(K) \hookrightarrow C_n(\widetilde{K}^n)$ *into the image of* $u_L : C_n(L) \hookrightarrow C_n(\widetilde{L}^n)$.

(iv) The composition $\overline{K} \overset{j_K}{\hookrightarrow} \widetilde{K}^n \overset{\widetilde{f}}{\longrightarrow} \widetilde{L}^n$ *is homotopic rel* $\widetilde{K}^{n-1}$ *to a map into* $\overline{L} \subset \widetilde{L}^n$.

Moreover, if these statements hold, then it follows from (ii) *that the pair* $(f, \widetilde{f})$ *is a morphism* $\left(K, Y_K, \overline{K}, q_K\right) \to \left(L, Y_L, \overline{L}, q_L\right)$ *in* $\boldsymbol{CW}^0_{n\supset\partial}$ *if and only if the induced map* $f_* : C_n(K) \to C_n(L)$ *satisfies* $f_*(Y_K) \subset Y_L$. *(This is exactly the condition required for morphisms in* $\boldsymbol{CW}_{n\supset\partial}$*.)*

Proof. $(i) \Rightarrow (ii)$. Take a basepoint $\widetilde{k}_0 \in \widetilde{K}^{n-1}$ and set $k_0 = p_K(\widetilde{k}_0)$, $l_0 = (f \circ p_K)(\widetilde{k}_0)$ and $\widetilde{l}_0 = \widetilde{f}(\widetilde{k}_0)$. Consider the following commutative diagram, where the equality signs are identifications via Hurewicz isomorphisms ($\widetilde{K}^{n-1}$ and $\widetilde{L}^{n-1}$ are simply connected):

$$\begin{array}{ccccccc} C_n(K) & \xleftarrow[\cong]{q_{K*}} & C_n(\overline{K}) = \pi_n(\overline{K}, \widetilde{K}^{n-1}, \widetilde{k}_0) & \xrightarrow{q_{K*}} & \pi_n(K^n, K^{n-1}, k_0) & \xleftarrow[\cong]{p^n_{K*}} & \pi_n(\widetilde{K}^n, \widetilde{K}^{n-1}, \widetilde{k}_0) = C_n(\widetilde{K}^n) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle \overline{f}_*}\downarrow & & \downarrow{\scriptstyle f^n_*} & & \downarrow{\scriptstyle \widetilde{f}_*} \\ C_n(L) & \xleftarrow[\cong]{q_{L*}} & C_n(\overline{L}) = \pi_n(\overline{L}, \widetilde{L}^{n-1}, \widetilde{l}_0) & \xrightarrow{q_{L*}} & \pi_n(L^n, L^{n-1}, l_0) & \xleftarrow[\cong]{p^n_{L*}} & \pi_n(\widetilde{L}^n, \widetilde{L}^{n-1}, \widetilde{l}_0) = C_n(\widetilde{L}^n). \end{array}$$

The map p^n_{K*} (analogously, p^n_{L*}) in the diagram is an isomorphism. (Apply the 5-lemma to the ladder of commutative squares between long exact homotopy sequences, which is induced by $p^n_K : (\widetilde{K}^n, \widetilde{K}^{n-1}, \widetilde{k}_0) \to (\widetilde{L}^n, \widetilde{L}^{n-1}, \widetilde{l}_0)$. Use that covering maps induce isomorphisms on higher homotopy groups.) The first line equals the inclusion $u_K : C_n(K) \hookrightarrow C_n(\widetilde{K}^n)$ (analogously, the second line is $u_L : C_n(L) \hookrightarrow C_n(\widetilde{L}^n)$). This follows from the following commutative diagram:

$$\begin{array}{ccccc} C_n(K) & \xleftarrow[\cong]{q_{K*}} & C_n(\overline{K}) = \pi_n(\overline{K}, \widetilde{K}^{n-1}, \widetilde{k}_0) & \xrightarrow{=} & \pi_n(\overline{K}, \widetilde{K}^{n-1}, \widetilde{k}_0) \\ {\scriptstyle u_K}\downarrow & & {\scriptstyle j_{K*}}\downarrow & & \downarrow{\scriptstyle q_{K*}} \\ C_n(\widetilde{K}^n) & \xleftarrow{=} & C_n(\widetilde{K}^n) = \pi_n(\widetilde{K}^n, \widetilde{K}^{n-1}, \widetilde{k}_0) & \xrightarrow[\cong]{p^n_{K*}} & \pi_n(K^n, K^{n-1}, k_0) \end{array}$$

(ii) ⇒ *(iii)*. This is clear by commutativity.
(iii) ⇒ *(iv)*. The composition $\widetilde{f} \circ j_K$ restricts to $\widetilde{f}^{n-1} : \widetilde{K}^{n-1} \to \widetilde{L}^{n-1} (\subset \overline{L})$ on $(n-1)$-skeletons. It suffices to show that the composition

$$(\overline{e}^n_\gamma, \partial \overline{e}^n_\gamma) \xrightarrow{\chi(\overline{e}^n_\gamma)} (\overline{K}, \widetilde{K}^{n-1}) \xrightarrow{j_K} (\widetilde{K}^n, \widetilde{K}^{n-1}) \xrightarrow{\widetilde{f}} (\widetilde{L}^n, \widetilde{L}^{n-1})$$

is homotopic rel $\partial \overline{e}^n_\gamma$ to a map into $\overline{L}$ for all γ. Proceeding analogously as in the proof of [1, Theorem 1.32, page 35ff], these homotopies can then be used to construct the desired rel $\widetilde{K}^{n-1}$ homotopy between $\widetilde{f} \circ j_K$ and a map into $\overline{L}$. Consulting the following part of the exact homotopy sequence of the triple $(\widetilde{L}^n, \overline{L}, \widetilde{L}^{n-1})$,

$$\pi_n(\overline{L}, \widetilde{L}^{n-1}) \xrightarrow{j_{L*}} \pi_n(\widetilde{L}^n, \widetilde{L}^{n-1}) \longrightarrow \pi_n(\widetilde{L}^n, \overline{L}),$$

it is sufficient to show that $[\widetilde{f} \circ j_K \circ \chi(\overline{e}^n_\gamma)] \in \pi_n(\widetilde{L}^n, \widetilde{L}^{n-1})$ lies in the image of j_{L*} for all γ. (Then, the composition $\widetilde{f} \circ j_K \circ \chi(\overline{e}^n_\gamma)$ represents zero in $\pi_n(\widetilde{L}^n, \overline{L}, \widetilde{l}_0)$ and is thus homotopic rel $\partial \overline{e}^n_\gamma$ to a map into $\overline{L}$.) Identify the relative homotopy groups $\pi_n(\overline{K}, \widetilde{K}^{n-1})$ and $\pi_n(\widetilde{K}^n, \widetilde{K}^{n-1})$ with the corresponding cellular chain groups via Hurewicz isomorphisms ($\widetilde{K}^{n-1}$ is simply connected) and do the same for L. Then the element $[\widetilde{f} \circ j_K \circ \chi(\overline{e}^n_\gamma)] \in \pi_n(\widetilde{L}^n, \widetilde{L}^{n-1})$ corresponds to the image of $\overline{e}^n_\gamma \in C_n\left(\overline{K}\right)$ under

$$C_n\left(\overline{K}\right) \xrightarrow{j_{K*}} C_n(\widetilde{K}^n) \xrightarrow{\widetilde{f}_*} C_n(\widetilde{L}^n).$$

By *(iii)*, $\widetilde{f}_*$ maps the image of u_K into the image of u_L. Note that the image of u_K equals the image of $j_{K*} : C_n\left(\overline{K}\right) \hookrightarrow C_n(\widetilde{K}^n)$ and the image of u_L equals the image of $j_{L*} : C_n\left(\overline{L}\right) \hookrightarrow C_n(\widetilde{L}^n)$. Thus, $\widetilde{f}_*(j_{K*}(\overline{e}^n_\gamma))$ lies in the image of $j_{L*} : C_n(\overline{L}) \hookrightarrow C_n(\widetilde{L}^n)$.
(iv) ⇒ *(i)*. By assumption, $\widetilde{f} \circ j_K$ is homotopic rel $\widetilde{K}^{n-1}$ to a map $\overline{f} : \overline{K} \to \overline{L}$. Therefore, $\overline{f}$ restricts to $\widetilde{f}^{n-1} : \widetilde{K}^{n-1} \to \widetilde{L}^{n-1}$ on $(n-1)$-skeletons. Now,

$$q_L \circ \overline{f} = p^n_L \circ j_L \circ \overline{f} \simeq p^n_L \circ \widetilde{f} \circ j_K = f^n \circ p^n_K \circ j_K = f^n \circ q_K \quad \text{rel } \widetilde{K}^{n-1}.$$

■

1.12 Lemma. *Let $q_K : \overline{K} \to K$ and $q_L : \overline{L} \to L$ be cellular maps between n-dimensional CW-complexes. Assume that $q_K \circ \chi(\overline{e}_\alpha) = \chi(e_\alpha)$ for all α, where $\{e_\alpha\}$ are the n-cells of K and $\{\overline{e}_\alpha\}$ are the n-cells of $\overline{K}$.*
(i) If $f^{n-1} : K^{n-1} \to L^{n-1}$ and $g : \overline{K} \to \overline{L}$ are cellular maps such that

$$\begin{array}{ccc} \overline{K}^{n-1} & \xrightarrow{g^{n-1}} & \overline{L}^{n-1} \\ \downarrow{\scriptstyle q_K^{n-1}} & & \downarrow{\scriptstyle q_L^{n-1}} \\ K^{n-1} & \xrightarrow{f^{n-1}} & L^{n-1} \end{array}$$

commutes, then f^{n-1} *extends to a map* $f : K \to L$ *such that* $f \circ q_K = q_L \circ g$.

(ii) *Let* $f_i : K \to L$, $i = 1, 2$, *be extensions of a cellular map* $f^{n-1} : K^{n-1} \to L^{n-1}$ *and let* $g_i : \overline{K} \to \overline{L}$, $i = 1, 2$, *be extensions of a cellular map* $g^{n-1} : \overline{K}^{n-1} \to \overline{L}^{n-1}$, *such that the following diagram commutes up to homotopy rel* $\overline{K}^{n-1}$ *for* $i = 1, 2$*:*

$$\begin{array}{ccc} \overline{K} & \xrightarrow{g_i} & \overline{L} \\ {\scriptstyle q_K}\downarrow & & \downarrow{\scriptstyle q_L} \\ K & \xrightarrow{f_i} & L \end{array}$$

If $g_1 \simeq g_2$ *rel* $\overline{K}^{n-1}$, *then* $f_1 \simeq f_2$ *rel* K^{n-1}.

(iii) *Assume that* $K^{n-1} = L^{n-1}$ *and* $\overline{K}^{n-1} = \overline{L}^{n-1}$ *and that* $q_L \circ \chi\left(\overline{e}'_\beta\right) = \chi\left(e'_\beta\right)$ *for all* β, *where* $\left\{e'_\beta\right\}$ *are the* n*-cells of* L *and* $\left\{\overline{e}'_\beta\right\}$ *are the* n*-cells of* $\overline{L}$. *Let* $f : K \to L$ *be an extension of* $\mathrm{id}_{K^{n-1}}$ *and let* $g : \overline{K} \to \overline{L}$ *be an extension of* $\mathrm{id}_{\overline{K}^{n-1}}$, *such that the following diagram commutes up to homotopy rel* $\overline{K}^{n-1}$*:*

$$\begin{array}{ccc} \overline{K} & \xrightarrow{g} & \overline{L} \\ {\scriptstyle q_K}\downarrow & & \downarrow{\scriptstyle q_L} \\ K & \xrightarrow{f} & L \end{array}$$

If g *is a homotopy equivalence rel* $\overline{K}^{n-1}$, *then* f *is a homotopy equivalence rel* K^{n-1}.

Proof. (i). Consider the following morphism of 3-diagrams of spaces:

$$\begin{array}{ccccc} K^{n-1} & \xleftarrow{\bigsqcup \chi(e_\alpha)|} & \bigsqcup_\alpha \partial e_\alpha = \bigsqcup_\alpha \partial \overline{e}_\alpha & \xrightarrow{\text{incl}} & \bigsqcup_\alpha \overline{e}_\alpha \\ {\scriptstyle \text{incl} \circ f^{n-1}}\downarrow & & \downarrow{\scriptstyle \text{incl} \circ f^{n-1} \circ \bigsqcup \chi(e_\alpha)|} & & \downarrow{\scriptstyle q_L \circ g \circ \bigsqcup \chi(\overline{e}_\alpha)} \\ L & \xleftarrow{=} & L & \xrightarrow{=} & L \end{array}$$

The right square commutes, since $q_L \circ g : \overline{K} \to L$ restricts to $q_L^{n-1} \circ g^{n-1} = f^{n-1} \circ q_K^{n-1} : \overline{K}^{n-1} \to L^{n-1}$ and $q_K^{n-1} \circ \chi\left(\overline{e}_\alpha\right)| = \chi\left(e_\alpha\right)|$ for all α. The realization of the first line is just K. Thus, the morphism of 3-diagrams of spaces induces a map $f : K \to L$. By construction, f extends f^{n-1} and is thus cellular. In order to show $f \circ q_K = q_L \circ g$, it remains to show that for every α and every $x \in \operatorname{int}\left(D^n\right)$ the following equality holds:

$$\left(q_L \circ g\right)\left(\chi\left(\overline{e}_\alpha\right)(x)\right) = \left(f \circ q_K\right)\left(\chi\left(\overline{e}_\alpha\right)(x)\right).$$

The left hand side is just the definition of $f\left(\chi\left(e_\alpha\right)(x)\right)$. This equals the right hand side, because $q_K\left(\chi\left(\overline{e}_\alpha\right)(x)\right) = \chi\left(e_\alpha\right)(x)$.

(ii). For every α and for $i = 1, 2$, observe that

$$f_i \circ \chi\left(e_\alpha\right) = f_i \circ q_K \circ \chi\left(\overline{e}_\alpha\right) \simeq q_L \circ g_i \circ \chi\left(\overline{e}_\alpha\right) \quad \text{rel } \partial e_\alpha = \partial \overline{e}_\alpha.$$

The assumption $g_1 \simeq g_2$ rel $\overline{K}^{n-1}$ yields $q_L \circ g_1 \circ \chi\left(\overline{e}_\alpha\right) \simeq q_L \circ g_2 \circ \chi\left(\overline{e}_\alpha\right)$ rel $\partial \overline{e}_\alpha$. Thus, for every α, we can fix a homotopy

$$H^\alpha : e_\alpha \times I \to L \quad \text{rel } \partial e_\alpha$$

between $H_0^\alpha = f_1 \circ \chi\left(e_\alpha\right)$ and $H_1^\alpha = f_2 \circ \chi\left(e_\alpha\right)$. Now let $H : K \times I \to L$ be the homotopy that is induced by the following morphism of 3-diagrams of spaces:

$$\begin{array}{ccccc}
\bigsqcup_\alpha e_\alpha \times I & \xleftarrow{\text{incl}} & \bigsqcup_\alpha \partial e_\alpha \times I & \xrightarrow{\bigsqcup \chi(e_\alpha)| \times \mathrm{id}_I} & K^{n-1} \times I \\
\Big\downarrow{\scriptstyle \bigsqcup H^\alpha} & & \Big\downarrow{\scriptstyle \bigsqcup H^\alpha|} & & \Big\downarrow{\scriptstyle \text{incl} \circ f^{n-1} \circ \text{proj}_1} \\
L & \xleftarrow{=} & L & \xrightarrow{=} & L
\end{array}$$

Then, H is a rel K^{n-1} homotopy between $H_0 = f_1$ and $H_1 = f_2$.
(iii). Choose a homotopy inverse $g' : \overline{L} \to \overline{K}$ of g which extends $\mathrm{id}_{\overline{K}^{n-1}}$ and such that $g \circ g' \simeq \mathrm{id}_{\overline{L}}$ rel $\overline{K}^{n-1}$ and $g' \circ g \simeq \mathrm{id}_{\overline{K}}$ rel $\overline{K}^{n-1}$. By (i), there exists an extension $f' : L \to K$ of $\mathrm{id}_{K^{n-1}}$ such that $f' \circ q_L = q_K \circ g'$. (Use that $q_K^{n-1} = q_L^{n-1}$, which follows from $q_L \circ g \simeq f \circ q_K$ rel $\overline{K}^{n-1}$ after restriction to $(n-1)$-skeletons.) By (ii), we have $f \circ f' \simeq \mathrm{id}_L$ rel K^{n-1} and $f' \circ f \simeq \mathrm{id}_K$ rel K^{n-1}. ∎

In order to define $\tau_{<n}^0$ on objects, one needs a counterpart of [1, Proposition 1.6, page 12] for path connected CW-complexes K:

1.13 Proposition. *Given an object $(K, Y_K, \overline{K}, q_K)$ in $\boldsymbol{CW}_{n\supset\partial}^0$, there exists a commutative diagram of n-dimensional CW-complexes and cellular maps*

$$\begin{array}{ccccc}
\overline{K}_{<n} & \xrightarrow{\overline{i}_K} & \overline{K}/n & \xrightarrow{\overline{h}_K} & \overline{K} \\
\Big\downarrow{\scriptstyle q_{K_{<n}}} & & \Big\downarrow{\scriptstyle q_{K/n}} & & \Big\downarrow{\scriptstyle q_K} \\
K_{<n} & \xrightarrow{i_K} & K/n & \xrightarrow{h_K} & K^n
\end{array}$$

such that the following properties are satisfied:

- *$(K, K/n, h_K, K_{<n})$ is an n-truncation structure and $Y_K = h_{K*} i_{K*} C_n\left(K_{<n}\right)$.*
- *$\overline{i}_K$ and $\overline{h}_K$ restrict to the identity map $\mathrm{id}_{\widetilde{K}^{n-1}}$ on $(n-1)$-skeletons.*
- *The following are morphisms in $\boldsymbol{CW}_{n\supset\partial}^0$:*

$$\begin{aligned}
(i_K, \widetilde{i}_K) &: \left(K_{<n}, C_n\left(K_{<n}\right), \overline{K}_{<n}, q_{K_{<n}}\right) \to \left(K/n, i_{K*}C_n\left(K_{<n}\right), \overline{K}/n, q_{K/n}\right), \\
(h_K, \widetilde{h}_K) &: \left(K/n, i_{K*}C_n\left(K_{<n}\right), \overline{K}/n, q_{K/n}\right) \to \left(K, Y_K, \overline{K}, q_K\right),
\end{aligned}$$

where $\widetilde{i}_K$ is the unique lift of $i_K \circ p_{K_{<n}}$ under $p_{K/n}$ and $\widetilde{h}_K$ is the unique lift of $h_K \circ p_{K/n}$ under p_K^n such that $\widetilde{i}_K$ and $\widetilde{h}_K$ restrict to $\mathrm{id}_{\widetilde{K}^{n-1}}$ on $(n-1)$-skeletons.

Proof. We start with the construction of the desired diagram. Choose bases $\{\zeta_\beta\}$ of $Z_n(K)$ and $\{\eta_\alpha\}$ of Y_K, which yields a basis $\{\zeta_\beta\} \cup \{\eta_\alpha\}$ of $C_n(K)$. It corresponds to a basis $\{\overline{\zeta}_\beta\} \cup \{\overline{\eta}_\alpha\}$ of $C_n(\overline{K})$ under the isomorphism $q_{K*} : C_n(\overline{K}) \xrightarrow{\cong} C_n(K)$. Now apply Proposition 1.1 to the path connected (even simply connected) n-dimensional CW-complex $\overline{K}$ and to the basis $\{\overline{\zeta}_\beta\} \cup \{\overline{\eta}_\alpha\}$ of $C_n\left(\overline{K}\right)$ to obtain

- an n-dimensional CW-complex $\overline{K}/n$ with $(n-1)$-skeleton $\widetilde{K}^{n-1}$.
- a cellular map $\overline{h}_K : \overline{K}/n \to \overline{K}$ which restricts to the identity map on $\widetilde{K}^{n-1}$ and which induces an isomorphism $\overline{h}_{K*} : C_n(\overline{K}/n) \stackrel{\cong}{\to} C_n\left(\overline{K}\right)$ sending the cell basis of $C_n(\overline{K}/n)$ to the basis $\{\overline{\zeta}_\beta\} \cup \{\overline{\eta}_\alpha\}$ of $C_n\left(\overline{K}\right)$.

Consequently, $\overline{h}_K$ induces an isomorphism $H_r(\overline{K}/n) \cong H_r\left(\overline{K}\right)$ for all integers r. Since $\widetilde{K}^{n-1}$ is simply connected $(n \geq 3)$, $\overline{K}$ and $\overline{K}/n$ are also simply connected. Therefore, $\overline{h}_K$ is a homotopy equivalence by the homological version of the Whitehead theorem. By [2, Proposition 0.19, page 16], $\overline{h}_K$ is a homotopy equivalence rel $\widetilde{K}^{n-1}$.
Let $\{\overline{f}^n_\delta\}$ be the n-cells of $\overline{K}/n$. For every δ set

$$d_\delta := p_K^{n-1} \circ \chi(\overline{f}^n_\delta)| : S^{n-1} \to K^{n-1}.$$

Taking new n-cells $\{f^n_\delta\}$, define the n-dimensional CW-complex

$$K/n := K^{n-1} \cup \bigcup_{d_\delta} f^n_\delta.$$

The morphism of 3-diagrams of spaces

$$\begin{array}{ccccc}
\widetilde{K}^{n-1} & \xleftarrow{\sqcup \chi(\overline{f}^n_\delta)|} & \bigsqcup_\delta \partial \overline{f}^n_\gamma & \xrightarrow{\text{incl}} & \bigsqcup_\delta \overline{f}^n_\delta \\
\downarrow{\scriptstyle p_K^{n-1}} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
K^{n-1} & \xleftarrow{\sqcup d_\delta} & \bigsqcup_\delta \partial f^n_\delta & \xrightarrow{\text{incl}} & \bigsqcup_\delta f^n_\delta
\end{array}$$

induces a cellular map $q_{K/n} : \overline{K}/n \to K/n$ such that $q_{K/n} \circ \chi(\overline{f}^n_\delta) = \chi\left(f^n_\delta\right)$ for all δ. Note that both q_K and $q_{K/n}$ restrict to p_K^{n-1} and $\overline{h}_K$ restricts to $\mathrm{id}_{\widetilde{K}^{n-1}}$ on $(n-1)$-skeletons. Thus, by Lemma 1.12 (i), there exists a cellular map $h_K : K/n \to K^n$ which extends the identity map on K^{n-1} and fits into the commutative diagram

$$\begin{array}{ccc}
\overline{K}/n & \xrightarrow{\overline{h}_K} & \overline{K} \\
\downarrow{\scriptstyle q_{K/n}} & & \downarrow{\scriptstyle q_K} \\
K/n & \xrightarrow{h_K} & K^n
\end{array}$$

Moreover, h_K is a homotopy equivalence rel K^{n-1} by Lemma 1.12 (iii), since $\overline{h}_K$ is a homotopy equivalence rel $\widetilde{K}^{n-1}$. The previous diagram induces a commutative diagram

$$
\begin{array}{ccccccc}
\{\overline{f}^n_\delta\} & \subset & C_n(\overline{K}/n) & \xrightarrow{\overline{h}_{K*}\ \cong} & C_n\left(\overline{K}\right) & \supset & \{\overline{\zeta}_\beta\} \cup \{\overline{\eta}_\alpha\} \\
 & & {\scriptstyle q_{K/n*}\ \cong}\big\downarrow & & \big\downarrow{\scriptstyle q_{K*}\ \cong} & & \\
\{f^n_\delta\} & \subset & C_n\left(K/n\right) & \xrightarrow{h_{K*}} & C_n\left(K\right) & \supset & \{\zeta_\beta\} \cup \{\eta_\alpha\}
\end{array}
$$

The diagram also shows the bases which correspond to each other under the marked isomorphisms. All in all, h_{K*} is an isomorphism sending the cell-basis $\{f^n_\delta\}$ of $C_n\left(K/n\right)$ to the basis $\{\zeta_\beta\} \cup \{\eta_\alpha\}$ of $C_n\left(K\right)$. The isomorphism $h_{K*} : C_n\left(K/n\right) \xrightarrow{\cong} C_n\left(K\right)$ restricts to an isomorphism $h_{K*}| : Z_n\left(K/n\right) \xrightarrow{\cong} Z_n\left(K\right)$ (this is shown as in the proof of Proposition 1.3, using that h_K restricts to the identity map on K^{n-1}). Thus, the basis $\{\zeta_\beta\}$ of $Z_n\left(K\right)$ corresponds under the inverse of h_{K*} to a basis $\left\{\zeta'_\beta\right\}$ of $Z_n\left(K/n\right)$ which is a subset of $\{f^n_\delta\}$ and the basis $\{\eta_\alpha\}$ of Y_K corresponds to the remaining n-cells $\{\eta'_\alpha\}$ in $\{f^n_\delta\}$. This shows that K/n has a basis of cells for its n-cycle group.
As always, define $i_K : K_{<n} \hookrightarrow K/n$ as the inclusion of the subcomplex which is obtained from K/n by taking away the n-cells $\left\{\zeta'_\beta\right\}$. Analogously, let $\overline{i}_K : \overline{K}_{<n} \hookrightarrow \overline{K}/n$ be the inclusion of the subcomplex which is obtained from $\overline{K}/n$ by taking away the n-cells that correspond to $\left\{\zeta'_\beta\right\}$ via $q_{K/n*}$. Then $q_{K/n}$ restricts to a map $q_{K_{<n}} : \overline{K}_{<n} \to K_{<n}$. This finishes the contruction of the desired diagram.
It remains to check the three stated properties. By construction, $(K, K/n, h_K, K_{<n})$ is an n-truncation structure (see Definition 1.5). The equation $h_{K*}i_{K*}C_n\left(K_{<n}\right) = Y_K$ follows, since the isomorphism $h_{K*} : C_n\left(K/n\right) \to C_n\left(K^n\right)$ restricts to an isomorphism between the subgroup $i_{K*}C_n\left(K_{<n}\right)$ of $C_n\left(K/n\right)$ spanned by the n-cells $\{\eta'_\alpha\}$ and Y_K. The second property is clear by construction of $\overline{i}_K$ and $\overline{h}_K$. Concerning the third property, all involved quadruples are objects in $\mathbf{CW}^0_{n\supset\partial}$ (see Definition 1.7). The pairs $(i_K, \widetilde{i}_K)$ and $(h_K, \widetilde{h}_K)$ satisfy property (i) of Proposition 1.11. Thus, they are morphisms in $\mathbf{CW}^0_{n\supset\partial}$ by the conclusion of the proposition. ∎

Now we can proceed to define $\tau^0_{<n}$ on objects (see Corollary 1.19). Given an object $(K, Y_K, \overline{K}, q_K)$ in $\mathbf{CW}^0_{n\supset\partial}$, we use Proposition 1.13 to choose a completion to a commutative diagram with the stated properties, and set

$$\tau^0_{<n}(K, Y_K, \overline{K}, q_K) = (K, K/n, h_K, K_{<n})\,.$$

In the special case that K is simply connected, the above completion just means to complete (K, Y_K) to an n-truncation structure, so in this case we are free to choose

$$\tau^0_{<n}(K, Y_K, \overline{K}, q_K) = \tau_{<n}(K, Y_K).$$

In the special case that K has a basis of cells for its group of n-cycles and Y_K is generated by those n-cells of K which are not cycles, one can choose $K/n = K^n$, $\overline{K}/n = \overline{K}$ and $h_K = \mathrm{id}_{K^n}$, $\overline{h}_K = \mathrm{id}_{\overline{K}}$, $q_{K/n} = q_K$ (the remaining spaces $K_{<n}$ and $\overline{K}_{<n}$ in the diagram are then uniquely determined and all properties in Proposition 1.13 are satisfied by the resulting diagram). As in the definition of $\tau_{<n}$ on objects in [1, page 29], we will assume in this case that

$$\tau^0_{<n}(K, Y_K, \overline{K}, q_K) = (K, K^n, \mathrm{id}_{K^n}, K_{<n})\,.$$

The following version of the compression theorem [1, Theorem 1.32, page 35] enables us to define the assignment $\tau^0_{<n}$ on morphisms:

1.14 Proposition. *Let* $(f, \widetilde{f}) : \left(K, Y_K, \overline{K}, q_K\right) \to \left(L, Y_L, \overline{L}, q_L\right)$ *be a morphism in* $\boldsymbol{CW}^0_{n\supset\partial}$*. If the involved objects are completed to commutative diagrams*

$$\begin{array}{ccccccccccc}
\overline{K}_{<n} & \xrightarrow{\bar{i}_K} & \overline{K}/n & \xrightarrow{\bar{h}_K} & \overline{K} & & \overline{L} & \xleftarrow{\bar{h}_L} & \overline{L}/n & \xleftarrow{\bar{i}_L} & \overline{L}_{<n} \\
\downarrow{\scriptstyle q_{K<n}} & & \downarrow{\scriptstyle q_{K/n}} & & \downarrow{\scriptstyle q_K} & & \downarrow{\scriptstyle q_L} & & \downarrow{\scriptstyle q_{L/n}} & & \downarrow{\scriptstyle q_{L<n}} \\
K_{<n} & \xrightarrow{i_K} & K/n & \xrightarrow{h_K} & K^n & \xrightarrow{f^n} & L^n & \xleftarrow{h_L} & L/n & \xleftarrow{i_L} & L_{<n}
\end{array}$$

with the properties of Proposition 1.13, then there exist cellular maps $f/n : K/n \to L/n$ *and* $f_{<n} : K_{<n} \to L_{<n}$ *such that:*

- *The following diagram commutes up to homotopy rel* K^{n-1}*:*

$$\begin{array}{ccccccc}
K_{<n} & \xrightarrow{i_K} & K/n & \xrightarrow{h_K} & K^n & \xrightarrow{\text{incl}} & K \\
\downarrow{\scriptstyle f_{<n}} & & \downarrow{\scriptstyle f/n} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f} \\
L_{<n} & \xrightarrow{i_L} & L/n & \xrightarrow{h_L} & L^n & \xrightarrow{\text{incl}} & L
\end{array}$$

  Thus, f/n *and* $f_{<n}$ *extend* f^{n-1} *and the following is a morphism in* $\boldsymbol{HoCW}^0_{\supset<n}$*:*

$$\left([f], [f^n], [f/n], [f_{<n}]\right) : (K, K/n, h_K, K_{<n}) \to (L, L/n, h_L, L_{<n}).$$

- *The following are morphisms in* $\boldsymbol{CW}^0_{n\supset\partial}$*:*

$$(f/n, \widetilde{f/n}) : \left(K/n, i_{K*}\left(C_n\left(K_{<n}\right)\right), \overline{K}/n, q_{K/n}\right) \to \left(L/n, i_{L*}\left(C_n\left(L_{<n}\right)\right), \overline{L}/n, q_{L/n}\right),$$
$$(f_{<n}, \widetilde{f}_{<n}) : \left(K_{<n}, C_n\left(K_{<n}\right), \overline{K}_{<n}, q_{K_{<n}}\right) \to \left(L_{<n}, C_n\left(L_{<n}\right), \overline{L}_{<n}, q_{L_{<n}}\right),$$

  where $\widetilde{f/n}$ *is the unique lift of* $f/n \circ p_{K/n}$ *under* $p_{L/n}$ *and* $\widetilde{f}_{<n}$ *is the unique lift of* $f_{<n} \circ p_{K_{<n}}$ *under* $p_{L_{<n}}$*, such that* $\widetilde{f/n}$ *and* $\widetilde{f}_{<n}$ *restrict to* $\widetilde{f}^{n-1}$ *on* $(n-1)$*-skeletons.*

Proof. It suffices to construct cellular maps $f/n : K/n \to L/n$ and $f_{<n} : K_{<n} \to L_{<n}$ with the following properties (where $\widetilde{f/n}$ and $\widetilde{f}_{<n}$ are defined as above):
(1) $f^n \circ h_K \simeq h_L \circ f/n$ rel K^{n-1}.
(2) $f/n \circ i_K \simeq i_L \circ f_{<n}$ rel K^{n-1}.
(3) $(\widetilde{f/n})_* : C_n(\widetilde{K}/n) \to C_n(\widetilde{L}/n)$ maps $u_{K/n} i_{K*}\left(C_n\left(K_{<n}\right)\right)$ into $u_{L/n} i_{L*}\left(C_n\left(L_{<n}\right)\right)$.
(4) $(\widetilde{f}_{<n})_* : C_n(\widetilde{K}_{<n}) \to C_n(\widetilde{L}_{<n})$ maps $u_{K_{<n}}\left(C_n\left(K_{<n}\right)\right)$ into $u_{L_{<n}}\left(C_n\left(L_{<n}\right)\right)$.
Choose a cellular homotopy inverse h'_L for h_L which restricts to the identity map on L^{n-1} and such that $h_L h'_L \simeq \mathrm{id}_{L^n}$ rel L^{n-1} and $h'_L h_L \simeq \mathrm{id}_{L/n}$ rel L^{n-1}. If we define

$$f/n := h'_L \circ f^n \circ h_K : K/n \to L/n,$$

then $h_L \circ f/n = h_L \circ h'_L \circ f^n \circ h_K \simeq f^n \circ h_K$ rel K^{n-1}, which is (1). Thus, if we choose $k_0 \in K^{n-1}$ and set $l_0 := f\left(k_0\right) \in L^{n-1}$, then the following diagram commutes (the map $f_{<n}$ will be constructed later; the dashed arrow does not yet exist):

$$
\begin{array}{ccccc}
\pi_n\left(K_{<n}, K^{n-1}, k_0\right) & \xrightarrow{i_{K*}} & \pi_n\left(K/n, K^{n-1}, k_0\right) & \xrightarrow{h_{K*}\ \cong} & \pi_n\left(K^n, K^{n-1}, k_0\right) \\
\Big\downarrow{\scriptstyle (f_{<n})_*} & & \Big\downarrow{\scriptstyle (f/n)_*} & & \Big\downarrow{\scriptstyle f^n_*} \\
\pi_n\left(L_{<n}, L^{n-1}, l_0\right) & \xrightarrow{i_{L*}} & \pi_n\left(L/n, L^{n-1}, l_0\right) & \xrightarrow{h_{L*}\ \cong} & \pi_n\left(L^n, L^{n-1}, l_0\right)
\end{array} \qquad (*)
$$

Recall that $\widetilde{i}_K$ is the unique lift of $i_K \circ p_{K_{<n}}$ under $p_{K/n}$ and $\widetilde{h}_K$ is the unique lift of $h_K \circ p_{K/n}$ under p^n_K, such that $\widetilde{i}_K$ and $\widetilde{h}_K$ restrict to $\mathrm{id}_{\widetilde{K}^{n-1}}$ on $(n-1)$-skeletons ($\widetilde{i}_L$ and $\widetilde{h}_L$ are defined analogously). Let $\widetilde{f}/n$ be the unique lift of $f/n \circ p_{K/n}$ under $p_{L/n}$, which restricts to $\widetilde{f}^{n-1}$ on $(n-1)$-skeletons. Let $\widetilde{k}_0 \in \widetilde{K}^{n-1}$ be a lift of k_0 under p^n_K and set $\widetilde{l}_0 := \widetilde{f}(\widetilde{k}_0) \in \widetilde{L}^{n-1}$. Under the inverses of the isomorphisms

$$
p_{X*} : \pi_n(\widetilde{X}, \widetilde{K}^{n-1}, \widetilde{k}_0) \xrightarrow{\cong} \pi_n\left(X, K^{n-1}, k_0\right),
$$
$$
p_{Y*} : \pi_n(\widetilde{Y}, \widetilde{L}^{n-1}, \widetilde{l}_0) \xrightarrow{\cong} \pi_n\left(Y, L^{n-1}, l_0\right),
$$

where $X \in \{K_{<n}, K/n, K^n\}$ and $Y \in \{L_{<n}, L/n, L^n\}$, the previous diagram corresponds to the following commutative diagram (the map $\widetilde{f}_{<n}$ will be constructed later; the dashed arrow does not yet exist):

$$
\begin{array}{ccccc}
\pi_n(\widetilde{K}_{<n}, \widetilde{K}^{n-1}, \widetilde{k}_0) & \xrightarrow{\widetilde{i}_{K*}} & \pi_n(\widetilde{K}/n, \widetilde{K}^{n-1}, \widetilde{k}_0) & \xrightarrow{\widetilde{h}_{K*}\ \cong} & \pi_n(\widetilde{K}^n, \widetilde{K}^{n-1}, \widetilde{k}_0) \\
\Big\downarrow{\scriptstyle (\widetilde{f}_{<n})_*} & & \Big\downarrow{\scriptstyle (\widetilde{f}/n)_*} & & \Big\downarrow{\scriptstyle \widetilde{f}_*} \\
\pi_n(\widetilde{L}_{<n}, \widetilde{L}^{n-1}, \widetilde{l}_0) & \xrightarrow{\widetilde{i}_{L*}} & \pi_n(\widetilde{L}/n, \widetilde{L}^{n-1}, \widetilde{l}_0) & \xrightarrow{\widetilde{h}_{L*}\ \cong} & \pi_n(\widetilde{L}^n, \widetilde{L}^{n-1}, \widetilde{l}_0)
\end{array} \qquad (**)
$$

If we identify the relative homotopy groups in this diagram with the corresponding cellular chain groups via Hurewicz isomorphisms ($\widetilde{K}^{n-1}$ and $\widetilde{L}^{n-1}$ are simply connected), then this diagram fits into the following commutative diagram (the commutativity of the four remaining squares follows from property (ii) of Proposition 1.11):

$$
\begin{array}{ccccc}
C_n\left(K_{<n}\right) & \xrightarrow{i_{K*}} & C_n\left(K/n\right) & \xrightarrow{h_{K*}\ \cong} & C_n\left(K^n\right) \\
\Big\downarrow{\scriptstyle u_{K_{<n}}} & & \Big\downarrow{\scriptstyle u_{K/n}} & & \Big\downarrow{\scriptstyle u_K} \\
C_n(\widetilde{K}_{<n}) & \xrightarrow{\widetilde{i}_{K*}} & C_n(\widetilde{K}/n) & \xrightarrow{\widetilde{h}_{K*}\ \cong} & C_n(\widetilde{K}^n) \\
\Big\downarrow{\scriptstyle (\widetilde{f}_{<n})_*} & & \Big\downarrow{\scriptstyle (\widetilde{f}/n)_*} & & \Big\downarrow{\scriptstyle \widetilde{f}_*} \\
C_n(\widetilde{L}_{<n}) & \xrightarrow{\widetilde{i}_{L*}} & C_n(\widetilde{L}/n) & \xrightarrow{\widetilde{h}_{L*}\ \cong} & C_n(\widetilde{L}^n) \\
\Big\uparrow{\scriptstyle u_{L_{<n}}} & & \Big\uparrow{\scriptstyle u_{L/n}} & & \Big\uparrow{\scriptstyle u_L} \\
C_n\left(L_{<n}\right) & \xrightarrow{i_{L*}} & C_n\left(L/n\right) & \xrightarrow{h_{L*}\ \cong} & C_n\left(L^n\right)
\end{array} \qquad (***)
$$

By assumption, $\widetilde{f}_*$ maps $u_K(Y_K) = u_K h_{K*} i_{K*}(C_n(K_{<n})) = \widetilde{h}_{K*} u_{K/n} i_{K*}(C_n(K_{<n}))$ into $u_L(Y_L) = u_L h_{L*} i_{L*}(C_n(L_{<n})) = \widetilde{h}_{L*} u_{L/n} i_{L*}(C_n(L_{<n}))$. Thus, $(\widetilde{f}/n)_* = (\widetilde{h}_{L*})^{-1} \widetilde{f}_* \widetilde{h}_{K*}$ maps $u_{K/n} i_{K*}(C_n(K_{<n})) = \widetilde{i}_{K*} u_{K_{<n}}(C_n(K_{<n}))$ into $u_{L/n} i_{L*}(C_n(L_{<n})) = \widetilde{i}_{L*} u_{L_{<n}}(C_n(L_{<n}))$, which shows (3). If $\{y_\alpha\}$ are the n-cells of $K_{<n}$ and $\{\overline{y}_\alpha\}$ are the corresponding n-cells of $\overline{K}_{<n} \subset \widetilde{K}_{<n}$, then we see in particular that $(\widetilde{f}/n)_* \widetilde{i}_{K*} u_{K_{<n}}(y_\alpha) = (\widetilde{f}/n)_* \widetilde{i}_{K*}(\overline{y}_\alpha)$ lies in the image of $\widetilde{i}_{L*}$ for all α. Setting $k_0 := \chi(y_\alpha)(s_0)$, this corresponds to the statement that in diagram $(*)$ the element

$$(f/n)_* i_{K*}[\chi(y_\alpha)] \in \pi_n\left(L/n, L^{n-1}, l_0\right)$$

lies in the image of $i_{L*} : \pi_n(L_{<n}, L^{n-1}, l_0) \to \pi_n(L/n, L^{n-1}, l_0)$ for all α. Following the proof of the original compression theorem, one can conclude that $f/n \circ i_K$ is homotopic rel K^{n-1} to a cellular map $f_{<n}$ into $L_{<n} \subset L/n$, which finishes the construction of $f_{<n}$ and shows (2). Consequently, the commutative diagram $(*)$ can now be extended by the dashed arrow induced by $f_{<n}$. Let $\widetilde{f}_{<n}$ be the unique lift of $f_{<n} \circ p_{K_{<n}}$ under $p_{L_{<n}}$, which restricts to $\widetilde{f}^{n-1}$ on $(n-1)$-skeletons. The commutative diagrams $(**)$ and $(***)$ can now be extended by the dashed arrows induced by $\widetilde{f}_{<n}$. By (3), we know that $(\widetilde{f}/n)_*$ maps $u_{K/n} i_{K*}(C_n(K_{<n})) = \widetilde{i}_{K*} u_{K_{<n}}(C_n(K_{<n}))$ into $u_{L/n} i_{L*}(C_n(L_{<n})) = \widetilde{i}_{L*} u_{L_{<n}}(C_n(L_{<n}))$. By commutativity of $(***)$, one concludes that

$$(\widetilde{f}/n)_* \widetilde{i}_{K*} u_{K_{<n}}(C_n(K_{<n})) = \widetilde{i}_{L*}(\widetilde{f}_{<n})_* u_{K_{<n}}(C_n(K_{<n})) \subset \widetilde{i}_{L*} u_{L_{<n}}(C_n(L_{<n})).$$

Since $\widetilde{i}_L$ is the inclusion $\widetilde{L}_{<n} \hookrightarrow \widetilde{L}/n$, the induced map $\widetilde{i}_{L*} : C_n(\widetilde{L}_{<n}) \to C_n(\widetilde{L}/n)$ is injective, so (4) holds. ∎

Given a morphism $(f, \widetilde{f}) : \left(K, Y_K, \overline{K}, q_K\right) \to \left(L, Y_L, \overline{L}, q_L\right)$ in $\mathbf{CW}^0_{n\supset\partial}$, we follow [1, page 48] to define $\tau^0_{<n}(f, \widetilde{f})$. If $(f, \widetilde{f})$ is the identity morphism, then we define

$$\tau^0_{<n}(f, \widetilde{f}) = \mathrm{id}_{\tau^0_{<n}\left(K, Y_K, \overline{K}, q_K\right)}.$$

Otherwise, we apply Proposition 1.14 to the completions of $\left(K, Y_K, \overline{K}, q_K\right)$ and $\left(L, Y_L, \overline{L}, q_L\right)$ to diagrams with the properties of Proposition 1.13, which were chosen in the definition of $\tau^0_{<n}$ on objects. Thus, we obtain cellular maps $f_{<n} : K_{<n} \to L_{<n}$ and $f/n : K/n \to L/n$ with the stated properties (by construction, $f/n = h'_L \circ f^n \circ h_K$, where h'_L is a homotopy inverse rel L^{n-1} for h_L) and define

$$\tau^0_{<n}(f, \widetilde{f}) = ([f], [f^n], [f/n], [f_{<n}]).$$

In the special case that K and L are simply connected, we are free to choose

$$\tau^0_{<n}(f, \widetilde{f}) = \tau_{<n}(f).$$

All in all, we have defined a covariant assignment $\tau^0_{<n} : \mathbf{CW}^0_{n\supset\partial} \to \mathbf{HoCW}^0_{\supset<n}$ of objects and morphisms, which extends $\tau_{<n} : \mathbf{CW}_{n\supset\partial} \to \mathbf{HoCW}_{\supset<n}$ (see Corollary 1.19). Although the definition $\tau^0_{<n}$ on objects depends on the choice of a completion of an object $(K, Y_K, \overline{K}, q_K)$ to a diagram with the properties of Proposition 1.13, the fourth component $K_{<n}$ constructed like this is well defined up to rel $(n-1)$-skeleton homotopy equivalence (compare [1, Scholium 1.26, page 33]):

1.15 Corollary. *If we choose two completions of an object* $(K, Y_K, \overline{K}, q_K)$ *in* $\boldsymbol{CW}^0_{n\supset\partial}$ *to commutative diagrams*

$$\begin{array}{ccccccccc}
\overline{K}_{<n} & \xrightarrow{\bar{i}_K} & \overline{K}/n & \xrightarrow{\bar{h}_K} & \overline{K} & \xleftarrow{\bar{h}'_K} & \overline{K}/n' & \xleftarrow{\bar{i}'_K} & \overline{K}'_{<n} \\
\downarrow{\scriptstyle q_{K_{<n}}} & & \downarrow{\scriptstyle q_{K/n}} & & \downarrow{\scriptstyle q_K} & & \downarrow{\scriptstyle q'_{K/n}} & & \downarrow{\scriptstyle q'_{K_{<n}}} \\
K_{<n} & \xrightarrow{i_K} & K/n & \xrightarrow{h_K} & K^n & \xleftarrow{h'_K} & K/n' & \xleftarrow{i'_K} & K'_{<n}
\end{array}$$

with the properties of Proposition 1.13, then $K_{<n}$ *and* $K'_{<n}$ *are homotopy equivalent rel* $(n-1)$*-skeleton.*

Proof. Application of Proposition 1.14 to the identity morphism

$$(\mathrm{id}_K, \mathrm{id}_{\widetilde{K}^n}) : (K, Y_K, \overline{K}, q_K) \to (K, Y_K, \overline{K}, q_K)$$

in $\mathbf{CW}^0_{n\supset\partial}$ yields cellular maps $f/n : K/n \to K/n'$ and $f_{<n} : K_{<n} \to K'_{<n}$ such that the diagram

$$\begin{array}{ccccc}
K_{<n} & \xrightarrow{i_K} & K/n & \xrightarrow{h_K} & K^n \\
\downarrow{\scriptstyle f_{<n}} & & \downarrow{\scriptstyle f/n} & & \downarrow{\scriptstyle =} \\
K'_{<n} & \xrightarrow{i'_K} & K/n' & \xrightarrow{h'_K} & K^n
\end{array}$$

commutes up to homotopy rel K^{n-1} and a morphism

$$(f_{<n}, \widetilde{f}_{<n}) : \left(K_{<n}, C_n\left(K_{<n}\right), \overline{K}_{<n}, q_{K_{<n}}\right) \to \left(K'_{<n}, C_n\left(K'_{<n}\right), \overline{K}'_{<n}, q_{K'_{<n}}\right)$$

in $\mathbf{CW}^0_{n\supset\partial}$, where $\widetilde{f}_{<n}$ is the unique lift of $f_{<n} \circ p_{K_{<n}}$ under $p_{K'_{<n}}$, such that $\widetilde{f}_{<n}$ restricts to $\mathrm{id}_{\widetilde{K}^{n-1}}$ on $(n-1)$-skeletons. We claim that $f_{<n}$ induces an isomorphism

$$f_{<n*} : C_n\left(K_{<n}\right) \xrightarrow{\cong} C_n\left(K'_{<n}\right).$$

Consider the commutative diagram

$$\begin{array}{ccccc}
C_n\left(K_{<n}\right) & \xrightarrow{i_{K*}} & C_n\left(K/n\right) & \xrightarrow{h_{K*}\ \cong} & C_n\left(K^n\right) \\
\downarrow{\scriptstyle f_{<n*}} & & \downarrow{\scriptstyle (f/n)_*} & & \downarrow{\scriptstyle =} \\
C_n\left(K'_{<n}\right) & \xrightarrow{i'_{K*}} & C_n\left(K/n'\right) & \xrightarrow{h'_{K*}\ \cong} & C_n\left(K^n\right)
\end{array}$$

Injectivity of $f_{<n*}$ follows from the injectivity of i_{K*} and h_{K*}. For the surjectivity of $f_{<n*}$, use $h_{K*}i_{K*}\left(C_n\left(K_{<n}\right)\right) = Y_K = h'_{K*}i'_{K*}\left(C_n\left(K'_{<n}\right)\right)$ and $(f/n)_* = (h'_{K*})^{-1} \circ h_{K*}$ to write

$$i'_{K*}f_{<n*}(C_n\left(K_{<n}\right)) = (f/n)_*\, i_{K*}\left(C_n\left(K_{<n}\right)\right) = i'_{K*}\left(C_n\left(K'_{<n}\right)\right).$$

By injectivity of i'_{K*} we obtain $f_{<n*}(C_n(K_{<n})) = C_n(K'_{<n})$.

Being a morphism in $\mathbf{CW}^0_{n\supset\partial}$, the pair $(f_{<n}, \widetilde{f}_{<n})$ satisfies $\widetilde{f}_{<n*} u_{K_{<n}}(C_n(K_{<n})) \subset u_{K'_{<n}}(C_n(K'_{<n}))$. This is property (iii) of Proposition 1.11. Thus, by property (i) of the same proposition, there exists an extension $\overline{f}_{<n} : \overline{K}_{<n} \to \overline{K}'_{<n}$ of $(\widetilde{f}_{<n})^{n-1} = \mathrm{id}_{\widetilde{K}^{n-1}}$, such that $q_{K'_{<n}} \circ \overline{f}_{<n} \simeq f_{<n} \circ q_{K_{<n}}$ rel $\widetilde{K}^{n-1}$. We claim that $\overline{f}_{<n}$ induces isomorphisms

$$\overline{f}_{<n*} : C_m(\overline{K}_{<n}) \xrightarrow{\cong} C_m(\overline{K}'_{<n}) \quad \forall\, m \geq 0.$$

For $m \neq n$ this is clear, because $\overline{K}_{<n}$ and $\overline{K}'_{<n}$ are n-dimensional and $\overline{f}_{<n}$ restricts to $\mathrm{id}_{\widetilde{K}^{n-1}}$ on $(n-1)$-skeletons. For $m = n$ this follows from commutativity of the following diagram:

$$\begin{array}{ccc} C_n(\overline{K}_{<n}) & \xrightarrow{\overline{f}_{<n*}} & C_n(\overline{K}'_{<n}) \\ \downarrow{\scriptstyle q_{K_{<n}*}\ \cong} & & \downarrow{\scriptstyle q'_{K_{<n}*}\ \cong} \\ C_n(K_{<n}) & \xrightarrow[]{f_{<n*}\ \cong} & C_n(K'_{<n}) \end{array}$$

Thus, $\overline{f}_{<n}$ induces isomorphisms on all homology groups. Since $\overline{K}_{<n}$ and $\overline{K}'_{<n}$ are simply connected, $\overline{f}_{<n}$ is a homotopy equivalence by the homological version of the Whitehead theorem. By [2, Proposition 0.19, page 16], $\overline{f}_{<n}$ is a homotopy equivalence rel $\widetilde{K}^{n-1}$. By Lemma 1.12 (iii), one can conclude from $q_{K'_{<n}} \circ \overline{f}_{<n} \simeq f_{<n} \circ q_{K_{<n}}$ rel $\widetilde{K}^{n-1}$ that $f_{<n}$ is a homotopy equivalence rel K^{n-1}. ∎

For morphisms in $\mathbf{HoCW}_{\supset<n}$, n-compression rigidity is defined in terms of eigenhomotopies, compare [1, Definition 1.33, page 40], which involves the concept of virtual cell groups, compare [1, Definition 1.10, page 18]. Although one could try to adapt these definitions to arbitrary path connected CW-complexes, the interpretation of virtual cell groups as sitting between two actual cellular chain groups gets lost, because it relies on Hurewicz isomorphisms, which are only available for simply connected spaces. For our purpose, n-compression rigidity for morphisms in $\mathbf{HoCW}^0_{\supset<n}$ is defined in the following way, which is equivalent to the original definition for morphisms in $\mathbf{HoCW}_{\supset<n}$ by [1, Proposition 1.34, page 40]:

Definition. 1.16 A morphism $([f], [f_n], [f/n], [f_{<n}]) : (K, K/n, h_K, K_{<n}) \to (L, L/n, h_L, L_{<n})$ in $\mathbf{HoCW}^0_{\supset<n}$ is called *n-compression rigid* if any two cellular maps $g_1, g_2 : K_{<n} \to L_{<n}$ such that the diagram

$$\begin{array}{ccc} K_{<n} & \xrightarrow{i_K} & K/n \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle f/n} \\ L_{<n} & \xrightarrow{i_L} & L/n \end{array}$$

homotopy commutes rel K^{n-1} for $i=1,2$ are homotopic rel K^{n-1}.
A subcategory $\mathbf{C}\subset\mathbf{CW}^0_{n\supset\partial}$ is called n-compression rigid, if every morphism in $\mathbf{C}$ has an n-compression rigid image under $\tau^0_{<n}$.

The following lemma is a variation of [1, Lemma 1.43, page 53].

1.17 Lemma. *Let X, Y and Y' be CW-complexes, where X is k-dimensional, $k\geq 1$, and Y, Y' are path connected. Assume that $g_1,g_2:X\to Y$ are two maps that agree on X^{k-1} and $f:Y\to Y'$ is a map such that the induced map $f_*:\pi_k(Y)\to\pi_k(Y')$ is injective (i.e. $f_*:\pi_k(Y,y_0)\to\pi_k(Y',f(y_0))$ is injective for one and hence for all $y_0\in Y$). If $f\circ g_1\simeq f\circ g_2$ rel X^{k-1}, then $g_1\simeq g_2$ rel X^{k-1}.*

Proof. The $(k+1)$-dimensional CW-complex $Z=X\times I$ has k-skeleton $Z^k=(X\times I)^k=(X\times\partial I)\cup\left(X^{k-1}\times I\right)$. Since $g_1|_{X^{k-1}}=g_2|_{X^{k-1}}$, one can define the map

$$g=(g_1\times\{0\}\cup g_2\times\{1\})\cup(g_1|_{X^{k-1}}\times\mathrm{id}_I):Z^k\to Y.$$

By assumption, there is a rel X^{k-1} homotopy $X\times I=Z\to Y'$ between $f\circ g_1$ and $f\circ g_2$. This homotopy restricts on Z^k to $f\circ g:Z^k\to Y'$. In other words, for every $(k+1)$-cell e^{k+1} of Z, the composition

$$S^k\xrightarrow{\chi\left(e^{k+1}\right)|}Z^k\xrightarrow{f\circ g}Y'$$

can be extended over D^{k+1} and is thus nullhomotopic rel $s_0\in S^k$. Therefore,

$$\left[f\circ g\circ\chi\left(e^{k+1}\right)|\right]=0\in\pi_k\left(Y',f\left(y_0\right)\right),$$

where y_0 denotes the image of the basepoint $s_0\in S^k$ under $g\circ\chi\left(e^{k+1}\right)|$. By injectivity of $f_*:\pi_k(Y,y_0)\to\pi_k(Y',f(y_0))$, one can conclude that $\left[g\circ\chi\left(e^{k+1}\right)|\right]$ is zero in $\pi_k(Y,y_0)$. Therefore, the composition

$$S^k\xrightarrow{\chi\left(e^{k+1}\right)|}Z^k\xrightarrow{g}Y$$

is nullhomotopic rel s_0 for all $(k+1)$-cells e^{k+1} of Z and can thus be extended over D^{k+1}. As a consequence, g can be extended to a rel X^{k-1} homotopy $Z\times I\to Y$ between g_1 and g_2. ∎

We record the following sufficient conditions for a morphism in $\mathbf{HoCW}^0_{\supset<n}$ to be n-compression rigid (see [1, Corollary 1.45, page 55] and [1, Corollary 1.49, page 58] for morphisms in $\mathbf{HoCW}_{\supset<n}$).

1.18 Proposition. *A morphism $([f],[f_n],[f/n],[f_{<n}]):(K,K/n,h_K,K_{<n})\to(L,L/n,h_L,L_{<n})$ in $\mathbf{HoCW}^0_{\supset<n}$ is n-compression rigid if one of the following holds:*

- *$i_{L*}:\pi_n(L_{<n})\to\pi_n(L/n)$ is injective.*
- *$\partial_n:C_n(K)\to C_{n-1}(K)$ is the zero map.*
- *$\partial_n:C_n(L)\to C_{n-1}(L)$ is injective.*

Proof. Assume that $g_1, g_2 : K_{<n} \to L_{<n}$ are two cellular maps such that $i_L \circ g_i \simeq f/n \circ i_K$ rel K^{n-1} for $i = 1, 2$. Thus, $i_L \circ g_1 \simeq i_L \circ g_2$ rel K^{n-1}. In particular, $i_L \circ g_1$ and $i_L \circ g_2$ agree on K^{n-1}. Since $i_L : L_{<n} \hookrightarrow L/n$ is an inclusion, one can conclude that g_1 and g_2 agree on $K^{n-1} = (K_{<n})^{n-1}$.

- For the first statement, the claim that $g_1 \simeq g_2$ rel K^{n-1} follows from Lemma 1.17.
- The second statement follows from the observation that $K_{<n} = K^{n-1}$ (since $\partial_n = 0$, all n-cells are cycle cells and $Y_K = 0$ is unique) and that g_1 and g_2 agree on K^{n-1} as shown above.
- For the proof of the third statement, note that $L/n = L_{<n}$ ($Z_n(L) = 0$ has a cell basis and $Y_L = C_n(L)$ is unique) and thus $i_L = \mathrm{id}_{L/n}$.

∎

The results of the present section can be summed up in the following

1.19 Corollary. *Let $n \geq 3$ be an integer. There is a covariant assignment*

$$\tau^0_{<n} : \boldsymbol{CW}^0_{n\supset\partial} \to \boldsymbol{HoCW}^0_{\supset<n}$$

of objects and morphisms, which restricts to $\tau_{<n}$ on $\boldsymbol{CW}_{n\supset\partial}$. Moreover, $\tau^0_{<n}$ is a functor on n-compression rigid subcategories $\boldsymbol{C} \subset \boldsymbol{CW}^0_{n\supset\partial}$ (the proof is analogous to the proof of [1, Corollary 1.40, page 50]).

Now, we proceed analogous to [1, page 50f]. Let $P^0_4 : \mathbf{HoCW}^0_{\supset<n} \to \mathbf{HoCW}_{n-1}$ be the functor given by projection to the fourth component. By composition with $\tau^0_{<n}$, we define the covariant assignment of objects and morphisms

$$t_{<n} = P^0_4 \circ \tau^0_{<n} : \mathbf{CW}^0_{n\supset\partial} \to \mathbf{HoCW}_{n-1}.$$

If $\left(K, Y_K, \overline{K}, q_K\right)$ is an object in $\mathbf{CW}^0_{n\supset\partial}$, then $\tau^0_{<n}\left(K, Y_K, \overline{K}, q_K\right) = (K, K/n, h_K, K_{<n})$ is an n-truncation structure. We define the natural transformation $\mathrm{emb}^0_n : t^0_{<n} \to t^0_{<\infty}$ on $\left(K, Y_K, \overline{K}, q_K\right)$ by the rel K^{n-1} homotopy class $K_{<n} \to K$ of the composition

$$K_{<n} \hookrightarrow K/n \xrightarrow{h_K} K^n \hookrightarrow K.$$

Finally, one arrives at the following counterpart of [1, Theorem 1.41, page 51]:

1.20 Theorem. *Let $n \geq 3$ be an integer. There is a spatial homology truncation assignment $t^0_{<n} : \boldsymbol{CW}^0_{n\supset\partial} \to \boldsymbol{HoCW}_{n-1}$ with natural transformation $\mathrm{emb}^0_n : t^0_{<n} \to t^0_{<\infty}$, which extend $t_{<n}$ and emb_n. Moreover, $t^0_{<n}$ is a spatial homology truncation functor on all n-compression rigid subcategories $\boldsymbol{C} \subset \boldsymbol{CW}^0_{n\supset\partial}$.*

Remark. 1.21 (dependence on choices) Let K be a path connected CW-complex. Let $(K, Y_K, \overline{K}, q_K)$ and $(K, Y'_K, \overline{K}', q'_K)$ be two completions to objects in $\mathbf{CW}^0_{n\supset\partial}$. What can be said about the connection between their images under $\tau^0_{<n}$? If K is simply connected, then [1, Proposition 1.25, page 30] characterizes algebraically when $K_{<n}$ and $K'_{<n}$ are homotopy equivalent rel K^{n-1}. In general, $K_{<n}$ and $K'_{<n}$ are not homotopy equivalent (even if K is simply connected, see Example 2.17). Now assume that $Y_K = Y'_K$. If $(\mathrm{id}_K, \mathrm{id}_{\widetilde{K}^n}) : (K, Y_K, \overline{K}, q_K) \to (K, Y_K, \overline{K}', q'_K)$ happens to be a morphism in $\mathbf{CW}^0_{n\supset\partial}$, then one can apply the proof of Corollary 1.15 to this morphism. In conclusion, $K_{<n}$ and $K'_{<n}$ are homotopy equivalent rel K^{n-1}.

1.3 Path Connected CW-Complexes and Normality

Following [6, Definition 8.1, page 53], a simply connected CW-complex K is called *normal*, if K has a filtration $K_2 \subset ... \subset K_n \subset ... \subset \cup_n K_n = K$ into simply connected subcomplexes such that (using the inclusions $i_r : K_r \hookrightarrow K$)

$$H_r(K_n) = 0, \quad \text{for } r > n,$$
$$i_{r*} : H_r(K_n) \xrightarrow{\cong} H_r(K), \quad \text{for } r \leq n.$$

It turns out that every simply connected CW-complex is homotopy equivalent to a normal CW-complex (see [6, Theorem 8.2, page 53]). The result of this section is that every path connected CW-complex K is homotopy equivalent rel 2-skeleton to a CW-complex L whose group of n-cycles has a cell-basis for all $n \geq 3$ (see Proposition 1.24). In consequence, L has a filtration $L_2 \subset ... \subset L_n \subset ... \subset \cup_n L_n = L$ into path connected subcomplexes with the above properties for homology groups (one obtains L_n for $n \geq 2$ by removing the $(n+1)$-cycle cells of L^{n+1}, see [1, Lemma 1.2, page 6]). The result is obtained by adapting the proof of [6, Theorem 8.2, page 53] to the case of path connected CW-complexes K. Roughly speaking, the n-truncation structures which can be constructed separately for every dimension $n \geq 3$ by Proposition 1.13 are now put together in a single space L. Under certain conditions involving fundamental groups, a given path connected CW-complex is even homotopy equivalent to a CW-complex whose group of n-cycles has a cell-basis for all $n \geq 2$ (see Remark 1.25).

The following lemma is a direct consequence of [6, Proposition 6.8, page 41].

1.22 Lemma. *Let $f : X \to Y$ be a homotopy equivalence between n-dimensional CW-complexes, $n \geq 0$. If X is the n-skeleton of an $(n+1)$-dimensional CW-complex X', then there exists an $(n+1)$-dimensional CW-complex Y' with n-skeleton Y and an extension $f' : X' \to Y'$ of f such that f' is a homotopy equivalence.*

The following lemma follows from Whitehead's theorem.

1.23 Lemma. *Let $f : X \to Y$ be a cellular map between path connected CW-complexes. If for all integers $n > 0$ there exists an integer $m > n$ such that the restriction $f^m : X^m \to Y^m$ to m-skeletons is a homotopy equivalence, then f is a homotopy equivalence.*

Proof. Choose a point $x_0 \in X$ and let $y_0 := f(x_0) \in Y$. It suffices to show that f induces isomorphisms $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ for all $n \geq 0$. Then the lemma follows from Whitehead's theorem [2, Theorem 4.5, page 346].
For $n = 0$ this is true, because $\pi_0(X, x_0)$ and $\pi_0(Y, y_0)$ vanish for path connected X and Y. For $n > 0$ choose an integer $m > n$ such that the restriction $f^m : X^m \to Y^m$ to m-skeletons is a homotopy equivalence. The map of pointed pairs $f : (X, X^m, x_0) \to (Y, Y^m, y_0)$ induces the following commutative diagram by naturality of the long exact sequence of homotopy groups ($i : X^m \hookrightarrow X$ and $j : Y^m \hookrightarrow Y$ are the inclusions):

$$\begin{array}{ccccccc}
\pi_{n+1}(X, X^m, x_0) & \xrightarrow{\partial} & \pi_n(X^m, x_0) & \xrightarrow{i_*} & \pi_n(X, x_0) & \xrightarrow{\text{incl}_*} & \pi_n(X, X^m, x_0) \\
 & & \downarrow f_*^m & & \downarrow f_* & & \\
\pi_{n+1}(Y, Y^m, y_0) & \xrightarrow{\partial} & \pi_n(Y^m, y_0) & \xrightarrow{j_*} & \pi_n(Y, y_0) & \xrightarrow{\text{incl}_*} & \pi_n(Y, Y^m, y_0)
\end{array}$$

As the pair (X, X^m) is m-connected, $\pi_{n+1}(X, X^m, x_0)$ and $\pi_n(X, X^m, x_0)$ vanish because of $m \geq n+1 > n > 0$. Thus, i_* is an isomorphism by exactness of the first row. Analogously, j_* is an isomorphism. Moreover, the homotopy equivalence $f^m : X^m \to Y^m$ induces an isomorphism $f_*^m : \pi_n(X^m, x_0) \to \pi_n(Y^m, y_0)$ by [2, Exercise 2, page 358]. Consequently, $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is also an isomorphism by commutativity. ∎

1.24 Proposition. *Given a path connected CW-complex K, there exist*

- *a CW-complex L with $L^2 = K^2$ and such that $Z_n(L)$ has a cell-basis for all $n \geq 3$.*
- *a homotopy equivalence $q : K \to L$ rel K^2 such that q is cellular and the restriction $q^n : K^n \to L^n$ to n-skeletons is the identity map for $n = 2$ and a homotopy equivalence for $n > 2$.*

Proof. In a first step, we construct inductively a sequence of CW-complexes

$$K = K_2 \xrightarrow{h_{<3}} K_3 \longrightarrow \ldots \longrightarrow K_n \xrightarrow{h_{<n+1}} K_{n+1} \longrightarrow \ldots$$

with the following properties for all $n \geq 2$:

- $(K_n)^n = (K_{n+1})^n$ and the group of $(n+1)$-cycles of K_{n+1} has a cell-basis.
- $h_{<n+1} : K_n \to K_{n+1}$ is a cellular map such that the restriction $(h_{<n+1})^m : (K_n)^m \to (K_{n+1})^m$ to m-skeletons is the identity map for $m = n$ and a homotopy equivalence for $m > n$.

Assume that the sequence has already been constructed up to K_n (for $n = 2$, take $K_2 = K$). We wish to construct K_{n+1} and $h_{<n+1} : K_n \to K_{n+1}$ with the desired properties. By Remark 1.9, K_n can be completed to an object in $\mathbf{CW}^0_{n+1\supset\partial}$. By Proposition 1.13, there exist

- an $(n+1)$-dimensional CW-complex $K_n/n+1$ with $(K_n/n+1)^n = (K_n)^n$ and such that its group of $(n+1)$-cycles has a cell-basis.
- a cellular map $h' : (K_n)^{n+1} \to K_n/n+1$ which restricts to the identity map on the common n-skeleton $(K_n)^n$ and which is a homotopy equivalence rel $(K_n)^n$.

Using Lemma 1.22, we expand h' inductively over the skeletons of K_n and obtain

- a CW-complex K_{n+1} with $(K_{n+1})^{n+1} = K_n/n+1$.
- a cellular map $h_{<n+1} : K_n \to K_{n+1}$ such that the restriction $(h_{<n+1})^m : (K_n)^m \to (K_{n+1})^m$ to m-skeletons is h' for $m = n+1$ and a homotopy equivalence for $m > n+1$.

One can check that K_{n+1} and $h_{<n+1}$ satisfy the desired properties:

- $(K_{n+1})^n = (K_n/n+1)^n = (K_n)^n$ and $Z_{n+1}(K_{n+1}) = Z_{n+1}(K_n/n+1)$ has a cell-basis.
- $h_{<n+1} : K_n \to K_{n+1}$ is cellular, $(h_{<n+1})^n = (h')^n$ is the identity map on $(K_n)^n$, $(h_{<n+1})^{n+1} = h'$ is a homotopy equivalence and $(h_{<n+1})^m$ is a homotopy equivalence for $m > n+1$.

In a second step, we construct L and $q : K \to L$ with the desired properties. Since $(K_n)^n = (K_{n+1})^n \subset (K_{n+1})^{n+1}$ for all $n \geq 2$, there exists a unique CW-complex L with $L^n = (K_n)^n$ for all $n \geq 2$. In particular, $L^2 = (K_2)^2 = K^2$. By construction, $Z_{n+1}(L) = Z_{n+1}(L^{n+1}) = Z_{n+1}\left((K_{n+1})^{n+1}\right)$ has a cell-basis for all $n \geq 2$. A homotopy equivalence $q : K \to L$ rel K^2 can be constructed as follows. For all $n \geq 2$, the composition

$$q_{n+1} : K^{n+1} = (K_2)^{n+1} \overset{(h_{<3})^{n+1}}{\longrightarrow} (K_3)^{n+1} \longrightarrow \dots \longrightarrow (K_n)^{n+1} \overset{(h_{<n+1})^{n+1}}{\longrightarrow} (K_{n+1})^{n+1} = L^{n+1}$$

is a homotopy equivalence satisfying

$$(q_{n+1})^n = \begin{cases} (h_{<n+1})^n \circ q_n = q_n & \text{for } n > 2, \\ \mathrm{id}_{K^2} & \text{for } n = 2. \end{cases}$$

Now define the cellular map $q : K \to L$ by $q^{n+1} = q_{n+1}$ for all $n \geq 2$. Thus, $q^n : K^n \to L^n$ is a homotopy equivalence for $n > 2$ and $q^2 = \mathrm{id}_{K^2}$. q is a homotopy equivalence by Lemma 1.23. (Note that L is path connected, because $L^2 = K^2$ is path connected.) q is a homotopy equivalence rel K^2 by [2, Proposition 0.19, page 16].

■

Remark. 1.25 We state conditions under which a path connected CW-complex K is homotopy equivalent to a CW-complex L such that $Z_n(L)$ has a cell-basis for all $n \geq 2$. Assume that K^2 is homotopy equivalent to a two-dimensional CW-complex P^2 such that $Z_2(P^2)$ has a cell-basis. Then, by Lemma 1.22 and Lemma 1.23, K is homotopy equivalent to a CW-complex P with 2-skeleton P^2 (thus, $Z_2(P)$ has a cell-basis). Finally, by Proposition 1.24, P is homotopy equivalent to the desired CW-complex L (note that $P^2 = L^2$).

- If the inclusion $K^1 \hookrightarrow K^2$ induces an isomorphism $\pi_1(K^1, k_0) \cong \pi_1(K^2, k_0)$ on fundamental groups (for one and hence for all basepoints $k_0 \in K^1$), then $Z_2(K^2) = C_2(K^2)$ (thus, $Z_2(K^2)$ has a cell-basis). To show this, let $\{e^2_\alpha\}$ be the 2-cells of K and consider the following portion of the exact homotopy sequence of the pointed pair (K^2, K^1, k_α) $(k_\alpha := \chi(e^2_\alpha)(s_0))$:

$$\pi_2\left(K^2, K^1, k_\alpha\right) \overset{\partial}{\longrightarrow} \pi_1\left(K^1, k_\alpha\right) \overset{\cong}{\longrightarrow} \pi_1\left(K^2, k_\alpha\right).$$

  Thus, $\partial([\chi(e^2_\alpha)]) = [\chi(e^2_\alpha)\,|] = 0$ in $\pi_1(K^1, k_\alpha)$ for all α. The connecting homomorphism $\delta : C_2(K) = H_2(K^2, K^1) \to H_1(K^1)$ satisfies

$$\delta\left(e^2_\alpha\right) = \delta\,\mathrm{Hur}\left(\left[\chi\left(e^2_\alpha\right)\right]\right) = \mathrm{Hur}\,\partial\left(\left[\chi\left(e^2_\alpha\right)\right]\right) = 0.$$

  Therefore, e^2_α is in the kernel of $\partial_2 : C_2(K) \to C_1(K)$ for all α.

- If K^2 is finite and $\pi_1(K^2)(=\pi_1(K))$ is finite abelian, one can write $\pi_1(K) = \mathbb{Z}/m_1 \times ... \times \mathbb{Z}/m_n$ with $m_i|m_{i+1}$ and $m_1 > 1$. By [7, page 123], K^2 is homotopy equivalent to a CW-complex

$$P^2 := K_r \vee \bigvee S^2,$$

where r is relatively prime to m_1 and K_r is the model of a twisted presentation

$$\mathcal{P}_r = \langle a_1, ..., a_r | a_1^{m_1}, ..., a_n^{m_n}, [a_1^r, a_2], [a_i, a_j] \text{ for } i < j, (i,j) \neq (1,2) \rangle$$

of $\pi_1(K)$ (see [7, page 108]). By definition, K_r is obtained from a bouquet of 1-spheres, where each 1-sphere corresponds to a generator of $\mathcal{P}_r$, by attaching for every relation in $\mathcal{P}_r$ a 2-sphere via the attaching map given by the relation. It is clear that the 2-cells corresponding to the commutator relations of $\mathcal{P}_r$ form a cell-basis of $Z_2(K_r)$. (Note that every non-trivial linear combination of 2-cells, which correspond to relations of the form $a_i^{m_i}$, has non-trivial image in $C_1(K_r)$ under the boundary operator.) Thus, $Z_2(P^2)$ has a cell-basis.
- If K^2 is compact and $\pi_1(K)$ is free, then K^2 is homotopy equivalent to a finite bouquet P^2 of one- and two-dimensional spheres by [7, Theorem 3.9, page 120]. Thus, $Z_2(P^2) = C_2(P^2)$ has a cell-basis.

Remark. 1.26 The fact that every simply connected CW-complex is homotopy equivalent to a normal CW-complex leads to the construction of a *homology decomposition* for a given homotopy type (compare [6, page 55ff]). The required k-invariants are obtained from [6, Theorem 7.1', page 47], where simple connectivity is assumed. The question whether a homology decomposition including k-invariants can also be obtained for path connected CW-complexes is not answered by the previous discussion.

2 Induced Maps between Intersection Spaces

Let $n \geq 2$ be an integer and let $\overline{p}$ be a perversity. Then the cut-off degree $k = n - 1 - \overline{p}(n)$ is an integer satisfying $1 \leq k \leq n-1$.

Definition. 2.1 The category $\mathbf{P}(n,\overline{p})$ consists of the following objects and morphisms:

- Objects are triples (X, Σ, Λ), where
  1. X is an n-dimensional compact topological pseudomanifold with only isolated singularities. Let $\sigma \subset X$ be the (finite) set of singular points of X and let λ be the set of corresponding links. This yields a bijection $\sigma \xrightarrow{\sim} \lambda$, $x \mapsto L_x$. We assume that all links $L \in \lambda$ are path connected.
  2. Σ is a set of pairs $(x, \text{cone}(L_x))$, such that every singularity $x \in \sigma$ has been equipped (in one way) with a small cone neighbourhood in X,
  $$x \quad \in \quad \text{int}(\text{cone}(L_x)) \quad \subset \quad \text{cone}(L_x) = (L_x \times I) / (L_x \times \{0\}) \quad \subset \quad X,$$
  and such that the neighbourhoods $\text{cone}(L_x) \subset X$ are pairwise disjoint. We use the notation $M := X - \bigsqcup_{L\in\lambda} \text{int}(\text{cone}(L))$ and $\partial M = \bigsqcup_{L\in\lambda} L$.
  3. Λ is a set of triples $(L, L_{<k}, f_L)$, such that every link $L \in \lambda$ has been equipped (in one way) with a CW-structure and with a spatial homology truncation $f_L : L_{<k} \to L$ in degree k (see Section 1.1). If $k = 1$ or if $k = 2$ and $L \in \lambda$ is simply connected, then we assume that f_L is the inclusion of a 0-cell $L_{<k} \hookrightarrow L$. This yields a valid spatial homology truncation of L in degree k (compare [1, Section 1.1.5, page 24]).
- Morphisms $(X, \Sigma, \Lambda) \to (X', \Sigma', \Lambda')$ are continuous maps $F : X \to X'$, such that $F(M) \subset M'$ and for every $L \in \lambda$ there is an $L'_L \in \lambda'$ with $F(L) \subset L'_L$.

Remark. 2.2 The links of an n-dimensional compact topological pseudomanifold X with isolated singularities are closed $(n-1)$-dimensional topological manifolds. It is a non-trivial requirement that they possess the structure of a CW-complex (compare [1, Remark 2.9, page 112]). But if a link L of X has been equipped with the structure of a CW-complex, then it can be completed to a triple $(L, L_{<k}, f_L)$ by Corollary 1.4.

The definition of $\mathbf{P}(n,\overline{p})$ is motivated as follows. Let (X, Σ, Λ) be an object in $\mathbf{P}(n,\overline{p})$. Let $j : \partial M \hookrightarrow M$ be the inclusion. The fixed CW-structures and spatial homology truncations $f_L : L_{<k} \to L$ for all links $L \in \lambda$ can be used to define the following map, which is a spatial homology truncation of ∂M in degree k:

$$f \quad : \quad (\partial M)_{<k} := \bigsqcup_{L\in\lambda} L_{<k} \quad \xrightarrow{\bigsqcup_{L\in\lambda} f_L} \quad \bigsqcup_{L\in\lambda} L = \partial M.$$

The intersection space construction yields the following assignment on the object level:

$$\mathrm{I} \; : \; \mathrm{Ob}\,\mathbf{P}(n,\overline{p}) \; \to \; \mathrm{Ob}\,\mathbf{HoTop}, \qquad \mathrm{I}(X,\Sigma,\Lambda) = \text{cone}(j \circ f).$$

The goal of the following sections is to define subcategories $\mathbf{P}_*(n,\overline{p})$ of $\mathbf{P}(n,\overline{p})$ and functors $\mathrm{I}_* : \mathbf{P}_*(n,\overline{p}) \to \mathbf{HoTop}$, which agree with I on objects. (In Section 2.5, objects and morphisms in $\mathbf{P}_*(n,\overline{p})$ will be equipped with some extra structure. In this case, $\mathbf{P}_*(n,\overline{p}) \to \mathbf{P}(n,\overline{p})$ denotes the forgetful functor instead of the inclusion functor.)

2.1 Induced Maps in Low Dimensions

This section treats the case of cut-off degree $k = 1$. Recall the construction of $\mathrm{I}(X, \Sigma, \Lambda)$ for an object (X, Σ, Λ) in $\mathbf{P}(n, \overline{p})$. For every link $L \in \lambda$ we take a copy I_L of the unit interval $[0,1]$. We identify in $M \bigsqcup_{L\in\lambda} I_L$ all endpoints $0 \in I_L$ to a single point $*$ and for every $L \in \lambda$ the point $1 \in I_L$ with the chosen 0-cell $L_{<1} \in L \subset \partial M$. In the following, we present two subcategories $\mathbf{P}_*(n, \overline{p})$ of $\mathbf{P}(n, \overline{p})$ together with functors $\mathrm{I}_* : \mathbf{P}_*(n, \overline{p}) \to \mathbf{HoTop}$, which are defined by $(X, \Sigma, \Lambda) \mapsto \mathrm{I}(X, \Sigma, \Lambda)$ on objects.

- Let $\mathbf{P}_\bullet(n, \overline{p})$ be the subcategory of $\mathbf{P}(n, \overline{p})$ with the same objects as in $\mathbf{P}(n, \overline{p})$ and whose morphisms $F : (X, \Sigma, \Lambda) \to (X', \Sigma', \Lambda')$ preserve the chosen 0-cells of the links, that is $F(L_{<1}) = (L'_L)_{<1}$ for all $L \in \lambda$ (recall that $F(L) \subset L'_L$). A continuous map $\overline{F} : \mathrm{I}(X, \Sigma, \Lambda) \to \mathrm{I}(X', \Sigma', \Lambda')$ is obtained by setting

$$\overline{F}(x) \quad = \quad \begin{cases} *', & \text{for} \quad x = *, \\ x \ (\in I_{L'_L}), & \text{for} \quad x \in (0,1) \subset I_L, \\ F(x) \ \ (\in M'), & \text{for} \quad x \in M. \end{cases}$$

  Setting $\mathrm{I}_\bullet(F) = \overline{F}$ obviously defines a functor $\mathrm{I}_\bullet : \mathbf{P}_\bullet(n, \overline{p}) \to \mathbf{Top}$.

- Let $\mathbf{P}_1(n, \overline{p})$ be the full subcategory of $\mathbf{P}(n, \overline{p})$, whose objects (X, Σ, Λ) have the property that all links $L \in \lambda$ are simply connected. Given a morphism $F : (X, \Sigma, \Lambda) \to (X', \Sigma', \Lambda')$ in $\mathbf{P}_1(n, \overline{p})$, choose for every $L \in \lambda$ a path $\phi_L : [0,1] \to L'_L$ between $\phi_L(0) = (L'_L)_{<1}$ and $\phi_L(1) = F(L_{<1}) (\in L'_L)$. A continuous map $\overline{F} : \mathrm{I}(X, \Sigma, \Lambda) \to \mathrm{I}(X', \Sigma', \Lambda')$ is obtained by setting

$$\overline{F}(x) \quad = \quad \begin{cases} *', & \text{for} \quad x = *, \\ 2x \ (\in I_{L'_L}), & \text{for} \quad x \in (0,1/2) \subset I_L, \\ \phi_L(2x-1) \ \ (\in L'_L), & \text{for} \quad x \in [1/2,1) \subset I_L, \\ F(x) \ \ (\in M'), & \text{for} \quad x \in M. \end{cases}$$

  The class $\left[\overline{F}\right]$ in $\mathbf{HoTop}$ does not depend on the choice of the paths ϕ_L. (If ϕ'_L is any other path for some $L \in \lambda$, then the paths $[1/2, 1] \to L'_L$ given by $x \mapsto \phi_L(2x-1)$ and $x \mapsto \phi'_L(2x-1)$ are homotopic rel endpoints, since L'_L is simply connected.) Sending F to the induced morphism $\left[\overline{F}\right]$ in $\mathbf{HoTop}$ defines a functor $\mathrm{I}_1 : \mathbf{P}_1(n, \overline{p}) \to \mathbf{HoTop}$. (If $H := G \circ F$ is the composition with a second morphism $G : (X', \Sigma', \Lambda') \to (X'', \Sigma'', \Lambda'')$, then $\overline{G} \circ \overline{F}$ and $\overline{H}$ agree on M and map $*$ to $*''$. For every $L \in \lambda$, the restrictions of $\overline{G} \circ \overline{F}$ and $\overline{H}$ to $[0,1] = I_L \subset \mathrm{I}(X, \Sigma, \Lambda)$ yield two paths in $L''_L \vee I_{L''_L} \subset \mathrm{I}(X'', \Sigma'', \Lambda'')$ between $*''$ and $G(F(L_{<1}))$. They are homotopic rel endpoints, since $L''_L \vee I_{L''_L} \simeq L''_L$ is simply connected. Given the identity morphism F on (X, Σ, Λ), choose $\phi_L = \mathrm{const}_{L_{<1}}$ for all $L \in \lambda$. The resulting map $\overline{F}$ is homotopic to the identity map on $\mathrm{I}(X, \Sigma, \Lambda)$.)

Remark. 2.3 The approach of Section 2.1 also applies to the case $k = 2$ and objects (X, Σ, Λ) in $\mathbf{P}(n, \overline{p})$ with simply connected links, since $L_{<2}$ are 0-cells of L for all $L \in \lambda$.

2.2 Induced Maps for Spaces with Exactly One Isolated Singularity

Let us consider objects in $\mathbf{P}\,(n,\overline{p})$ with exactly one isolated singularity.

Definition. 2.4 The subcategory $\mathbf{P}_{\text{one}}\,(n,\overline{p})$ of $\mathbf{P}\,(n,\overline{p})$ consists of the following objects and morphisms:

- Objects (X,Σ,Λ) are objects in $\mathbf{P}\,(n,\overline{p})$, such that X has exactly one singularity. (We use the notation $\sigma=\{x\}$ and $\lambda=\{L\}$.)
- Morphisms $F:(X,\Sigma,\Lambda)\to(X',\Sigma',\Lambda')$ are morphisms in $\mathbf{P}\,(n,\overline{p})$, which are either identity morphisms or satisfy $F\,(X)\subset X'-x'$.

Let us check that $\mathbf{P}_{\text{one}}\,(n,\overline{p})$ is closed under composition $G\circ F$ of morphisms $F:(X,\Sigma,\Lambda)\to(X',\Sigma',\Lambda')$ and $G:(X',\Sigma',\Lambda')\to(X'',\Sigma'',\Lambda'')$. This is clear if G is an identity morphism. Otherwise, $(G\circ F)(X)\subset G(X')\subset X''-x''$.
In the following, we construct a functor

$$\mathrm{I}_{\text{one}}:\mathbf{P}_{\text{one}}\,(n,\overline{p})\to\mathbf{HoTop},$$

which agrees with I on objects. In order to define I_{one} on morphisms, we will use the following maps for a given object (X,Σ,Λ) in $\mathbf{P}_{\text{one}}\,(n,\overline{p})$:

- Let $q:X-x\to M$ be the projection map, which restricts to the identity map on M and is given on $L\times(0,1)\subset X-x$ by $q(a,t)=(a,1)\in L\times\{1\}\subset M$. Let $i:M\hookrightarrow X$ and $l:X-x\hookrightarrow X$ be the inclusions. Note that the composition $i\circ q$ is homotopic to l.
- The *canonical maps*

  $$b\;:\;M\;\to\;\mathrm{I}\,(X,\Sigma,\Lambda)\,,\qquad c\;:\;\mathrm{I}\,(X,\Sigma,\Lambda)\;\to\;\widehat{X}\;(:=X/\sigma=X),$$

  are defined as follows (compare [1, page 157]). b is just the inclusion $M\hookrightarrow\mathrm{I}\,(X,\Sigma,\Lambda)=\mathrm{cone}(j\circ f_L)$. (Note that $f=f_L$.) $c:\mathrm{I}\,(X,\Sigma,\Lambda)=\mathrm{cone}(j\circ f_L)\to\mathrm{cone}(j)=X$ is induced by the following 3-diagram of spaces:

  $$\begin{array}{ccccc}
  \mathrm{cone}(L_{<k}) & \xleftarrow{\text{at }1} & L_{<k} & \xrightarrow{j\circ f_L} & M\\
  \downarrow{\scriptstyle \mathrm{cone}(f_L)} & & \downarrow{\scriptstyle f_L} & & \downarrow{\scriptstyle =}\\
  \mathrm{cone}(L) & \xleftarrow{\text{at }1} & L & \xrightarrow{j} & M
  \end{array}$$

  Note that the canonical maps fit into the commutative diagram

  $$\begin{array}{ccc}
  M & \xrightarrow{b} & \mathrm{I}\,(X,\Sigma,\Lambda)\\
  & {\scriptstyle i}\searrow & \downarrow{\scriptstyle c}\\
  & & X.
  \end{array}$$

Let $F:(X,\Sigma,\Lambda)\to(X',\Sigma',\Lambda')$ be a morphism in $\mathbf{P}_{\text{one}}\,(n,\overline{p})$.
If F is the identity morphism on (X,Σ,Λ), then we set

$$\mathrm{I}_{\text{one}}\,(F)=[\mathrm{id}_{\mathrm{I}(X,\Sigma,\Lambda)}]\in\mathbf{HoTop}.$$

Otherwise, if F is not an identity morphism, then it restricts to a map $F| : X \to X' - x'$. Define $\mathrm{I}_{\text{one}}(F)$ to be the homotopy class in **HoTop** of the composition

$$\mathrm{I}(X, \Sigma, \Lambda) \xrightarrow{c} X \xrightarrow{F|} X' - x' \xrightarrow{q'} M' \xrightarrow{b'} \mathrm{I}(X', \Sigma', \Lambda').$$

In order to check functoriality, let $G : (X', \Sigma', \Lambda') \to (X'', \Sigma'', \Lambda'')$ be a second morphism in $\mathbf{P}_{\text{one}}(n, \overline{p})$. One has to show that $\mathrm{I}_{\text{one}}(G \circ F) = \mathrm{I}_{\text{one}}(G) \circ \mathrm{I}_{\text{one}}(F)$. This is clear if at least one of the morphisms F and G is an identity morphism. Otherwise, one has

$$\begin{aligned}
\mathrm{I}_{\text{one}}(G) \circ \mathrm{I}_{\text{one}}(F) &= [b'' \circ q'' \circ G| \circ (c' \circ b') \circ q' \circ F| \circ c] \\
&= [b'' \circ q'' \circ G| \circ (i' \circ q') \circ F| \circ c] \\
&= [b'' \circ q'' \circ (G| \circ l' \circ F|) \circ c] \\
&= [b'' \circ q'' \circ (G \circ F)| \circ c] \\
&= \mathrm{I}_{\text{one}}(G \circ F).
\end{aligned}$$

Let $J : \mathbf{P}_{\text{one}}(n, \overline{p}) \to \mathbf{Top}$ be the forgetful functor. (J maps objects to the underlying spaces and morphisms to the underlying continuous maps.) Let $p : \mathbf{Top} \to \mathbf{HoTop}$ be the natural projection functor. We construct a natural transformation

$$\mathrm{I}_{\text{one}} \longrightarrow p \circ J.$$

We map a given object (X, Σ, Λ) in $\mathbf{P}_{\text{one}}(n, \overline{p})$ to the homotopy class of the canonical map $[c] : \mathrm{I}(X, \Sigma, \Lambda) \to X$ in **HoTop**. It remains to show that for every morphism $F : (X, \Sigma, \Lambda) \to (X', \Sigma', \Lambda')$ in $\mathbf{P}_{\text{one}}(n, \overline{p})$, the following diagram commutes in **HoTop**:

$$\begin{array}{ccc}
\mathrm{I}(X, \Sigma, \Lambda) & \xrightarrow{\mathrm{I}_{\text{one}}(F)} & \mathrm{I}(X', \Sigma', \Lambda') \\
\downarrow{\scriptstyle [c]} & & \downarrow{\scriptstyle [c']} \\
X & \xrightarrow{[F]} & X'
\end{array}$$

This is clear, if F is an identity morphism. Otherwise, the diagram factorizes as

$$\begin{array}{ccccccccc}
\mathrm{I}(X, \Sigma, \Lambda) & \xrightarrow{[c]} & X & \xrightarrow{[F|]} & X' - x' & \xrightarrow{[q']} & M' & \xrightarrow{[b']} & \mathrm{I}(X', \Sigma', \Lambda') \\
\downarrow{\scriptstyle [c]} & & \downarrow{\scriptstyle [\mathrm{id}_X]} & & \downarrow{\scriptstyle [l']} & & \downarrow{\scriptstyle [i']} & & \downarrow{\scriptstyle [c']} \\
X & \xrightarrow{[\mathrm{id}_X]} & X & \xrightarrow{[F]} & X' & \xrightarrow{[\mathrm{id}_{X'}]} & X' & \xrightarrow{[\mathrm{id}_{X'}]} & X'
\end{array}$$

2.3 Induced Maps for Links Having a Basis of Cells for Their k-Cycle Group

In the present section, we will restrict our attention to objects in $\mathbf{P}(n, \overline{p})$, such that every link is equipped with a CW-structure, whose group of k-cycles has a basis of cells. In this case, the spatial homology truncations $f_L : L_{<k} \to L$ of the links L can be taken to be inclusions. The definition of the appropriate subcategory $\mathbf{P}_{\mathrm{Cyl}}(n, \overline{p})$ of $\mathbf{P}(n, \overline{p})$ is based on a category $\mathbf{Cyl}(k)$. The latter is supposed to model the morphisms in $\mathbf{P}_{\mathrm{Cyl}}(n, \overline{p})$ near the isolated singularities of the pseudomanifolds:

Definition. 2.5 The category $\mathbf{Cyl}(k)$ consists of the following objects and morphisms:

- Objects in $\mathbf{Cyl}(k)$ are path connected CW-complexes P, such that the group $Z_k(P)$ of k-cycles of P has a basis of k-cells. Let $e_P : P_{<k} \hookrightarrow P$ be the inclusion of the CW-complex $P_{<k}$ obtained from P^k by removing all k-cycle cells.
- Morphisms $F : P \to Q$ in $\mathbf{Cyl}(k)$ are maps of triples

$$F : (P \times I, P \times \{0\}, P \times \{1\}) \to (Q \times I, Q \times \{0\}, Q \times \{1\}),$$

which satisfy $F\left(P^{k-1} \times I\right) \subset Q^{k-1} \times I$ and $F(P_{<k} \times \{0\}) \subset Q_{<k} \times \{0\}$.

Example. 2.6 (objects in $\mathbf{Cyl}(k)$) A CW-complex K certainly has a basis of cells for its group of k-cycles, if the boundary map $\partial_k : C_k(K) \to C_{k-1}(K)$ is either injective or the zero map, since then the kernel of ∂_k is either 0 or $C_k(K)$. In particular, this is the case if K has at most one k-cell e^k. (If the image of e^k under ∂_k is not zero, then ∂_k is injective, as $C_{k-1}(K)$ is free abelian.) Moreover, the product complex of two CW-complexes, whose boundary maps in the cellular chain complex are all zero, has again this property. The following oriented closed path connected manifolds admit CW-structures with at most one cell in every dimension:

- m-spheres for $m \geq 1$: $S^m = e^0 \cup e^m$.
- lens spaces L_p for $p \geq 2$: $L_p = e^0 \cup e^1 \cup_p e^2 \cup e^3$.
- real projective spaces of odd dimension $m \geq 1$: $\mathbb{R}P^m = e^0 \cup ... \cup e^m$.
- complex projective spaces of dimension $2m$, $m \geq 1$: $\mathbb{C}P^m = e^0 \cup e^2 \cup ... \cup e^{2m}$.

Example. 2.7 (morphisms in $\mathbf{Cyl}(k)$) Let $k \geq 3$ and let P and Q be objects in $\mathbf{Cyl}(k)$. Choose completions $\left(P, Y_P, \overline{P}, q_P\right)$ and $\left(Q, Y_Q, \overline{Q}, q_Q\right)$ to objects in $\mathbf{CW}^0_{k \supset \partial}$ (see Definition 1.7), such that $Y_P \subset C_k(P)$ and $Y_Q \subset C_k(Q)$ are generated by those k-cells, which are not cycle cells. We show that every morphism $(f, \widetilde{f}) : \left(P, Y_P, \overline{P}, q_P\right) \to \left(Q, Y_Q, \overline{Q}, q_Q\right)$ in $\mathbf{CW}^0_{k \supset \partial}$ induces a morphism $F : P \to Q$ in $\mathbf{Cyl}(k)$, such that the restriction $F| : P = P \times \{1\} \to Q \times \{1\} = Q$ is equal to f. By Section 1.2, there is a cellular map $f_{<k} : P_{<k} \to Q_{<k}$, such that the following diagram commutes up to homotopy rel $(k-1)$-skeleton:

$$\begin{array}{ccc} P_{<k} & \xrightarrow{e_P} & P \\ \downarrow{\scriptstyle f_{<k}} & & \downarrow{\scriptstyle f} \\ Q_{<k} & \xrightarrow{e_Q} & Q. \end{array}$$

($\mathrm{emb}_k^0 : t_{<k}^0 \to t_{<\infty}^0$ is a natural transformation by Theorem 1.20. By construction of $\tau_{<k}^0$, we have $\mathrm{emb}_k^0\left(P, Y_P, \overline{P}, q_P\right) = [e_P]$ and $\mathrm{emb}_k^0\left(Q, Y_Q, \overline{Q}, q_Q\right) = [e_Q]$. Thus, one can take $f_{<k}$ as a representative of $t_{<k}^0(f, \widetilde{f})$.) Choose a rel P^{k-1} homotopy $H : P_{<k} \times I \to Q$ between $H_0 = e_Q \circ f_{<k}$ and $H_1 = f \circ e_P$. By the homotopy extension property of the CW-pair $(P, P_{<k})$, the map

$$P \times \{1\} \cup P_{<k} \times I \xrightarrow{f \cup H} Q$$

can be extended to a map $F' : P \times I \to Q$. The following map has all desired properties:

$$F : P \times I \to Q \times I, \quad F(p,t) = \left(F'(p,t), t\right).$$

Definition. 2.8 The subcategory $\mathbf{P}_{\mathrm{Cyl}}(n, \overline{p})$ of $\mathbf{P}(n, \overline{p})$ consists of the following objects and morphisms:

- Objects in $\mathbf{P}_{\mathrm{Cyl}}(n, \overline{p})$ are objects (X, Σ, Λ) in $\mathbf{P}(n, \overline{p})$, such that every link $L \in \lambda$ is an object in $\mathbf{Cyl}(k)$ and the chosen spatial homology truncations f_L are the inclusions $e_L : L_{<k} \hookrightarrow L$, which were introduced in Definition 2.5.
- Morphisms in $\mathbf{P}_{\mathrm{Cyl}}(n, \overline{p})$ are morphisms $F : (X, \Sigma, \Lambda) \to (X', \Sigma', \Lambda')$ in $\mathbf{P}(n, \overline{p})$, which satisfy the following property: For every $L \in \lambda$ there is a morphism $F_L : L \to L'_L$ in $\mathbf{Cyl}(k)$, such that F restricts for every L to the map $\widehat{F_L} : \mathrm{cone}(L) \to \mathrm{cone}(L'_L)$, which is induced from F_L by collapsing the ends of the cylinders at 0 to points. (By continuity, the maps $\widehat{F_L}$ determine the maps F_L uniquely.)

In order to define a functor $\mathrm{I}_{\mathrm{Cyl}} : \mathbf{P}_{\mathrm{Cyl}}(n, \overline{p}) \to \mathbf{HoTop}$, which agrees with I on objects, we need some definitions. Given $e_P : P_{<k} \to P$, let π_P denote the projection

$$\pi_P : P \bigsqcup (P_{<k} \times I) \to \left(P \bigsqcup (P_{<k} \times I)\right) / \left(e_P(x) \sim (x, 1) \ \forall x \in P_{<k}\right) = \mathrm{cyl}(e_P).$$

Define the following subspaces of $\mathrm{cyl}(e_P)$:

$$\begin{aligned} A_P &:= \pi_P\left(P_{<k} \times \{0\}\right), \\ B_P &:= \pi_P(P), \\ C_P &:= \pi_P\left(P^{k-1} \times I\right). \end{aligned}$$

Note that π_P restricts to a homeomorphism

$$\sigma_P : P \xrightarrow{\cong} B_P.$$

The restriction of π_P to $P_{<k} \times I$ will be denoted by

$$\rho_P : \left(P_{<k} \times I, P_{<k} \times \{0\}, P_{<k} \times \{1\}\right) \hookrightarrow \left(\mathrm{cyl}(e_P), A_P, B_P\right).$$

Note that the following diagram commutes:

$$\begin{array}{ccc} P_{<k} & \xrightarrow{=} & P_{<k} \times \{1\} \\ \downarrow{\scriptstyle e_P} & & \downarrow{\scriptstyle \rho_P|} \\ P & \xrightarrow{\sigma_P} & B_P. \end{array}$$

2.9 Lemma. *For every object P in $\mathbf{Cyl}(k)$, the inclusion $E_P : \mathrm{cyl}(e_P) \hookrightarrow P \times I$ is a homotopy equivalence.*

Proof. The inclusions $i : P \xrightarrow{\sigma_P} B_P \hookrightarrow \mathrm{cyl}(e_P)$ and $E_P \circ i : P = P \times \{1\} \hookrightarrow P \times I$ are homotopy equivalences. Homotopy inverses are given by the projection $q : P \times I \to P$ to the first component for $E_P \circ i$ and $q \circ E_P : \mathrm{cyl}(e_P) \to P$ for i. Thus, E_P is a homotopy equivalence with homotopy inverse $i \circ q$. ■

For the following discussion, we introduce some appropriate categories. The category **HoCyl** models "cylinders" as spaces with two fixed disjoint subspaces, which represent the two ends of the cylinder. Morphisms in **HoCyl** are homotopy classes of maps which preserve the ends of the cylinders and are rel the second subspace. This enables us to glue larger spaces to the second subspaces in a functorial way, which is implemented by the category **HoGlue** and the functor glue : $\mathbf{HoGlue} \to \mathbf{HoCyl}_{\mathrm{refl}}$. The category $\mathbf{HoCyl}_{\mathrm{refl}}$ will be used in Section 2.6 to construct maps between reflective diagrams. Let $F : (X, \Sigma, \Lambda) \to (X', \Sigma', \Lambda')$ be a morphism in $\mathbf{P}_{\mathrm{Cyl}}(n, \overline{p})$. The above categories are needed to explain how a collection of suitable maps $\{L \times I \to L'_L \times I\}_{L \in \lambda}$ in **HoCyl**, which will be assigned to F, induces a continuous map $\mathrm{I}(X, \Sigma, \Lambda) \to \mathrm{I}(X', \Sigma', \Lambda')$. (First, the functor glue : $\mathbf{HoGlue} \to \mathbf{HoCyl}_{\mathrm{refl}}$ is used to glue the manifolds M and M' along their boundaries to the second subspaces of the cylinders $\{L \times I\}_{L \in \lambda}$ and $\{L' \times I\}_{L' \in \lambda'}$. Afterwards, the upper subspaces of the cylinders are collapsed to a point by using a collapsing functor coll : $\mathbf{HoCyl}_{\mathrm{refl}} \to \mathbf{HoTop}$.) This will reduce the construction of a functor $\mathrm{I}_{\mathrm{Cyl}} : \mathbf{P}_{\mathrm{Cyl}}(n, \overline{p}) \to \mathbf{HoTop}$, which agrees with I on objects, to the problem of constructing a functor $\mathrm{T}_k : \mathbf{Cyl}(k) \to \mathbf{HoCyl}$ with suitable properties (see Proposition 2.14).

Definition. 2.10 The category **HoCyl** consists of the following objects and morphisms: Objects are triples (X, A, B) of topological spaces, where A and B are disjoint (possibly empty) subspaces of X. Morphisms $(X, A, B) \to (X', A', B')$ are rel B homotopy classes $[F]$ of maps $F : (X, A, B) \to (X', A', B')$ (i.e. F and $\widetilde{F}$ are equivalent if and only if there exists a homotopy rel B between F and $\widetilde{F}$ which maps A into A' at all times). The composition of two morphisms $[F] : (X, A, B) \to (X', A', B')$ and $[F'] : (X', A', B') \to (X'', A'', B'')$ is given by $[F'] \circ [F] := [F' \circ F]$.

Definition. 2.11 The category $\mathbf{HoCyl}_{\mathrm{refl}}$ consists of the following objects and morphisms: Objects are quadruples (X, A, B, C) of topological spaces, where A, B and C are subspaces of X, such that $B \subset A$. Morphisms $(X, A, B, C) \to (X', A', B', C')$ are rel C homotopy classes $[F]$ of maps $F : (X, A, B, C) \to (X', A', B', C')$.

Definition. 2.12 The category **HoGlue** consists of the following objects and morphisms: Objects are quadruples (X_0, A, B_0, B), where (X_0, A, B_0) is an object in **HoCyl** and $B_0 \subset B$ is a subspace. Morphisms $([F_0], \rho) : (X_0, A, B_0, B) \to (X'_0, A', B'_0, B')$ consist of a morphism $[F_0] : (X_0, A, B_0) \to (X'_0, A', B'_0)$ in **HoCyl** and a morphism $\rho : B \to B'$ in **Top**, which restricts to $\rho| = F_0| : B_0 \to B'_0$. The composition with a second morphism $([F'_0], \rho') : (X'_0, A', B'_0, B') \to (X''_0, A'', B''_0, B'')$ is given by $([F'_0], \rho') \circ ([F_0], \rho) := ([F'_0] \circ [F_0], \rho' \circ \rho)$.

Now, define the functor glue : $\mathbf{HoGlue} \to \mathbf{HoCyl}_{\text{refl}}$

- on objects by $\text{glue}(X_0, A, B_0, B) = (X, X_0, A, B)$, where X denotes the realization of the 3-diagram $X_0 \hookleftarrow B_0 \hookrightarrow B$ of spaces.
- on morphisms $([F_0], \rho) : (X_0, A, B_0, B) \to (X_0', A', B_0', B')$ by $\text{glue}([F_0], \rho) = [F]$, where $F : (X, X_0, A, B) \to (X', X_0', A', B')$ is induced by

$$
\begin{array}{ccccc}
X_0 & \xleftarrow{\text{incl}} & B_0 & \xrightarrow{\text{incl}} & B \\
\downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle F_0|} & & \downarrow{\scriptstyle \rho} \\
X_0' & \xleftarrow{\text{incl}} & B_0' & \xrightarrow{\text{incl}} & B'.
\end{array}
$$

Assume that there is a covariant functor

$$
\mathrm{T}_k \quad : \quad \mathbf{Cyl}(k) \quad \longrightarrow \quad \mathbf{HoCyl},
$$

such that the following two properties are satisfied:

- For every object P in $\mathbf{Cyl}(k)$ one has $\mathrm{T}_k(P) = (\text{cyl}(e_P), A_P, B_P)$.
- For every morphism $F : P \to Q$ in $\mathbf{Cyl}(k)$, the following diagram commutes:

$$
\begin{array}{ccc}
P \times \{1\} = P & \xrightarrow{\sigma_P \ \cong} & B_P \\
{\scriptstyle F|}\downarrow & & \downarrow{\scriptstyle \mathrm{T}_k(F)|} \\
Q \times \{1\} = Q & \xrightarrow{\sigma_Q \ \cong} & B_Q.
\end{array}
$$

Given a morphism $F : (X, \Sigma, \Lambda) \to (X', \Sigma', \Lambda')$ in $\mathbf{P}_{\text{Cyl}}(n, \overline{p})$, one has

$$
\begin{aligned}
\mathrm{T}_k(L) &= (\text{cyl}(e_L), L_{<k}, L) \quad \forall L \in \lambda, \\
\mathrm{T}_k(L') &= (\text{cyl}(e_{L'}), L'_{<k}, L') \quad \forall L' \in \lambda'.
\end{aligned}
$$

The following diagram commutes in $\mathbf{Top}$:

$$
\begin{array}{ccc}
\bigsqcup_{L \in \lambda} L = \partial M & \xrightarrow{\text{incl}} & M \\
{\scriptstyle \bigsqcup \mathrm{T}_k(F_L)| = \bigsqcup F_L|}\downarrow & & \downarrow{\scriptstyle F|} \\
\bigsqcup_{L' \in \lambda'} L' = \partial M' & \xrightarrow{\text{incl}} & M'.
\end{array}
$$

All in all, F gives rise to a morphism

$$
\left(\bigsqcup_{L \in \lambda} \mathrm{T}_k(F_L), F| \right) : \big(\text{cyl}(f), (\partial M)_{<k}, \partial M, M\big) \to \big(\text{cyl}(f'), (\partial M')_{<k}, \partial M', M'\big)
$$

in $\mathbf{HoGlue}$, where $f = \bigsqcup_{L \in \lambda} e_L$ and $(\partial M)_{<k} = \bigsqcup_{L \in \lambda} L_{<k} \times \{0\}$ $(\subset \text{cyl}(f))$.
Since T_k is a functor, this yields a functor

$$
\overline{\mathrm{T}}_k \quad : \quad \mathbf{P}_{\text{Cyl}}(n, \overline{p}) \quad \longrightarrow \quad \mathbf{HoGlue},
$$

which is given

- on objects (X,Σ,Λ) in $\mathbf{P}_{\mathrm{Cyl}}(n,\overline{p})$ by $\overline{\mathrm{T}}_k(X,\Sigma,\Lambda) = \big(\mathrm{cyl}(f), (\partial M)_{<k}, \partial M, M\big)$.
- on morphisms $F : (X,\Sigma,\Lambda) \to (X',\Sigma',\Lambda')$ in $\mathbf{P}_{\mathrm{Cyl}}(n,\overline{p})$ by $\overline{\mathrm{T}}_k(F) = \big(\bigsqcup_{L\in\lambda} \mathrm{T}_k(F_L), F|\big)$.

The composition of functors

$$\widetilde{\mathrm{I}}_{\mathrm{Cyl}(k)} = \mathrm{glue} \circ \overline{\mathrm{T}}_k \ : \ \mathbf{P}_{\mathrm{Cyl}}(n,\overline{p}) \ \to \ \mathbf{HoCyl}_{\mathrm{refl}}$$

is given on objects by

$$\widetilde{\mathrm{I}}_{\mathrm{Cyl}(k)}(X,\Sigma,\Lambda) = \big(\mathrm{cyl}(j\circ f), \mathrm{cyl}(f), (\partial M)_{<k}, M\big),$$

where $j : \partial M \hookrightarrow M$ is the inclusion. Define the collapsing functor

$$\mathrm{coll} \quad : \quad \mathbf{HoCyl}_{\mathrm{refl}} \quad \longrightarrow \quad \mathbf{HoTop}$$

- on objects (X,A,B,C) in $\mathbf{HoCyl}_{\mathrm{refl}}$ by $\mathrm{coll}(X,A,B,C) = X/B$.
- on morphisms $[\varphi] : (X,A,B,C) \to (X',A',B',C')$ in $\mathbf{HoCyl}_{\mathrm{refl}}$ by $\mathrm{coll}([\varphi]) = [\overline{\varphi}]$, where $\overline{\varphi} : X/B \to X'/B'$ is induced by φ after passing to quotient spaces.

Finally, define the covariant functor

$$\mathrm{I}_{\mathrm{Cyl}} = \mathrm{coll} \circ \widetilde{\mathrm{I}}_{\mathrm{Cyl}(k)} \quad : \quad \mathbf{P}_{\mathrm{Cyl}}(n,\overline{p}) \quad \longrightarrow \quad \mathbf{HoTop}.$$

Note that $\mathrm{I}_{\mathrm{Cyl}}$ agrees with I on objects.

In the following, we will construct such a functor T_k with the desired properties. By the first property, T_k is already given on objects. We go on to define T_k on morphisms in $\mathbf{Cyl}(k)$. For every object P in $\mathbf{Cyl}(k)$, one has the inclusions

$$A_P \cup B_P \cup C_P \subset \mathrm{cyl}(e_P) \overset{E_P}{\hookrightarrow} P \times I.$$

Note that every morphism $F : P \to Q$ in $\mathbf{Cyl}(k)$ satisfies $F(A_P) \subset A_Q$, $F(B_P) \subset B_Q$ and $F(C_P) \subset C_Q$. Consequently, $F(A_P \cup B_P \cup C_P) \subset A_Q \cup B_Q \cup C_Q$ $(\subset \mathrm{cyl}(e_Q))$. Given a morphism $F : P \to Q$ in $\mathbf{Cyl}(k)$, we will construct a map

$$F^{<} : \mathrm{cyl}(e_P) \to \mathrm{cyl}(e_Q),$$

such that the diagram

$$\begin{array}{ccc} \mathrm{cyl}(e_P) & \xrightarrow{E_P = \text{ incl}} & P \times I \\ {\scriptstyle F^{<}} \downarrow & & \downarrow {\scriptstyle F} \\ \mathrm{cyl}(e_Q) & \xrightarrow{E_Q = \text{ incl}} & Q \times I \end{array}$$

commutes up to homotopy rel $A_P \cup B_P \cup C_P$. Note that this implies that $F^{<}$ agrees with F on $A_P \cup B_P \cup C_P$. (E_P and E_Q are inclusions.) Thus, $F^{<}(A_P) \subset A_Q$, $F^{<}(B_P) \subset B_Q$, $F^{<}(C_P) \subset C_Q$ and $F^{<}(A_P \cup B_P \cup C_P) \subset A_Q \cup B_Q \cup C_Q$. In particular, $F^{<}$ induces a morphism $[F^{<}] : (\mathrm{cyl}(e_P), A_P, B_P) \to (\mathrm{cyl}(e_Q), A_Q, B_Q)$ in $\mathbf{HoCyl}$. This will be used as the definition of T_k on morphisms. First, we use the compression lemma to construct a map $F^{<}$ as above. In Proposition 2.13 we show that $[F^{<}]$ is a well-defined morphism in **HoCyl**. Finally, we will show in Proposition 2.14 that this defines a covariant functor $\mathrm{T}_k : \mathbf{Cyl}\,(k) \to \mathbf{HoCyl}$ with the desired properties.

In order to construct $F^{<}$, we apply the compression lemma [2, Lemma 4.6, page 346]: Consider the map

$$F \circ E_P : (\mathrm{cyl}\,(e_P)\,, A_P \cup B_P \cup C_P) \to (Q \times I, \mathrm{cyl}\,(e_Q))$$

between CW-pairs, where $\mathrm{cyl}\,(e_Q) \neq \emptyset$ is path connected. The complement

$$\mathrm{cyl}\,(e_P) - A_P \cup B_P \cup C_P = (P_{<k} \times I) - (P_{<k} \times I)^k$$

has only cells of dimension $(k+1)$. Since the inclusion $E_Q : \mathrm{cyl}\,(e_Q) \hookrightarrow Q \times I$ is a homotopy equivalence by Lemma 2.9, it follows from the long exact homotopy sequence of the pair $(Q \times I, \mathrm{cyl}\,(e_Q))$ that $\pi_{k+1}\,(Q \times I, \mathrm{cyl}\,(e_Q)) = 0$. Thus, there is a map

$$F^{<} : \mathrm{cyl}\,(e_P) \to \mathrm{cyl}\,(e_Q)\,,$$

such that $F \circ E_P$ is homotopic rel $A_P \cup B_P \cup C_P$ to $E_Q \circ F^{<}$. The map $F^{<}$ induces a well-defined morphism in **HoCyl**:

2.13 Proposition. *Let $F : P \to Q$ be a morphism in $\mathbf{Cyl}\,(k)$. Suppose that $F^{<} : \mathrm{cyl}\,(e_P) \to \mathrm{cyl}\,(e_Q)$ is a continuous map such that the following diagram commutes up to homotopy rel $A_P \cup B_P \cup C_P \subset \mathrm{cyl}\,(e_P)$:*

$$\begin{array}{ccc} \mathrm{cyl}\,(e_P) & \xrightarrow{E_P = \textit{incl}} & P \times I \\ {\scriptstyle F^{<}}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ \mathrm{cyl}\,(e_Q) & \xrightarrow{E_Q = \textit{incl}} & Q \times I \end{array}$$

*If $G^{<}$ is a second map with the same property, then $[F^{<}] = [G^{<}]$ in **HoCyl**. (Here, we take the homotopy classes of the maps $F^{<}, G^{<} : (\mathrm{cyl}\,(e_P)\,, A_P, B_P) \to (\mathrm{cyl}\,(e_Q)\,, A_Q, B_Q)$.)*

Proof. By assumption, the compositions

$$E_Q \circ F^{<},\ E_Q \circ G^{<} : \mathrm{cyl}\,(e_P) \to Q \times I$$

are homotopic rel $A_P \cup B_P \cup C_P$, since they are both homotopic to $F \circ E_P$ rel $A_P \cup B_P \cup C_P$. Thus, $E_Q \circ F^{<}$ and $E_Q \circ G^{<}$ agree on $A_P \cup B_P \cup C_P$. Since E_Q is injective, it follows that $F^{<}$ and $G^{<}$ agree on $A_P \cup B_P \cup C_P$. Using that $F : P \to Q$ is a morphism in $\mathbf{Cyl}\,(k)$, we showed above that $F^{<}\,(A_P) \subset A_Q$, $F^{<}\,(B_P) \subset B_Q$ and $F^{<}\,(C_P) \subset C_Q$ (and analogous for $G^{<}$). In particular, it follows from $F^{<}\,(B_P)\,, G^{<}\,(B_P) \subset B_Q$, that $F^{<}$ and $G^{<}$ restrict to the same map

$$D : B_P \to B_Q.$$

Next, we apply Lemma 1.17 to the following setting. Define $X := P_{<k} \times I$, $Y := \mathrm{cyl}\,(e_Q)$ and $Y' := Q \times I$. X is an $(k+1)$-dimensional CW-complex and the CW-complexes Y and Y' are path connected. By composition with the inclusion $\rho_P : P_{<k} \times I \hookrightarrow \mathrm{cyl}\,(e_P)$, define the maps

$$g_1 := F^{<} \circ \rho_P,\ g_2 := G^{<} \circ \rho_P\ :\ X\ \to\ Y.$$

Note that $\rho_P\left(X^k\right) \subset A_P \cup B_P \cup C_P$. (This follows from $X^k = P_{<k} \times \{0,1\} \cup P^{k-1} \times I$ and $\rho_P\left(P_{<k} \times \{0\}\right) = A_P$, $\rho_P\left(P_{<k} \times \{1\}\right) \subset B_P$ and $\rho_P\left(P^{k-1} \times I\right) = C_P$.) Since $F^{<}$ and $G^{<}$ agree on $A_P \cup B_P \cup C_P$, one can conclude that g_1 and g_2 agree on X^k. The map $f := E_Q : Y \to Y'$ induces an isomorphism $f_* : \pi_{k+1}(Y) \to \pi_{k+1}(Y')$, because E_Q is a homotopy equivalence (see Lemma 2.9). Composition with f yields the maps

$$f \circ g_1,\ f \circ g_2 : X \to Y',$$

which are homotopic rel X^k, because $E_Q \circ F^{<}$ is homotopic to $E_Q \circ G^{<}$ rel $A_P \cup B_P \cup C_P$ and $\rho_P\left(X^k\right) \subset A_P \cup B_P \cup C_P$. Therefore, by Lemma 1.17, $g_1 = F^{<} \circ \rho_P$ and $g_2 = G^{<} \circ \rho_P$ are homotopic via a rel X^k homotopy. On $P_{<k} \times \{1\} \subset X^k$, the maps g_1 and g_2 restrict to the same map

$$P_{<k} \times \{1\} \overset{\rho_P|}{\hookrightarrow} B_P \overset{D}{\to} B_Q.$$

Hence, the homotopy rel X^k between g_1 and g_2 can be extended to a homotopy rel $X^k \cup B_P \subset \mathrm{cyl}(e_P)$ between $g_1 \cup D : \mathrm{cyl}(e_P) \to \mathrm{cyl}(e_Q)$ and $g_2 \cup D : \mathrm{cyl}(e_P) \to \mathrm{cyl}(e_Q)$. (If $J : X \times I \to Y$ denotes the rel X^k homotopy between g_1 and g_2, then the desired homotopy rel $X^k \cup B_P$ between $g_1 \cup D$ and $g_2 \cup D$ is at $t \in I$ induced by the morphism

$$\begin{array}{ccccc} P_{<k} \times I & \xleftarrow{\text{incl}} & P_{<k} \times \{1\} & \xrightarrow{\rho_P|} & B_P \\ \downarrow{\scriptstyle J_t} & & \downarrow{\scriptstyle D \circ \rho_P|} & & \downarrow{\scriptstyle D} \\ \mathrm{cyl}(e_Q) & \xleftarrow{\text{incl}} & B_Q & \xrightarrow{=} & B_Q \end{array}$$

of 3-diagrams.) Since $g_1 \cup D = F^{<}$, $g_2 \cup D = G^{<}$ and $X^k \cup B_P = A_P \cup B_P \cup C_P$, we can conclude that $F^{<}, G^{<} : (\mathrm{cyl}\,(e_P), A_P, B_P) \to (\mathrm{cyl}\,(e_Q), A_Q, B_Q)$ are equal in **HoCyl**. ∎

Let us now construct the desired functor $\mathrm{T}_k : \mathbf{Cyl}\,(k) \to \mathbf{HoCyl}$. Given an object P in $\mathbf{Cyl}\,(k)$, the first property requires the definition

$$\mathrm{T}_k\,(P) := (\mathrm{cyl}\,(e_P), A_P, B_P)\,.$$

Given a morphism $F : P \to Q$ in $\mathbf{Cyl}\,(k)$, the above construction of $F^{<}$ and Proposition 2.13 yield a well-defined morphism in **HoCyl**:

$$\mathrm{T}_k\,(F) := [F^{<}] : \mathrm{T}_k\,(P) \to \mathrm{T}_k\,(Q)\,.$$

The second property is also satisfied, because $F^{<}$ agrees with F on B_P.

2.14 Proposition. T_k *is a covariant functor with the desired properties.*

Proof. If $F : P \to P$ is an identity morphism, then $F^{<}$ can also be chosen to be the identity map on $\mathrm{cyl}(e_P)$. Thus, $\mathrm{T}_k(F) = [F^{<}]$ is also an identity morphism.
It remains to show that for any two morphisms $F : P \to Q$ and $G : Q \to R$ in $\mathbf{Cyl}(k)$ with composition $H := G \circ F$, one has

$$\mathrm{T}_k(H) = \mathrm{T}_k(G) \circ \mathrm{T}_k(F).$$

By construction of T_k, the morphisms $\mathrm{T}_k(F)$, $\mathrm{T}_k(G)$ and $\mathrm{T}_k(H)$ in **HoCyl** can be represented by maps

$$\begin{aligned}
F^{<} &: (\mathrm{cyl}(e_P), A_P, B_P) \to (\mathrm{cyl}(e_Q), A_Q, B_Q),\\
G^{<} &: (\mathrm{cyl}(e_Q), A_Q, B_Q) \to (\mathrm{cyl}(e_R), A_R, B_R),\\
H^{<} &: (\mathrm{cyl}(e_P), A_P, B_P) \to (\mathrm{cyl}(e_R), A_R, B_R),
\end{aligned}$$

such that there are homotopies

$$\begin{aligned}
\alpha &: E_Q \circ F^{<} \simeq F \circ E_P && \mathrm{rel}\ A_P \cup B_P \cup C_P,\\
\beta &: E_R \circ G^{<} \simeq G \circ E_Q && \mathrm{rel}\ A_Q \cup B_Q \cup C_Q,\\
\gamma &: E_R \circ H^{<} \simeq H \circ E_P && \mathrm{rel}\ A_P \cup B_P \cup C_P.
\end{aligned}$$

We have to show that $[H^{<}] = [G^{<} \circ F^{<}]$ in **HoCyl**. The first two homotopies α and β imply that $E_R \circ G^{<} \circ F^{<}$ is homotopic to $H \circ E_P$ rel $A_P \cup B_P \cup C_P$. Such a homotopy can be constructed as follows. First, we use β to obtain a rel $A_P \cup B_P \cup C_P$ homotopy $(E_R \circ G^{<}) \circ F^{<} \simeq (G \circ E_Q) \circ F^{<}$. (Note that $F^{<}(A_P \cup B_P \cup C_P) \subset A_Q \cup B_Q \cup C_Q$ and β is rel $A_Q \cup B_Q \cup C_Q$.) Second, we use α to obtain a rel $A_P \cup B_P \cup C_P$ homotopy $G \circ (E_Q \circ F^{<}) \simeq G \circ (F \circ E_P)$. Using γ, we find that both $E_R \circ H^{<}$ and $E_R \circ G^{<} \circ F^{<}$ are homotopic to $H \circ E_P$ rel $A_P \cup B_P \cup C_P$. Thus, the claim follows from the application of Proposition 2.13 to $H^{<}$ and $G^{<} \circ F^{<}$. ∎

Remark. 2.15 Our construction of the functor $\mathrm{I}_{\mathrm{Cyl}} : \mathbf{P}_{\mathrm{Cyl}}(n, \overline{p}) \to$ **HoTop** does not directly use that all links of objects in $\mathbf{P}_{\mathrm{Cyl}}(n, \overline{p})$ have a basis of cells for their k-cycle groups. We only use that $L^{k-1} \subset L_{<k} \subset L^k$ for all links $L \in \lambda$ of objects (X, Σ, Λ) in $\mathbf{P}_{\mathrm{Cyl}}(n, \overline{p})$ (and choose the spatial homology truncations $f_L : L_{<k} \to L$ to be inclusions.) This is equivalent to saying that there is a direct sum complement Y of $Z_k(L)$ in $C_k(L)$, such that Y has a basis of k-cells for all links L. Thus, this is more general than to require that all links of an object in $\mathbf{P}_{\mathrm{Cyl}}(n, \overline{p})$ have a basis of k-cells for their k-cycle group. (In this case, the remaining k-cells form a basis of a direct sum complement of $Z_k(L)$.) However, there is in general no canonical choice for the subcomplex $L_{<k} \subset L$. But if we require that the k-cycle group of L has a cell-basis, then $L^{k-1} \subset L_{<k} \subset L^k$ is unique by [1, Proposition 1.3, page 7]. If we work with the weaker assumptions, then we have to give up this kind of uniqueness. Thus, we have to record the choice of $L_{<k}$ in the definition of objects in $\mathbf{Cyl}(k)$. (The definition of morphisms in $\mathbf{Cyl}(k)$ is based on the knowledge of the truncations $L_{<k}$.)

2.4 Dependence on Choices

Assume that $\mathcal{X} = (X, \Sigma, \Lambda)$ and $\mathcal{X}' = (X, \Sigma, \Lambda')$ are two objects in $\mathbf{P}(n, \overline{p})$. Note that the only difference lies in the chosen CW-structures and spatial homology truncations for the links of X. Thus, $\mathrm{id}_X : \mathcal{X} \to \mathcal{X}'$ and $\mathrm{id}_X : \mathcal{X}' \to \mathcal{X}$ are isomorphisms in $\mathbf{P}(n, \overline{p})$, which are inverse to each other. Assume that $\mathcal{X}$ and $\mathcal{X}'$ are objects of a subcategory $\mathbf{P}_*(n, \overline{p})$ of $\mathbf{P}(n, \overline{p})$, such that there is a functor

$$\mathrm{I}_* : \mathbf{P}_*(n, \overline{p}) \to \mathbf{HoTop},$$

which agrees on objects with the assignment I. If $\mathrm{id}_X : \mathcal{X} \to \mathcal{X}'$ and $\mathrm{id}_X : \mathcal{X}' \to \mathcal{X}$ are morphisms in $\mathbf{P}_*(n, \overline{p})$, then $\mathcal{X}$ and $\mathcal{X}'$ are isomorphic objects in $\mathbf{P}_*(n, \overline{p})$. Thus, $\mathrm{I}_*(\mathcal{X})$ and $\mathrm{I}_*(\mathcal{X}')$ are isomorphic objects in **HoTop**. This means that the choices of CW-structures and homology truncations for the links of X, which complete (X, Σ, λ) to the objects $\mathcal{X}$ and $\mathcal{X}'$ in $\mathbf{P}_*(n, \overline{p})$, result in homotopy equivalent intersection spaces of X. If all completions of (X, Σ, λ) to objects in $\mathbf{P}(n, \overline{p})$ would be objects in $\mathbf{P}_*(n, \overline{p})$ and if we knew that for all completions $\mathcal{X}$ and $\mathcal{X}'$ of (X, Σ, λ) to objects in $\mathbf{P}(n, \overline{p})$, $\mathrm{id}_X : \mathcal{X} \to \mathcal{X}'$ and $\mathrm{id}_X : \mathcal{X}' \to \mathcal{X}$ were morphisms in $\mathbf{P}_*(n, \overline{p})$, then we could conclude that the homotopy type of the intersection space of X was independent of the choices involved in its construction.

Example. 2.16 Recall that the *interleaf category* **ICW** is the full subcategory of $\mathbf{CW}^1$, whose objects have finitely generated even-dimensional homology and vanishing odd-dimensional homology for any coefficient group (see [1, Definition 1.62, page 71]). Let $\mathbf{P}_{\mathbf{ICW}}(n, \overline{p})$ be the subcategory of $\mathbf{P}(n, \overline{p})$, which consists of the following objects and morphisms:

- Objects (X, Σ, Λ) are objects in $\mathbf{P}(n, \overline{p})$, such that all links $L \in \lambda$ are objects in **ICW** and all homology truncations $L_{<k}$ are simply connected. (Hence, the maps $f_L : L_{<k} \to L$ satisfy the properties (T1)-(T3) of [1, page 132]. By [1, Lemma 2.25, page 132], the truncations $L_{<k}$ are also objects in **ICW**.)
- The set of morphisms from an object (X, Σ, Λ) to an object (X', Σ', Λ') in $\mathbf{P}_{\mathbf{ICW}}(n, \overline{p})$ consists of id_X, if $X = X'$ and $\Sigma = \Sigma'$, and is else empty.

Using the proof of [1, Theorem 2.26, page 132], one can define a functor

$$\mathrm{I}_{\mathbf{ICW}} : \mathbf{P}_{\mathbf{ICW}}(n, \overline{p}) \to \mathbf{HoTop},$$

which agrees on objects with the assignment $\mathrm{I} : \mathrm{Ob}\,\mathbf{P}(n, \overline{p}) \to \mathrm{Ob}\,\mathbf{HoTop}$. For every object (X, Σ, Λ) in $\mathbf{P}_{\mathbf{ICW}}(n, \overline{p})$, the proof of the theorem yields a reference model $\mathrm{I}_{\mathrm{ref}}(X, \Sigma, \lambda)$ for the perversity $\overline{p}$ intersection space of X. The construction of $\mathrm{I}_{\mathrm{ref}}(X, \Sigma, \lambda)$ uses that all links of X are in the interleaf category, but it does not make use of the fixed CW-structures and homology truncations of the links. Moreover, the proof yields a homotopy equivalence $\mathrm{I}(X, \Sigma, \Lambda) \to \mathrm{I}_{\mathrm{ref}}(X, \Sigma, \lambda)$, because all homology truncations $f_L : L_{<k} \to L$ satisfy the properties (T1)-(T3) of [1, page 132]. Now we proceed as follows. For all objects (X, Σ, Λ) in $\mathbf{P}_{\mathbf{ICW}}(n, \overline{p})$, we fix a homotopy equivalence $h(X, \Sigma, \Lambda) : \mathrm{I}(X, \Sigma, \Lambda) \to \mathrm{I}_{\mathrm{ref}}(X, \Sigma, \lambda)$ and a homotopy inverse $\overline{h}(X, \Sigma, \Lambda)$. If $F = \mathrm{id}_X : (X, \Sigma, \Lambda) \to (X, \Sigma, \Lambda')$ is a morphism in $\mathbf{P}_{\mathbf{ICW}}(n, \overline{p})$, then the following definition yields the desired functor:

$$\mathrm{I}_{\mathbf{ICW}}(F) = \left[\overline{h}(X, \Sigma, \Lambda') \circ h(X, \Sigma, \Lambda)\right] : \mathrm{I}(X, \Sigma, \Lambda) \to \mathrm{I}(X, \Sigma, \Lambda').$$

Example. 2.17 The proof of [1, Theorem 2.26, page 132] shows that for an object L in the interleaf category and an integer $k > 0$, any two simply connected spatial homology truncations of L in degree k are homotopy equivalent. (More precisely, a homotopy equivalence $\varepsilon : L \to E(L)$ to a finite CW-complex $E(L)$ with only even-dimensional cells is used in the the proof. It is then shown that if a map $f : L_{<k} \to L$ satisfies properties (T1)-(T3) of [1, page 132] with respect to k and L, then there is a homotopy equivalence $\widetilde{e} : L_{<k} \to E(L)^{k-1}$.) However, the following example shows that in general the homotopy type of a spatial homology truncation $t_{<m}(K, Y)$ of a simply connected CW-complex K in a dimension $m \geq 3$ does depend on the completion of K to an object (K, Y) in $\mathbf{CW}_{m \supset \partial}$.
Define the 5-dimensional simply connected CW-complex

$$K = \left(S^3 \vee S^4\right) \cup_\alpha e_a^5 \cup_\beta e_b^5,$$

where the attaching maps are defined by the compositions

$$\alpha : \partial e_a^5 = S^4 \xrightarrow{2} S^4 \hookrightarrow S^3 \vee S^4,$$
$$\beta : \partial e_b^5 = S^4 \xrightarrow{c} S^4 \vee S^4 \xrightarrow{\gamma \vee 2} S^3 \vee S^4,$$

where c collapses the equator $S^3 \subset S^4$ to a point and the homotopy class of $\gamma : S^4 \to S^3$ is the generator of $\pi_4(S^3) = \mathbb{Z}/2$. Obviously, $\partial_5 e_a^5 = \partial_5 e_b^5 = 2e^4$ and $Z_5(K) = \mathbb{Z}(e_a^5 - e_b^5) \subset C_5(K) = \mathbb{Z}e_a^5 \oplus \mathbb{Z}e_b^5$. Two possible choices for direct sum complements are given by $Y_a = \mathbb{Z}e_a^5$ and $Y_b = \mathbb{Z}e_b^5$. This yields two completions of K to objects (K, Y_a) and (K, Y_b) in $\mathbf{CW}_{5 \supset \partial}$. Following the construction of the 5-truncations $t_{<5}(K, Y_a)$ and $t_{<5}(K, Y_b)$ (see the proof of [1, Proposition 1.6, page 12]), we are free to choose the cell-bases $\{\eta^{(a)}\} = \{e_a^5\}$ for Y_a and $\{\eta^{(b)}\} = \{e_b^5\}$ for Y_b and obtain

$$K_{<5}^{(a)} = t_{<5}(K, Y_a) = \left(S^3 \vee S^4\right) \cup_\alpha e_a^5 = S^3 \vee \left(S^4 \cup_2 e_a^5\right),$$
$$K_{<5}^{(b)} = t_{<5}(K, Y_b) = \left(S^3 \vee S^4\right) \cup_\beta e_b^5.$$

We claim that $K_{<5}^{(a)}$ and $K_{<5}^{(b)}$ are not homotopy equivalent. The attaching maps α and β yield the same boundary maps in the cellular chain complexes, so the homotopy types of $K_{<5}^{(a)}$ and $K_{<5}^{(b)}$ cannot be kept apart by comparing homology groups. Since $K_{<5}^{(a)}$ and $K_{<5}^{(b)}$ have the same 4-skeleton, they have identical homotopy groups in dimensions ≤ 3. We consider their homotopy groups in dimension 4. Using the list of elementary complexes in [4, page 129] and the table of their homotopy groups [4, page 133],

- $S^4 \cup_2 e_a^5$ is of type 8 with $n = 3$ and $q' = 1$, so $\pi_4(S^4 \cup_2 e_a^5) = \mathbb{Z}/2$.
- $(S^3 \vee S^4) \cup_\beta e_b^5$ is of type 10 with $n = 3$ and $q' = 1$, so $\pi_4(K_{<5}^{(b)}) = \mathbb{Z}/4$.

Application of [4, Theorem 4.2, page 131] with $r = 2$, $n = 3$ and $s = 4$ to $K_{(1)} = S^3$ and $K_{(2)} = S^4 \cup_2 e_a^5$ (note that $K_{<5}^{(a)} = K_{(1)} \vee K_{(2)}$ and $1 < s < 2n - 1$) finally yields

$$\pi_4(K_{<5}^{(a)}) = \pi_4\left(S^3\right) \oplus \pi_4\left(S^4 \cup_2 e_a^5\right) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \neq \mathbb{Z}/4 = \pi_4(K_{<5}^{(b)}).$$

2.5 Induced Maps for Links in a Compression Rigid Category

We assume $k \geq 3$ for the cut-off degree. For every object $\left(P, Y_P, \overline{P}, q_P\right)$ in $\mathbf{CW}^0_{k\supset\partial}$ (see Definition 1.16) we fix a representative

$$e_P \quad : \quad P_{<k} = t^0_{<k}\left(P, Y_P, \overline{P}, q_P\right) \quad \longrightarrow \quad t^0_{<\infty}\left(P, Y_P, \overline{P}, q_P\right) = P$$

of the homotopy class $\mathrm{emb}^0_k\left(P, Y_P, \overline{P}, q_P\right)$ in $\mathbf{HoCW}_{k-1}$. ($\mathrm{emb}^0_k : t^0_{<k} \to t^0_{<\infty}$ is the natural transformation between the covariant assignment $t^0_{<k} : \mathbf{CW}^0_{k\supset\partial} \to \mathbf{HoCW}_{k-1}$ and the natural projection functor $t^0_{<\infty} : \mathbf{CW}^0_{k\supset\partial} \to \mathbf{HoCW}_{k-1}$, see Theorem 1.20.)
We fix a k-compression rigid subcategory $\mathbf{C} \subset \mathbf{CW}^0_{k\supset\partial}$, such that all objects $\left(P, Y_P, \overline{P}, q_P\right)$ in $\mathbf{C}$ satisfy $\pi_{k+1}(P) = 0$. Based on the choice of $\mathbf{C}$, we will define a category $\mathbf{P_C}(n, \overline{p})$ (see Definition 2.19) together with a forgetful functor

$$\mathcal{F}_{\mathbf{C}} : \mathbf{P_C}(n, \overline{p}) \to \mathbf{P}(n, \overline{p}).$$

Objects $(X, \Sigma, \Lambda_{\mathbf{C}})$ in $\mathbf{P_C}(n, \overline{p})$ come from suitable objects (X, Σ, Λ) in $\mathbf{P}(n, \overline{p})$ by completion of all links of X to objects in $\mathbf{C}$. Morphisms in $\mathbf{P_C}(n, \overline{p})$ are morphisms in $\mathbf{P}(n, \overline{p})$ which preserve these completions of the links. We will then use the covariant *functor* $t^0_{<k} : \mathbf{C} \to \mathbf{HoCW}_{k-1}$ ($\mathbf{C}$ is k-compression rigid!) to construct a functor

$$\mathrm{I}_{\mathbf{C}} : \mathbf{P_C}(n, \overline{p}) \to \mathbf{HoTop},$$

such that $\mathrm{I}(\mathcal{F}_{\mathbf{C}}(X, \Sigma, \Lambda_{\mathbf{C}})) = \mathrm{I}_{\mathbf{C}}(X, \Sigma, \Lambda_{\mathbf{C}})$ for all objects in $\mathbf{P_C}(n, \overline{p})$. By an analogous argument as in Section 2.3, it suffices to construct a covariant functor

$$\mathrm{T}_{\mathbf{C}} : \mathbf{C} \to \mathbf{HoCyl},$$

such that the following conditions are satisfied:

- For every object $\left(P, Y_P, \overline{P}, q_P\right)$ in $\mathbf{C}$, one has $\mathrm{T}_{\mathbf{C}}\left(P, Y_P, \overline{P}, q_P\right) = (\mathrm{cyl}(e_P), A_P, B_P)$.
- For every morphism $(f, \widetilde{f}) : \left(P, Y_P, \overline{P}, q_P\right) \to \left(Q, Y_Q, \overline{Q}, q_Q\right)$ in $\mathbf{C}$, the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\sigma_P \ \cong} & B_P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{T}_{\mathbf{C}}(f,\widetilde{f})|} \\ Q & \xrightarrow{\sigma_Q \ \cong} & B_Q. \end{array}$$

The first condition already defines $\mathrm{T}_{\mathbf{C}}$ on objects. The assumptions made for the subcategory $\mathbf{C} \subset \mathbf{CW}^0_{k\supset\partial}$ are used in the construction of $\mathrm{T}_{\mathbf{C}}$ on morphisms as follows. $\mathrm{T}_{\mathbf{C}}$ will be well-defined on morphisms, because $\pi_{k+1}(P) = 0$ for all objects $\left(P, Y_P, \overline{P}, q_P\right)$ in $\mathbf{C}$. To show that $\mathrm{T}_{\mathbf{C}}$ is a functor, we will finally use that $t^0_{<n}$ is a functor on $\mathbf{C}$.

Example. 2.18 The following are closed orientable aspherical topological manifolds, which have a CW-structure, such that all boundary maps vanish:

- the one-sphere $S^1 = e^0 \cup e^1$.
- the closed oriented surface of genus $g \geq 1$: $X_g = e^0 \cup e^1_{a_1} \cup e^1_{b_1} \cup \ldots \cup e^1_{a_g} \cup e^1_{b_g} \cup e^2$.

Finite products of these spaces are again closed orientable aspherical topological manifolds, which have a CW-structure, such that all boundary maps vanish. If we choose $\mathbf{C}$ as a subcategory of $\mathbf{CW}^0_{k\supset\partial}$, such that for all objects $\left(P, Y_P, \overline{P}, q_P\right)$ of $\mathbf{C}$ the CW-complex P is of this type, then $\mathbf{C}$ is a k-compression rigid category by Proposition 1.18. Moreover, for all objects $\left(P, Y_P, \overline{P}, q_P\right)$ in $\mathbf{C}$ we have $\pi_{k+1}(P) = 0$, since P is aspherical.

Definition. 2.19 The category $\mathbf{P}_{\mathbf{C}}(n, \overline{p})$ consists of the following objects and morphisms:

- Objects in $\mathbf{P}_{\mathbf{C}}(n, \overline{p})$ are triples $(X, \Sigma, \Lambda_{\mathbf{C}})$ with the following property: There exist an object (X, Σ, Λ) in $\mathbf{P}(n, \overline{p})$ and for every $L \in \lambda$ a completion $\left(L, Y_L, \overline{L}, q_L\right)$ to an object in $\mathbf{C}$, such that $\Lambda_{\mathbf{C}}$ is the set of these completions and f_L is the fixed representative $e_L : L_{<k} \to L$ of the class $\mathrm{emb}^0_k\left(L, Y_L, \overline{L}, q_L\right)$ for all $L \in \lambda$. In particular, (X, Σ, Λ) is uniquely determined by $(X, \Sigma, \Lambda_{\mathbf{C}})$ and will be denoted by $\mathcal{F}_{\mathbf{C}}(X, \Sigma, \Lambda_{\mathbf{C}})$. This yields an assignment on the object level:
$$\mathcal{F}_{\mathbf{C}} : \mathrm{Ob}\,\mathbf{P}_{\mathbf{C}}(n, \overline{p}) \to \mathrm{Ob}\,\mathbf{P}(n, \overline{p}).$$
- Morphisms $(X, \Sigma, \Lambda_{\mathbf{C}}) \to (X', \Sigma', \Lambda'_{\mathbf{C}})$ in $\mathbf{P}_{\mathbf{C}}(n, \overline{p})$ are pairs (F, Φ), where $F : \mathcal{F}_{\mathbf{C}}(X, \Sigma, \Lambda_{\mathbf{C}}) \to \mathcal{F}_{\mathbf{C}}(X', \Sigma', \Lambda'_{\mathbf{C}})$ is a morphism in $\mathbf{P}(n, \overline{p})$ and Φ is a set of completions
$$(f_L, \widetilde{f}_L) : \left(L, Y_L, \overline{L}, q_L\right) \longrightarrow (L'_L, Y_{L'_L}, \overline{L'_L}, q_{L'_L})$$
of the restrictions $f_L := F| : L \to L'_L$ to morphisms in $\mathbf{C}$. The composition with a second morphism $(F', \Phi') : (X', \Sigma', \Lambda'_{\mathbf{C}}) \to (X'', \Sigma'', \Lambda''_{\mathbf{C}})$ is given by $(F' \circ F, \Phi'')$, where Φ'' is the set of all compositions $(f'_{L'_L} \circ f_L, \widetilde{f}'_{L'_L} \circ \widetilde{f}_L)$ in $\mathbf{C}$. The projection to the first component $\mathcal{F}_{\mathbf{C}}(F, \Phi) = F$ yields a forgetful functor
$$\mathcal{F}_{\mathbf{C}} : \mathbf{P}_{\mathbf{C}}(n, \overline{p}) \to \mathbf{P}(n, \overline{p}).$$

Let $(f, \widetilde{f}) : \left(P, Y_P, \overline{P}, q_P\right) \to \left(Q, Y_Q, \overline{Q}, q_Q\right)$ be a morphism in $\mathbf{C}$. Choose a representative $f_{<k} : P_{<k} \to Q_{<k}$ of the rel $(k-1)$-skeleton homotopy class $t^0_{<k}(f, \widetilde{f})$. As $\mathrm{emb}^0_k : t^0_{<k} \to t^0_{<\infty}$ is a natural transformation, the following diagram commutes up to homotopy rel $(k-1)$-skeleton:

$$\begin{array}{ccc} P_{<k} & \xrightarrow{e_P} & P \\ \downarrow{\scriptstyle f_{<k}} & & \downarrow{\scriptstyle f} \\ Q_{<k} & \xrightarrow{e_Q} & Q. \end{array}$$

Since e_P and e_Q restrict to $\mathrm{id}_{P^{k-1}}$ and $\mathrm{id}_{Q^{k-1}}$ on $(k-1)$-skeletons, $f_{<k}$ agrees with the cellular map f^{k-1} on P^{k-1}. Hence, $f_{<k}$ is cellular. (The CW-complexes $P_{<k}$ and $Q_{<k}$ are k-dimensional.) Let $H : P_{<k} \times [1/2, 1] \to Q$ be a rel P^{k-1} homotopy between $H_{1/2} = e_Q \circ f_{<k}$ and $H_1 = f \circ e_P$. Let

$$F^{<} : (\mathrm{cyl}(e_P), A_P, B_P) \to (\mathrm{cyl}(e_Q), A_Q, B_Q)$$

be the map that is induced by the following morphism of 3-diagrams of spaces:

$$\begin{array}{ccccc}
P_{<k} \times [0,1/2] & \xleftarrow{\text{at } 1/2} & P_{<k} & \xrightarrow{\text{at } 1/2} & P \cup_{e_P} P_{<k} \times [1/2,1] \\
\downarrow{\scriptstyle f_{<k} \times (t \mapsto 2t)} & & \downarrow{\scriptstyle f_{<k}} & & \downarrow{\scriptstyle f \cup_{e_P} H} \\
Q_{<k} \times I & \xleftarrow{\text{at } 1} & Q_{<k} & \xrightarrow{e_Q} & Q
\end{array}$$

(Here the space $P \cup_{e_P} P_{<k} \times [1/2,1]$ is the realization of $P_{<k} \times [1/2,1] \xleftarrow{\text{at } 1} P_{<k} \xrightarrow{e_P} P$.) Now define the following morphism in **HoCyl**:

$$\mathrm{T}_{\mathbf{C}}(f, \widetilde{f}) = [F^{<}] \, .$$

2.20 Proposition. *The construction of $\mathrm{T}_{\boldsymbol{C}}(f, \widetilde{f})$ is independent of all choices (namely the choice of the representative $f_{<k}$ of $t^0_{<k}(f, \widetilde{f})$ and the choice of the homotopy H). Moreover, $\mathrm{T}_{\boldsymbol{C}} : \boldsymbol{C} \to$ **HoCyl** is a functor.*

Proof. Given a morphism $(f, \widetilde{f}) : \left(P, Y_P, \overline{P}, q_P\right) \to \left(Q, Y_Q, \overline{Q}, q_Q\right)$ in $\mathbf{C}$, we show first that $\mathrm{T}_{\mathbf{C}}(f, \widetilde{f})$ does not depend on the choices of $f_{<k}$ and H. Suppose that $f'_{<k}$ is a second representative of the rel $(k-1)$-skeleton homotopy class $t^0_{<k}(f, \widetilde{f})$ and that $H' : P_{<k} \times [1/2, 1] \to Q$ is a rel P^{k-1} homotopy between $H'_{1/2} = e_Q \circ f'_{<k}$ and $H'_1 = f \circ e_P = H_1$. Then the construction above yields a map $F^{<'} : (\mathrm{cyl}\,(e_P), A_P, B_P) \to (\mathrm{cyl}\,(e_Q), A_Q, B_Q)$ and one has to show that $[F^{<}] = \left[F^{<'}\right]$ in **HoCyl**.
As $[f_{<k}] = t^0_{<k}(f, \widetilde{f}) = [f'_{<k}]$ in $\mathbf{HoCW}_{k-1}$, there exists a rel $(k-1)$-skeleton homotopy $E : P_{<k} \times [0, 1/2] \to Q_{<k}$ between $E_0 = f'_{<k}$ and $E_{1/2} = f_{<k}$. The rel P^{k-1} homotopies $e_Q \circ E : P_{<k} \times [0, 1/2] \to Q$ and $H : P_{<k} \times [1/2, 1] \to Q$ satisfy $(e_Q \circ E)_{1/2} = e_Q \circ f_{<k} = H_{1/2}$. Therefore, one can fix a homotopy

$$L : (P_{<k} \times [1/2, 1]) \times I \longrightarrow Q$$

between $L_0 = H$ and the "concatenation" $L_1 = (e_Q \circ E) * H$. (L_1 is a rel P^{k-1} homotopy between $(L_1)_{1/2} = (e_Q \circ E)_0 = e_Q \circ f'_{<k}$ and $(L_1)_1 = H_1 = f \circ e_P$.) Explicitly, we set

$$L_s\,(x, t) = \begin{cases} (e_Q \circ E)_{2t - \frac{1}{2} - \frac{s}{2}}\,(x)\,, & \text{for} \quad 1/2 \le t \le s/4 + 1/2, \\ H_{\frac{2t-s}{2-s}}\,(x)\,, & \text{for} \quad s/4 + 1/2 \le t \le 1. \end{cases}$$

Let $M : \mathrm{cyl}\,(e_P) \times I \to \mathrm{cyl}\,(e_Q)$ be the homotopy which is given for every $s \in I$ by the map that is induced by the following morphism of 3-diagrams of spaces:

$$\begin{array}{ccccc}
P_{<k} \times [0,1/2] & \xleftarrow{\text{at } 1/2} & P_{<k} & \xrightarrow{\text{at } 1/2} & P \cup_{e_P} P_{<k} \times [1/2,1] \\
\downarrow{\scriptstyle E_{\frac{1-s}{2}} \times (t \mapsto 2t)} & & \downarrow{\scriptstyle E_{\frac{1-s}{2}}} & & \downarrow{\scriptstyle f \cup_{e_P} L_s} \\
Q_{<k} \times I & \xleftarrow{\text{at } 1} & Q_{<k} & \xrightarrow{e_Q} & Q
\end{array}$$

The map $f \cup_{e_P} L_s$ is well-defined, because $(L_s)_1 = H_1 = f \circ e_P$. The right square commutes, because $(L_s)_{1/2} = (e_Q \circ E)_{\frac{1-s}{2}} = e_Q \circ E_{\frac{1-s}{2}}$. The homotopy M is rel B_P and satisfies $M_s(A_P) \subset A_Q$ and $M_s(B_P) \subset B_Q$ for every $s \in I$. Therefore, $[F^<] = [M_0] = [M_1]$ in **HoCyl**. (Note that $M_0 = F^<$, because $L_0 = H$ and $E_{1/2} = f_{<k}$.)
Now, we apply [1, Lemma 1.43, page 53] to the maps $L_1, H' : P_{<k} \times [1/2, 1] \to Q$. The CW-complex $P_{<k} \times [1/2, 1]$ is of dimension $(k+1)$. The CW-complex Q is $(k+1)$-simple because of the condition $\pi_{k+1}(Q) = 0$. Moreover, L_1 and H' agree on the k-skeleton $(P_{<k} \times [1/2,1])^k = P^{k-1} \times [1/2,1] \cup P_{<k} \times \{1/2\} \cup P_{<k} \times \{1\}$, because $H'_{1/2} = e_Q \circ f'_{<k} = (L_1)_{1/2}$ and $H'_1 = H_1 = (L_1)_1$ and for every $t \in [1/2, 1]$ one has $H'_t|_{P^{k-1}} = H'_1|_{P^{k-1}} = (L_1)_1|_{P^{k-1}} = (L_1)_t|_{P^{k-1}}$ (the homotopies H' and L_1 are rel P^{k-1}). The obstruction cocycle

$$\omega(L_1, H') \in C^{k+2}\left((P_{<k} \times [1/2,1]) \times I; \pi_{k+1}(Q)\right)$$

vanishes, as $\pi_{k+1}(Q) = 0$. Consequently, there exists a rel $(P_{<k} \times [1/2,1])^k$ homotopy

$$N : (P_{<k} \times [1/2,1]) \times I \to Q, \quad N_0 = L_1, \quad N_1 = H'.$$

Let $M' : \operatorname{cyl}(e_P) \times I \to \operatorname{cyl}(e_Q)$ be the homotopy which is given for each $s \in I$ by the map that is induced by the following morphism of 3-diagrams of spaces:

$$\begin{array}{ccccc}
P_{<k} \times [0,1/2] & \xleftarrow{\text{at } 1/2} & P_{<k} & \xrightarrow{\text{at } 1/2} & P \cup_{e_P} P_{<k} \times [1/2,1] \\
\downarrow{\scriptstyle f'_{<k} \times (t \mapsto 2t)} & & \downarrow{\scriptstyle f'_{<k}} & & \downarrow{\scriptstyle f \cup_{e_P} N_s} \\
Q_{<k} \times I & \xleftarrow{\text{at } 1} & Q_{<k} & \xrightarrow{e_Q} & Q
\end{array}$$

The map $f \cup_{e_P} N_s$ is well-defined, because N is rel $P_{<k} \times \{1\} \subset (P_{<k} \times [1/2,1])^k$ and thus $N_s|_{P_{<k} \times \{1\}} = N_0|_{P_{<k} \times \{1\}} = (L_1)_1 = f \circ e_P$ for every $s \in I$. The right square commutes, because N is rel $P_{<k} \times \{1/2\} \subset (P_{<k} \times [1/2,1])^k$ and hence $N_s|_{P_{<k} \times \{1/2\}} = N_0|_{P_{<k} \times \{1/2\}} = (L_1)_{1/2} = e_Q \circ f'_{<k}$ for every $s \in I$. Note that $M'_0 = M_1$ and $M'_1 = F^{<'}$. Moreover, $M'_s(A_P) \subset A_Q$ and $M'_s(B_P) \subset B_Q$ for every $s \in I$ and M' is rel B_P. Therefore, $[F^<] = [M_1] = [M'_0] = [M'_1] = \left[F^{<'}\right]$ in **HoCyl**.
It remains to show that $\mathrm{T}_{\mathbf{C}}$ is a functor. For an identity morphism $\mathrm{id}_{(P, Y_P, \overline{P}, q_P)} = (\mathrm{id}_P, \mathrm{id}_{\widetilde{P}^k})$ in $\mathbf{C}$, one has $t^0_{<k}(\mathrm{id}_P, \mathrm{id}_{\widetilde{P}^k}) = \left[\mathrm{id}_{P_{<k}}\right]$ by definition of $\tau^0_{<k}$ on identity morphisms. Thus, one can choose $\mathrm{id}_{P_{<k}}$ as a representative of $t^0_{<k}(\mathrm{id}_P, \mathrm{id}_{\widetilde{P}^k})$ and $H = e_P \times \mathrm{id}_I$. The resulting map $F^< : \operatorname{cyl}(e_P) \to \operatorname{cyl}(e_P)$ is homotopic rel $A_P \cup B_P$ to $\mathrm{id}_{\operatorname{cyl}(e_P)}$. Thus, $\mathrm{T}_{\mathbf{C}}(\mathrm{id}_P, \mathrm{id}_{\widetilde{P}^k}) = [\mathrm{id}_{(\operatorname{cyl}(e_P), A_P, B_P)}]$ in **HoCyl**. Now suppose that $(f, \widetilde{f}) : (P, Y_P, \overline{P}, q_P) \to (Q, Y_Q, \overline{Q}, q_Q)$ and $(g, \widetilde{g}) : (Q, Y_Q, \overline{Q}, q_Q) \to (R, Y_R, \overline{R}, q_R)$ are morphisms in $\mathbf{C}$ with composition $(h, \widetilde{h}) = (g \circ f, \widetilde{g} \circ \widetilde{f}) : (P, Y_P, \overline{P}, q_P) \to (R, Y_R, \overline{R}, q_R)$. We choose representatives $f_{<k}$ and $g_{<k}$ of the homotopy classes $t^0_{<k}(f, \widetilde{f})$ and $t^0_{<k}(g, \widetilde{g})$ in $\mathbf{HoCW}_{k-1}$. By Theorem 1.20, the assignment $t^0_{<k} : \mathbf{CW}^0_{k \supset \partial} \to \mathbf{HoCW}_{k-1}$ restricts to a functor on the compression rigid subcategory $\mathbf{C}$. Therefore,

$$t^0_{<k}(h, \widetilde{h}) = t^0_{<k}(g \circ f, \widetilde{g} \circ \widetilde{f}) = t^0_{<k}(g, \widetilde{g}) \circ t^0_{<k}(f, \widetilde{f}) = [g_{<k}] \circ [f_{<k}] = [g_{<k} \circ f_{<k}]$$

in $\mathbf{HoCW}_{k-1}$. This shows that $g_{<k} \circ f_{<k}$ represents the rel $(k-1)$-skeleton homotopy class $t^0_{<k}(h, \widetilde{h})$. If we choose a rel P^{k-1} homotopy $e_Q \circ f_{<k} \simeq f \circ e_P$ and a rel Q^{k-1} homotopy $e_R \circ g_{<k} \simeq g \circ e_Q$, then the construction above yields maps $F^< : \mathrm{cyl}\,(e_P) \to \mathrm{cyl}\,(e_Q)$ and $G^< : \mathrm{cyl}\,(e_Q) \to \mathrm{cyl}\,(e_R)$. We define the composition

$$D \quad : \quad P_{<k} \times [1/4, 1] \quad \overset{\alpha}{\to} \quad \mathrm{cyl}\,(e_P) \quad \xrightarrow{G^< \circ F^<} \quad \mathrm{cyl}\,(e_R)\,.$$

The map α is given by the composition

$$P_{<k} \times [1/4, 1] \quad \hookrightarrow \quad P \bigsqcup P_{<k} \times I \quad \overset{\pi_P}{\longrightarrow} \quad P \cup_{e_P} P_{<k} \times I = \mathrm{cyl}\,(e_P)\,.$$

In fact, we have defined a map $D : P_{<k} \times [1/4, 1] \to R$ ($\subset \mathrm{cyl}\,(e_R)$), because

$$(G^< \circ F^<)(P \cup_{e_P} P_{<k} \times [1/4, 1]) \quad \subset \quad G^<(Q \cup_{e_Q} Q_{<k} \times [1/2, 1]) \quad \subset \quad R \quad (\subset \mathrm{cyl}\,(e_R)).$$

We observe that $D : P_{<k} \times [1/4, 1] \to R$ is a homotopy rel P^{k-1} between $D_{1/4} = e_R \circ (g_{<k} \circ f_{<k})$ and $D_1 = g \circ f \circ e_P = h \circ e_P$. (To see that D is rel P^{k-1}, we use that e_P restricts to $\mathrm{id}_{P^{k-1}}$ on $(k-1)$-skeletons. Thus, we have $P^{k-1} \times I \subset \mathrm{cyl}(e_P)$ (and analogous for Q). Since $(f_{<k})^{k-1} = f^{k-1}$, the map $F^< : \mathrm{cyl}\,(e_P) \to \mathrm{cyl}\,(e_Q)$ restricts to

$$P^{k-1} \times I \to Q^{k-1} \times I, \quad (x, t) \mapsto \begin{cases} (f(x), 2t), & \text{for } 0 \le t \le 1/2, \\ (f(x), 1), & \text{for } 1/2 < t \le 1 \end{cases}$$

(and analogous for $G^<$). Thus, $D(x, t) = (g \circ f)(x)$ for $(x, t) \in P^{k-1} \times [1/4, 1]$.)
Let $K : \mathrm{cyl}\,(e_P) \times I \to \mathrm{cyl}\,(e_Q)$ be the homotopy which is given for each $s \in I$ by the map that is induced by the following morphism of 3-diagrams of spaces:

$$\begin{array}{ccccc} P_{<k} \times [0, \frac{s+1}{4}] & \xleftarrow{\text{at } \frac{s+1}{4}} & P_{<k} & \xrightarrow{\text{at } \frac{s+1}{4}} & P \cup_{e_P} P_{<k} \times [\frac{s+1}{4}, 1] \\ \Big\downarrow {\scriptstyle (g_{<k} \circ f_{<k}) \times (t \mapsto \frac{4}{s+1} t)} & & \Big\downarrow {\scriptstyle g_{<k} \circ f_{<k}} & & \Big\downarrow {\scriptstyle h \cup_{e_P} (D \circ \beta_s)} \\ R_{<k} \times I & \xleftarrow{\text{at } 1} & R_{<k} & \xrightarrow{e_R} & R. \end{array}$$

Here, the map $\beta_s : P_{<k} \times [\frac{s+1}{4}, 1] \to P_{<k} \times [1/4, 1]$ is defined by $\beta_s(x, t) = (x, \frac{3t-s}{3-s})$. The map $h \cup_{e_P} (D \circ \beta_s)$ is well-defined, because $(D \circ \beta_s)_1 = D_1 = h \circ e_P$. The right square commutes, because $D_{1/4} = e_R \circ (g_{<k} \circ f_{<k})$. We have constructed a rel $A_P \cup B_P$ homotopy $K : \mathrm{cyl}\,(e_P) \times I \to \mathrm{cyl}\,(e_Q)$ between $K_0 = G^< \circ F^<$ and $H^< := K_1$. Note that $\mathrm{T}_{\mathbf{C}}(h, \widetilde{h}) = [H^<]$ by construction of $\mathrm{T}_{\mathbf{C}}$. Therefore, in **HoCyl**, we get

$$\mathrm{T}_{\mathbf{C}}(h, \widetilde{h}) = [H^<] \overset{K}{=} [G^< \circ F^<] = [G^<] \circ [F^<] = \mathrm{T}_{\mathbf{C}}(g, \widetilde{g}) \circ \mathrm{T}_{\mathbf{C}}(f, \widetilde{f}).$$

■

Remark. 2.21 The part of the previous proof that shows the independence of $\mathrm{T}_{\mathbf{C}}$ of all choices does not make use of the assumption that $\mathbf{C}$ is a compression-rigid category. Therefore, the only condition that is needed to show that $\mathrm{I}_{\mathbf{C}}$ is well-defined on morphisms is the condition that π_{k+1} of the links vanishes. In this case, one still gets covariant assignments $\mathrm{T}_{\mathbf{C}}$ and $\mathrm{I}_{\mathbf{C}}$.

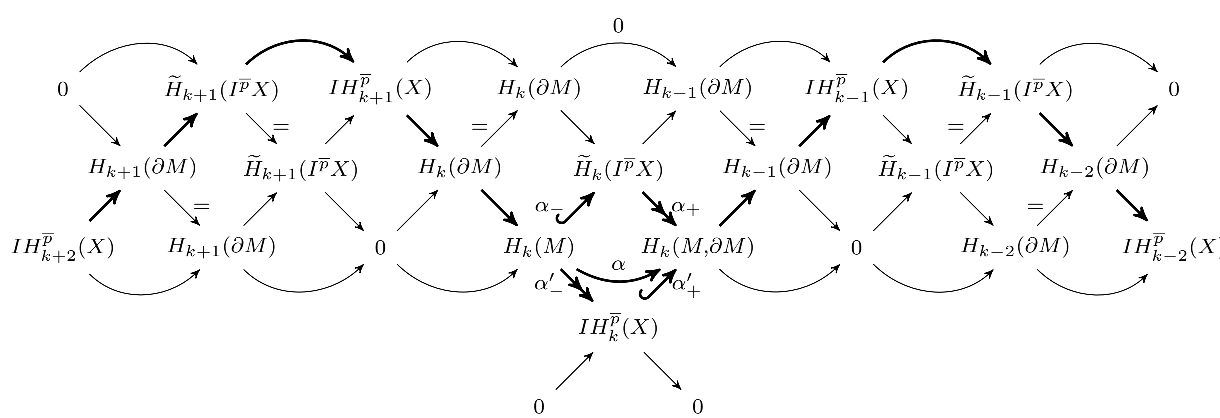
2.6 Induced Morphisms between Reflective Diagrams

Let $F : (X, \Sigma, \Lambda) \to (X', \Sigma', \Lambda')$ be a morphism in $\mathbf{P}(n, \overline{p})$. On the one hand, we have seen in the previous sections that if F lies in a suitable subcategory $\mathbf{P}_*(n, \overline{p})$ of $\mathbf{P}(n, \overline{p})$, then we can induce a well-defined homotopy class of continuous maps $\mathrm{I}_*(F) : I^{\overline{p}}X \to I^{\overline{p}}X'$ in **HoTop**. This class will furthermore induce homomorphisms $\widetilde{H}_*(I^{\overline{p}}X) \to \widetilde{H}_*(I^{\overline{p}}X')$ between reduced integral homology groups. On the other hand, the intersection homology of the n-dimensional compact topological pseudomanifold X with only isolated singularities is given by (see Remark 2.23)

$$IH_r^{\overline{p}}(X) \quad = \quad \begin{cases} H_r(M), & r < k, \\ \mathrm{im}(\alpha), & r = k, \\ H_r(M, \partial M), & r > k, \end{cases}$$

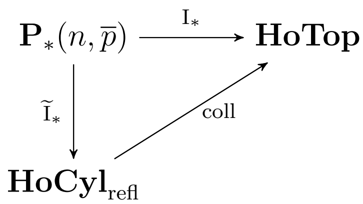
where $\alpha : H_k(M) \to H_k(M, \partial M)$ is induced by the inclusion $M \hookrightarrow (M, \partial M)$. Since $F : X \to X'$ is a continuous map which satisfies $F(M) \subset M'$ and $F(\partial M) \subset \partial M'$, F induces homomorphisms $IH_r^{\overline{p}}(X) \to IH_r^{\overline{p}}(X')$ for all r. (If $r = k$, then we can restrict the induced homomorphism $H_k(M, \partial M) \to H_k(M', \partial M')$ to a homomorphism $\mathrm{im}(\alpha) \to \mathrm{im}(\alpha')$.) What can be said about the relation between the induced homomorphisms on reduced homology groups of intersection spaces and on intersection homology groups?

Following the proof of [1, Theorem 2.12, page 114ff], we assign to every object (X, Σ, Λ) in $\mathbf{P}(n, \overline{p})$ a k-reflective diagram (see [1, Definition 2.1, page 107f]) written as a braid



The thick arrows indicate the k-reflective diagram in its original form. The intersection homology groups $IH_*^{\overline{p}}(X)$ are calculated as above. We will show that (under a certain factorization condition for $\mathrm{I}_* : \mathbf{P}_*(n, \overline{p}) \to$ **HoTop**) morphisms in $\mathbf{P}_*(n, \overline{p})$ induce morphisms (see [1, Definition 2.2, page 109]) between the associated k-reflective diagrams. The latter morphisms will then assemble the induced homomorphisms of our interest (see Proposition 2.22). In the proof, we will start with an induced morphism between braid diagrams. Among the induced homomorphisms between corresponding homology groups we will analyze those which are required to obtain a morphism between k-reflective diagrams.

Some of the functors I_* of Chapter 2 factorize over $\mathbf{HoCyl}_{\mathrm{refl}}$ (see Definition 2.11):



(∗).

Here, $\widetilde{\mathrm{I}}_* : \mathbf{P}_*(n,\overline{p}) \to \mathbf{HoCyl}_{\mathrm{refl}}$ is a covariant functor, such that

- $\widetilde{\mathrm{I}}_*(X,\Sigma,\Lambda) = (\mathrm{cyl}(j\circ f), \mathrm{cyl}(f), (\partial M)_{<k}, M)$ for all objects (X,Σ,Λ) in $\mathbf{P}_*(n,\overline{p})$.
- $\widetilde{\mathrm{I}}_*(F)|_M = F|_M$ for all morphisms $F : (X,\Sigma,\Lambda) \to (X',\Sigma',\Lambda')$ in $\mathbf{P}_*(n,\overline{p})$.

Moreover, coll : $\mathbf{HoCyl}_{\mathrm{refl}} \to \mathbf{HoTop}$ is the covariant functor defined in Section 2.3:

- $\mathrm{coll}(X,A,B,C) = X/B$ for all objects (X,A,B,C) in $\mathbf{HoCyl}_{\mathrm{refl}}$.
- $\mathrm{coll}([\varphi]) = [X/B \xrightarrow{\overline{\varphi}} X'/B']$ for all morphisms $[\varphi] : (X,A,B,C) \to (X',A',B',C')$.

The factorization (∗) applies to the functors $\mathrm{I}_\bullet$ and I_1 of Section 2.1, to the functor $\mathrm{I}_{\mathrm{Cyl}}$ of Section 2.3 and to the functors $\mathrm{I}_{\mathbf{C}}$ of Section 2.5 (use $\mathcal{F}_{\mathbf{C}}$ in the notation).

2.22 Proposition. *Let $\mathrm{I}_* : \boldsymbol{P}_*(n,\overline{p}) \to \boldsymbol{HoTop}$ be a functor, which factorizes as in (∗). Then every morphism $F : (X,\Sigma,\Lambda) \to (X',\Sigma',\Lambda')$ in $\mathbf{P}_*(n,\overline{p})$ induces a morphism from the k-reflective diagram associated to (X,Σ,Λ) to the k-reflective diagram associated to (X',Σ',Λ'), such that the homomorphisms*

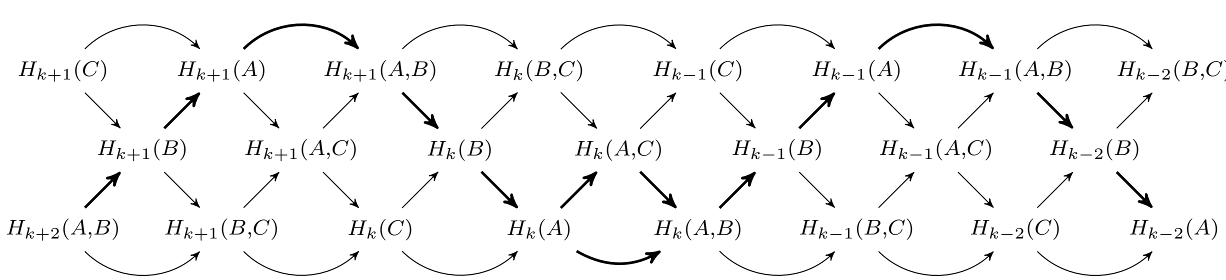
- *$\widetilde{H}_*(I^{\overline{p}}X) \to \widetilde{H}_*(I^{\overline{p}}X')$ are induced by $\mathrm{I}_*(F) : I^{\overline{p}}X \to I^{\overline{p}}X'$.*
- *$IH^{\overline{p}}_*(X) \to IH^{\overline{p}}_*(X')$ are induced by $F : X \to X'$ (as explained above).*
- *$H_*(\partial M) \to H_*(\partial M')$, $\quad H_k(M) \to H_k(M')$, $\quad H_k(M,\partial M) \to H_k(M',\partial M')$ are induced by the following restrictions of F: $\partial M \to \partial M'$, $\quad M \to M'$, $\quad (M,\partial M) \to (M',\partial M')$.*

This assignment is obviously functorial.

Proof. Let $F : (X,\Sigma,\Lambda) \to (X',\Sigma',\Lambda')$ be a morphism in $\mathbf{P}_*(n,\overline{p})$. We have

$$\widetilde{\mathrm{I}}_*(X,\Sigma,\Lambda) = (\mathrm{cyl}(j\circ f), \mathrm{cyl}(f), (\partial M)_{<k}, M),$$
$$\widetilde{\mathrm{I}}_*(X',\Sigma',\Lambda') = (\mathrm{cyl}(j'\circ f'), \mathrm{cyl}(f'), (\partial M')_{<k}, M').$$

The braid diagram of a triple (A,B,C) of spaces with $C \subset B \subset A$ is given by



(The thick arrows already indicate the construction of the above k-reflective diagram.) Following the first half of the proof of [1, Theorem 2.12, page 114], the k-reflective diagrams associated to (X, Σ, Λ) and (X', Σ', Λ') are constructed from the braids of

$$(A, B, C) := (\mathrm{cyl}(j \circ f), \mathrm{cyl}(f), (\partial M)_{<k}),$$
$$(A', B', C') := (\mathrm{cyl}(j' \circ f'), \mathrm{cyl}(f'), (\partial M')_{<k}).$$

In fact, these braids agree with the braids which are considered in the original proof:

- $H_r(f) = H_r(B, C)$ and $H_r(j \circ f) = H_r(A, C)$ by definition.
- $H_r((\partial M)_{<k}) = H_r(C)$ remains unchanged.
- The following identifications are induced by inclusions:

  $$H_r(M) = H_r(A), \quad H_r(\partial M) = H_r(B), \quad \text{and} \quad (H_r(j) =)\, H_r(M, \partial M) = H_r(A, B).$$

  (These inclusions are homotopy equivalences, whose homotopy inverses are the obvious projections.)

We choose a representative of the homotopy class $\widetilde{\mathrm{I}}_*(F)$ in $\mathbf{HoCyl}_{\mathrm{refl}}$:

$$\widetilde{F} : \quad (A, B, C, M) \quad \longrightarrow \quad (A', B', C', M).$$

The map $\widetilde{F} : (A, B, C) \longrightarrow (A', B', C')$ of triples induces a well-defined morphism between braid diagrams. (This is a consequence of naturality of long exact homology sequences for pairs and triples). We investigate the induced homomorphisms between objects of the thick subdiagrams:

- $H_r(B) \to H_r(B')$ and $H_r(A) \to H_r(A')$:
  By assumption, we have the following commutative diagrams:

  $$\begin{CD} \partial M @>{\widetilde{F}| = F|}>> \partial M' \\ @V{\mathrm{incl}}VV @VV{\mathrm{incl}}V \\ B @>{\widetilde{F}|}>> B' \end{CD} \qquad \text{and} \qquad \begin{CD} M @>{\widetilde{F}| = F|}>> M' \\ @V{\mathrm{incl}}VV @VV{\mathrm{incl}}V \\ A @>{\widetilde{F}}>> A'. \end{CD}$$

  All vertical inclusions are homotopy equivalences. Thus, they induce isomorphisms on homology groups. Under their inverses, the induced maps

  $$\widetilde{F}|_* : H_r(B) \to H_r(B') \qquad \text{and} \qquad \widetilde{F}_* : H_r(A) \to H_r(A')$$

  correspond to

  $$F|_* : H_r(\partial M) \to H_r(\partial M') \qquad \text{and} \qquad F|_* : H_r(M) \to H_r(M').$$

- $H_r(A, B) \to H_r(A', B')$:
  By assumption, we have the following commutative diagram:

  $$\begin{CD} (M, \partial M) @>{\widetilde{F}| = F|}>> (M', \partial M') \\ @V{\mathrm{incl}}VV @VV{\mathrm{incl}}V \\ (A, B) @>{\widetilde{F}|}>> (A', B'). \end{CD}$$

By the 5-lemma and the previous item, the vertical inclusions of pairs induce isomorphisms on homology groups. Under their inverses, the induced map

$$\widetilde{F}|_* : H_r(A, B) \to H_r(A', B')$$

corresponds to

$$F|_* : H_r(M, \partial M) \to H_r(M', \partial M').$$

- $H_r(A, C) \to H_r(A', C')$: (Take note of the identification $H_r(A, C) = H_r(A)$ for $r > k$ and $H_r(A, C) = H_r(A, B)$ for $r < k$!)
  We have the following commutative diagram in **HoTop**:

$$\begin{array}{ccc} (A, C) & \xrightarrow{[\widetilde{F}]} & (A', C') \\ \Big\downarrow{\scriptstyle [\mathrm{proj}]} & & \Big\downarrow{\scriptstyle [\mathrm{proj}]} \\ (\mathrm{I}(X, \Sigma, \Lambda), *) & \xrightarrow{\mathrm{coll}([\widetilde{F}])} & (\mathrm{I}(X', \Sigma', \Lambda'), *'). \end{array}$$

  The vertical quotient maps induce isomorphisms on homology groups (see [2, Proposition 2.22, page 124]). Under these isomorphisms, the induced map

$$\widetilde{F}_* : H_r(A, C) \to H_r(A', C')$$

  corresponds to

$$\mathrm{coll}([\widetilde{F}])_* = \mathrm{coll}(\widetilde{\mathrm{I}}_*(F))_* \overset{(*)}{=} (\mathrm{I}_*(F))_* : \widetilde{H}_r(I^{\overline{p}} X) \to \widetilde{H}_r(I^{\overline{p}} X').$$

- Finally, in dimension k, the commutative diagram

$$\begin{array}{ccc} H_k(M) & \xrightarrow{\alpha} & H_k(M, \partial M) \\ \Big\downarrow{\scriptstyle F|_*} & & \Big\downarrow{\scriptstyle F|_*} \\ H_k(M') & \xrightarrow{\alpha'} & H_k(M', \partial M') \end{array}$$

  factorizes as

$$\begin{array}{ccccc} H_k(M) & \xrightarrow{\alpha|} & \mathrm{im}(\alpha) & \xrightarrow{\mathrm{incl}} & H_k(M, \partial M) \\ \Big\downarrow{\scriptstyle F|_*} & & \Big\downarrow{\scriptstyle (F|_*)|} & & \Big\downarrow{\scriptstyle F|_*} \\ H_k(M') & \xrightarrow{\alpha'|} & \mathrm{im}(\alpha') & \xrightarrow{\mathrm{incl}} & H_k(M', \partial M'), \end{array}$$

  where $\mathrm{im}(\alpha) = IH_k^{\overline{p}}(X)$ and $\mathrm{im}(\alpha') = IH_k^{\overline{p}}(X')$.

■

Remark. 2.23 *(intersection homology and placid maps)* Let (X, Σ, Λ) be an object in $\mathbf{P}(n, \overline{p})$. We assume that X is equipped with the stratification

$$X = X_n \supset X_{n-1} = ... = X_0 = \sigma \supset X_{-1} = \emptyset,$$

where $\sigma \subset X$ denotes the (finite) set of singular points of X. (Thus, the strata of X are given by $X - \sigma$ and the points of σ.) In [8, Proposition 4.4.1, page 55], it is deduced from the definition of intersection homology that

$$IH_r^{\overline{p}}(X) = \begin{cases} H_r(X-\sigma), & r < k, \\ \operatorname{im}(\beta), & r = k, \\ H_r(X), & r > k, \end{cases}$$

where $\beta : H_k(X-\sigma) \to H_k(X)$ is induced by the inclusion $X - \sigma \hookrightarrow X$.
In [8, Definition 4.8.1, page 61], a continuous map $F : X \to X'$ between topologically stratified spaces is called *placid*, if for every stratum T of X' the preimage $F^{-1}(T)$ is a union of strata of X, and $\operatorname{codim} F^{-1}(T) \geq \operatorname{codim} T$.
If (X, Σ, Λ) and (X', Σ', Λ') are objects in $\mathbf{P}(n, \overline{p})$, then a continuous map $F : X \to X'$ is placid if and only if $F^{-1}(\sigma') \subset \sigma$. (If F is placid, then we have $\operatorname{codim} F^{-1}(x') \geq \operatorname{codim} x' = n$ for all $x' \in \sigma'$. Thus, $F^{-1}(x')$ must be a union of strata of X with codimension $\geq n$. These are points in σ. Conversely, assume that $F^{-1}(\sigma') \subset \sigma$. Then, the claim is clear for all $x' \in \sigma'$. Since $\operatorname{codim}(X' - \sigma') = 0$, it remains to show that $F^{-1}(X' - \sigma')$ is a union of strata of X. This follows from $X - \sigma \subset X - F^{-1}(\sigma') = F^{-1}(X' - \sigma')$.) The condition $F^{-1}(\sigma') \subset \sigma$ is equivalent to $F(X - \sigma) \subset X' - \sigma'$.
Now, let us assume that $F : X \to X'$ is placid, i.e. $F(X - \sigma) \subset X' - \sigma'$. Then, F induces homomorphisms $IH_r^{\overline{p}}(X) \to IH_r^{\overline{p}}(X')$ for all r. (If $r = k$, then we can restrict the induced homomorphism $H_k(X) \to H_k(X')$ to a homomorphism $\operatorname{im}(\beta) \to \operatorname{im}(\beta')$.)
How is this related to the intersection homology groups which were used in the considerations above? Using excision and homotopy invariance, one can show that we have in fact used the same intersection homology groups in Proposition 2.22 (see [8, Remark 4.4.2, page 56] and compare to the diagram below). Now, we will show that all identifications are compatible with the homomorphisms induced by F between intersection homology groups, if we assume that $F(C) \subset C'$, where $C := \bigsqcup_{L \in \lambda} \operatorname{cone}(L) \subset X$.
Let $F : (X, \Sigma, \Lambda) \to (X', \Sigma', \Lambda')$ be a morphism in $\mathbf{P}(n, \overline{p})$, such that F is placid and satisfies $F(C) \subset C'$. Thus, the continuous map $F : X \to X'$ satisfies:

$$F(M) \subset M', \quad F(\partial M) \subset \partial M', \quad F(C) \subset C', \quad F(X - \sigma) \subset X' - \sigma'.$$

All claimed compatibilities will result from the following commutative diagram:

$$\begin{array}{ccccccccc}
H_r(M) & \xrightarrow{\cong\ \varphi_1} & H_r(X-\sigma) & \xrightarrow{\varphi_2} & H_r(X) & \xrightarrow{\cong\ \varphi_3} & H_r(X,C) & \xleftarrow{\cong\ \varphi_4} & H_r(M,\partial M) \\
\downarrow{\scriptstyle F|_*} & & \downarrow{\scriptstyle F|_*} & & \downarrow{\scriptstyle F_*} & & \downarrow{\scriptstyle F_*} & & \downarrow{\scriptstyle F|_*} \\
H_r(M') & \xrightarrow{\cong\ \varphi_1'} & H_r(X'-\sigma') & \xrightarrow{\varphi_2'} & H_r(X') & \xrightarrow{\cong\ \varphi_3'} & H_r(X',C') & \xleftarrow{\cong\ \varphi_4'} & H_r(M',\partial M').
\end{array}$$

All horizontal maps are induced by inclusions. φ_1 is an isomorphism by homotopy invariance. φ_3 is an isomorphism for $r > 0$. (The homology groups of C in the long exact homology sequence of the pair (X, C) vanish in positive degrees.) φ_4 is an isomorphism by excision and homotopy invariance. (Excise open cone neighbourhoods of the singular points.) If $r = k$, then $\varphi_2 = \beta$ and the composition of the first line is α.

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